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Theory Behind Regulatory Capital Formulae

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The purpose of bank regulation is to ensure that institutions in the financial market remain solvent in an event of unexpected credit loss peaks. Since the widespread banking crises in the 1970s and 1980s, Basel Committee on Banking Supervision has introduced regulation accords that determine the supervisory capital requirements for banks. A revised framework, Basel II, replaced the strongly criticized, original Basel Accord in 2004.

Regulation is targeted for banks of all sizes and locations. Therefore it is important that the framework treats banks equally. In Basel II, the credit risk model is required to be portfolio-invariant, meaning that the loss of a loan should not depend on the structure of the portfolio to which the loan is added.

The theoretical model behind the Basel II supervisory capital requirements is based on a specific Asymptotic Single-Risk Factor (ASRF) credit risk model that provides the desired portfolio-invariance. This thesis presents the mathematical background of the ASRF credit risk model and the regulatory capital formulae.

The 2008 global financial crisis unveiled problems in the Basel II model for supervisory capital requirements. Although the revised framework supplies a more sophisticated approach on bank capital regulation than the original accord, the regulatory capital model is of procyclical nature. The thesis discusses the procyclicality and other deficiencies of the Basel II regulatory capital. We conclude the thesis with future prospects.

Keywords: Basel II, regulatory capital, bank regulation, Asymmetric Single-Risk Factor credit risk model

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Pankkisääntelyllä pyritään varmistamaan, että pankit pystyvät selviytymään maksukykyisinä myös odottamattomista luottotappioista. 1970–1980-lukujen laajalle ulottuneiden pankkikriisien jälkeen luotiin viitekehys pankeilta vaadittavan riskipääoman määrälle. Alkuperäisen Basel-sopimuksen lähestymistapaa kritisoitiin liian yksinkertaiseksi, minkä vuoksi se päätettiin korvata vuonna 2004 julkaistulla Basel II -viitekehyksellä.

Koska pankkitoimialaa säännellään maailmanlaajuisesti, on tärkeää, ettei viitekehys aseta pankkeja eriarvoiseen asemaan, vaan sitä on helppo soveltaa niin pienten kuin suurten pankkien tarpeisiin maasta riippumatta. Basel II -viitekehyksen käyttämän luottoriskimallin haluttiinkin olevan portfolioinvariantti, mikä tarkoittaa, että lainaportfolion rakenne ei vaikuta portfolioon lisättävän lainan laskennalliseen luottotappioon.

Basel II -malli riskipääomavaatimuksille perustuu tiettyyn Asymptotic Single-Risk Factor -luottoriskimalliin (ASRF), joka täyttää vaatimuksen portfolioinvarianttiudesta. Tässä diplomityössä käsitellään matemaattista teoriaa ASRF-mallin sekä pankeilta vaaditun riskipääoman taustalla.

Basel II on monella tapaa alkuperäistä Basel-viitekehystä hienostuneempi lähestymistapa pankkien riskipääoman sääntelyyn. Vuoden 2008 finanssikriisin myötä Basel II -viitekehyksessä havaittiin kuitenkin heikkouksia, jotka ilmenivät muun muassa voimistuneena suhdannevaihteluna, mikä on osaltaan häirinnyt talouden elpymistä. Diplomityössä pohditaan Basel II -riskipääomavaatimuksiin liittyviä ongelmakohtia sekä tulevaisuuden näkymiä.

Asiasanat: Basel II, sääntely, riskipääoma, pankkisääntely

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“All things are so very uncertain, and that’s exactly what makes me feel reassured.”

Tove Jansson, *Moominland Midwinter*

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Chapter 1

Introduction

The motive behind bank regulation is to ensure that institutions in the financial market hold enough capital to overcome unexpected peaks of credit losses that interest rates do not suffice to wholly compensate. The global economy faced widespread banking crises in the 1970s and 1980s. In the US, almost a third of savings and loan associations failed during the Savings and loan crisis.

Financial institutions hold a special position in society. History has shown that banking crises connected to sudden changes in the value and quality of assets, or bank runs, often originate from the internal relationships in the financial system [5]. It is common to have short-term liabilities and bank deposits to fund assets with longer maturity. However, in the case of an asset value downgrade or a bank run, banks are in danger of running out of capital to fund other liabilities. Unfortunately, banking crises emerging from such systemic nature are often linked to large-scale economic disruption and recessions. The impact of banking business on the economy and also on social welfare is significant. Thus, it is in the interest of policy makers to regulate the bank capital and require high standards of corporate governance.[5]

After the severe crises of 1970s and 1980s, Basel Committee on banking supervision introduced the Basel Accord framework that determine the minimum capital requirements for banks [21]. The framework was revised in 2004, with the essential goal to develop a framework that would ensure even more sound and stable international banking system. The additional objective of the revision was to remove the potential source of competitive inequality among internationally active banks that arose from the original capital adequacy regulation [23]. In the case of commercial loans, for instance, the regulatory capital requirements based on the 1988 Basel Accord framework depend only on the loan face value, thus discarding the financial strength of the obligor as well as the quality of the collateral [14].

Significant credit exposures drive financial institutions into progressively utilizing different risk models to manage credit risk at the portfolio level. In 2003, credit risk models had become popular also among commercial banks that often had maintained a more passive approach to portfolio management. Compared to some traditional lending limit controls, the benefits of credit risk models are the possibilities of recognizing concentrations of risk and opportunities for diversification. The approach improved liquidity in the credit market, and offered a relatively objective framework. Using credit risk models to determine the capital requirements was widely considered a rather appealing solution to the regulatory problem related to first Basel Accord.[14]

The purpose of this thesis is to describe the theoretical model behind the Basel II supervisory capital requirements. The thesis presents the assumptions and credit risk models on which the Basel II regulatory capital charges are based. We also discuss how well the model reaches its goals, as well as the advantages or weakness that relate to the formulae.

This thesis is divided into five chapters. In chapter 2 we introduce the concept and reasoning of regulatory capital. Chapter 3 focuses on the theory and the mathematical model behind the formulae for capital requirements. In chapter 4 we discuss the benefits and problems that relate to the regulatory capital of the Basel II framework. Chapter 5 is devoted to discussion on future prospects.

Chapter 2

Regulatory Capital

Banks differ from other companies in a sense that they operate in the credit business and issue loans to individuals and companies. Due to the nature of credit business, losses are common and take place frequently. There is always a possibility that some of these obligors can not meet their obligations and therefore default on their loans. Consequently, banks suffer losses. The actual entity of losses varies yearly and depends on the amount and severity of default events.

Surely, banks do not know in advance the actual losses they will suffer in a given year but they can forecast the average level of credit losses that they can legitimately expect to face. Financial institutions manage these expected losses through, for instance, the pricing of credit exposures and provisioning.

However, banks occasionally experience losses that exceed the level of expected losses. These losses are often referred to as unexpected losses. Risk premia and interest rates that are charged can absorb some parts of unexpected losses but not all of them. In fact, the market does not support such prices that can cover all unexpected losses. Instead, banks are required to hold special capital reserves to cover the peak losses. The purpose of bank capital is to protect a bank's debt holders from unexpected peak losses. [11]

However, for a bank it is not optimal to maximize its capital reserves because that weakens its investment opportunities. The more money is reserved as capital, the less can be exchanged to assets. Hence, banks are keen on keeping capital as minimal as suitable while it still satisfies the supervisory amount. This means there is a desire to find an optimal solution to a multicriteria optimization problem.

Regulatory capital equals the amount of capital that a financial regulator requires a financial institution to hold [11]. For this reason, regulatory capital is often referred to as the minimal capital requirement. The objective is to ensure that financial institutions are able to overcome potentially large unexpected losses.

Before Basel II framework, regulatory capital requirement on commercial bank loans was based on the 1988 Basel Accord that determines the capital charge on commercial lending to be a uniform percentage of loan face value [21]. However, this approach pays no interest on borrowers' financial strength or the quality of the security. Financial institutions vary extensively when it comes to the composition and size of portfolios.

In a detailed study [22] of how banks' internal models could be utilized in setting capital requirements, The Basel Committee on Bank Supervision noted that a more accurate measure of portfolio credit risk could be obtained with a "carefully specified and calibrated model" than with any rule-based system. However, the Committee also highlighted importance of comparability across institutions.

As economic capital can be allocated to individual instruments, it describes a shadow price on the cost of holding each position. The allocated amounts can also be called capital charges on individual instruments.

2.1 Value-at-Risk

Value-at-Risk is defined as the q-th quantile of the distribution of loss. The idea behind the Value-at-Risk concept: Financial institutions have target ratings for their own debt. In order to maintain its target rating, an institution must hold capital to assure the survival over given time horizon. The target rating is associated with a probability of survival q over the horizon (for instance, 99.9 percent over a year), also called *target survival probability.* Financial institutions must hold enough reserves and equity capital that can cover up to the q-th quantile of the distribution of portfolio loss.

2.2 Asymptotic Single Risk Factor credit portfolio model

Credit risk models are basically functions that map market-level parameters and asset-level characteristics to a distribution of portfolio credit losses over a preferred time horizon [14]. Among other attributes, the model outputs the amount of capital required to support the portfolio against credit losses.

Generally, portfolio models do not produce portfolio-invariant capital charges. Many models rely on the assumption that correlations across obligors are a consequence of common dependence on a set of systematic risk factors [14]. "A natural property" of such models is that the marginal capital required for an asset is dependent on how it affects diversification, which means that required capital depends on the composition of the portfolio.

In order to maintain applicability, the Basel II model was required to be portfolio invariant. Thus, only the risk of a loan should affect the required amount of capital, not the portfolio the loan is added to. This restriction was made to ensure that the new framework is applicable to a wide range of countries and institutions. More advanced credit risk models take into account the composition of a portfolio when capital for each loan is determined. However, this approach would have been a too complex task for most financial institutions and supervisors. [11]

The desire for portfolio invariance of required capital affects strongly the structure of the portfolio model. In fact, it can be shown that the Asymptotic Single Risk Factor (ASRF) credit portfolio models are the only actually portfolio invariant models [14]. ASRF credit portfolio models can be interpreted as simplified credit risk models that are constructed by applying the law of large numbers on more general credit portfolio models.

In Chapter 3, we will show that in order to guarantee portfolio-invariance, the portfolio needs to be asymptotically fine-grained and there needs to be at most a single systematic risk factor, hence the name.

'Asymptotic' refers to the need for the portfolio to be asymptotically fine-grained. This means that the portfolio does not include any exposure that would account for more than extremely small share of portfolio exposure in total. In the case of a portfolio consisting of a large number of (relatively) small exposures, idiosyncratic risks related to individual exposures are likely to cancel out one-another. Idiosyncratic risk is a risk that is asset specific and has little or no correlation with market risk. Also, only systematic risks affecting numerous liabilities have a significant effect on portfolio losses.

'Single-risk factor' on the other hand refers to all systematic risk being modelled with only one systematic risk factor. In the ASRF model, all systematic risks such as regional or industry risks are combined and treated as one. The single systematic risk factor could be described as a reflection of the state of the global economy.

The supervisory capital requirements for unexpected losses are derived from the Basel risk weight functions on the ASRF model in Basel II framework. As mentioned before, the model needs to be portfolio invariant. This means that the capital required for each loan only depends on the risk of that loan, in order to ensure the applicability of the framework world-wide. For most banks and supervisors, it would have been too challenging to take into account the actual portfolio composition when determining capital for loans.

However, this makes institution-specific diversification effects hard to recognize within the IRB framework. After all, the diversification effects depend on how well a new loan fits into an existing portfolio. Due to this aspect, the Revised Basel II framework is adjusted to well-diversified banks.

In the internal-ratings based approach financial institutions are allowed to use their internal measures for key drivers of credit risk subject to meeting certain conditions and to explicit supervisory approval. [11]

The Basel II framework introduced a new, internal-ratings based approach to bank regulation. Banks are allowed to independently choose the credit risk models to use for their internal risk measurement and risk management needs. Institutions are also permitted to determine the borrowers' probabilities of default.

Borrowers default if they are unable to fully meet their obligations at a fixed point of time because they cannot pay back their liabilities. Average probabilities of default are transformed into conditional probabilities of default with a mapping function adapted from Merton's [19] single asset model to credit portfolios. In Merton's model, the value of borrower's assets is a variable with a value changing over time. The change in the value of assets is considered a random variable following normal distribution.

Vasicek [28] [29] [30] showed that under certain conditions Merton's model can be extended to a specific ASRF credit portfolio model. Gordy 2003 further shows that ASRF framework has features that make it a relatively favorable credit risk model. Basel Committee adopted the assumptions of systematic and idiosyncratic risk factors being normally distributed.

Chapter 3

Theory Behind Regulatory Capital Formulae

In this chapter we will provide insight to theory behind the formulae for capital regulations in Basel II framework. First, we will derive the Asymmetric Single-Risk Factor (ASRF) model and also show why the ASRF framework is the only framework satisfying the essential requirement, the portfolio invariance. Then, we will present how to take into account the common scenario where credit portfolios consist of assets with different maturities. This is achieved by introducing maturity adjustments. We will also explain how correlations between assets are modelled in the Basel II framework. Last, by combining these theories, we will derive the formula for regulatory capital in Basel II.

For practicality reasons, we consider the concept of loss as credit loss that occurs only if the obligor defaults, meaning that we ignore changes in market value that result from rating downgrade or upgrade.

3.1 Asymptotic Single Risk Factor credit portfolio model

3.1.1 Merton's Single Asset Model to Credit Portfolio

Merton extended the Black-Scholes - Merton theory of option pricing model ([4], [18]) to cover also other corporate liabilities.

The pricing model is developed under following assumptions: [19]

1. There are no transaction costs nor taxes. Also, indivisibilities of assets do not generate problems.

2. There are sufficient number of investors. The investors have proportionate levels of wealth such that every investor believes she can buy and sell any amount of an asset she wants at the market price.

3. There exists an exchange market for borrowing and lending at the same interest rate.

4. Short-sales of all assets is allowed with full use of the proceeds. This means that an investor is allowed to immediately use the proceeds received from a sale if desired.

5. Trading in assets takes place continuously in time.

6. The Modigliani-Miller theorem [20]: The value of the firm is not affected by the capital structure of the firm.

7. The Term-Structure of interest rates (= the yield curve) is "flat" and known with certainty. This means that the price of a risk-free discount bond, which promises a one dollar payment at time τ in the future, is $P(\tau) = e^{-r\tau}$ where r is the instantaneous risk-free rate of interest and stays the same for all time.

8. The dynamics for the value of the firm, V, through time can be described by a diffusion-type stochastic process with stochastic differential equation
$$dV = (\mu V - C)dt + \sigma V dW(t) \tag{3.1}$$
where μ is the instantaneous expected rate of return on the firm per unit time, C is the total dollar payouts by the firm per unit time to either its shareholders or liabilities-holders if positive, and it is the net

dollars received by the firm from new financing if negative; σ^2 is the instantaneous variance of the return on the firm per unit time; $W(t)$ is a standard Wiener-Gaussian process.

In this case, we can ignore the payouts C and concentrate on describing the dynamics of the firm value with

$$dV = \mu V dt + \sigma V dW(t). \tag{3.2}$$

Of the given assumptions, only 5 and 8 can be considered critical. We want the market for the securities to be open for trading most of time. This is guaranteed under assumption 5. We also require that price movements are continuous and that returns on the securities are serially independent. consistent with the efficient market hypothesis, meaning that the asset prices fully reflect all information available [17].

Now, suppose that there exists a security with a market value Y that at any point in time can be expressed as a function F of the value of the firm and time

$$Y = F(V, t). \tag{3.3}$$

The dynamics of this security's value can be written

$$dY = \mu_y Y dt + \sigma_y Y dW_y(t) \tag{3.4}$$

where μ_y is the instantaneous expected rate of return per unit time on this security; σ_y^2 is the instantaneous variance of the return per unit time; $W_y(t)$ is the standard Wiener-Gaussian process.

However, given equation (3.3) and Itô's Lemma (Appendix A, (A.4)), we can express the dynamics of the market value Y also in the form

$$\begin{aligned} dY &= dF(V,t) \\ &= F_t dt + F_v dV + \frac{1}{2} F_{vv} (dV)^2 \\ &= F_t dt + F_v \left(\mu V dt + \sigma V dW(t)\right) + \frac{1}{2} F_{vv} \left(\mu V dt + \sigma V dW(t)\right)^2 \\ &= \left(F_t + \mu V F_v + \frac{1}{2}\sigma^2 V^2 F_{vv}\right) dt + \sigma F_v V dW(t) \end{aligned} \tag{3.5}$$

When we compare equations (3.4) and (3.5) (coefficients of stochastic and non-stochastic components), we obtain

$$\begin{cases} \mu_y Y = \mu_y F = F_t + \mu V F_v + \frac{1}{2}\sigma^2 V^2 F_{vv} \\ \sigma_y Y = \sigma_y F = \sigma F_v V \\ dW_y(t) = dW(t). \end{cases} \tag{3.6}$$

Let us now compose a portfolio that consists of three instruments: the firm, the particular security, and risk-free debt, which have values V, Y and $D = e^{rt}$, respectively. The portfolio is composed so that the aggregate investment equals zero.

Let w_1 denote dollars of the portfolio invested in the firm, w_2 denote dollars of the portfolio invested in the security, and w_3 denote dollars of the portfolio invested in risk-free debt. As the total investment is set to zero, we can express the number of dollars w_3 invested in risk-free debt as $w_3 = -(w_1 + w_2)$.

The return to the portfolio equals the sum of the returns to each instrument in the portfolio

$$\begin{aligned} \Delta x &= w_1 \frac{\Delta V}{V} + w_2 \frac{\Delta Y}{Y} + w_3 \frac{\Delta D}{D} && (3.7) \\ &= w_1 \frac{\Delta V}{V} + w_2 \frac{\Delta Y}{Y} + w_3 \frac{\Delta (e^{rt})}{e^{rt}}. && (3.8) \end{aligned}$$

Now, we obtain the expected *instantaneous* dollar return to the portfolio by differentials

$$\begin{aligned} dx &= w_1 \frac{dV}{V} + w_2 \frac{dY}{Y} + w_3 \frac{d(e^{rt})}{e^{rt}} \\ &= w_1 \frac{dV}{V} + w_2 \frac{dY}{Y} + w_3 r dt \\ &= w_1(\mu dt + \sigma dW(t)) + w_2(\mu_y dt + \sigma_y dW_y(t)) - (w_1 + w_2) r dt \\ &= w_1(\mu - r)dt + w_2(\mu_y - r)dt + w_1 \sigma dW(t) + w_2 \sigma_y dW_y(t) \\ &\overset{(3.6)}{=} w_1(\mu - r)dt + w_2(\mu_y - r)dt + (w_1\sigma + w_2\sigma_y)dW(t). && (3.9) \end{aligned}$$

Assume that the portfolio is composed strategically so that the expected instantaneous dollar return dx is non-stochastic, meaning that the coefficient for $dW(t)$ in equation (3.9) is required to be zero. On the other hand, we defined earlier that the net investment in portfolio needs to equal zero.

Therefore, following equations must hold

$$\begin{cases} w_1(\mu - r) + w_2(\mu_y - r) = 0 \\ w_1\sigma + w_2\sigma_y = 0. \end{cases} \tag{3.10}$$

The equation pair provides two possible ways to express w_1, from which we further derive the relation between μ, μ_y, σ, and σ_y.

$$\begin{aligned} w_1 = -\frac{w_2(\mu_y - r)}{\mu - r} &= -\frac{w_2\sigma_y}{\sigma} \\ \Rightarrow \frac{\mu_y - r}{\mu - r} &= \frac{\sigma_y}{\sigma} \\ \Leftrightarrow \frac{\mu - r}{\sigma} &= \frac{\mu_y - r}{\sigma_y} \end{aligned} \tag{3.11}$$

As we now substitute μ_y and σ_y with values from equation (3.6), we get

$$\begin{aligned} \frac{\mu - r}{\sigma} &= \frac{\frac{1}{F}(F_t + \mu V F_v + \frac{1}{2}\sigma^2 V^2 F_{vv}) - r}{\frac{1}{F}\sigma F_v V} \\ &= \frac{F_t + \mu V F_v + \frac{1}{2}\sigma^2 V^2 F_{vv} - rF}{\sigma F_v V}. \end{aligned} \tag{3.12}$$

Manipulating the equation further we eventually arrive to an appealing form.

$$\begin{aligned} (\mu - r)F_v V &= F_t + \mu V F_v + \frac{1}{2}\sigma^2 V^2 F_{vv} - rF \\ \Leftrightarrow 0 &= F_t + rVF_v + \frac{1}{2}\sigma^2 V^2 F_{vv} - rF. \end{aligned} \tag{3.13}$$

It is noteworthy that the partial differential equation (3.13) only depends on the value of the firm V, time t, interest rate r, the volatility of firm's value σ. On the other hand, the equation does not include the expected rate of return on the firm, nor the risk-preferences of investors. Any security whose value can be written as a function of the value of the firm and time $F(V, t)$ must satisfy the partial differential equation (3.13).

An essential feature of PDE (3.13) is that even if two investors had different utility functions and different expectations for the firm's future but agreed on the volatility σ of the firm's value, they would, for a given interest rate and current firm value, agree on the value of the particular security F. A complete solution of the PDE additionally requires boundary conditions and an initial value. It is the boundary conditions that distinguish securities from each other.

3.1.2 Extending Merton's model to ASRF

In his 2002 paper [30], Vasicek shows that it is possible to extend Merton's model to a specific Asymptotic Single Risk Factor credit portfolio model. Here, it is assumed that a loan defaults if the borrower cannot meet their obligations because the value of the borrower's assets at maturity T falls below the contractual value D of payable obligations.

Let A_i be the value of the i-th borrower's assets whose dynamics through time can be described with a process

$$dA_i = \mu_i A_i dt + \sigma_i A_i dW_i(t), \tag{3.14}$$

where $W_i(t)$ denotes a Wiener process, and μ_i and σ_i represent the asset-specific drift and volatility, respectively.

It can be shown that the stochastic differential equation (3.14) is satisfied by a stochastic process following geometric Brownian motion. It is important to note that for Wiener processes W_i apply $W_i(0) = 0$. In addition, Wiener processes have independent and normally distributed increments: $W_i(t) - W_i(s) \sim N(0, t-s)$, $t > s$. Thus, it holds that $W_i(t) \sim N(0, t)$. From here we can derive the solution for SDE (3.14) by first dividing both sides by A_i:

$$\frac{dA_i}{A_i} = \mu_i dt + \sigma_i dW_i(t) \tag{3.15}$$

$$\Rightarrow \int_0^t \frac{dA_i}{A_i} = \mu_i t + \sigma_i W_i(t) \tag{3.16}$$

Because A_i is an Itô process, we can use Itô's formula (A.5) for $d(\ln A_i)$

$$\begin{aligned}
d(\ln A_i) &= \frac{dA_i}{A_i} - \frac{1}{2}\frac{dA_i dA_i}{A_i^2} \\
&= \frac{dA_i}{A_i} - \frac{1}{2}\frac{\sigma_i^2 A_i^2 dt}{A_i^2} \\
&= \frac{dA_i}{A_i} - \frac{1}{2}\sigma_i^2 dt \qquad (3.17) \\
\Rightarrow \ln A_i - \ln A_i(0) &= \int_0^t \frac{dA_i}{A_i} - \frac{1}{2}\sigma_i^2 t \qquad (3.18)
\end{aligned}$$

From equations (3.16) and (3.18) we can derive the following solution

$$\begin{aligned}
\Rightarrow \ln A_i &= \ln A_i(0) + \mu_i t - \frac{1}{2}\sigma_i^2 t + \sigma_i W_i(t) \qquad (3.19) \\
&= \ln A_i(0) + (\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i \sqrt{t} X_i(t), \qquad (3.20)
\end{aligned}$$

where $X_i(t)$ is a random variable following the standard normal distribution. Now, the probability of the i-th loan defaulting at maturity time T is the probability of the value of assets A_i being less than the contractory value D_i.

$$\begin{aligned}
\mathbb{P}(A_i(T) < D_i) &= \mathbb{P}(X_i(T) > c_i) \\
&= \mathcal{N}(-c_i), \qquad (3.21)
\end{aligned}$$

where

$$c_i = \frac{\ln A_i(0) - \ln D_i + (\mu_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i \sqrt{T}}, \qquad (3.22)$$

and $\mathcal{N}$ denotes the cumulative normal distribution function.

Let us now consider a portfolio consisting of n loans in equal dollar amounts. Let the probability of default on each loan be p. In addition, assume that the asset values of the borrowing companies are correlated with a coefficient ρ for any two firms and that all the loans have the same term T.

Let L_i denote the gross loss on the i-th loan so that $L_i = 1$ if the i-th borrower defaults and $L_i = 0$ if it does not. In other words, we can express the gross loss as

$$L_i = \mathbb{1}_{\{X_i < c_i\}} = \begin{cases} 1 \text{ when } X_i < c_i, \\ 0 \text{ otherwise.} \end{cases} \quad (3.23)$$

Now, we can express the portfolio percentage gross loss L by

$$L = \frac{1}{n} \sum_{i=1}^{n} L_i. \quad (3.24)$$

By the central limit theorem, the distribution of the arithmetic mean of a large number of independent and identically distributed random variables would converge to a normal distribution. However, as the events of default are not independent, the conditions are not satisfied, and thus L is not asymptotically normally distributed.

However, as mentioned, asset values $A_i(t)$ of the borrowing companies are pairwise correlated with a coefficient ρ for any two firms and all the loans have the same term T. It follows that also variables X_i have pairwise correlation ρ. Thus we can express X_i with jointly standard normal variables (Y, Z_i)

$$X_i = aY + b_i Z_i, \quad (3.25)$$

where Y and Z_i, $i = 1, ..., n$, are mutually independent variables following standard normal distribution. The value for coefficients a and b_i follow from X_i being standard normally distributed and pairwise correlated and Y, Z_i being mutually independent.

We begin the derivation of coefficient a by examining the correlation of

X_i and X_j.

$$\begin{aligned}
\operatorname{corr}(X_i, X_j) = \frac{\operatorname{Cov}(X_i, X_j)}{\sigma_{X_i}\sigma_{X_j}} &= \frac{\mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])]}{\sigma_{X_i}\sigma_{X_j}} &(3.26)\\
&= \frac{\mathbb{E}[X_i X_j]}{1 \cdot 1}\\
&= \mathbb{E}[(aY + b_i Z_i)(aY + b_j Z_j)]\\
&= a^2\mathbb{E}[Y^2] + ab_i\mathbb{E}[Y]\mathbb{E}[Z_i]\\
&\quad + ab_j\mathbb{E}[Y]\mathbb{E}[Z_j] + b_i b_j\mathbb{E}[Z_i]\mathbb{E}[Z_j]\\
&= a^2\mathbb{E}[Y^2]\\
&= a^2(\operatorname{Var}[Y] + \mathbb{E}[Y]^2)\\
&= a^2 &(3.27)\\
\Rightarrow a &= \sqrt{\operatorname{corr}(X_i, X_j)} = \sqrt{\rho} &(3.28)
\end{aligned}$$

Values for coefficients b_i are derived using the variance of X_i.

$$\begin{aligned}
\operatorname{Var}[X_i] &= \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 &(3.29)\\
&= a^2\mathbb{E}[Y^2] + 2ab_i\mathbb{E}[Y]\mathbb{E}[Z_i] + b_i^2\mathbb{E}[Z_i^2]\\
&= a^2 + b_i^2 &(3.30)\\
\Rightarrow b_i &= \sqrt{\operatorname{Var}[X_i] - a^2} = \sqrt{1-\rho} &(3.31)
\end{aligned}$$

So variables X_i are can be expressed in the form

$$X_i = \sqrt{\rho}\, Y + \sqrt{1-\rho}\, Z_i. \tag{3.32}$$

Unconditional default probability describes the probability of obligor defaulting before some time horizon, considering all information that is currently observable. Conditional default probability, on the other hand, represents the default probability in the case where we also know the realized value of systematic risk factors at the time horizon. Hereby, we could also express the unconditional probability of default as the average value of the conditional probability across all possible realizations of the systematic risk at the horizon.

The portfolio common factor Y is an aspect affecting all the loans in the portfolio and can be therefore interpreted as an economic index, for instance, describing economic circumstances. We evaluate the probability of the portfolio loss as the expectation over the common factor Y of the conditional probability given Y. This can be viewed as assuming numerous scenarios for the economy and determining the probability of the portfolio loss under each scenario.

Given a fixed common factor Y, we can represent the conditional probability of any one loan defaulting as

$$\begin{aligned} p(Y) = \mathbb{P}[L_i = 1 \mid Y] =& \mathbb{P}[X_i < c_i \mid Y] && (3.33) \\ =& \mathbb{P}\left[Z_i < \frac{c_i - \sqrt{\rho}\, Y}{\sqrt{1-\rho}} \mid Y\right] && (3.34) \\ =& \mathcal{N}\left(\frac{c_i - \sqrt{\rho}\, Y}{\sqrt{1-\rho}}\right). && (3.35) \end{aligned}$$

Recalling that the probability of a loan defaulting is defined both as p and $P[X_i < c_i] = \mathcal{N}(c_i)$, we can substitute the variable c_i in Equation (3.35). Now the conditional probability of any one loan defaulting is of form

$$p(Y) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(p) - \sqrt{\rho}\, Y}{\sqrt{1-\rho}}\right). \tag{3.36}$$

Here $p(Y)$ represents the loan default probability under the given scenario Y. The expected value $p(Y)^*$ is the average of probabilities over all the scenarios Y weighted by their probabilities of occurrence.

Conditional on Y, the variables L_i are independent identically distributed variables with a finite variance. By the law of large numbers, the sample average converges in probability towards the expected value when sample size $n \to \infty$.

$$\begin{aligned} \mathbb{E}[L \mid Y] =& \mathbb{E}\left[\sum \frac{1}{n} L_i \mid Y\right] && (3.37) \\ =& \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[L_i \mid Y] && (3.38) \\ =& \mathbb{E}[L_i \mid Y] && (3.39) \end{aligned}$$

Thus, as the number n of loans in portfolio grows without bound, the portfolio loss L conditional on Y converges to its expectation $p(Y)$. This means that we can express the probability of portfolio loss being less than x as

$$\mathbb{P}[L \leq x] = \mathbb{P}[p(Y) \leq x]. \tag{3.40}$$

The following steps are of quite technical nature. By the result given in Vasicek (1991), we can derive the following distribution for portfolio loss

$$\mathbb{P}[L \leq x] = \mathcal{N}\left(\frac{\sqrt{1-\rho}\,\mathcal{N}^{-1}(x) - \mathcal{N}^{-1}(p)}{\sqrt{\rho}}\right). \tag{3.41}$$

The proof is rather technical and is therefore not included in this thesis. For details, see [29].

Let us now study the case of portfolios with unequally weighted loans. Let $w_1, w_2, \ldots, w_n$ denote the weights in the loan portfolio for which holds $\sum w_i = 1$. Then the porfolio percentage gross loss equals

$$L = \sum_{i=1}^{n} w_i L_i. \tag{3.42}$$

By Kolmogorov's law of large numbers: suppose $\{X_n, n \geq 1\}$ is a sequence of independent random variables. If

$$\sum_{i=1}^{\infty} \mathrm{Var}[X_i] < \infty, \tag{3.43}$$

then

$$\sum_{i=1}^{\infty} (X_i - \mathbb{E}[X_i]) \tag{3.44}$$

converges almost surely.

Suppose $U_i = w_i L_i$, and note that its variance is finite.

$$\begin{aligned} \mathrm{Var}[U_i] =& \mathbb{E}[U_i^2] - \mathbb{E}[U_i]^2 \\ \leq& \mathbb{E}[U_i^2] \\ =& \mathbb{E}[w_i^2 L_i^2] \\ =& w_i^2 \mathbb{E}[L_i^2] \\ \leq& w_i^2 < \infty \end{aligned} \tag{3.45}$$

Therefore, the expectation of loss L conditional on Y converges to $p(Y)$:

$$\begin{aligned}\mathbb{E}[L \mid Y] &= \mathbb{E}\left[\sum_{i=1}^{n} w_i L_i \mid Y\right] \\ &= \sum_{i=1}^{n} w_i \mathbb{E}[L_i \mid Y] \\ &\to \ \mathbb{E}[L_i|Y] = p(Y). \end{aligned} \tag{3.46}$$

Suppose the bank selects its portfolio as the first n elements of an infinite sequence of lending opportunities. To guarantee that idiosyncratic risk vanishes as more assets are added to the portfolio, the sequence of exposure sizes must neither blow up nor shrink to zero too quickly. To ensure that the portfolio loss L conditional on Y asymptotically approaches its expected value, we require its variance approach zero as $n \to \infty$.

$$\begin{aligned}\mathrm{Var}[L \mid Y] =&\mathrm{Var}\left[\sum_{i=1}^{n} w_i L_i \mid Y\right] \\ =&\sum_{i=1}^{n} w_i^2 \mathrm{Var}[L_i \mid Y] \end{aligned} \tag{3.47}$$

As mentioned, conditional on Y the variables L_i are independent with finite variance. It is therefore required that

$$\lim_{n\to\infty} \sum_{i=1}^{n} w_i^2 = 0, \tag{3.48}$$

for the variance of L conditional on Y to vanish

$$\lim_{n\to\infty} \sum_{i=1}^{n} w_i^2 \mathrm{Var}[L_i \mid Y] = 0. \tag{3.49}$$

Thus, when $\sum w_i^2 \to 0$, the portfolio percentage gross loss L conditional on Y converges to its expectation $p(Y)$

$$L \mid Y = \sum_{i=1}^{n} w_i L_i \mid Y \to \mathbb{E}[L \mid Y] = p(Y). \tag{3.50}$$

The result implies that in the case where the portfolio includes sufficiently large number of loans and no loan would dominate in size, then the result in (3.41) holds also with unequal weights on loans.

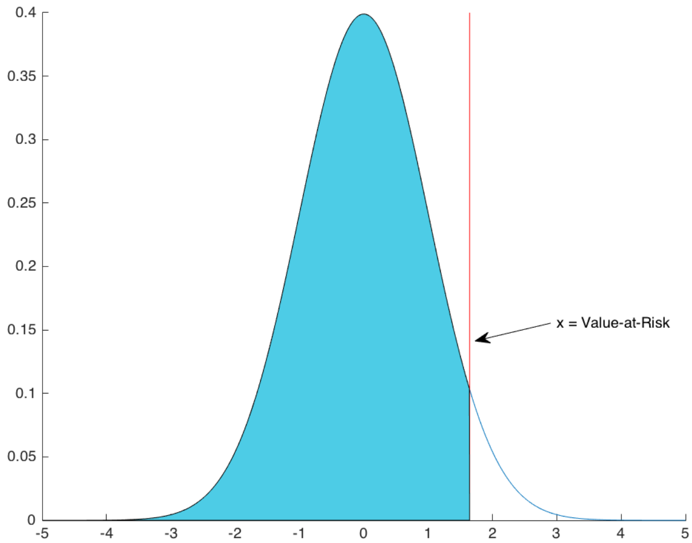
3.1.3 Portfolio-invariance

For a given $q \in (0, 1)$, Value-at-Risk is defined as the q-th quantile of the distribution of loss, and is denoted $\mathrm{VaR}_q[L_n]$. Let $\alpha_q(Y)$ denote the q-th quantile of the distribution of random variable Y.

$$\alpha_q(Y) \equiv \inf\{y : \mathbb{P}[Y \leq y] \geq q\} \tag{3.51}$$

In terms of this more general notation we now have

$$\mathrm{VaR}_q[L_n] = \alpha_q(L_n). \tag{3.52}$$



Assumption 1. The portfolio percentage loss is defined as

$$L = \sum_{i=1}^{n} w_i L_i. \tag{3.53}$$

Assumption 2. The weights w_i form a sequence of positive constants such that

$$\sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} w_i^2 \to 0. \tag{3.54}$$

The assumptions on L, L_i, and w_i are similar to properties presented in previous subsection.

We have shown that if Assumptions 1 and 2 hold, then conditional on the common factor Y, $L - \mathbb{E}[L|Y] \to 0$ almost surely. We will now show that if Assumptions 1 and 2 hold, then

$$\mathrm{Var}[L] - \mathrm{Var}[\mathbb{E}[L|Y]] \to 0. \tag{3.55}$$

By the law of total variance

$$\mathrm{Var}[L] = \mathbb{E}[\mathrm{Var}[L|Y]] + \mathrm{Var}[\mathbb{E}[L|Y]], \tag{3.56}$$

we can substitute the left hand side of Equation (3.55) with $\mathbb{E}[\mathrm{Var}[L|Y]]$, resulting in a requirement

$$\mathbb{E}[\mathrm{Var}[L \mid Y]] \to 0. \tag{3.57}$$

Now,

$$\begin{aligned} \mathbb{E}[\mathrm{Var}[L \mid Y]] =& \mathbb{E}\left[\sum_{i=1}^{n} w_i^2 \mathrm{Var}[L_i|Y]\right] \\ =& \sum_{i=1}^{n} w_i^2 \mathbb{E}[\mathrm{Var}[L_i|Y]]. \end{aligned} \tag{3.58}$$

Given that $\sum_{i=1}^{n} w_i^2 \to 0$, it needs to hold that

$$\mathbb{E}[\mathrm{Var}[L_i|Y]] < \infty. \tag{3.59}$$

As we can see from Equation (3.23), the variance of variable L_i conditional on Y is finite by definition. We can now move to an even more desirable result. We want that for any $q \in (0,1)$, the q-th quantile of the unconditional loss distribution approaches the unconditional distribution of $\mathbb{E}[L|Y]$ as $n \to \infty$. In other words, we desire to have

$$\alpha_q(L) - \alpha_q(\mathbb{E}[L|Y]) \to 0. \tag{3.60}$$

For technical reasons we have to limit the result on its "slightly restricted" variant. Let F_n denote the cumulative distribution function of portfolio loss L. Now, under Assumptions 1 and 2, we can show that for any $\epsilon > 0$

$$\begin{cases} F_n(\alpha_q(\mathbb{E}[L|Y]) + \epsilon) \to [q, 1], \\ F_n(\alpha_q(\mathbb{E}[L|Y]) - \epsilon) \to [0, q]. \end{cases} \tag{3.61}$$

From equation (3.50), we know that conditional on $Y = y$, $L - \mathbb{E}[L \mid Y] \to 0$ almost surely. This implies convergence in probability and can be expressed as

$$\mathbb{P}[|L - \mathbb{E}[L \mid Y]| < \epsilon \mid Y = y] \to 1. \tag{3.62}$$

We can express the probability of the absolute value as the sum of probabilities of two events.

$$\mathbb{P}[L - \mathbb{E}[L \mid Y] < \epsilon \mid Y = y] - \mathbb{P}[L - \mathbb{E}[L \mid Y] < -\epsilon \mid Y = y] \to 1 \tag{3.63}$$

$$\Leftrightarrow \quad \mathbb{P}[L < \mathbb{E}[L \mid Y] + \epsilon \mid Y = y] - \mathbb{P}[L < \mathbb{E}[L \mid Y] - \epsilon \mid Y = y] \to 1 \tag{3.64}$$

Given that F_n is the cumulative distribution function of L, it follows that

$$F_n(\mathbb{E}[L \mid Y] + \epsilon \mid Y = y) - F_n(\mathbb{E}[L \mid Y] - \epsilon \mid Y = y) \to 1. \tag{3.65}$$

The cumulative distribution function F_n is bounded in $[0, 1]$, so it must hold that

$$\begin{cases} F_n(\mathbb{E}[L \mid Y] + \epsilon \mid Y = y) \to 1 \\ F_n(\mathbb{E}[L \mid Y] - \epsilon \mid Y = y) \to 0. \end{cases} \tag{3.66}$$

Now, let S^+ denote the set of realizations y of Y such that the conditional expectation $\mathbb{E}[L \mid Y]$ is less than or equal to its q-th quantile:

$$S^+ \equiv \{y : \mathbb{E}[L \mid Y = y] \leq \alpha_q(\mathbb{E}[L \mid Y = y])\}. \tag{3.67}$$

The definition (3.67) ensures that $\mathbb{P}[y \in S^+] \geq q$. Using the rules for conditional probability, we have

$$\begin{aligned} F_n(\alpha_q(\mathbb{E}[L \mid Y]) + \epsilon) =& F_n(\alpha_q(\mathbb{E}[L \mid Y]) + \epsilon \mid Y \in S^+)\mathbb{P}[Y \in S^+] \\ &+ F_n(\alpha_q(\mathbb{E}[L \mid Y]) + \epsilon \mid Y \notin S^+)\mathbb{P}[Y \notin S^+] \quad (3.68) \\ \geq& F_n(\alpha_q(\mathbb{E}[L \mid Y]) + \epsilon \mid Y \in S^+)\mathbb{P}[Y \in S^+] \\ \geq& F_n(\alpha_q(\mathbb{E}[L \mid Y]) + \epsilon \mid Y \in S^+)q. \quad (3.69) \end{aligned}$$

By construction (3.67), for all $y \in S^+$ it holds that

$$F_n(\alpha_q(\mathbb{E}[L \mid y]) + \epsilon) \mid y) \geq F_n(\mathbb{E}[L \mid y] + \epsilon \mid y) \overset{(3.66)}{\to} 1. \quad (3.70)$$

It follows from the dominated convergence theorem [3] that

$$\begin{aligned} F_n(\alpha_q(\mathbb{E}[L \mid Y]) + \epsilon \mid Y \in S^+) \to 1 \quad (3.71) \\ \Rightarrow F_n(\alpha_q(\mathbb{E}[L \mid Y]) + \epsilon) \geq q. \quad (3.72) \end{aligned}$$

Thus, we have the desired result

$$F_n(\alpha_q(\mathbb{E}[L \mid Y]) + \epsilon) \in [q, 1]. \quad (3.73)$$

The other half: Let S^- denote the set of realizations y of Y such that the conditional expectation $\mathbb{E}[L \mid Y]$ is greater than or equal to its q-th quantile, i.e.

$$S^- \equiv \{y : \mathbb{E}[L \mid Y = y] \geq \alpha_q(\mathbb{E}[L \mid Y = y])\}, \quad (3.74)$$

so $\mathbb{P}[y \in S^-] \geq 1 - q$. Then,

$$\begin{aligned} F_n(\alpha_q(\mathbb{E}[L \mid Y]) - \epsilon) =& F_n(\alpha_q(\mathbb{E}[L \mid Y]) - \epsilon \mid Y \notin S^-)\mathbb{P}[Y \notin S^-] \\ &+ F_n(\alpha_q(\mathbb{E}[L \mid Y]) - \epsilon \mid Y \in S^-)\mathbb{P}[Y \in S^-] \\ \leq& q + F_n(\alpha_q(\mathbb{E}[L \mid Y]) - \epsilon \mid Y \in S^-)q. \quad (3.75) \end{aligned}$$

For all y in S^-, we have

$$F_n(\alpha_q(\mathbb{E}[L \mid Y] - \epsilon \mid Y = y) \leq F_n(\mathbb{E}[L \mid Y] - \epsilon \mid Y = y) \overset{(3.66)}{\to} 0, \quad (3.76)$$

so by the dominated convergence theorem,

$$F_n(\alpha_q(\mathbb{E}[L \mid Y]) - \epsilon \mid Y \in S^-) \to 0. \quad (3.77)$$

Now, from equation (3.75) we get the desired

$$F_n(\alpha_q(\mathbb{E}[L \mid Y]) - \epsilon) \to [0, q]. \tag{3.78}$$

In practical sense, the requirement (3.61) ensures the relation (3.60). Literally interpreted, the same equation (3.61) claims that if the capital required is strictly greater than the q-th quantile of $\mathbb{E}[L|Y]$, then it is guaranteed, in the limit, to cover or come arbitrarily close to covering q or more of the distribution. Similarly, the required capital fails to cover q (or more) of the distribution L if it is strictly less than the q-th quantile of $\mathbb{E}[L|Y]$.

Equation (3.61) is significant because it allows us to substitute the quantiles of portfolio loss L with the respective quantiles of $\mathbb{E}[L|Y]$ when the portfolio grows large. In many cases, the quantiles of $\mathbb{E}[L|Y]$ are easier to calculate than the quantiles of L.

It is important to note that we have so far made very few restrictions on the composition of the portfolio as well as on the nature of credit risk in the model.

Adding two restrictions, we obtain a desirably simple and asymptotic form for the expectation $\mathbb{E}[L|Y]$.

Assumption 3. The systematic risk factor Y is one-dimensional.

Assumption 4. There is an open interval B containing $\alpha_q(Y)$ and a real number $n_0 < \infty$ such that

(i) for all i, the conditional expectation $\mathbb{E}[L_i \mid Y = y]$ is continuous in y on B,

(ii) $\mathbb{E}[L \mid Y = y]$ is nondecreasing in y on B for all $n > n_0$, and

(iii) $\inf \{\mathbb{E}[L \mid Y = y] : y \in B\} \geq \sup \{\mathbb{E}[L \mid Y = y] : y \leq \inf B\}$ and $\sup \{\mathbb{E}[L \mid Y = y] : y \in B\} \leq \inf \{\mathbb{E}[L \mid Y = y] : y \geq \sup B\}$ for all $n > n_0$.

The assumption 3 can be interpreted so that there is a single global economic cycle that accounts for all dependence across exposures. Assumption 4 brings some very desirable features to the model. Part (iii) sets all the values on the left side of interval B smaller than or equal to values on B, and all the values on the right side of B greater than or equal to values on B. In other words, the areas outside interval B are not required to be monotonic. By assuming 4, we also ensure that the environment of the q-th quantile of

$\mathbb{E}[L \mid Y]$ is linked to a unique environment of the q-th quantile of Y. Some L_i are allowed to be negatively connected with Y as long as, asymptotically and in the overall picture, the monotonic dependence of losses on Y stays unaffected. In practice, this means that portfolios are allowed to include instruments and obligors that have counter-cyclical credit risk. For practical purposes, we introduce a notation $\mu_i(y) \equiv \mathbb{E}[L_i \mid Y = y]$, followed by a notation

$$M_n(x) \equiv \mathbb{E}[L \mid Y = y] = \mathbb{E}\left[\sum_{i=1}^{n} w_i L_i \mid Y = y\right] \tag{3.79}$$

$$= \sum_{i=1}^{n} w_i \mathbb{E}[L_i \mid Y = y] \tag{3.80}$$

$$= \sum_{i=1}^{n} w_i \mu_i(y). \tag{3.81}$$

Following assumptions 3 and 4, we can show that for $n > n_0$, it holds that

$$\alpha_q(\mathbb{E}[L \mid Y]) = \mathbb{E}[L \mid \alpha_q(Y)] = M_n(\alpha_q(Y)). \tag{3.82}$$

Fix $n > n_0$. If $Y \leq \alpha_q(Y)$, then, by assumption 4, $M_n(Y) \leq M_n(\alpha_q(Y))$. Therefore, we can state that

$$\mathbb{P}[M_n(Y) \leq M_n(\alpha_q(Y))] \ \geq \ \mathbb{P}[Y \leq \alpha_q(Y)] \ \geq \ q. \tag{3.83}$$

The latter inequation follows directly from the definition of α_q (Eq. (3.51)).

Now, fix any $z < M_n(\alpha_q(Y))$, and let

$$\hat{y} = \sup\{y : M_n(y) \leq z\}. \tag{3.84}$$

Because $\hat{y}$ is the least upper bound for y that satisfies the condition $M_n(y) \leq z$, it follows that

$$M_n(Y) \leq z \quad \Rightarrow \quad Y \leq \sup\{y : M_n(y) \leq z\} = \hat{y}. \tag{3.85}$$

Thus, given that Y follows standard normal distribution,

$$\mathbb{P}[M_n(Y) \leq z] \ \leq \ \mathbb{P}[Y \leq \hat{y}] = \mathbb{P}[Y < \hat{y}]. \tag{3.86}$$

Assumption 4 ensures that $\hat{y} < \alpha_q(Y)$, so

$$\mathbb{P}[Y < \hat{y}] < \mathbb{P}[Y < \alpha_q(Y)] < q. \tag{3.87}$$

Via inequation (3.83), we know that $\mathbb{P}[M_n(Y) \le M_n(\alpha_q(Y))] \ge q$. On the other hand, we also have showed that $\mathbb{P}[M_n(Y) \le z] < q$ for any $z < M_n(\alpha_q(Y))$. Thus,

$$\alpha_q(M_n(Y)) = \inf\{z : \mathbb{P}[M_n(Y) \le z] \ge q\} = M_n(\alpha_q(Y)). \quad \square \tag{3.88}$$

The result allows us to utilize the linearity of the expectations operator, which simplifies the computing of quantiles. In comparison to the generally complicated $\alpha_q(\mathbb{E}[L \mid Y])$, the term $\mathbb{E}[L \mid \alpha_q(Y)]$ can be expressed as

$$\mathbb{E}[L \mid \alpha_q(Y)] = \sum_{i=1}^{n} w_i \mathbb{E}[L_i \mid \alpha_q(Y)] \tag{3.89}$$

$$= \sum_{i=1}^{n} w_i \mu_i(\alpha_q(Y)). \tag{3.90}$$

We can now determine capital requirements through a relatively simple way. Let allocated capital per dollar value for asset i be

$$c_i \equiv \mu_i(\alpha_q(Y)) + \epsilon, \tag{3.91}$$

where ϵ is some arbitrarily small number. Portfolio losses exceed capital only if

$$\sum_{i=1}^{n} w_i L_i > \sum_{i=1}^{n} w_i c_i. \tag{3.92}$$

Given our definitions for c_i and $L = \sum_{i=1}^{n} w_i L_i$, we have

$$\mathbb{P}\left[\sum_{i=1}^{n} w_i L_i > \sum_{i=1}^{n} w_i c_i\right] = \mathbb{P}\left[L > \sum_{i=1}^{n} w_i(\mu_i(\alpha_q(Y)) + \epsilon)\right] \tag{3.93}$$

$$= \mathbb{P}\big[L > \mathbb{E}\big[L|\alpha_q(Y)\big] + \epsilon\big] \tag{3.94}$$

$$= \mathbb{P}\left[L > \alpha_q(\mathbb{E}[L|Y]) + \epsilon\right] \tag{3.95}$$

$$= 1 - \mathbb{P}[L \le \alpha_q(\mathbb{E}[L|Y]) + \epsilon] \tag{3.96}$$

$$\overset{(3.61)}{\rightarrow} [0, 1 - q]. \tag{3.97}$$

The result implies that, in the limit, the probability of portfolio capital failing to cover portfolio credit losses is less than $1 - q$. It is important to notice that the suggested capital charge depends solely on the characteristics of the asset i. Hence, this feature establishes the desired portfolio-invariance.

3.1.4 The Unexpected Loss of a Portfolio

In the Basel II framework, the expected loss of a portfolio is assumed to comprise of three elements [11]. They are the proportion of borrowers that might default within a given time frame, the unresolved exposure at default, and the loss given default that describes the percentage of credit exposure the financial institution might lose if the obligor defaults.

Let PD denote the probability of default per rating grade. It gives the average percentage of obligors that default in that rating grade in the course of one year. Let LGD then denote the loss given default that describes the percentage of credit exposure the financial institution might lose if the obligor defaults. Finally, let EAD denote the exposure at default that represents an estimate of the amount outstanding if the obligor defaults.

The expected loss (in currency amounts) can then be expressed as follows

$$EL = PD \cdot EAD \cdot LGD. \tag{3.98}$$

Now, considering the ASRF framework, we can estimate the sum of the expected and unexpected losses related to each credit exposure by "calculating the conditional expected loss for an exposure given an appropriately conservative value of the single systematic risk factor". In the particular case of the ASRF model applied in Basel II, the conditional expected loss is formulated as a product of the probability of default PD and the loss-given-default LGD.

The suitable default threshold for average default conditions: apply a reverse of the Merton model to the average PD's. Merton's model connects the default threshold and the obligor's PD through normal distribution function. Thus, the default threshold can be derived from the inverse normal distribution function applied to the average PD. Similarly, we can get a hold of the "appropriately conservative value" of the systematic risk factor by applying the inverse normal distribution function to the predetermined supervisory confidence level. As we weigh the sum of default threshold and the conservative value of the systematic factor with asset correlation, we get conditional default threshold.

Let us re-examine the derived distribution for the portfolio percentage gross loss in (3.41):

$$\mathbb{P}[L \leq x] = \mathcal{N}\left(\frac{\sqrt{1-\rho}\,\mathcal{N}^{-1}(x) - \mathcal{N}^{-1}(p)}{\sqrt{\rho}}\right) \tag{3.99}$$

Here, x denotes a "limit" for the portfolio percentage gross loss, the so-called default threshold.

When we apply the normal distribution function to the conditional default threshold we get the probability to the value of assets being less than the given default threshold. So, rearranging the terms we get

$$\mathcal{N}^{-1}(x) = \frac{\sqrt{\rho}\,\mathcal{N}^{-1}(\mathbb{P}[L \leq x]) + \mathcal{N}^{-1}(p)}{\sqrt{1-\rho}} \tag{3.100}$$

and further (3.101)

$$x = \mathcal{N}\left(\sqrt{\frac{\rho}{1-\rho}}\mathcal{N}^{-1}(\mathbb{P}[L \leq x]) + \frac{1}{\sqrt{1-\rho}}\,\mathcal{N}^{-1}(p)\right). \tag{3.102}$$

LGD multiplied with the default threshold describes the total capital required to cover the Value-at-Risk. The Value-at-Risk consists of both Expected and unexpected losses.

The term $PD \cdot LGD$ describes the expected loss of a portfolio. It is deducted from the required capital value given by ASRF model because the ASRF only delivers the total capital amount.

Basel II framework utilizes average PD's. They reflect expected default rates under normal business conditions. Banks estimate the average PD's themselves, and, in order to obtain the conditional expected loss, these average PD's reported by banks are derived into conditional probabilities of default.

The conditional probabilities of default illustrate default rates given an appropriately conservative value of the systematic risk factor.

3.2 Asset correlations

In the Basel II framework, the state of the overall economy is modelled as the single systematic risk (SSR) factor. As mentioned, this risk factor affects all assets and could be interpreted so that it links all the obligors to each other. The asset correlation R describes how much a particular asset is exposed to this systematic risk. As all the common risk between assets in the ASRF model comes from the SSR factor, the asset correlation can be interpreted to describe how asset values depend on each other.

In the ASRF distribution formula (3.41), the risk weight formulas

$$\sqrt{\frac{1}{1-\rho}} \quad \text{and} \quad \sqrt{\frac{\rho}{1-\rho}} \tag{3.103}$$

are shaped by the asset correlation ρ. The asset correlations, and thus also risk weights, depend on the asset class. Different asset classes have different degrees of dependence on the state of the global economy. For instance, large corporations are often more sensitive to the state of the world economy than smaller local businesses. Therefore, assets in large corporation loan books tend to have higher asset correlation than retail loan portfolios. The stronger the asset correlation the higher the variation in loss rates.

There has been some empirical evidence that asset correlations seem to decrease with increasing PD's [16]. The effect can also be reasoned with intuition. For instance, higher probability of default indicate higher idiosyncratic, borrower-specific risk components, meaning that the default risk is more subject to the company-specific factors and less subject to overall economic conditions, the common factors.

The asset correlations in Basel II risk weight formulas (3.103) are determined by the function

$$\rho = 0.12\,\frac{1-e^{-kPD}}{1-e^{-k}} + 0.24\Big(1-\frac{1-e^{-kPD}}{1-e^{-k}}\Big) - 0.04\Big(1-\frac{S-5}{45}\Big), \tag{3.104}$$

where k denotes the asset class dependent k-factor, PD denotes the probability of default, and S denotes the firm's annual sales [11].

The k-factor determines how fast the exponential function decreases and is set at 50 for corporate exposures.

The formula also takes into account the size of the company. The part

$$-0.04\Big(1-\frac{S-5}{45}\Big) \tag{3.105}$$

includes the company's annual sales as its parameter. The firm size adjustment has two limit points: at $S=5$ the term equals -0.04, and at $S=50$ zero. The size adjustment ensures that asset correlations are set lower for small companies than for large corporates. Also this could be reasoned with intuition: the larger the firm, the more dependent it is on the state of world economy.

3.3 Maturity adjustments

Credit portfolios typically consist of instruments with different maturities. Intuitively, long-term credits can be considered riskier than short-term credits. There are also empirical studies that support this view [18]. Interest rates have a greater probability to rise within a longer period of time than within a shorter period. When interest rates get higher, it affects a bond's market price negatively. The increase in risk indicates that the required capital should also increase with maturity.

The face value of an asset is the nominal value the asset holds. The market value of an asset, on the other hand, is determined by demand and supply. How well the face and market values correlate depends on the market conditions.

Two loans with the same face value can be notably different in the market value if the default probabilities differ from another. Loans with high default probabilities have lower market value today than loans with low default probabilities because investors take the expected loss into consideration in asset pricing.

The potential downgrades and loss of market-value become the more significant the longer the maturity. With higher market values, low PD loans are more probable to face down-gradings than high PD loans. Therefore, the Basel II maturity adjustments depend on the default probability as well. The maturity adjustments are, in relative terms, higher for low PD loan books than for high PD books.

The Basel II framework maturity adjustments are a function of maturity M and probability of default PD. The actual form is derived by implementing a specific Mark-to-Market (MTM) credit risk model [11].

The portfolio model outputs a grid of Value-at-Risk measures for a range of maturities and rating grades. The rating grades represent the probabilities of default.

The model sets the standard maturity to 2.5 years. Maturity adjustments are obtained by computing ratios of each VaR value in the grid to the VaR of the standard maturity for each rating grade. In order to get the maturity adjustments in function form, a grid of these VaR ratios is smoothed by a statistical regression model.

The full maturity adjustment in the formula for capital requirement (3.108) comprises of the part

$$\frac{1 + (M - 2.5) \cdot b(PD)}{1 - 1.5b(PD)}, \tag{3.106}$$

where M denotes the maturity, and PD denotes the probability of default.

The element $b(PD)$ describes the smoothed maturity adjustment over probabilities of default PD, and is formally expressed

$$b(PD) =[0.11852 - 0.05478 \cdot \log(PD)]^2. \tag{3.107}$$

As we can see from equations (3.106) and (3.107), the regression model is chosen so that it holds certain features. The adjustments are linear and increasing in the maturity M. For a maturity of one year, the function yields the value 1, which ensures that the resulting capital requirement corresponds to the output of the ASRF model. In addition, the slope of the adjustment function with respect to M decreases as the PD increases.

3.4 Capital requirement

Let ρ denote asset correlation, describing the degree of exposure to the systematic risk, $\mathcal{N}$ denote the cumulative normal distribution function, $\mathcal{N}^{-1}$ denote the inverse normal distribution function, and $b(PD)$ denote smoothed maturity adjustment over PD. Let LGD denote the loss given default, and M denote the effective maturity.

Now, given the result (3.102) we can write the formula for capital requirement K as

$$\begin{aligned} K =& \Big(LGD \cdot \mathcal{N} \left[\sqrt{\frac{1}{1-\rho}} \cdot \mathcal{N}^{-1}(PD) + \sqrt{\frac{\rho}{1-\rho}} \cdot \mathcal{N}^{-1}(0.999) \right] \\ & - PD \cdot LGD \Big) \cdot \frac{1 + (M - 2.5) \cdot b(PD)}{1 - 1.5 b(PD)}. \end{aligned} \tag{3.108}$$

LGD multiplied with the default threshold given by the ASRF model describes the total capital required to cover the Value-at-Risk. The Value-at-Risk consists of both the expected and unexpected losses.

The term $PD \cdot LGD$ describes the expected losses of a portfolio. It is deducted from the value given by the ASRF model so that the value of required capital charges concentrates exactly to cover unexpected losses. The 0.999 in the formula represents the confidence level that by the likelihoof of 99.9 percent the portfolio loss is less than the regulatory capital.

Chapter 4

Benefits and Problems Related to the Regulatory Capital

The original Basel Accord [21] was in need of revision due to its relatively simplistic approach to supervisory capital requirements. Basel II framework [23] offers a technologically more sophisticated solution to regulatory needs. This chapter discusses both theoretical and realized benefits from Basel II regulatory capital charges, as well as problems related to the framework.

4.1 Benefits

4.1.1 Regulatory arbitrage

One of the most noteworthy weaknesses related to the Basel I framework is the possibility of regulatory arbitrage. The supervisory rules on capital charges include loopholes, which enable banks to relatively easily circumvent the regulation. This allowed many banks to control the amount of required capital by suitably moving assets on and off balance sheet. However, although the capital was sufficient on paper it often did not provide enough protection for bank portfolios. Thus, the realization of the regulation was far from the desired outcome. [5],[9]

4.1.2 Small and medium sized enterprises (SME)

Altman and Sabato [1] study how Basel II capital requirements affect small and medium sized enterprises (SME). Small and medium sized enterprises play globally a significant role in the economy: for instance, in OECD countries, over 97 % of all firms are considered SME. Due to their rather simple structure SMEs can respond to changing economic conditions quickly.

Borrowings from commercial banks form the main source of financing for small and medium sized enterprises. Banks, on the other hand, could potentially think that SMEs carry higher risk than large corporations. As Basel II introduced its risk-calibrating approach, concerns arose if the lending activity towards SMEs was to reduce. However, under Basel, II banks are able to consider SMEs as retail or as corporate entities. The retail entities are assumed to correlate less with systemic risk and be less sensitive to the business cycle. The SME companies classified as retail can experience significantly lower regulatory capital than companies considered as corporate. Altman and Sabato [1] find that along this new classification, SME firms might face even easier lending opportunities.

However, retail classification requires organizationally and technologically more than the corporate classification. Thus, banks may be inclined to classify all SMEs as corporate to avoid additional costs. In the end, banks face a trade-off between lower regulatory capital and higher technology costs. On the other hand, banks might also be motivated to implement more advanced internal models to lower the technology costs.

4.2 Problems

Especially after the 2008 global financial crisis, there has been much discussion on the deficiencies in the Basel II framework that the crisis unveiled. However, it needs to be noted that Basel II capital charges had not been fully implemented when the crisis hit the economy.

4.2.1 Single measure of systematic risk

One of the concerns regarding Basel II regulatory capital relates to the Asymptotic Single-Risk Factor model that the framework is based on. There is on-going discussion on whether the assumption of single systemic risk factor can be defended [7], [10].

For instance, Borio (2011) argues that it is not realistic to assume a single measure of systemic risk to fit all purposes. The 2008 crisis revealed alternative sources for systemic risk: a common shock that results in financial institutions defaulting simultaneously, and the "informational spillovers" where the spread news about one bank defaulting affect other banks in form of increased refinancing costs [10].

4.2.2 Procyclicality

In the context of economic policy, procyclicality refers to any aspect of policy that could enhance the fluctuations in economy. An economic or financial policy is called countercyclical if it works against the cyclical tendencies in the economy. Countercyclical policies cool down the economy, when it is in upswing, and stimulate it in downturn.

Basel II regulation insists banks to increase their capital ratio when the risks get higher. However, as risk factors tend to grow during recession, the increasing capital ratios weaken the lending opportunities. Hence, the economic downturn can lack stimulation, which in turn could make recession even deeper and weaken the recovery of the economy.

Bongaerts and Charlier (2009) among others addresses concern that rises as banks' internal credit models become more risk-sensitive.

As Blundell-Wignall and Atkinson (2010) point out, the Basel II risk-weighting approach could be considered failed at its goals in the event of economic turmoil. It turned out that banks with higher capital before the 2008 crisis faced higher losses than banks with lower capital buffer, quite opposite to the intended outcome.

4.2.3 Value-at-Risk

Value-at-Risk is an essential part of the Basel II regulatory capital model. It could be considered one of the most standard measures in financial risk management thanks to its relatively straightforward concept with desirable computing facilities. However, VaR has received criticism on whether it offers a comprehensive enough approach [6].

Conditional Value-at-Risk (CVaR), also known as expected shortfall or mean excess loss, is a risk measure that takes into account losses exceeding the Value-at-Risk [27]. The CVaR is derived as a weighted average of the Value-at-Risk and losses that exceed the Value-at-Risk.

From mathematical point of view, Value-at-Risk can be difficult to optimize as it generally lacks convexity and sub-additivity. CVaR on the other hand, is convex and sub-additive, and can be therefore optimized in relatively non-complex ways [27].

In addition, empirical studies show that risk managers that use Value-at-Risk in managing market-risk exposure tend to choose larger exposures to risky assets, which in case of losses occurring leads to larger losses. [2]

Some advantages of CVaR approach appear to be the improved ability to quantify threats beyond VaR, shortcuts in optimization, and numerical efficiency. Contrary to VaR, CVaR is not only constrained to percentiles of the distribution but can quantify also the "tail" of the loss distribution [26]. In addition to the mentioned shortcomings of VaR, some models currently in industrial use assume the loss given default LGD to be known and non-stochastic. In addition, when it comes to the most popular credit Value-at-Risk models, those that allow for stochasticity of LGD nevertheless consider the business recovery risk to be purely idiosyncratic. However, in practice the loss given default may not only be profoundly uncertain but also be exposed to systematic risk. For more, see [14].

4.2.4 Banks' internal models

Time series are often too short to provide sufficient information about the underlying processes. Making estimations based on inadequate data can lead to unreliable results and to wrong kind of suggestions. Hence, financial institutions should be aware of their data quality. Also, it should be noted that credit risk events are rare in nature and are therefore difficult to assess. All this can cause problems when banks estimate asset correlations. [15]

Banks often have a narrower view on the financial system than the models assume. Banks are not able to hold all the relevant information regarding the network structure of the system, which weakens the opportunity to estimate the portfolio correlations correctly. The underestimation of asset correlations makes it eventually impossible to reach optimal risk management. It has been suggested that the supervisory authorities take a more active role in determining the systematic risk as they hold better comprehension of the macro-level structure of the financial network. For further discussion, see [10]. In addition, it can vary quite a lot how different banks perceive the concepts of probability of default PD, loss given default LGD, and exposure at default EAD [9]. This can in turn make the banks' risk measures inaccurate and unsuitable for comparison.

Studies illustrate that a bank's chosen methodology for internal rating has impact on how volatile the regulatory capital is [8]. This also impacts how big the capital buffers should be. Peura and Jokivuolle (2004) show numerical results that, due to increased volatility, rating sensitive regulatory capital will call for higher capital buffers than Basel I suggested.

It is noteworthy that at a certain point the increasing capital will incite banks to move more capital into the unregulated shadow banking sector. For more, see [5] and [10].

Lastly, empirical studies conflict on the relationship between asset correlations and default probabilities. [13]

Chapter 5

Future prospects

The global financial crisis in 2008 showed the short-comings that Basel II framework has with how it regulates the systemic risk. The crisis also unveiled a moral hazard related to systemically important financial institutions. Although the Basel II was not yet fully implemented, the supervisory parties quickly agreed that the regulation framework needed revision. [10]

Several problems related to Basel II regulatory capital are addressed in the new Basel III framework [12]. Overall, the Basel III regulatory capital aims at better quality and quantity of buffer capital. It introduces new regulatory measures, such as leverage ratio that is designed to prevent the regulatory arbitrage.

However, in essence, the new framework has only small changes in the risk-calibration approach of Basel II. The new revision is not meant to overrun the Basel I and II recommendations but rather provide additional guidelines to sufficient bank capital. For this reason, Basel III faces criticism for not introducing alternatives to the Basel II risk weighting approach that could eventually prevent the procyclicality. [5]

There are proposals that suggest changes in the financial architecture. Some suggest that in the future the financial supervision should focus on protecting the financial "platforms" rather than banks [25]. These platforms could comprise of different financial markets as well as payment systems. The scale of structural changes needed in the system, however, might be too large for this approach to be executable [10]. Some papers on the other hand propose that the whole risk-calibration of capital requirement should be removed and instead the amount of regulatory capital should be considerably increased [15]. This approach would address the problem related to procyclicality of the Basel II and III frameworks but might also lack approval of the banking sector [10]. Another aspect that might need further discussion is the proper way to measure systemic distress.

The more advanced and accurate system cannot properly function until the more risk-sensitive measures are widely adopted. For this reason it is of high importance that the regulation also incites the banks to use more advanced risk-sensitive approaches. The banks are allowed to gain higher returns if they use better risk measures. However, too high additional gains could result in reduced competition as they might drive smaller banks out of business. [6],[23]

In the future, supervisors and experts might play a more significant role in regulation process and work more interactively with banks to improve the realization of regulation framework. It is also important to note that raising the level of regulation might come at a price. In case the amount of different reportable risk measures increases, supervisory parties should be prepared to prevent financial institutions from getting incentives to shift their capital into the unregulated shadow banking sector.

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Appendices

Appendix A

Itô's Lemma, informal proof

Assume that $X(t)$ is an Itô drift-diffusion process that satisfies the stochastic differential equation

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \tag{A.1}$$

where $W(t)$ is a standard Wiener-Gaussian process.

Let $f(x,t)$ be a twice-differentiable scalar function of which Taylor expansion can be written as

$$df = \frac{\partial f(x,t)}{\partial t}dt + \frac{\partial f(x,t)}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 + \dots . \tag{A.2}$$

Let us now substitute x with $X(t)$ and dx with $dX(t)$ from A.1

$$\begin{aligned} df =& \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\left(\mu(t)dt + \sigma(t)dW(t)\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\mu(t)dt + \sigma(t)dW(t)\right)^2 + \dots \\ =& \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\left(\mu(t)dt + \sigma(t)dW(t)\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\Big(\mu(t)^2 dt^2 \\ & + 2\mu(t)\sigma(t)dtdW(t) + \sigma(t)^2 dW(t)^2\Big) + \dots \end{aligned} \tag{A.3}$$

In the limit $dt \to 0$, the terms dt^2 and $dtdW(t)$ approach 0 faster than $dW(t)^2$ that is $\mathcal{O}(dt)$. Now if we substitute dt^2 and $dtdW(t)$ with 0, and $dW(t)^2$ with dt, we get

$$df = \left(\frac{\partial f}{\partial t} + \mu(t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma(t)^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma(t)\frac{\partial f}{\partial x}dW(t). \tag{A.4}$$

For a diffusion-type stochastic differential equation (A.1), Itô's lemma can also be expressed as

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))\sigma^2 dt. \tag{A.5}$$