

# Memoisation: Purely, Left-recursively, and with (Continuation Passing) Style

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## Abstract

Memoisation, or tabling, is a well-known technique that yields large improvements in the performance of some recursive computations. Tabled resolution in Prologs such as XSB and B-Prolog can transform so called *left-recursive* predicates from non-terminating computations into finite and well-behaved ones. In the functional programming literature, memoisation has usually been implemented in a way that does not handle left-recursion, requiring supplementary mechanisms to prevent non-termination. A notable exception is Johnson’s (1995) continuation passing approach in Scheme. This, however, relies on mutation of a memo table data structure and coding in explicit continuation passing style. We show how Johnson’s approach can be implemented purely functionally in a modern, strongly typed functional language (OCaml), presented via a monadic interface that hides the implementation details, yet providing a way to return a compact representation of the memo tables at the end of the computation.

## 1 Introduction

Memoisation (Michie, 1968; Norvig, 1991) is a well known technique for speeding up computations involving repeated copies of the same sub-problem by storing the results of solving such sub-problems and then referring to these stored results later rather than recomputing them, thus trading space for time. As such, it is a form of dynamic programming, and is especially effective for computing certain recursive functions, which may have exponential time complexity when implemented directly but are reduced to polynomial or linear complexity when implemented with memoisation. The classical example is the function for finding the  $n^{\text{th}}$  number in the Fibonacci sequence, which starts  $0, 1, 1, 2, 3 \dots$ , with each successive number the sum of the previous two. Hence, the  $n^{\text{th}}$  number is

$$\text{fib } n = \begin{cases} n & \text{if } n \in \{0, 1\} \\ \text{fib}(n-2) + \text{fib}(n-1) & \text{otherwise.} \end{cases} \quad (1)$$

Implemented directly in a language supporting recursive functions, the time taken to compute  $\text{fib } n$  grows exponentially with  $n$ , because very many computations

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\*The bulk of this note was written in 2013 while the author was affiliated with UCL. Sections 7 and 8 were written in July 2017 while the author was affiliated with Jukedeck Ltd.

are repeated. For example, to compute *fib* 8, we must compute *fib* 6 and *fib* 7. Assuming we compute *fib* 6 first, we must then compute *fib* 7 = *fib* 5 + *fib* 6. But we have already computed *fib* 6, and therefore end up duplicating that computation. Memoisation improves the situation dramatically by removing these duplicates.

The Fibonacci function is a deterministic computation. If we expand our scope to *nondeterministic* computations, which can produce zero or more answers (Wadler, 1985), then it becomes possible to define *left recursive* computations that not only call themselves recursively, but call themselves recursively *with the same arguments*. Implemented directly, such computations do not terminate. Precisely this situation crops up when working with grammars (such as grammars for natural languages) which include left-recursive production rules. These can often be the most succinct way of describing certain grammatical constructions. A left-recursive grammar can always be transformed into one which is not left-recursive one which recognises the same language (Moore, 2000; Johnson and Roark, 2000) but doing so can force a more convoluted programming style and make it harder to build a semantic representation of sentences while parsing. It may also be impossible to do this for probabilistic grammars without affecting the resulting distribution over sentences. Hence, the ability to handle left recursion can be useful feature.

## 1.1 Tabling in logic programming

In the logic programming community, memoisation is often referred to as *tabling*, and is a feature of several Prolog implementations, including XSB, B-Prolog and YAP. By tabling the relevant predicates, both deterministic and nondeterministic recursive predicates can be written that would be exponentially slow or non-terminating without tabling. For example, in B-Prolog we can compute the transitive closure of an *edge*/2 relation as follows:

```
% left recursive transitive closure of edge/2.
:- table path/2.

path(X,Z) :- path(X,Y), path(Y,Z).
path(X,Z) :- edge(X,Z).

edge(a,b).
edge(b,c).
```

Prolog's standard depth-first search would immediately go into an infinite recursion on the first clause of *path*/2, but with tabling, this presents no difficulties: entering a query *path*(*a*,*X*) produces the solutions *b* and *c* with no duplicates.

Given Prolog's close association with natural language applications, a relationship between tabling and efficient chart parsing algorithms was quickly noticed (Warren, 1975; Pereira and Warren, 1983). Chart parsers build a compact representation of the results of parsing various subsequences of the target sequence, which can then be used to build higher levels of the syntax tree while avoid repeated computations. These charts are essentially memo tables. The correspondence is even clearer if we use Prolog's definite clause grammar (DCG) syntax to write the grammar, yielding a succinct, executable specification. Without tabling, applying the top level predicate to an input sequence executes a

recursive descent, backtracking search for a valid parses. As such it is vulnerable to exponential time complexity or non-termination with left-recursive clauses. As Warren (1995) states, when you write a DCG, you get a parser “for free”, but not necessarily a very good one, but with tabling, the parser you get “for free” is a pretty good one, essentially the same as Earley’s algorithm (Earley, 1970). Indeed, the process of tabled resolution in Prolog is sometimes called “Earley deduction” (Porter, 1986).

Thus, tabling in Prolog is a powerful tool, but in most cases, it requires low-level support within the Prolog engine and cannot be modified by the programmer. Attempts have been made to reduce the reliance on low-level support (Ramesh and Chen, 1997; De Guzmán et al., 2008), but in these cases, a foreign library (e.g. in C) is still required to achieve acceptable performance.<sup>1</sup>

## 1.2 Memoisation in functional programming

In the functional programming world, memoisation has been tackled in an embedded way, by writing code in the language rather than by implementing new primitives in the compiler or interpreter.

Norvig (1991) showed how to define a higher-order function in LISP that, given any function, produces a memoised version of it. Crucially it relied on mutation effects, both to manage the memo tables using updatable variables and to modify the symbol table mapping symbols to functions so that any recursive calls in the memoised function are redirected to the memoised version. The system was demonstrated in a program for parsing context free grammars, using lists to represent non-determinism as described by Wadler (1985).

Johnson (1995) extended this approach, this time working in Scheme, introducing what are essentially parser combinators, and transforming the program into a continuation passing style (CPS). Having done this, Johnson showed how the memoisation process could accommodate left-recursion by effectively suspending the branches of the computation which are left recursive and allowing the other branches to proceed until they produce a result, at which point, suspended branches can resume and consume the newly generated result. In this approach, non-determinism is not represented with lists of results, but by calling a captured continuation multiple times, once with each alternative. Notably, it essentially the same as the most commonly used tabling strategies in Prolog, OLDT resolution (Tamaki and Sato, 1986) and SLG resolution (Chen and Warren, 1993).

Frost (1994) showed how a memoising combinator can be defined in a pure lazy functional language (Miranda, a precursor of Haskell) by explicitly threading a state (containing the memo tables) through all memoised computations. The framework was applied to defining functional parsers and parser combinators, this time using lists to represent non-determinism, but could not handle left recursive grammars or left-recursion in general. In subsequent publications, Frost and his co-workers developed the idea, adopting a monadic approach to state threading, this time in Haskell (Frost, 2003), developing a way to handle left-recursive grammars, though not by using Johnson’s method, but by managing and limiting the depth of left-recursive calls (Frost and Hafiz, 2006), and adding the ability to return a compact representation of multiple parse trees (Frost et al., 2007, 2008).

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<sup>1</sup>Since the bulk of this note was written, a Prolog implementation of tabling as a library (Desouter et al., 2015) based on delimited continuations (Schrijvers et al., 2013) has become available.

Frost *et al* acknowledge Johnson’s continuation passing approach but do not adopt it; indeed, they express surprise that it has not received much attention and suggest that that this may be because the approach is “somewhat convoluted and extending it to return packed representations of parse trees [...] could be too complicated.”

In this note, we show how Johnson’s CPS memoisation solution can indeed be expressed in a functional style without relying on any mutable state or side effects to any significant degree. We hide the implementation behind a monadic interface that provides not only memoisation, but also nondeterminism, as computation effects. With a slight complication of the interface, we can also return a compact representation of the memo tables built during parsing, which can then be interrogated to produce all possible parse trees. The code presented below is in OCaml, which is not strictly a pure functional language, but we restrict ourselves to a pure functional fragment. The only exception to this is that mutable references are used to implement a universal type, but the visibility of this impurity is confined to a very limited scope.

In the following sections, we define a framework for using monads in OCaml (§2), deduce suitable types for memoisation and discuss open recursion and fixed point combinators (§3) before presenting our code for memoisation of left recursive functions in a continuation passing style §5. We then extend this to allow the memo tables to be extracted at the end of the computation (§6). The system is compared with other memoising parser frameworks in §7 before concluding in §8. Supporting code is provided in the appendix.

## 2 Monads in OCaml

As we intend to present complete working code, we start by defining some utility functions corresponding to functions of the same name in Haskell:

---

```

let id x = x                (* identity function *)
let (∘) f g x = f (g x)      (* function composition *)
let cons x xs = x :: xs       (* prepend item to list *)
let curry f x y = f (x,y)

```

---

It is well known that a wide variety of computational effects can be structured using monads (Wadler, 1992). Unlike Haskell, OCaml lacks a standard monad library, so we provide some module interfaces for a few monad classes:

---

```

module type MONAD = sig
  type α m
  val return : α → α m
  val bind : α m → (α → β m) → β m
end

module type MONADPLUS = sig
  include MONAD
  val mzero : unit → α m
  val mplus : α m → α m → α m
end

```

---

```

module type MONADREF = sig
  include MONAD
  type  $\alpha$  ref
  val new_ref :  $\alpha \rightarrow \alpha$  ref  $m$ 
  val get_ref :  $\alpha$  ref  $\rightarrow \alpha$   $m$ 
  val put_ref :  $\alpha$  ref  $\rightarrow \alpha \rightarrow \text{unit}$   $m$ 
end

```

---

*MONADREF* is modelled on Haskell's *MonadRef* type class, and provides operators for managing polymorphic mutable references, much like OCaml's built in  $\alpha$  **ref** type. Given any monad implementing the primitive *return* and *bind* functions, we can provide some useful derived functions equivalent to those defined in the Haskell monad library:

---

```

module MonadOps ( $M : \text{MONAD}$ ) = struct
  open  $M$ 

  let ( $\gg=$ ) = bind
  let ( $\gg$ )  $m1\ m2$  = bind  $m1$  (fun  $_ \rightarrow m2$ )

  let liftM  $op\ m$  =  $m \gg= \text{return} \circ op$ 
  let liftM2  $op\ m\ n$  =  $m \gg= \text{fun } x \rightarrow n \gg= (\text{return} \circ op\ x)$ 

  let rec mapM  $f$  = function
    | []  $\rightarrow \text{return []}$ 
    |  $x :: xs \rightarrow \text{liftM2 cons } (f\ x)\ (\text{mapM } f\ xs)$ 
end

```

---

We will also be using monad transformers (Liang et al., 1995) to combine monadic effects, in particular, layering nondeterminism over a base monad, for which we use a port of the standard Haskell *ListT* monad transformer, adhering to the *MONADPLUS* interface:

---

```

module ListT ( $M : \text{MONAD}$ ) = struct
  type  $\alpha\ m = \alpha\ \text{list } M.m$ 

  module MO = MonadOps ( $M$ )
  open MO

  let return  $x$  =  $M.\text{return } [x]$ 
  let bind  $m\ f$  =  $m \gg= \text{mapM } f \gg= (M.\text{return} \circ \text{List.concat})$ 
  let lift  $m$  =  $M.\text{bind } m\ \text{return}$ 
  let mzero () =  $M.\text{return []}$ 
  let mplus  $f\ g$  = liftM2 (@)  $f\ g$ 
end

```

---

This represents the result of a nondeterministic computation as a list of possible values, with *mzero* () denoting a computation that fails and *mplus*  $a\ b$  denoting a nondeterministic choice between two computations  $a$  and  $b$ .

### 3 Recursion and fixed-point combinators

If we were to follow Norvig's or Johnson's approaches (Norvig, 1991; Johnson, 1995), we might try to define a high order function  $memo : (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$  that takes an ordinary (pure) function and returns a memoised version of it. There are two problems with this.

Firstly, since our aim is to do purely functional memoisation, using monads to represent computational effects, and maintaining the memo table requires state handling effects, the memoised function must have a monadic type. In addition, if a recursive function is to be able to call a memoised version of itself, it too must be lifted into the monad. This might lead us to propose  $memo : (\alpha \rightarrow \beta \ m) \rightarrow (\alpha \rightarrow \beta \ m)$  as our memoising operator, where  $m$  is the type constructor of the monad which will carry the necessary effects. However, if we attempt to write such a function, we find that, because functions cannot be compared for equality, there is no way to implement the functions `get_table_for` and `modify_table_for` below:<sup>2</sup>

```
type ( $\alpha, \beta$ ) table (* table for a function of type  $\alpha \rightarrow \beta$  *)
val lookup :  $\alpha \rightarrow (\alpha, \beta)$  table  $\rightarrow \beta$  option
val insert :  $\alpha \rightarrow \beta \rightarrow (\alpha, \beta)$  table  $\rightarrow (\alpha, \beta)$  table

let memo f x =
  get_table_for f >>= fun table  $\rightarrow$  (* NOT POSSIBLE *)
  match lookup x table with
  Some y  $\rightarrow$  return y
  None  $\rightarrow$  f x >>= fun y  $\rightarrow$ 
    modify_table_for f (insert x y) > (* NOT POSSIBLE *)
    return y
```

One reasonable solution is to add an identifier as a parameter to `memo`, yielding something like this:

```
type id = string

val memo : id  $\rightarrow (\alpha \rightarrow \beta \ m) \rightarrow \alpha \rightarrow \beta \ m$ 
let memo id f x =
  get_table id >>= fun table  $\rightarrow$ 
  match lookup x table with
  Some y  $\rightarrow$  return y
  None  $\rightarrow$  f x >>= fun y  $\rightarrow$ 
    mod_table id (insert x y) >>
    return y

let rec fib x =
  memo "fib" (function
    | 0  $\rightarrow$  return 0
    | 1  $\rightarrow$  return 1
    | n  $\rightarrow$  liftM2 (+) (fib (n-1)) (fib (n-2))) x
```

<sup>2</sup>The grey bars to the left of this and other examples below denote non-functioning or incomplete code.

where, for the sake of brevity, we have used strings as identifiers. Behind the scenes, *get\_table* and *mod\_table* can maintain an associative mapping between ids and memo tables (though doing so polymorphically in a type-safe manner presents some difficulties). This is essentially the approach taken by Frost (2003), who restrict themselves to memoising parsing functions of uniform type, and so do not have to deal with the polymorphism issue. One potential problem is that it is now the programmer's responsibility to keep track of the ids, most importantly to avoid duplicates. Also, in some applications, it might become a distracting burden to have to invent an identifier for each memoised function. What is required is an *allocation* step, where a unique identifier is generated for each function to be memoised. Allocation of resources is an effectful operation, so the preparation of a memoised function will itself need to be a monadic operation. Rather than exposing the programmer to the details of resource allocation, perhaps the safest option is to hide all of this in a memoising operator of type  $memo : (\alpha \rightarrow \beta\ m) \rightarrow (\alpha \rightarrow \beta\ m)\ m$ . This has two advantages over Frost's approach: (a) it is no longer possible to make an error handling the memo table ids and (b) each memo table can be properly initialised, which may improve the performance of later operations. However, it also means we can no longer use an ordinary **let rec** binding to define  $fib : int \rightarrow int\ m$ .

This brings us to the second problem: with a monadic memoising combinator, there is no recursive binding construct available that will let us refer to the result of the computation produced by *memo* in the argument to *memo*. The solution is to adopt the *open* style of recursion and use an explicit fixed-point combinator. For example, in open-recursive style, the monadic Fibonacci function is

```
val fib' : (int → int m) → int → int m
let fib' f = function | 0 → return 0 | 1 → return 1
                  | n → liftM2 (+) (f (n-1)) (f (n-2))
```

A fixed-point combinator,  $fix : ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)) \rightarrow \alpha \rightarrow \beta$ , is most straightforwardly written in OCaml using a **let rec** binding:

---

```
let fix f = (let rec fp x = f fp x in fp)
```

---

This closes the recursion and allows the (un-memoised) Fibonacci function to be written as  $fix\ fib' : int \rightarrow int\ m$ . For defining two or more *mutually* recursive functions, we need to generalise the idea of an open-recursive function to allow the first argument to be a data structure containing the fixed points of *all* the recursive functions in the set; for example, a *dyadic* fixed-point combinator is

---

```
let fix2 ( (f : (α → β) × (γ → δ) → α → β),
          (g : (α → β) × (γ → δ) → γ → δ) ) =
  let rec fp x = f (fp, gp) x
  and    gp x = g (fp, gp) x
  in (fp, gp)
```

---

Note the types of the open-recursive functions *f* and *g* here: higher arity fixed-point combinators will require correspondingly elaborated types for the first argument of each open-recursive function. With a few tricks (Kiselyov, 2003) one can also write variadic fix-point combinators that work with an arbitrary number of functions, but we will not pursue that here.

## 4 Monadic memoisation

Using open-recursion, it is relatively easy to write memoising fixed-point combinators that prepare the memo tables and tie up the recursion while inserting the appropriate code for checking the memo tables on each call. Instead of doing this, we will follow McAdam (1997) and consider transformations of open-recursive functions (or “functionals”, as McAdam calls them). In this scheme, a “wrapper” is a high-order function that takes an open-recursive function and returns a new one that may do something interesting to intervene in the operation of the original function each time it is called. McAdam showed how memoisation can be handled by a wrapper, leaving the job of tying up the recursion to a separate fixed-point combinator.

We make two modifications to McAdam’s idea. Firstly, to accommodate the possibility of memoising mutually recursive functions, we generalise the type of open-recursive functions to  $\gamma \rightarrow \alpha \rightarrow \beta$ , where  $\gamma$  is the type of the data structure holding the fixed points of all the relevant open-recursive functions, as discussed in the previous section. Incidentally, non-recursive functions can be accommodated by setting  $\gamma = \text{unit}$ . Secondly, since we are not using mutable data structures to manage the memo tables, the creation of a memoising wrapper needs to be a monadic operation of type  $(\gamma \rightarrow \alpha \rightarrow \beta \ m) \rightarrow (\gamma \rightarrow \alpha \rightarrow \beta \ m) \ m$ . Armed with such a wrapper, any number of mutually recursive functions can be memoised using the appropriate fixed-point combinators. A simple memoising framework equivalent to the one we have been attempting to write above can now be sketched out:

```

type ( $\alpha, \beta$ ) id

val new_memo : ( $\alpha, \beta$ ) id m
val get_table : ( $\alpha, \beta$ ) id  $\rightarrow$  ( $\alpha, \beta$ ) table m
val mod_table : ( $\alpha, \beta$ ) id  $\rightarrow$  (( $\alpha, \beta$ ) table  $\rightarrow$  ( $\alpha, \beta$ ) table)  $\rightarrow$  unit m
val memo : ( $\gamma \rightarrow \alpha \rightarrow \beta \ m$ )  $\rightarrow$  ( $\gamma \rightarrow \alpha \rightarrow \beta \ m$ ) m

let memo fn =
  new_memo  $\gg=$  fun id  $\rightarrow$ 
  return (fun fp x  $\rightarrow$ 
    get_table id  $\gg=$  fun table  $\rightarrow$ 
    match lookup x table with
    | Some y  $\rightarrow$  return y
    | None  $\rightarrow$  fn fp x  $\gg=$  fun y  $\rightarrow$ 
      mod_table id (insert x y)  $\gg$ 
      return y)

let test_fib n = memo fib'  $\gg=$  fun f  $\rightarrow$  fix f n

```

This leads us to propose the following *MONADMEMO* as a general interface for any memoising monad and an accompanying functor for defining useful operations for any memoising monad, including *mem* :  $(\alpha \rightarrow \beta \ m) \rightarrow (\alpha \rightarrow \beta \ m) \ m$  for memoising non-recursive functions and *memrec* to get the memoised fixed point of an open-recursive monadic computation:



---

```

module type MONADMEMO = sig
  include MONAD

  val memo : ( $\gamma \rightarrow \alpha \rightarrow \beta$  m)  $\rightarrow$  ( $\gamma \rightarrow \alpha \rightarrow \beta$  m) m
end

module MemoOps (M : MONADMEMO) = struct
  include MonadOps(M)
  open M

  let mem f = memo (fun () x  $\rightarrow$  f x)  $\gg=$  fun mf  $\rightarrow$  return (mf ())
  let memrec f = liftM fix (memo f)
  let memrec2 (f,g) = liftM2 (curry fix2) (memo f) (memo g)
end

```

---

Note that *liftM* and *liftM2* are required to apply the ordinary functions *fix* and *fix2* to the results of the monadic memoisation operator *memo*.

## 5 Nondeterminism and left-recursion

Nondeterminism, using lists to represent multiple success (Wadler, 1985), can already be dealt with using the memoiser sketched out in the previous section, simply by memoising functions of type  $\gamma \rightarrow \alpha \rightarrow \beta$  *list m*. This results in a memo table where each input of type  $\alpha$  is associated with a list of results instead of just one, and is equivalent to the methods of both Norvig (1991) and Frost (1994).

In order to deal with left-recursion, however, we will lift both nondeterminism and memoisation into a continuation passing monad, modelled on Haskell's *ContT* monad transformer, and adapt Johnson's (1995) method. The key to this is to notice that the continuation monad provides *delimited* (or composable) continuations, which, as Filinski (1994, 1999) showed, can be used to implement the computational effects of any monad or combination of monads. For our purposes, we will define a *ContT* functor in OCaml, parameterised by a fixed answer type and a base monad:

---

```

module type TYPE = sig type t end

module ContT (W : TYPE) (M : MONAD) = struct
  type  $\alpha$  m = {run: ( $\alpha \rightarrow W.t$  M.m)  $\rightarrow$  W.t M.m}

  let return x = {run = fun k  $\rightarrow$  k x}
  let bind m f = {run = fun k  $\rightarrow$  m.run (fun x  $\rightarrow$  (f x).run k)}
  let shift f = {run = fun k  $\rightarrow$  (f (return  $\circ$  k)).run id}
  let lift m = {run = fun k  $\rightarrow$  M.bind m k}
end

```

---

In addition, because of the fixed answer type (*W.t*) of *ContT*, we will borrow Filinski's (1999) *Dynamic* module implementing a universal type to enable sufficient polymorphism when running computations in the memoising monad:

---

```

module Dynamic = struct
  exception Dynamic
  type t = Dyn of (unit → unit)

  let newdyn () : ( $\alpha \rightarrow t$ )  $\times$  ( $t \rightarrow \alpha$ ) =
    let r = ref None in
      ( (fun a → Dyn (fun () → r := Some a)),
        (fun (Dyn d) → r := None; d ())
          match !r with
            | Some a → a
            | None → raise Dynamic))
end

```

---

This module uses an OCaml reference as a channel to communicate polymorphically across a monomorphic interface and is the only place where effectful OCaml constructs are used. The rest of the program is completely insulated from these effects and so the system can still be considered pure—alternative implementations could use type coercions or delimited continuations with full answer type polymorphism (Asai and Kameyama, 2007).

Using *ContT*, we can write a functor *MemoT* parameterised by an arbitrary monad of type *MONADREF* for providing typed references. This module provides memoisation and nondeterminism using the *ListT* monad transformer. Generalising it to use an arbitrary monad transformer for nondeterminism would be relatively straightforward.

---

```

module MemoT (Ref : MONADREF) : sig
  type  $\alpha$  m

  val memo : ( $\alpha \rightarrow \beta \rightarrow \gamma$  m) → ( $\alpha \rightarrow \beta \rightarrow \gamma$  m) m
  val return :  $\alpha \rightarrow \alpha$  m
  val bind :  $\alpha$  m → ( $\alpha \rightarrow \beta$  m) →  $\beta$  m
  val mzero : unit →  $\alpha$  m
  val mplus :  $\alpha$  m →  $\alpha$  m →  $\alpha$  m
  val run :  $\alpha$  m →  $\alpha$  list Ref.m

end = struct
  module ND = ListT (Ref)
  module CC = ContT (Dynamic) (ND)
  module CO = MonadOps (CC)
  include CC

  let run m =
    let (ind, outd) = Dynamic.newdyn () in
      ND.bind (m.run (ND.return  $\circ$  ind)) (ND.return  $\circ$  outd)

  let liftRef m = lift (ND.lift m)
  let mzero () = {run = fun k → ND.mzero ()}
  let mplus f g = {run = fun k → ND.mplus (f.run k) (g.run k)}
  let msum = List.fold_left mplus (mzero ())

```

```

let memo fop = let open CO in
  liftRef (Ref.new_ref BatMap.empty) >>= (fun loc →
    let update x e t = liftRef (Ref.put_ref loc (BatMap.add x e t)) in
    return (fun p x →
      liftRef (Ref.get_ref loc) >>= fun table →
      try let (res, conts) = BatMap.find x table in
        shift (fun k → update x (res, k::conts) table >>
          msum (List.map k res))
      with Not_found →
        shift (fun k → update x ([], [k]) table >>
          fop p x >>= fun y →
            liftRef (Ref.get_ref loc) >>= fun table' →
              let (res, conts) = BatMap.find x table' in
              if List.mem y res then mzero ()
              else update x (y::res, conts) table' >>
                msum (List.map (fun k → k y) conts))))))
end

```

---

Some comments on the code are appropriate here: the basic framework is a stack of two monad transformers on a base monad of type *MONADREF*, which provides mutable references for storing the memo tables. The stateful operations from *MONADREF* are lifted through the two layers using *liftRef*. The module *ND* provides nondeterminism layered over state such that the state is shared across alternative branches of execution. However, this nondeterminism is exposed (via *mplus* and *mzero*) by capturing the current continuation, using it twice or not at all, and combining the results using operations from *ND*.

The memo tables are implemented using the polymorphic map module from OCaml With Batteries; for a function of type  $\alpha \rightarrow \beta$  *m*, the memo table is of type  $(\alpha, \beta \text{ list} \times (\beta \rightarrow \text{Dyn.t } m) \text{ list}) \text{ BatMap.t}$ .

When *memo* is applied to open-recursive function *fop*, a new, empty memo table is allocated and a function implementing the memoised computation is returned. This retrieves the memo table and attempts to look up the argument *x*. If it is found, *shift* is used to capture the continuation, which is added to the list of continuations associated with *x* in the memo table and then called for each result in the memo table entry, combining the results using *msum*.

If *x* is not found in the table, meaning this is the first time the memoised function has been applied to *x*, a table entry is created, containing no results and one continuation, after which *fop* is called. Then, for each result produced by *fop* that is not already in the memo table, the entry is updated and all the continuations registered for *x* are called with the new value, with the results of each again combined using *msum*.

We can try out the module on the Fibonacci function and the transitive closure program given in § 1.1.

---

```

module Fibonacci (M : MONAD) = struct
  include MonadOps (M)
  let fib f = function | 0 → M.return 0 | 1 → M.return 1
    | n → liftM2 (+) (f (n-1)) (f (n-2))
end

```

```

module TransClose (M : MONADPLUS) = struct
  open M
  let edge = function | "a" → return "b"
                    | "b" → return "c"
                    | _   → mzero ()
  let path p x = mplus (edge x) (bind (p x) p)
end

module Test = struct
  module MM = MemoT (Ref)
  module FF = Fibonacci (MM)
  module TC = TransClose (MM)
  module MO = MemoOps (MM)
  open MO

  let test_fib n = Ref.run (MM.run (memrec FF.fib >>= fun fib → fib n))
  let test_path x = Ref.run (MM.run (memrec TC.path >>= fun path → path x))
end

```

---

The *Ref* monad given in the appendix is used to provide mutable references. Using the run function from *MemoT* yields a computation in the *Ref* monad which is then run using *Ref.run*: in *test\_fib*, this returns a single integer, while in *test\_path*, it returns a list of strings, e.g., *Test.test\_path* "a" returns ["b", "c"].

## 6 Getting access to the memo tables

If the memoising monad *MemoT* is used for parsing, the resulting memo tables contain all the information held in the charts used by efficient chart parsing algorithms, providing a compact representation of all the parse trees. As it stands, there is no way to get hold of them—they are buried inside the *Ref* monad. One way to obtain them is to modify the *memo* combinator so that, as well as returning the memoised function, it also returns a monadic operator to return the current memo table for that function. Suppose that, for a memoised function of type  $\alpha \rightarrow \beta$  *m*, the type of the memo table is  $(\alpha, \beta)$  *table*. Then we might try  $memo : (\gamma \rightarrow \alpha \rightarrow \beta \text{ } m) \rightarrow ((\alpha, \beta) \text{ table } m \times (\alpha \rightarrow \beta \text{ } m)) \text{ } m$ , where second element of the pair produced by *memo* is the memoised function as before, but the first is a computation that returns the memo table. This will not do: since the *MemoT* monad includes nondeterminism as an effect, a parsing computation which ends with reading and returning a memo table will result in *multiple* memo tables, one for each successful parse, even though the memo table is shared across nondeterministic alternatives.

Instead we need memo table initialisation and extraction to to operate in the base *Ref* monad, rather than in the nondeterministic *ContT* monad. The work-flow will consist of preparing memoised functions in the *Ref* monad, running the nondeterministic computation in the memoising monad layered over *Ref*, and then retrieving the memo tables after dropping back into the *Ref* monad. A suitable interface for managing this is *MONADMEMOTABLE*, along with an accompanying functor implementing memoising fixed point operators:

---

```

module type MONADMEMOTABLE = sig
  include MONAD
  module Nondet : MONADPLUS

  type ( $\alpha, \beta$ ) table = ( $\alpha \times \beta$  list) list
  val run :  $\alpha$  Nondet.m  $\rightarrow$   $\alpha$  list m
  val memo : ( $\gamma \rightarrow \alpha \rightarrow \beta$  Nondet.m)  $\rightarrow$ 
    (( $\alpha, \beta$ ) table m  $\times$  ( $\gamma \rightarrow \alpha \rightarrow \beta$  Nondet.m)) m
end

module MemoTabOps (M : MONADMEMOTABLE) = struct
  module MO = MonadOps (M)
  open M
  open MO

  let mem f = memo (fun () x  $\rightarrow$  f x)  $\gg=$  fun (g, mf)  $\rightarrow$  return (g, mf ())
  let memrec f = memo f  $\gg=$  fun (g, mf)  $\rightarrow$  return (g, fix mf)
  let memrec2 (f, g) = memo f  $\gg=$  fun (get_f, mf)  $\rightarrow$ 
    memo g  $\gg=$  fun (get_g, mg)  $\rightarrow$ 
    let (fp, gp) = fix2 (mf, mg) in
    return ((get_f, get_g), (fp, gp))
end

```

---

For the sake of concreteness, we have fixed the representations of nondeterministic alternatives and the memo tables to use lists, but it would also be possible to parameterise the module type over alternative representations. The *MONADMEMOTABLE* signature presents *two* monadic interfaces: the nested *Nondet* module is for use by memoised nondeterministic computations, while the outer one provides the memoising operator and a function to run a memoised computation. The implementation is similar to *MemoT* except for this repackaging into an outer and an inner module. The inner module *Nonedet* implements the *MONADPLUS* interface using continuations. The only substantive addition is the operator to return a memo table: the *BatMap* used internally is converted to a list of pairs, and the *sanitize* function removes the list of continuations associated with each entry before returning it.

---

```

module MemoTabT (Ref : MONADREF) = struct
  module ND = ListT (Ref)
  include Ref

  module Nondet = struct
    include ContT (Dynamic) (ND)
    let mzero () = {run = fun k  $\rightarrow$  ND.mzero ()}
    let mplus f g = {run = fun k  $\rightarrow$  ND.mplus (f.run k) (g.run k)}
  end

  module RefO = MonadOps (Ref)
  module CCO = MonadOps (Nondet)

```

```

type ( $\alpha, \beta$ ) table = ( $\alpha \times \beta$  list) list

let run ( $m : \alpha$  Nondet.m) :  $\alpha$  list Ref.m =
  let (ind, outd) = Dynamic.newdyn () in
  ND.bind (m.run (ND.return  $\circ$  ind)) (ND.return  $\circ$  outd)

let memo fop = let open RefO in
  new_ref BatMap.empty  $\gg$  (fun loc  $\rightarrow$ 

    let liftRef m = Nondet.lift (ND.lift m) in
    let sanitize ( $x, (s, \_)$ ) = ( $x, BatSet.fold$  cons s []) in
    let update x e t = liftRef (put_ref loc (BatMap.add x e t)) in

    return (
      get_ref loc  $\gg$  return  $\circ$  List.map sanitize  $\circ$  BatMap.bindings,
      ( fun p x  $\rightarrow$  let open CCO in let open Nondet in
        liftRef (get_ref loc)  $\gg$  fun table  $\rightarrow$ 
        try let (res, conts) = BatMap.find x table in
          shift (fun k  $\rightarrow$  update x (res, k::conts) table  $\gg$ 
            BatSet.fold (mplus  $\circ$  k) res (mzero ()))
        with Not_found  $\rightarrow$ 
          shift (fun k  $\rightarrow$  update x (BatSet.empty, [k]) table  $\gg$ 
            fop p x  $\gg$  fun y  $\rightarrow$ 
              liftRef (get_ref loc)  $\gg$  fun table'  $\rightarrow$ 
                let (res, conts) = BatMap.find x table' in
                if BatSet.mem y res then mzero ()
                else update x (BatSet.add y res, conts) table'  $\gg$ 
                  List.fold_right (fun k  $\rightarrow$  mplus (k y))
                    conts (mzero ())))))
    )
  )
end

```

---

Note that the results are now being collected in a data structure optimised for fast lookups (*BatSet*). We can use this module to implement a simple parser which, thanks to memoisation, is equivalent to an Earley parser. First, we need a parser combinator library (Hutton, 1990): parsers are represented as nondeterministic monadic computations of type  $\alpha$  *list*  $\rightarrow$   $\alpha$  *list m*, taking a list of tokens to parse and producing, if successful, the list of remaining tokens. The operators  $\ast>$  and  $\diamond$  combine two parsers, in sequence or as alternatives, respectively. The primitive  $\epsilon$  parses an empty sequence and *term x* matches a single token *x*.

```

module Parser ( $M : MONADPLUS$ ) = struct
  let ( $\ast>$ ) f g xs = M.bind (f xs) g
  let ( $\diamond$ ) f g xs = M.mplus (f xs) (g xs)
  let  $\epsilon$  xs = M.return xs
  let term x = function | y::ys when  $x=y$   $\rightarrow$  M.return ys
    |  $\_$   $\rightarrow$  M.mzero ()
end

```

---

Then we can write a small grammar, equivalent to Johnson's (1995) example of a left recursive grammar.

---

```

module Test2 = struct
  module MM = MemoTabT (Ref)
  include Parser (MM.Nondet)
  include MonadOps (MM)
  include MemoTabOps (MM)
  open MM

  let v   = term "likes" <math>\diamond</math> term "knows"
  let pn  = term "Kim" <math>\diamond</math> term "Sandy"
  let det = term "every" <math>\diamond</math> term "no"
  let n   = term "student" <math>\diamond</math> term "professor"
  let np' = fun np  $\rightarrow$  pn <math>\diamond</math> det * $\triangleright$  n <math>\diamond</math> np * $\triangleright$  term "'s" * $\triangleright$  n
  let vp' = fun np (vp,s)  $\rightarrow$  v * $\triangleright$  np <math>\diamond</math> v * $\triangleright$  s
  let s'  = fun np (vp,s)  $\rightarrow$  np * $\triangleright$  vp

  let success = function | []  $\rightarrow$  true | _  $\rightarrow$  false
  let parse input =
    memrec np' >>= fun (get_np,np)  $\rightarrow$ 
    memrec2 (vp' np,s' np) >>= fun ((get_vp,get_s),(vp,s))  $\rightarrow$ 
    run (s input) >>= fun results  $\rightarrow$ 
    get_s >>= fun s_memo  $\rightarrow$ 
    get_np >>= fun np_memo  $\rightarrow$ 
    get_vp >>= fun vp_memo  $\rightarrow$ 
    return (List.exists success results,
            ["s",s_memo; "np",np_memo; "vp",vp_memo])
end

```

---

The non-recursive nonterminals  $v$ ,  $pn$ ,  $det$  and  $n$  match single words, while the open-recursive nonterminals  $np'$ ,  $vp'$  and  $s'$  match phrases and sentences and are defined as functions taking the fixed points of their recursions as arguments. Note that  $np$  is left recursive, to allow for noun phrases such as “Kim’s professor”.

The function `parse` returns a computation in the *Ref* monad. It ties up the recursive parsers using the memoising fixed point operators `memrec` and `memrec2`, processes a list of strings and returns **true** or **false** to indicate whether or not the whole input was parsed, along with a list containing the memo tables for the three recursive rules  $s$ ,  $np$  and  $vp$ . It can be run in the OCaml top-level interpreter using `Ref.run`, for example:

```

# Ref.run (Test2.parse ["Sandy"; "'s"; "professor"; "knows"; "Kim"]);
- : bool  $\times$  (string  $\times$  (string list, string list) Test2.MM.table) list =
(true,
 [("s", [(["Kim"], []];
           (["Sandy"; "'s"; "professor"; "knows"; "Kim"], [ [] ])]);
  ("np", [(["Kim"], [ [] ]];
           (["Sandy"; "'s"; "professor"; "knows"; "Kim"],
            [ ["knows"; "Kim"];
              ["'s"; "professor"; "knows"; "Kim"] ])]);
  ("vp", [([], []];
           (["'s"; "professor"; "knows"; "Kim"], []];
           (["knows"; "Kim"], [ [] ])))]])

```

In this case, the sentence was successfully parsed, and the memo tables show that, for example, ["Kim"] could not be parsed as a sentence, but was parsed as a noun phrase, and also that ["Sandy"; "s"; "professor"; "knows"; "Kim"] admits of two partial parses as a noun phrase, the first consuming only the first token "Sandy" and the second consuming the three tokens ["Sandy"; "s"; "professor"] and making use of the left-recursive production rule for noun phrases.

## 7 Comparison with previous work

Functional approaches to parsing, including parser combinators, have been studied for several decades (Burge, 1975; Fairbairn, 1987; Frost and Launchbury, 1989; Hutton, 1990). Both Norvig (1991) and Leermakers (1993) use memoisation to improve efficiency, but Norvig forbids left recursive rules, while Leermakers avoids the problem of left recursion by using a ‘recursive ascent’ strategy, sacrificing the modularity of top-down approaches (Koskimies, 1990). Johnson’s (1995) continuation-based system, the basis for the one developed here, was written in Scheme without the benefit of a strong type system, and relied on mutation side effects to manage the memo tables. It also required the code to be written in explicit continuation passing style, as opposed to using the monadic interface *ContT* described here. The possibility of a monadic interface to parser combinators was recognised by Wadler (1990).

Lickman (1995) takes a pure functional and monadic approach to parsing left recursive grammars, including memoisation. He relies on defining a fixed point operator for recursive parsers which is in turn defined in terms of a fixed point operator for set-to-set functions. While mathematically elegant, the resulting implementation suffers from potentially exponential time complexity. In comparison, the continuation-based approach described here computes the fixed point incrementally, since each *new* solution from a memoised parser is fed back into its context, which may be itself if the parser is recursive, until no more new solutions are produced.

Frost (1993) proposes ‘guarded attribute grammars’ as a way to handle left-recursion: each left recursive rule is ‘guarded’ by a non-left recursive recogniser, which delimits the segment to which the left recursive parser can then safely be applied. However, the time complexity is still exponential in the depth of the left recursion. The later work of Frost and Hafiz (2006) (see § 1.2) improves on this, reaching  $O(n^4)$  time complexity in the length of the input for left recursive grammars. In comparison, the system described here handles left recursion without having to look ahead to the end of the input sequence to limit the depth of left recursion, and achieves the same  $O(n^3)$  theoretical time complexity as Earley’s chart parser. Another difference is that Frost et al’s system requires each memoised parser to be given a label, whereas the proposed system does not. Finally, Frost’s system is implemented in Haskell, which supports arbitrary recursive binding constructs without any special effort, whereas in OCaml, it was necessary to use open recursion and explicit fixed point operators.

Frost et al. (2007) presented some timings of their system on a small set of abstract, highly ambiguous grammars, some involving left recursion. The three memoised parsers, encoded below, are *sm* (recursive); *sml* (left recursive) and *smml* (composed of two mutually recursive rules, one of which is also left recursive).



input length	Proposed system			Frost et al, 2007		
	<i>sm</i>	<i>sml</i>	<i>smml</i>	<i>sm</i>	<i>sml</i>	<i>smml</i>
12	0.001	<b>0.002</b>	<b>0.002</b>	<b>0.001</b>	0.004	0.003
24	0.008	<b>0.008</b>	<b>0.01</b>	<b>0.005</b>	0.02	0.02
48	0.08	<b>0.09</b>	<b>0.11</b>	<b>0.02</b>	0.3	0.3
72	0.39	<b>0.42</b>	<b>0.52</b>	<b>0.10</b>	2.4	2.5
96	1.2	<b>1.3</b>	<b>1.6</b>	<b>0.26</b>	8.1	8.7

Table 1: Execution times (in seconds) comparing the system presented here with that of Frost et al. (2007) parsing sequences of the token “a” of various lengths (first column) using four highly ambiguous grammars. The best performance for each test case is indicated in boldface.

---

```

module Ambig (MM: MONADMEMOTABLE) = struct
  include Parser (MM.Nondet)
  include MonadOps (MM)
  include MemoTabOps (MM)
  open MM

  let rec sentence n = function 0 → [] | n → "a"::sentence (n-1)

  let sm = memrec (fun sm → term "a" *> sm *> sm <◇ ε)
  let sml = memrec (fun sml → sml *> sml *> term "a" <◇ ε)
  let smml = memrec2 ((fun (smml, aux) → smml *> aux <◇ ε),
    (fun (smml, aux) → smml *> term "a"))
end

```

---

In the above module, each grammar is represented as a monadic operation that will produce a memo table extractor and the memoised parser itself. All three recognise arbitrary length sequences of the token "a", which can be generated using the *sentence* function.

Frost et al’s Haskell code<sup>3</sup> was modified to (a) reduce the effect of laziness by traversing the entire resulting data structure (derived from the memo tables) and (b) compute the total execution time including the final traversal but not any printing or writing to files. The programs for both systems were fully compiled, rather than being run in the OCaml or Haskell interactive environments. The results, obtained using a 2012 Macbook Pro with a 2.5 GHz Intel Core i5 CPU and 8 GB of memory, are shown in Table 1. The overall picture that emerges is that Frost et al’s system performs very well for non left recursive grammars, partly, one suspects, due to Haskell’s laziness and sophisticated optimising compiler, which can eliminate many the overheads associated with data structures and high-order function manipulations, but there may be other factors, such as the use of integers to represent the parsing state (i.e., the index of the next token to be processed) as opposed to using the tail of the input sequence in the present system. However, in all cases, the continuation based system handles left recursion more effectively.

<sup>3</sup>Available at <http://hafiz.myweb.cs.uwindsor.ca/xsaiga/imp.html>.

## 8 Conclusions

A purely functional, continuation-based system for memoising recursive and left recursive nondeterministic computations, including those involved in parsing left recursive grammars, has been presented in the form of a complete implementation in the functional programming language OCaml. The three computational effects required: statefulness, nondeterminism and delimited continuation capture, were implemented as a stack of monads. The system was compared with that of Frost et al. (2007), which shares many of the same aims and tools, but uses a different method to handle left recursion, as well as differing in a number of other respects, such as the need to assign labels to memoised parsers and the representation of the parser state. It was found that the continuation based system was more efficient asymptotically for left recursive grammars, but was slower for shorter input sequences and non-left recursive grammars, possibly due to the overheads introduced by the stack of monads. Hence, one strategy for improving the performance is to eliminate the stack of monads and investigate the use of delimited continuations as a primitive mechanism and implement the computational effects directly in delimited control operators.

Filinski (1994) showed the close relationship between monads and delimited continuations: the present system relies on a monad to represent delimited control operators, as previously by Dyvbig et al. (2005). Conversely, it is also possible to implement layered monadic effects using delimited control as a primitive (Filinski, 1999). Kiselyov (2012) describes his *delimcc* library (written in 2001) which implements delimited continuations for OCaml efficiently in so-called ‘direct style’, that is, without introducing the data structures and associated overhead required in the monadic approach. The system described here could easily be transformed into a ‘direct style’ implementation, using delimited control operators to implement the required effects (statefulness, nondeterminism and memoisation) when required, but executing ordinary OCaml code directly and without overhead for the pure functional parts of the computation.

Another possible optimisation would be to replace the list used to collect the results from each memoised function with a more efficient associative map, since the cost of looking up each new result to see if it has already been seen might be significant when there are many results.

All the code used to generate the results in this note, including the modified version of Frost et al’s code, is available at <http://github.com/samer--/cpsmemo>.

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## A Implementation of supporting modules

To implement *MONADREF* without using OCaml mutable references, we use a state monad where the state is a polymorphic key-value store implemented using the *BatMap* functor from OCaml With Batteries, with integer-valued keys. This uses a balanced binary tree internally, with lookup and insertion costs of  $O(\log n)$ .

where  $n$  is the number of items in the store. The key type  $\alpha \text{ loc}$  is opaque to users of the module. The following code fragment must be included between the definitions of the *ContT* and *MemoT* functors.

---

```

module Store : sig
  type t
  type  $\alpha \text{ loc}$ 

  val empty : t
  val new_loc : t  $\rightarrow$   $\alpha \text{ loc} \times t$ 
  val get :  $\alpha \text{ loc} \rightarrow t \rightarrow \alpha$ 
  val put :  $\alpha \text{ loc} \rightarrow \alpha \rightarrow t \rightarrow t$ 
  val upd :  $\alpha \text{ loc} \rightarrow (\alpha \rightarrow \alpha) \rightarrow t \rightarrow t$ 

end = struct
  module M = BatMap.Make(BatInt)

  type t = int  $\times$  Dynamic.t M.t
  type  $\alpha \text{ loc}$  = int  $\times$  ( $\alpha \rightarrow$  Dynamic.t)  $\times$  (Dynamic.t  $\rightarrow$   $\alpha$ )

  let empty = (0, M.empty)
  let get (j, _, outd) (_, m) = outd (M.find j m)
  let put (j, ind, _) x (i, m) = (i, M.add j (ind x) m)
  let upd (j, ind, outd) f (i, m) = (i, M.modify j (ind  $\circ$  f  $\circ$  outd) m)
  let new_loc (i, m) = let (ind, outd) = Dynamic.newdyn () in
    ((i, ind, outd), (i+1, m))

end

module StateM (State : TYPE) = struct
  type  $\alpha \text{ m}$  = State.t  $\rightarrow$   $\alpha \times$  State.t
  type state = State.t

  let return x s = (x, s)
  let bind m f s = let (x, s') = m s in f x s'
  let get s = (s, s)
  let put s _ = ((), s)
  let upd f s = ((), f s)

end

module Ref = struct
  include StateM(Store)
  type  $\alpha \text{ ref}$  =  $\alpha$  Store.loc

  let put_ref loc x = upd (Store.put loc x)
  let upd_ref loc f = upd (Store.upd loc f)
  let get_ref loc = bind get (return  $\circ$  Store.get loc)
  let new_ref x = bind Store.new_loc (fun loc  $\rightarrow$ 
    bind (put_ref loc x) (fun _  $\rightarrow$  return loc))
  let run m = fst (m Store.empty)

end

```

---

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