Chapter – 2

NUMBERS – II

Last digit of any power

The last digits of the powers of any number follow a cyclic pattern - i.e., they repeat after certain number of steps. If we find out after how many steps the last digit of the powers of a number repeat, then we can find out the last digit of any power of any number.
Let us look at the powers of 2.
Last digit of 2^1 is 2
Last digit of 2^2 is 4
Last digit of 2^3 is 8
Last digit of 2^4 is 6
Last digit of 2^5 is 2

Since last digit of 2^5 is the same as the last digit of 2^1, then onwards the last digit will start repeating, i.e. digits of 2^5, 2^6, 2^7, 2^8 will be the same as those of 2^1, 2^2, 2^3, 2^4. Then the last digit of 2^9 is again the same as the last digit of 2^1 and so on. So, we have been able to identify that for powers of 2 the last digits repeat after every 4 steps. In other words whenever the power is a multiple of 4, the last digit of that number will be the same as the last digit of 2^4.

Suppose we want to find out the last digit of 2^{67}, we should look at a multiple of 4 which is less than or equal to the power 67. Since 64 is a multiple of 4, the last digit of 2^{64} will be the same as the last digit of 2^4.

Then the last digits of 2^{65}, 2^{66}, 2^{67} will be the same as the last digits of 2^1, 2^2, 2^3 respectively. Hence the last digit of 2^{67} is the same as the last digit of 2^3 i.e. 8.

Similarly, we can find out the last digit of 3^{74} by writing down the pattern of the powers of 3.
Last digit of 3^1 is 3
Last digit of 3^2 is 9
Last digit of 3^3 is 7
Last digit of 3^4 is 1
Last digit of 3^5 is 3

The last digit repeats after 4 steps (like in the case of powers of 2)

To find the last digit of 3^{74}, we look for a multiple of 4 which is less than or equal to 74. Since 72 is multiple of 4, the last digit of 3^{72} will be the same as that of 3^4. Hence the last digit of 3^{74} will be the same as the last digit of 3^2, i.e. 9.

Last digit of a sum or product

The problem consists of finding the last digit of the sum of two numbers each of which is a power of some integer. For example, you may be asked to find out the last digit of the sum $2^{67} + 3^{74}$

In general, when we want to find out the last digit of the sum of two numbers, we can just take the last digit of the two numbers and add them up. That will be the last digit of the sum. The last digit of 243 + 456 will be the same as the sum of the last digits of the two numbers, i.e., the sum of 3 and 6, which is 9. Similarly, in the case of $2^{67} + 3^{74}$ also, the last digit will be equal to the sum of the last digits of the two terms 2^{67} and 3^{74}.

We have already looked at finding out the last digit of powers like 2^{67} and 3^{74}. Hence the last digit of $2^{67} + 3^{74}$ is 8 + 9 i.e. 7.

Similarly, the last digit of a product will be equal to the last digit of the product of the last digits of the two given numbers.

For example, the last digit of the product $2^{67} \times 3^{74}$ will be equal to the last digit of the product of the last digit of 2^{67} and the last digit of 3^{74}, i.e. the last digit of 8 x 9, i.e., 2. Hence the last digit of $2^{67} \times 3^{74}$ is 2.

2.01. Find the units digit of $14^{124} \times 29^{123}$.

Sol: Units digit of $(14^{124} \times 29^{123})$
= Units digit of $(4^{124} \times 9^{123})$
The units digit of any power of 4 is 4 if the exponent is odd and 6 if the exponent is even. The units digit of any exponent of 9 is 9 if the exponent is odd and 1 if the exponent is even.
$\therefore$ The required units digit is 4.

Finding the remainder in divisions involving powers of numbers

It is explained below with the help of an example.

2.02. Find the remainder of 3^{43} when divided by 4.

Sol: Let us find the pattern that the remainders follow when the successive powers of 3 are divided by 4.
Remainder of 3^1 when divided by 4 = 3
Remainder of 3^2 when divided by 4 = 1
Remainder of 3^3 when divided by 4 = 3
$\therefore$ The remainder repeats after 2 steps and it is 3 when the exponent of 3 is odd and it is 1 when the exponent of 3 is even.
$\therefore$ Required remainder = 3
(since the power of 3 is odd)

Pattern method

Similar to the last digit of the powers of a number repeating in a certain pattern, the remainders of powers of a number also follow a certain pattern. If we identify the pattern in which the remainders repeat, we can find out the remainder of any division given
To solve the example given above, let us find the pattern that remainders follow when various powers of 2 are divided by 7.

Remainder when 2^1 is divided by 7 is 2
Remainder when 2^2 is divided by 7 is 4
Remainder when 2^3 is divided by 7 is 1
Remainder when 2^4 is divided by 7 is 2
We find that the remainder repeats in the fourth step, i.e., <u>after 3 steps</u>. So,
- the remainder of 2^4 when divided by 7 is the same as that when 2^1 is divided by 7, i.e., 2
- the remainder of 2^5 when divided by 7 is the same as that when 2^2 is divided by 7, i.e., 4

- the remainder of 2^6 when divided by 7 is the same as that when 2^3 is divided by 7, i.e., 1
- the remainder of 2^7 when divided by 7 is the same as that when 2^1 is divided by 7, i.e., 2
  and so on.

If we take 2^{54}, since 54 is divisible by 3, 2^{54} itself completes a cycle of 3 steps and hence the remainder when 2^{54} is divided by will be the same as that when 2^3 is divided by 7. Hence the remainder is 1.

Remainder Theorem

We can apply Remainder Theorem to find the remainder in problems like the one discussed above. Let us first look at Remainder Theorem and understand it.

Remainder Theorem states that when f(x), a polynomial function in x is divided by x – a, the remainder is f(a).

A polynomial function in x is a function where x will appear only in the form of x^n and not in any other form, where n is a positive integer.

Let us take an example to understand Remainder Theorem.

When the function $x^2 + 2x - 3$ is divided by x – 1, the remainder will be f(1). This is because, as per Remainder Theorem, when the divisor is (x – a), the remainder is f(a). Here the divisor is x-1 and hence the remainder is f(1). To get f(1), we should substitute x = 1 in the given equation. As we get f(1) = 0, the remainder in this case is 0. {Note that when f(x) is divided by x – a, if the remainder is 0, then x – a will be a factor of f(x). So, in this case, (x – 1) is a factor of $x^2 + 2x - 3$}.

When the function $x^2 + 2x + 3$ is divided by x + 1, the reminder will be f (–1) which is $(-1)^2 + 2(-1) + 3$, i.e., 2.

Now let us take the example of finding the remainder when 2^{54} is divided by 7 (which was solved by the Pattern Method above) and solve it by Remainder Theorem Method.
In the division $2^{54}/7$, the dividend is 2^{54} and the divisor is 7. Since the numerator is in terms of powers of 2, express the denominator also in terms of powers of 2. In this case, 7 can be written as 8 – 1 which is $2^3 - 1$. So, now the denominator is in terms of 2^3, the numerator, i.e., the dividend should be rewritten in terms of 2^3 which will be $(2^3)^{18}$. Now, the given problem reduces to finding out the remainder when $(2^3)^{18}$ is divided by $2^3 - 1$. Here, if we consider 2^3 as x, it is equivalent to finding out the remainder when x^{18} is divided by (x – 1) which, as per Remainder Theorem, is f(1), i.e., the remainder is obtained by substituting 1 in place of x. So, the remainder will be $(1)^{18}$, i.e., 1.

2.03. Find the remainder when 2^{99} is divided by 9.

Sol: $$\frac{2^{99}}{9} = \frac{\left(2^3\right)^{33}}{2^3 - (-1)}$$

By remainder theorem, remainder is $(-1)^{33} = -1$
$\Rightarrow -1 + 9 = 8$
Note: When a negative remainder is obtained, add the divisor to get the equivalent positive remainder.

2.04. Find the remainder when 2^{70} is divided by 7.

Sol: $$\frac{2^{70}}{7} = \frac{2\left(2^3\right)^{23}}{2^3 - 1}$$

By remainder theorem, remainder is $2\,(1)^{23} = 2$

2.05. Find the remainder when 2^{97} is divided by 15.

Sol: Let us find the pattern that the remainders follow when successive powers of 2 are divided by 15.
Remainder when 2^1 is divided by 15 = 2
Remainder when 2^2 is divided by 15 = 4
Remainder when 2^3 is divided by 15 = 8
Remainder when 2^4 is divided by 15 = 1
Remainder when 2^5 is divided by 15 = 2
$\therefore$ The remainder repeats after 4 steps.
$\therefore$ Required remainder

= Remainder of $\frac{2^1}{15} = 2$ (since 97 is 4 (24) + 1)

2.06. Find the remainder when 2^{201} is divided by 5.

Sol: From the above example, the units digit of powers of 2 repeats after 4 steps.
$\therefore$ Required remainder = Remainder of $\frac{2^1}{5} = 2$

Remainder theorem method

$$\frac{2^{201}}{5} = \frac{2\left(2^2\right)^{100}}{2^2 - (-1)}$$

By remainder theorem, required remainder
= $2\,(-1)^{100} = 2$

2.07. Find the remainder when 3^{101} is divided by 10.

Sol: Let us find the pattern that remainders follow when the successive powers of 3 are divided by 10.
Remainder when 3^1 is divided by 10 is 3
Remainder when 3^2 is divided by 10 is 9
Remainder when 3^3 is divided by 10 is 7
Remainder when 3^4 is divided by 10 is 1
Remainder when 3^5 is divided by 10 is 3
$\therefore$ The remainder repeats after 4 steps.

$\therefore$ Required remainder = Remainder of $\frac{3^1}{10} = 3$

As is evident from the above examples, the remainder theorem is more suited to cases where the denominator (i.e., the divisor) can be written in the form of one more or one less than some power of the base in the numerator. For example, in case of $2^{54}/7$, since the base in the numerator is 2, the denominator 7 has to be written as one more or one less than some power of 2. In this case it can be written as $2^3 - 1$. In cases where it is not possible to write it in this manner, then applying the Pattern Method is the easiest method.

Last two digits of a^m

The terms of any Geometric progression (GP) leave a cyclic pattern of remainders when divided by any divisor. The sequence of powers of the base ‘a’ is a GP with common ratio equal to ‘a’.

If we take the divisor as 100, the remainder is simply the last two digits. We’ll find it convenient to consider the following 4 cases separately.

(1) The base ends in 0
(2) The base ends in 5
(3) The base ends in 1, 3, 7 or 9
(4) The base ends in 2, 4, 6 or 8

The first two cases are very simple.

(1) If a ends in 0, the square and all higher powers end in at least 2 zeroes.
(2) If a ends in 5, the powers either all end in 25 or end alternately in 25 and 75, depending on whether the tens digit of a is even or odd.
(3) If the base ends in 1, 3, 7 or 9, there is a cycle of at the most 20 distinct remainders. The twentieth power ends in 01. (The cycle length could also be some factor of 20 ie.1, 2, 4, 5 or 10)
(4) If the base ends in 2, 4, 6 or 8, there is a cycle of at the most 20 distinct remainders. The twentieth power ends in 76. The cycle length could also be some factor of 20.

(4.1) Moreover, if a = 4k, the second set and all the subsequent sets of 20 remainders are exactly the same as the first set.

(4.2) But if a = 4k + 2, it is not possible to get 4k + 2 as the last two digits in any higher power. All such powers are multiples of 4. Consequently, of the forty ‘two-digit’ numbers (02, 04, 06, 08, 12, 14, 16, 18 ----- 92, 94, 96, 98) only twenty, viz 04, 08, 12, 16, 24, …., 92, 96 can occur as the last two digits in the higher powers. If 02, 06, 14 etc do occur, they can occur only as the first power. We find that the last two digits of a^{21} are obtained by adding 50 to 4k + 2 (For example, 2^{21} ends in 52, 6^{21} ends in 56, 14^{21} ends in 64 etc). Therefore, while the second set of 20 remainders is almost the same as the first set (differing only in the first remainder), all subsequent sets are exactly the same as the second set. The examples below will illustrate these points.

Consider point (3) above
The last two digits of successive powers of 13 are 13, 69, 97, 61 …. 01; 13, 69, 97,61….01 etc

Consider point (4.1) above
The last two digits of successive powers of 4 are 04, 16, 64, 56, ….76; 04,16,….76; etc.
Consider point (4.2) above

The last two digits of successive powers of 2 are 02, 04, 08, 16, ….76; 52,04,08,16, ….76(instead of the 02, we get 52)
These 6 points, 1, 2, 3, 4, 4.1, 4.2 (whichever is applicable) should be used in all problems on the last two digits.

2.08: Find the remainder when N = 817^{673} is divided by 100. Alternatively, find the last two digits of N.

Sol: We are interested only in the last two digits of N. We need to consider only the last two digits of 817, i.e., 17. Successive powers of 17(or any other number) show a cyclic pattern, when divided by 100(or any other divisor). We can list these remainders until we discover the point, where the repetition starts.

17	57	97	37	77
89	69	49	29	09
13	73	33	93	53
21	41	61	81	01

$17^1 = 17$

To get the next number, we take only the last two digits of 17^2, i.e. 89. To get the next number, we take only the last two digits of 17(89). We need not perform the complete multiplication. We need only the units and tens digits. The units digit is 3 and there are 3 parts to the tens digit – the carry over of 6, the units digit of 9(1) and 8(7) i.e. 6 + 9 + 6. Again we need only the units digit of this which is 1.

$\therefore$ The last two digits of 17^3 are 13. Similarly, we can work out the other numbers.

It is convenient to break the column after every 4 steps (the units digit is found to be the same in each row. This serves as a check to our calculations). After we get 01, the next 20 powers show the same pattern.
In the given example, as 673 = 20 (33) + 13.

$\therefore$The 13^{th} number in the list, i.e. 37, is our answer.
In general, we find that if we are interested in the last 2 digits, we need to go up to at most 20 steps. In some cases the period may be some factor of 20 (1, 2, 4, 5 or 10).

Consider the powers of 01. The pattern is 01;01 etc.
The period is 1.

Consider powers of 49, 51 or 99. The patterns are
49, 01; 49, 01; etc
51, 01; 51, 01; etc
99, 01; 99, 01; etc, i.e. the period is 2.
Consider powers of 07, 43, 57 or 93. The patterns are
07, 49, 43, 01; etc
43, 49, 07, 01; etc
57, 49, 93, 01; etc
93, 49, 57, 01; etc, i.e. the period is 4.

Consider powers of 21,
The pattern is 21, 41, 61, 81, 01; etc. The period is 5.

Consider powers of 29, 71 or 79. The patterns are
29, 41, 89, 81, 49, 21, 09, 61, 69, 01; etc
71, 41, 11, 81, 51, 21, 91, 61, 31, 01; etc
79, 41, 39, 81, 99, 21, 59, 61, 19, 01; etc
The period is 10.

2.09: What are the last two digits of 37^{12345}?

Sol: N = $37^{12345} = 37^{12340}\,37^5 = 37^{20(617)}.37^5$
37^{20} ends in 01,
while $37^5 \equiv (37)^4\,37 \equiv (1369)^2\,(37) \equiv (69)^2\,(37) \equiv (4761)\,(37)$
$\equiv (61)\,(37) = 2257$. $\therefore$ N ends in 57

Note: a ≡ b means a – b is divisible by the considered divisor

2.10: Find the remainder when 164^{359} is divided by 100

Sol: N = 164^{359} We need the last two digits. These digits for successive powers form a pattern of cycle length 20. As 359 = 340 + 19 and 164 = 100 + 64, we can think of $64^{19} = 2^{114}$
Now 2^{14} = 16384, It ends in 84.
$\therefore$ N also ends in 84

2.11: Find the last two digits of 282^{822}

Sol: N = 282^{822}. We can think of 82^2 = 6724
$\therefore$ N ends in 24

Some important theorems

Binomial Theorem: For any natural number n, $(a + b)^n = {}^nC_0a^n + {}^nC_1 a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1} ab^{n-1} + {}^nC_nb^n$.
where ${}^nC_r$ is the number of ways of choosing r objects out of n distinct objects and is given by

$${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1(2)(3)\ldots..(r)} = \frac{n!}{r!(n-r)!}$$

It can be observed that $(a + b)^n = a^n$ + (A multiple of b) = (A multiple of a) + b^n

2.12: Show that $(a + b)^7 - a^7 - b^7$ is a multiple of 7 for all positive integral values of a and b.

Sol: $(a + b)^7$
$= {}^7C_0a^7 + {}^7C_1\, a^6\, b + {}^7C_2\, a^5\, b^2 + {}^7C_3\, a^4\, b^3 + \ldots + {}^7C_7\, b^7$
$\therefore (a + b)^7 - a^7 - b^7$
$= {}^7C_1\, a^6\, b + {}^7C_2\, a^5\, b^2 + {}^7C_3\, a^4\, b^3 + \ldots\ldots + {}^7C_7\, b^7$ ---- (1)

If p is any prime number,

${}^pC_r = \dfrac{p(p-1)\ldots..(p-r+1)}{1(2)\ldots..(r)}$ = p(an integer) for all r < p.

$\therefore$ The RHS of (1) (and hence the LHS of (1)) is a multiple of 7.

Note: When n is prime, $(a + b)^n = a^n + b^n$ + (a multiple of n)

2.13: Find the remainder when 2^{1000} is divided by 33

Sol: $2^{1000} = (2^5)^{200} = (33 - 1)^{200} = (33^{200} + {}^{200}C_1 (33)^{199} (-1) + {}^{200}C_2 (33)^{198}(-1)^2 + \ldots + {}^{200}C_{199} (33)(-1)^{199} + (-1)^{200} = M(33) + (-1)^{200}$ =
(A multiple of 33) + 1

Fermat's little theorem: If p is prime and HCF (a, p) = 1, then $a^{p-1} - 1$ is a multiple of p.

For example, take p = 5, a = 3. From the theorem, $3^4 - 1$ or 80 is a multiple of 5.

If we take successive powers of 3, we get all the possible remainders.

$3^1 = 3$, $3^2 = 4$, $3^3 = 2$, $3^4 = 1$ (also $3^5 = 3$, $3^6 = 3^2$, $3^7 = 3^3$ etc). At a certain stage, we get a remainder of 1 and after that, the pattern repeats. In this example, the pattern is 3, 4, 2, 1; 3,4,2,1; etc. The pattern length is 4. In general, it would be (p – 1) or some factor of (p – 1).

2.14: What is the remainder when 5^{119} is divided by 59?

Sol: N = 5^{119} We need $\text{Rem}\dfrac{N}{59}$
By Fermat's Little Theorem, 5^{58} = 59k + 1 (where k is a natural number)
5^{59} = 59 (5k) + 5 or $5^{59} \equiv 5$
$\therefore 5^{118} \equiv 25$ and $5^{119} \equiv 125 \equiv 7$

2.15: Find the remainder when 26^{57} is divided by 29

Sol: $\text{Rem}\dfrac{26^{57}}{29} = \text{Rem}\dfrac{(26)(26)^{56}}{29}$

$= \left\{\text{Rem}\dfrac{26}{29}\right\}\left\{\text{Rem}\dfrac{26^{56}}{29}\right\}$

= {26} {1} = 26. $\therefore$ The remainder is 26.

Wilson's Theorem: If p is prime, (p – 1)! +1 is a multiple of p.

For example,
(2 – 1)! + 1= 2(1), (3-1)! + 1 = 3(1), (5-1)! + 1 = 5(5),
(7 – 1)! + 1 = 721 = 7(103) and so on.

2.16: What is the remainder when 28! is divided by 29?

Sol: By Wilson's theorem, $\text{Rem}\dfrac{28!+1}{29} = 0$

$\Rightarrow \text{Rem}\dfrac{28!}{29} = -1$ or $-1 + 29 = 28$

Remainder of a number when divided by $10^n \pm 1$

This is best illustrated with examples:

2.17: Find the remainder when 123,123, … (up to 300 digits) is divided by 999.

Sol: To find the remainder when some number (say N) is divided by 9 (or $10^1 - 1$), we add up all the digits of N to get (say S_1) and divide S_1 by 9
Similarly to find the remainder when N is divided by 99(or $10^2 - 1$), we start at the right end of N, group the digits two at a time and add up all the groups to get, say S_2. Then we find the remainder of S_2/99.
In general to find the remainder when N is divided by D_n = 99 . . . . 9(n nines) or (10^n - 1), we start at the right end of N, group the digits n at a time and add up all the groups to get say S_n.

$\text{Rem}\dfrac{N}{D_n} = \text{Rem}\dfrac{S_n}{D_n}$

Similarly we can start with the remainder rule for 11 and work out the corresponding rules for 101, 1001, 10001 etc. All this is an application of Remainder theorem.

Here, N = 123, 123, ..., 123 (a total of 300 digits or 100 groups) = 123 (1000^{99}) + 123 (1000^{98}) + . . . + 123 (1000^1) + 123

Now, let N = f(1000); When N or f(1000) is divided by 999 or (1000 – 1), the remainder is f(1) i.e., 123(100) by remainder theorem.[i.e. S_3 = 123 (100)]

$\therefore \text{Rem}\dfrac{N}{999} = \text{Rem}\dfrac{12300}{999} = \text{Rem}\dfrac{12+300}{999}$
= 312

2.18: Let N = 345345345 …. upto 300 digits. What is the remainder when N is divided by 999? Also find the remainder when N is divided by 1001.

Sol: N = 345, 345, ……… 345 (upto 300 digits or 100 groups of 3 digits) = 345 $[10^{3(99)} + 10^{3(98)} + 10^{3(97)} + \ldots + 10^3 + 1]$

$$\text{Rem}\frac{N}{999} = \text{Rem}\left(\frac{N}{(10^3-1)}\right) = \text{Rem}\frac{(345)(100)}{999}$$

($\because$ By remainder theorem)

$$= \text{Rem}\frac{34{,}500}{999} = \text{Rem}\frac{34+500}{999} = 534$$

To get $\text{Rem}\frac{N}{1001}$, we need U and Th, where U is the sum of all the alternate groups starting with the rightmost (the group containing the units digit) and Th is the sum of all the alternate groups starting with the second rightmost (the group consisting of the thousands digit)

U = 345(50) = 17250 and Th = 345(50) = 17250

$$\therefore \text{Rem}\frac{N}{1001} = \text{Rem}\frac{U-Th}{1001} = 0$$

Rules pertaining to $a^n + b^n$ or $a^n - b^n$

Sometimes, there will be problems involving **numbers that can be written in the form** $a^n + b^n$ or $a^n - b^n$ which can be simplified using simple rules. Let us first look at the rules pertaining to both $a^n + b^n$ and $a^n - b^n$, a, b and n being positive integers.

The following rules should be remembered for numbers in the form of $\mathbf{a^n - b^n}$.

1. It is always (i.e. when n is even as well as odd) divisible by a – b.
2. When n is even it is also divisible by a + b.
3. When n is odd it is divisible by a + b, if a + b is a factor of $2.b^n$.

The following rules should be remembered for numbers in the form of $\mathbf{a^n + b^n}$.

1. When n is odd it is divisible by a + b.
2. When n is odd, it is divisible by a – b, when a – b is a factor of $2.b^n$.
3. When n is even, it is divisible by a + b, if a + b is a factor of $2b^n$.

Some important identities:

$a^N - b^N = (a - b)(a^{N-1} + a^{N-2}b + a^{N-3}b^2 + \ldots + a^2 b^{N-3} + ab^{N-2} + b^{N-1})$ for all positive integer values of N.

$a^N + b^N = (a+b)(a^{N-1} - a^{N-2}b + a^{N-3}b^2 - a^{N-4}b^3 + \ldots - a^3 b^{N-4} + a^2 b^{N-3} - ab^{N-2} + b^{N-1})$ for all odd positive integer values of N.

Concept Review Questions

Directions for questions 1 to 25: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. $11^{103} + 14^{103}$ is a multiple of _______.
(A) 15 (B) 25 (C) 35 (D) 45

2. The greatest number which divides $29^{2n} – 11^{2n}$ for all positive integral values of n is _______.

3. A number when divided by 48 leaves a remainder of 31. Find the remainder if the same number is divided by 24.
(A) 5 (B) 7 (C) 9 (D) 11

4. A number when divided by 18 leaves a remainder of 15. Which of the following could be the remainder when it is divided by 72?
(A) 33 (B) 51
(C) 15 (D) All the above

5. What is the remainder when 18^{168} is divided by 19?

6. What is the last digit in the product of the first 20 odd natural numbers?

7. What is the last but one digit in the product of the first 10 even natural numbers?
(A) 2 (B) 4 (C) 6 (D) 0

8. What is the remainder when 95674321 is divided by 9?

9. What is the remainder when 47235674837 is divided by 25?
(A) 2 (B) 12 (C) 7 (D) 17

10. What is the largest 4-digit number that when divided by 19 leaves a remainder of 7?

11. Is the five-digit number pqr26 a perfect square?
(A) Yes (B) No (C) Cannot say

12. Is the six - digit number 1a4b75 a perfect square?
(A) Yes (B) No (C) Cannot say

13. Is the four-digit number AB36 a perfect square?
(A) Yes (B) No (C) Cannot say

14. Is the three-digit number C36 a perfect square?
(A) Yes (B) No (C) Cannot say

15. (a) If the three-digit number P6Q is a perfect square, is P odd?
(A) Yes (B) No (C) Cannot say

(b). If the three-digit number A5B is a perfect square, is A odd?
(A) Yes (B) No (C) Cannot say

(c). If the three-digit number AB1 is a perfect square, is A odd?
(A) Yes (B) No (C) Cannot say

16. Is the number $3^9\ 5^{11}\ 15^{13}$ a perfect square?
(A) Yes (B) No (C) Cannot say

17. The product of a nine-digit number and a ten-digit number must have ___ digits.
(A) Only 18 (B) Only 19
(C) Either (A) or (B)

18. The product of a six-digit number, eight-digit number and a ten-digit number must have ____ digits.
(A) Only 22
(B) Only 23
(C) Only 24
(D) Either (A) or (B) or (C)

19. Find the number of digits in the square root of a thirteen digit number.
(A) Only 7 (B) Only 6
(C) Either (A) or (B)

20. How many digits are there in $(3PQR)^4$ where 3PQR is a four digit number?
(A) Only 14 (B) Only 15
(C) Either (A) or (B)

21. Find the number of digits in the cube root of a 25-digit number.
(A) Only 9 (B) Only 10
(C) Either (A) or (B)

22. The units digits of the sum of the factorials of the first 10 natural numbers is

23. Find the units digit of the following.
(a) $8858 \times 7234 \times 842 \times 763$
(A) 2 (B) 4 (C) 3 (D) 6

(b) $7^{48} \times 3^{56} \times 165^{35}$
(A) 0 (B) 5 (C) 1 (D) 7

(c) $8^{4n} \times 6^n \times 9^{2n}$, n being any natural number.
(A) 8 (B) 4
(C) 6 (D) Cannot be determined

(d) $31 \times 32 \times 33 \times ..... \times 39$.
(A) 0 (B) 1 (C) 2 (D) 3

24. For what values of n the following statement is true (where n is a natural number)?
$2^{3n} –1$ is divisible by 7
(A) Even values of n (B) Odd values of n
(C) All values of n (D) Cannot say

25. What is the remainder when 2^{63} is divided by 7?

Exercise – 2(a)

Directions for questions 1 to 25: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. (a) Find the units digit of 8^{173}.
(A) 2 (B) 4 (C) 8 (D) 6

(b) What is the last digit of $518^{163} + 142^{157}$?
(A) 2 (B) 4 (C) 6 (D) 8

(c) Find the last digit of $1567^{143} \times 1239^{197} \times 2566^{1027}$
(A) 2 (B) 3 (C) 4 (D) 6

2. If n is a positive integer, then $43^{5n} – 21^{5n}$ is always divisible by _______.
(A) 11 (B) 18 (C) 25 (D) 64

3. Find the remainder when 1! + 2! + 3! + 4! + 5! + . . . + 49! is divided by 7.

4. What is the remainder when 3^{147} is divided by 11?
(A) 4 (B) 9 (C) 5 (D) 1

5. Find the remainder when $21^3 + 23^3 + 25^3 + 27^3$ is divided by 96.

6. $N = 10^{51} – 750$
Consider the following statements.
I. The remainder of N when divided by 11 is 8.
II. The remainder of N when divided by 7 is 5.
Which of the following can be concluded?
(A) Only I is true
(B) Only II is true
(C) Both I and II are true
(D) Neither I nor II is true

7. If n is a natural number, the remainder of $5^{8n+4} + 4^{4n+2} – 10$ divided by 641 is _______.
(A) 611 (B) 589
(C) 631 (D) cannot be determined

8. P and Q are positive integers. P leaves a remainder of 1 when divided by 20. Q leaves a remainder of 2 when divided by 20.
Consider the following statements.
I. $2^P – 2P$ is divisible by 10.
II. $8^Q + 8Q$ is divisible by 10.
Which of the following can be concluded?
(A) Only I is definitely true.
(B) Only II is definitely true.
(C) Both I and II are definitely true.
(D) Neither I nor II is true.

9. N = (4711) (4713) (4715). Find the remainder when N is divided by 48.

10. $N = 161^3 – 77^3 – 84^3$, which of the following statements is not true?
(A) N is divisible by 4 and 23.
(B) N is divisible by 23 and 11.
(C) N is divisible by 4 and 7.
(D) N is divisible by 8 and 11.

11. What is the remainder when 792379237923 …. upto 400 digits is divided by 101?

12. What is the remainder when 24242424 ….. upto 300 digits is divided by 999?
(A) 333 (B) 666 (C) 242 (D) 757

13. What is the remainder when 73^{382} is divided by 100?
(A) 71 (B) 51 (C) 29 (D) 49

14. What is the remainder when 787^{777} is divided by 100?
(A) 17 (B) 57 (C) 87 (D) 67

15. What is the remainder when 948^{728} is divided by 100?

16. What are the last two digits of 674^{586}?
(A) 36 (B) 76 (C) 56 (D) 96

17. What is the remainder when $98^{100} + 100^{100}$ is divided by 99?
(A) 2 (B) 0 (C) 1 (D) 98

18. What is the sum of the coefficients in the expansion of $(1 – 3x + x^2)^{55}$?
(A) 1 (B) –1 (C) 5^{55} (D) 2^{55}

19. What is the remainder when 2^{123} is divided by 61?

20. What is the remainder when 10^{400} is divided by 199?
(A) 149 (B) 5 (C) 198 (D) 50

21. What is the remainder when 14^{400} is divided by 1393?
(A) 14 (B) 805 (C) 185 (D) 115

22. What is the remainder when 100! is divided by 97^2?
(A) 582 (B) 8148 (C) 1261 (D) 8827

23. What is the remainder when 45! is divided by 47?

24. Find the remainder when $81(64^{25})$ is divided by 9^4.
(A) 2079 (B) 2997 (C) 2103 (D) 2719

25. N is a positive integer not more than 100. If $7^N + N^3$ has a units digit of 0, how many values can N take?

Exercise – 2(b)

Directions for questions 1 to 35: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Find the units digit of $314^{779} + 149^{138}$.

2. Which of the following is the remainder when 2^{69} is divided by 5?
(A) 2 (B) 4 (C) 8 (D) 3

3. How many prime numbers less than 100 leave an odd remainder when divided by 5?

4. What is the remainder when 2^{125} is divided by 11?
(A) 7 (B) 10 (C) 8 (D) 6

5. What is the remainder when 987654987654987654 … upto 750 digits is divided by 999?
(A) 666 (B) 333 (C) 330 (D) 669

6. What is the remainder when 445445445 … up to 525 digits is divided by 1001?
(A) 223 (B) 334 (C) 445 (D) 556

7. Find the remainder when 7^{1000} is divided by 50.

8. What is the remainder when 767^{1009} is divided by 25?
(A) 17 (B) 23 (C) 12 (D) 22

9. What are the last two digits of 36465676^{7897}?
(A) 76 (B) 56 (C) 36 (D) 16

10. What is the remainder when 768^{1234} is divided by 100?
(A) 24 (B) 44 (C) 64 (D) 84

11. What is the remainder when 994^{499} is divided by 100?

12. What is the sum of the coefficients in the expansion $(3 + 2x)^{99}$?
(A) 5^{99} (B) 3^{99} (C) 2^{99} (D) 6^{99}

13. What is the remainder when 624^{739} is divided by 125?

14. What is the remainder when 17^{325} is divided by 109?
(A) 1 (B) 17 (C) 108 (D) 92

15. What is the remainder when 12^{433} is divided by 438?
(A) 12 (B) 6 (C) 432 (D) 426

16. What is the remainder when 10^{2000} is divided by 19?
(A) 10 (B) 5 (C) 9 (D) 14

17. What is the remainder when 19! is divided by 361?

18. What is the remainder when 27! is divided by 29?
(A) 27 (B) 28 (C) 1 (D) 2

19. Find the remainder when $7^{900} – 908$ is divided by 8.
(A) 1 (B) 3 (C) 5 (D) 7

20. What is the remainder of $(25^3 + 27^3 + 29^3 + 31^3)$ divided by 112?
(A) 84 (B) 56 (C) 28 (D) 0

21. Let A be the set of all the prime numbers less than 100. B denotes the product of all the elements of A. Find the number of consecutive zeroes with which B ends.

22. Find the units digit of the sum of the factorials of the first 100 natural numbers.

23. Find the remainder when 209! is divided by 422.
(A) 212 (B) 218 (C) 224 (D) 206

24. The last two digits of 2^{924} are ______.
(A) 16 (B) 36 (C) 76 (D) 96

25. Q is an odd number satisfying $P^2 + 7^Q = 2^6\,5^6$, where P is an integer. The number of values P can take is

26. Find the remainder when 2^{168} is divided by 105.
(A) 7 (B) 15 (C) 3 (D) 1

27. If $x = \dfrac{40^{74} - 39^{74}}{\left(40^{36} + 39^{36}\right)\left(40^{37} + 39^{37}\right)}$, then it follows that ______.
(A) $x \le 0.5$
(B) $0.5 \le x \le 0.75$
(C) $0.75 \le x \le 1$
(D) $x > 1$

28. The remainder of $2^2(1!) + 3^2(2!) + 4^2(3!) + \ldots\ 18^2(17!)$ divided by 19 is _____.

29. The rightmost non-zero digit of $(70^{20})^{118} + (80^{40})^{59}$ is _______.
(A) 9 (B) 3 (C) 7 (D) 1

30. Find the remainder when the sum of the factorials of the first 150 positive integers is divided by 18.
(A) 9 (B) 12 (C) 15 (D) 6

31. Find the remainder when $7777777777 + 7^{262}$ is divided by 16.
(A) 8 (B) 4 (C) 6 (D) 2

32. If α = (188^3 + 200^3 + 211^3 + 299^3) leaves a remainder r when divided by 80, then r = _____.
(A) 23 (B) 37
(C) 9 (D) None of these

33. N is the number formed by writing all the positive integers from 75 to 120 one after another. Find the remainder when N is divided by 9.

34. Find the remainder when 7^{1000} is divided by 5.
(A) 1 (B) 2 (C) 3 (D) 4

35. (a) The square of a prime number when divided by 6 cannot leave a remainder of _______.
(A) 4 (B) 3 (C) 1 (D) 5

(b) The cube of a prime number when divided by 6 cannot leave a remainder of _______.
(A) 1 (B) 2 (C) 3 (D) 4

Key

Concept Review Questions

1. B	6. 5	11. B	(b) B	19. A	(b) B
2. 720	7. D	12. B	(c) C	20. C	(c) C
3. B	8. 1	13. C	16. A	21. A	(d) A
4. D	9. B	14. B	17. C	22. 3	24. C
5. 1	10. 9982	15 (a) A	18. D	23. (a) A	25. 1

Exercise – 2(a)

1. (a) C	4. B	9. 21	14. D	19. 8	24. B
(b) B	5. 0	10. D	15. 56	20. D	25. 10
(c) A	6. C	11. 56	16. B	21. B	
2. A	7. C	12. A	17. A	22. D	
3. 5	8. C	13. C	18. B	23. 1	

Exercise – 2(b)

1. 5	7. 1	13. 124	19. C	25. 0	31. D
2. A	8. D	14. B	20. D	26. D	32. D
3. 12	9. A	15. A	21. 1	27. D	33. 3
4. B	10. A	16. B	22. 3	28. 17	34. A
5. C	11. 4	17. 342	23. A	29. C	35. (a) D
6. C	12. A	18. C	24. A	30. A	(b) D