CS/COE 1501

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Greedy Algorithms and Dynamic Programming

Consider the change making problem

- What is the minimum number of coins needed to make up a given value k?
- If you were working as a cashier, what would your algorithm be to solve this problem?

This is a *greedy algorithm*

- At each step, the algorithm makes the choice that seems to be best at the moment
- Have we seen greedy algorithms already this term?
    - Yes!
        - Building Huffman trees
        - Nearest neighbor approach to travelling salesman

… But wait …

- Nearest neighbor doesn't solve travelling salesman
  - Does not produce an optimal result
- Does our change making algorithm solve the change making problem?
  - For US currency…
  - But what about a currency composed of pennies (1 cent), thrickels (3 cents), and fourters (4 cents)?
    - What denominations would it pick for k=6?

So what changed about the problem?

- For greedy algorithms to produce optimal results, problems must have two properties:
  - Optimal substructure
    - Optimal solution to a subproblem leads to an optimal solution to the overall problem
  - The greedy choice property
    - Globally optimal solutions can be assembled from locally optimal choices
- Why is optimal substructure not enough?

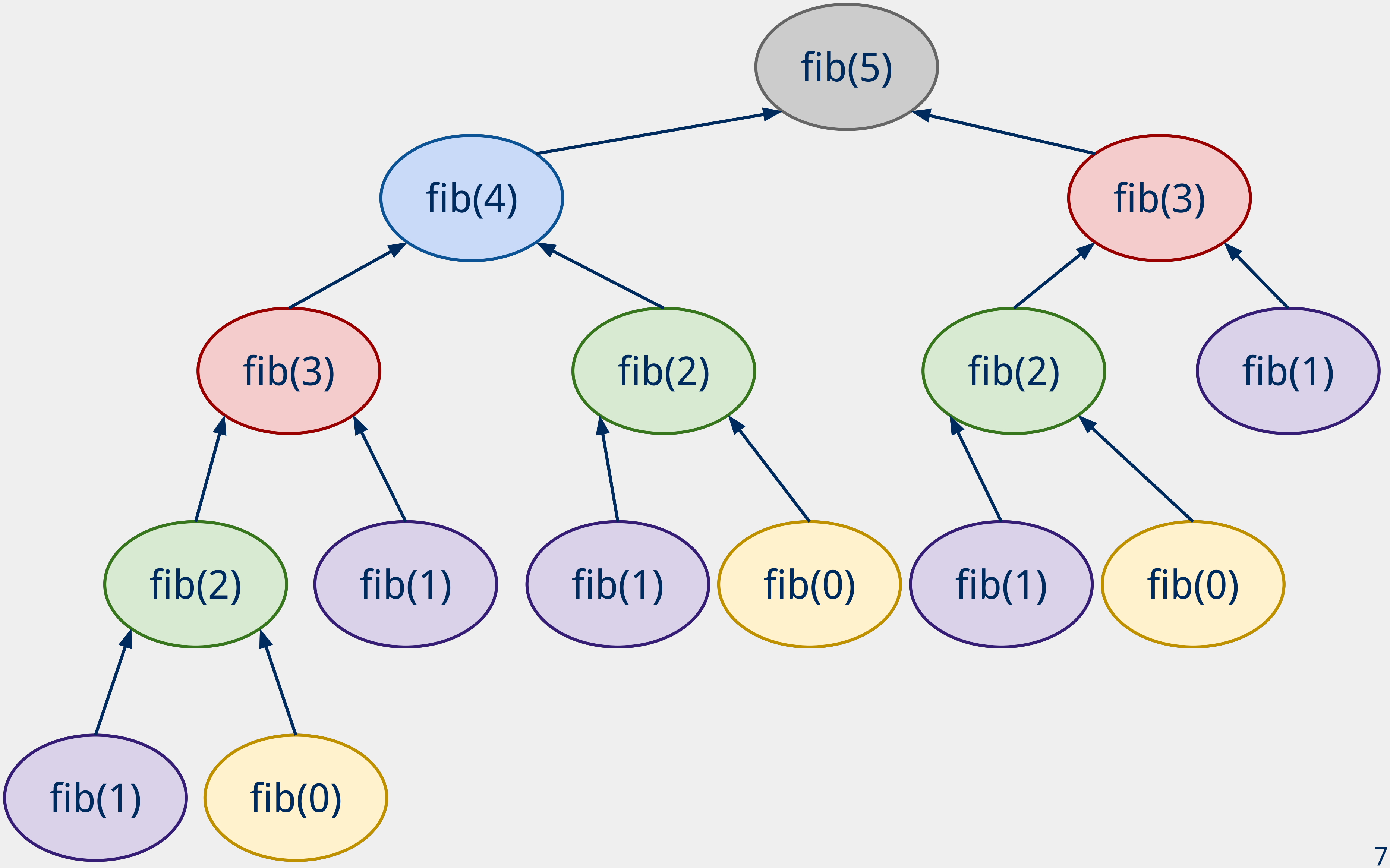
Finding all subproblems solutions can be inefficient

- Consider computing the Fibonacci sequence:

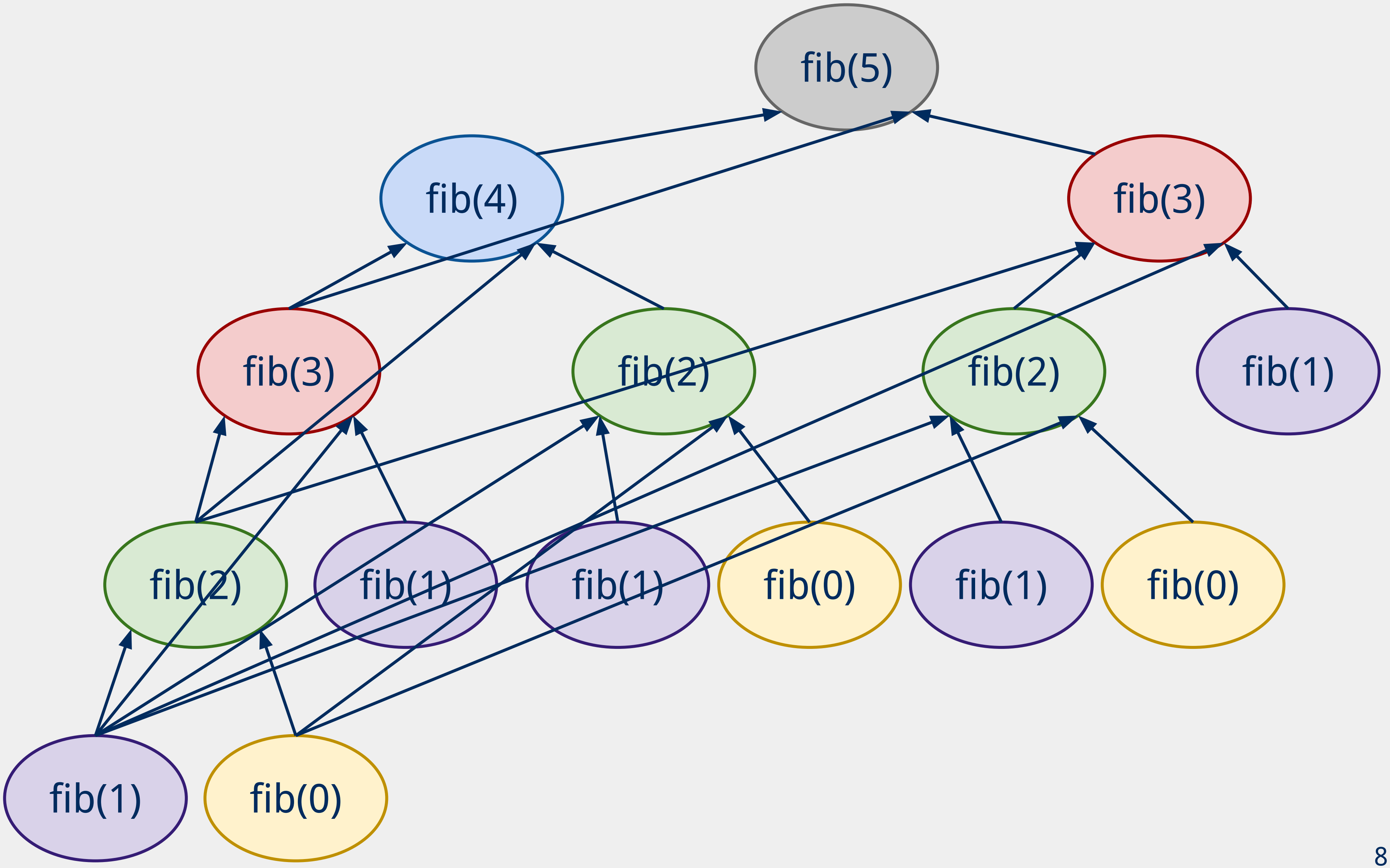
```
int fib(n) {
    if (n == 0) { return 0 };
    else if (n == 1) { return 1 };
    else {
        return fib(n - 1) + fib(n - 2);
    }
}
```

- What does the call tree for n = 5 look like?

fib(5)


fib(5)
fib(4)
fib(3)
fib(3)
fib(2)
fib(2)
fib(1)
fib(2)
fib(1)
fib(1)
fib(0)
fib(1)
fib(0)
fib(1)
fib(0)

How do we improve?



Memoization

```
int[] F = new int[n+1];
F[0] = 0;
F[1] = 1;
for(int i = 2; i <= n; i++) { F[i] = -1 };


int dp_fib(x) {
    if (F[x] == -1) {
        F[x] = dp_fib(x-1) + dp_fib(x-2);
    }
    return F[x];
}
```

Note that we can also do this bottom-up

```
int bottomup_fib(n) {
    if (n == 0)
        return 0;
    int[] F = new int[n+1];
    F[0] = 0;
    F[1] = 1;
    for(int i = 2; i <= n; i++) {
        F[i] = F[i-1] + F[i-2];
    }
    return F[n];
}
```

Can we improve this bottom-up approach?

```
int improve_bottomup_fib(n) {
    int prev = 0;
    int cur = 1;
    int new;
    for (int i = 0; i < n; i++) {
        new = prev + cur;
        prev = cur;
        cur = new;
    }
    return cur;
}
```

Where can we apply dynamic programming?

- To problems with two properties:
  - Optimal substructure
    - Optimal solution to a subproblem leads to an optimal solution to the overall problem
  - Overlapping subproblems
    - Naively, we would need to recompute the same subproblem multiple times

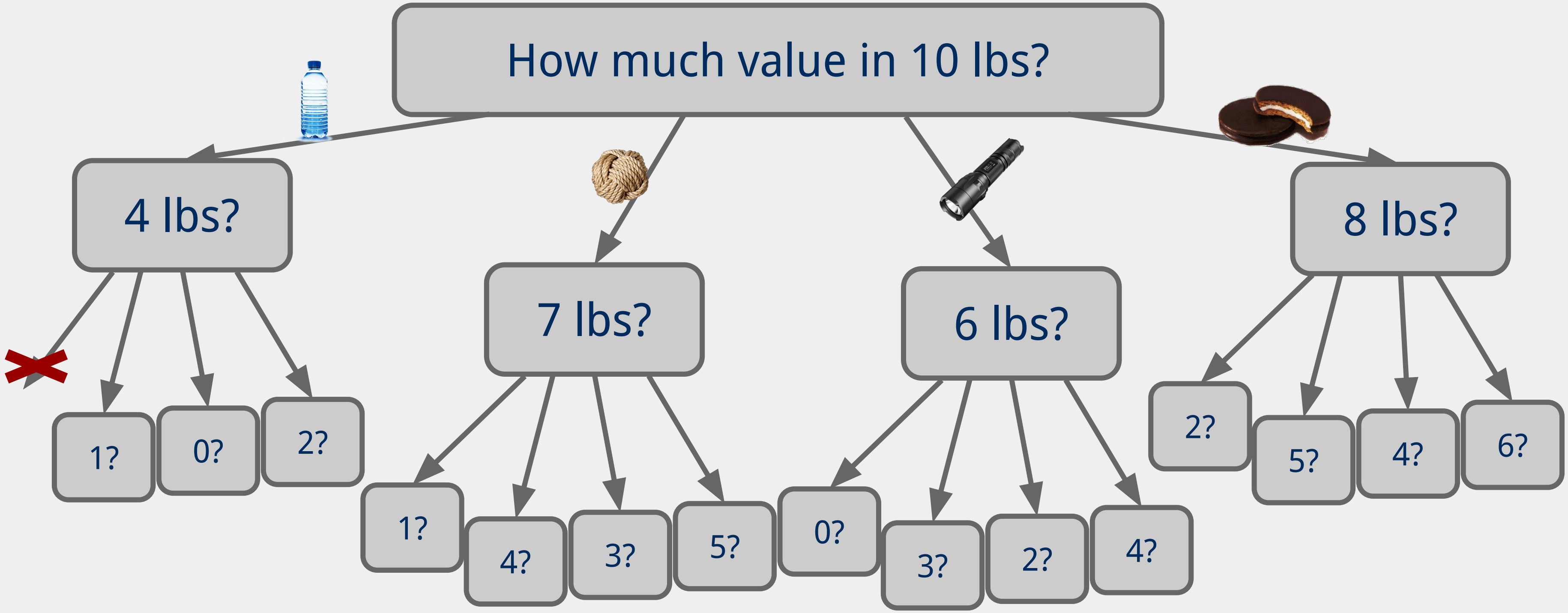
The unbounded knapsack problem

- Given a knapsack that can hold a weight limit L, and a set of n types items that each has a weight (w_i) and value (v_i), what is the maximum value we can fit in the knapsack if we assume we have unbounded copies of each item?

Recursive example

weight:	6	3	4	2
value:	30	14	16	9

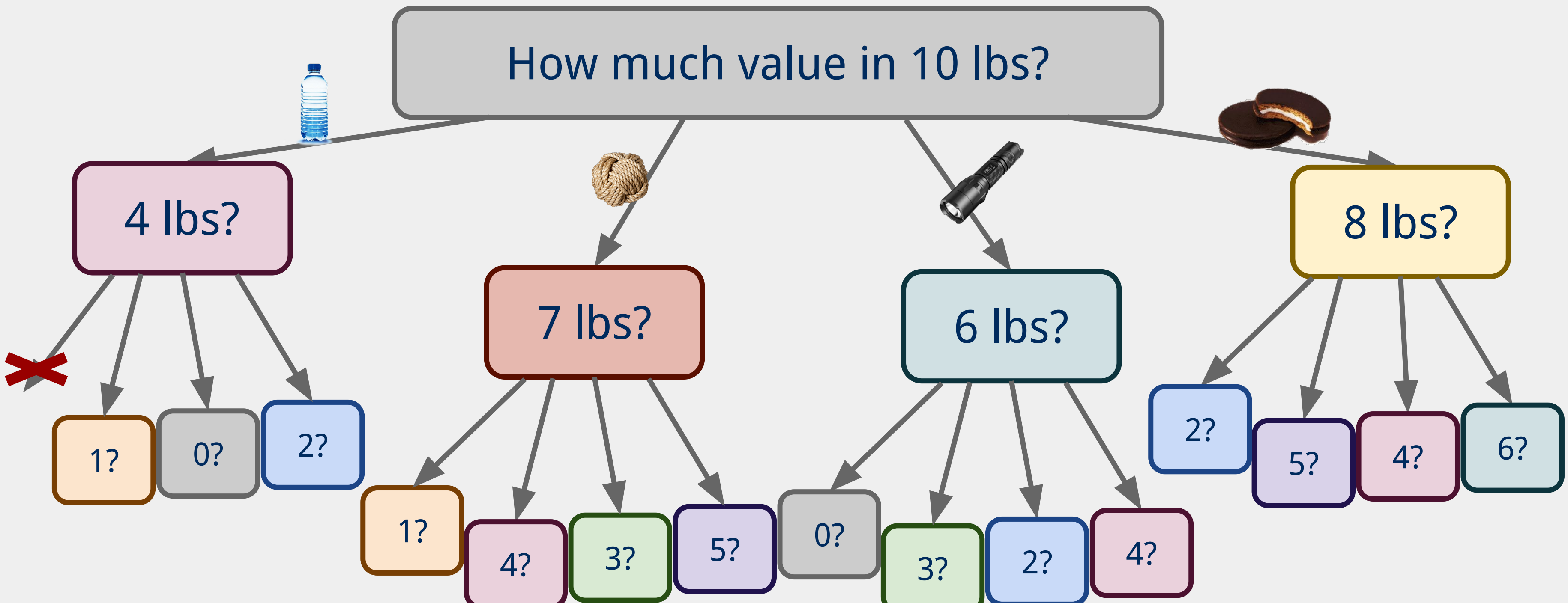
10 lb.
capacity



Recursive example

weight:	6	3	4	2
value:	30	14	16	9

10 lb. capacity



Bottom-up example

				
weight:	6	3	4	2
value:	30	14	16	9

Size:	0	1	2	3	4	5	6	7	8	9	10
Max val:	0	0	9	14	18	23	30	32	39	44	48

Bottom-up solution

```
K[0] = 0
for (l = 1; l <= L; l++) {
    int max = 0;
    for (i = 0; i < n; i++) {
        if (w_i <= l && v_i + K[l - w_i]) > max) {
            max = v_i + K[l - w_i];
        }
    }
    K[l] = max;
}
```

What would have happened with a greedy approach?

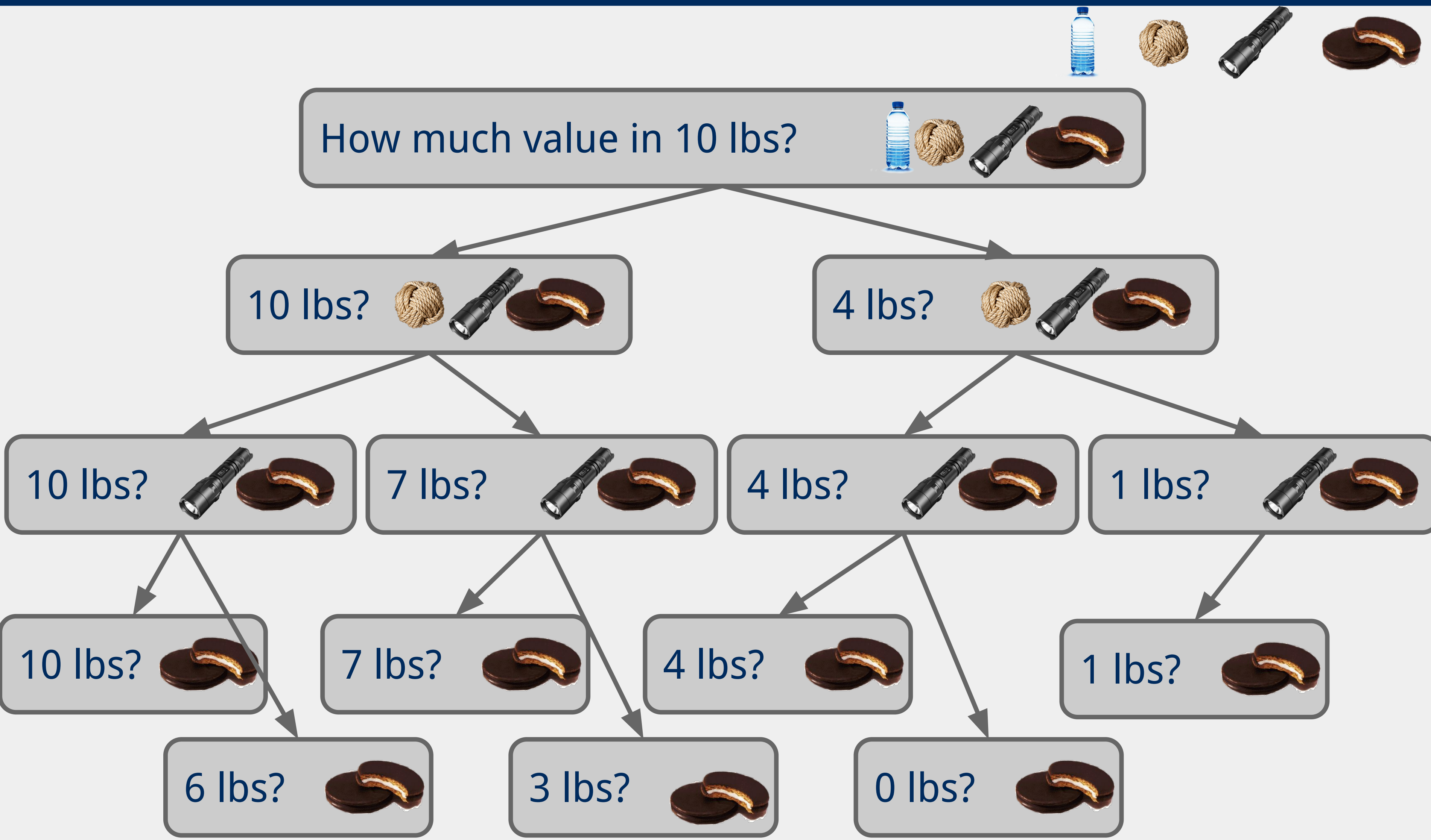
- Try adding as many copies of highest value per pound item as possible:
  - Water: 30/6 = 5
  - Rope: 14/3 = 4.66
  - Flashlight: 16/4 = 4
  - Moonpie: 9/2 = 4.5
- Highest value per pound item? Water
  - Can fit 1 with 4 space left over
- Next highest value per pound item? Rope
  - Can fit 1 with 1 space left over
- No room for anything else
- Total value in the 10 lb knapsack?
  - 44
    - Bogus!

The 0/1 knapsack problem

- What if we have a finite set of items that each has a weight and value?
    - Two choices for each item:
        - Goes in the knapsack
        - Is left out

0/1 Recursive example

weight:	6	3	4	2
value:	30	14	16	9



Recursive solution

```
int knapSack(int[] wt, int[] val, int L, int n) {
    if (n == 0 || L == 0) { return 0 };
    if (wt[n-1] > L) {
        return knapSack(wt, val, L, n-1)
    }
    else {
        return max( val[n-1] + knapSack(wt, val, L-wt[n-1], n-1),
                    knapSack(wt, val, L, n-1)
                   );
    }
}
```

The 0/1 knapsack dynamic programming example

```
wt  = [ 2, 3, 4, 5 ]
val = [ 3, 4, 5, 6 ]
```

i\l	0	1	2	3	4	5
0	0	0	0	0	0	0
1						
2						
3						
4						

The 0/1 knapsack dynamic programming example

```
wt  = [ 2, 3, 4, 5 ]
val = [ 3, 4, 5, 6 ]
```

i\l	0	1	2	3	4	5
0	0	0	0	0	0	0
1						
2						
3						
4						

The 0/1 knapsack dynamic programming example

```
wt  = [ 2, 3, 4, 5 ]
val = [ 3, 4, 5, 6 ]
```

i\l	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2						
3						
4						

The 0/1 knapsack dynamic programming example

```
wt  = [ 2, 3, 4, 5 ]
val = [ 3, 4, 5, 6 ]
```

i\l	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3						
4						

The 0/1 knapsack dynamic programming example

```
wt  = [ 2, 3, 4, 5 ]
val = [ 3, 4, 5, 6 ]
```

i\l	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4						

The 0/1 knapsack dynamic programming solution

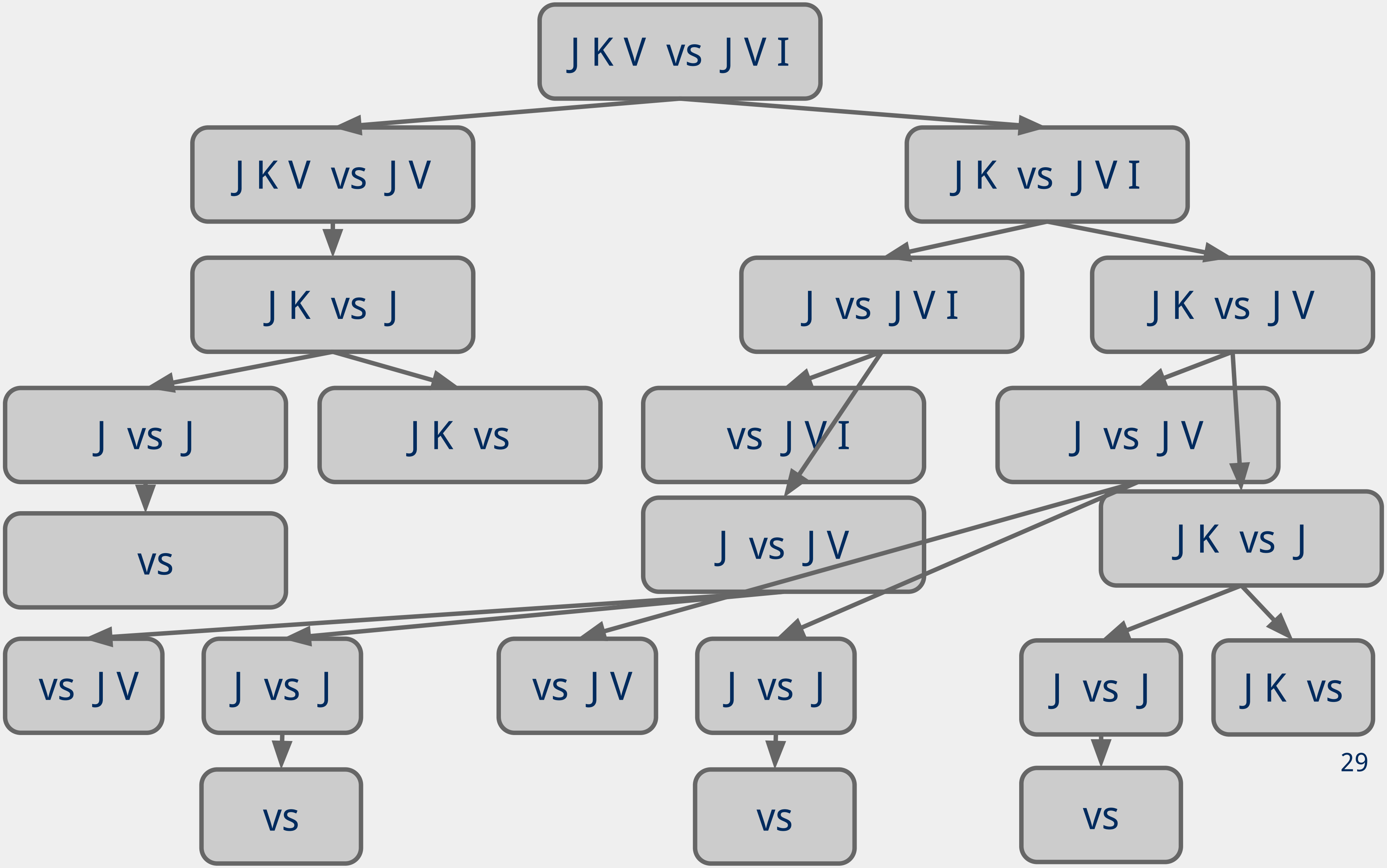
```
int knapSack(int wt[], int val[], int L, int n) {
    int[][] K = new int[n+1][L+1];
    for (int i = 0; i <= n; i++) {
        for (int l = 0; l <= L; l++) {
            if (i==0 || l==0){ K[i][l] = 0 };
            else if (wt[i-1] > l){ K[i][l] = K[i-1][l] };
            else {
                K[i][l] = max(val[i-1] + K[i-1][l-wt[i-1]],
                              K[i-1][l]);
            }
        }
    }
    return K[n][L];
}
```

Longest Common Subsequence

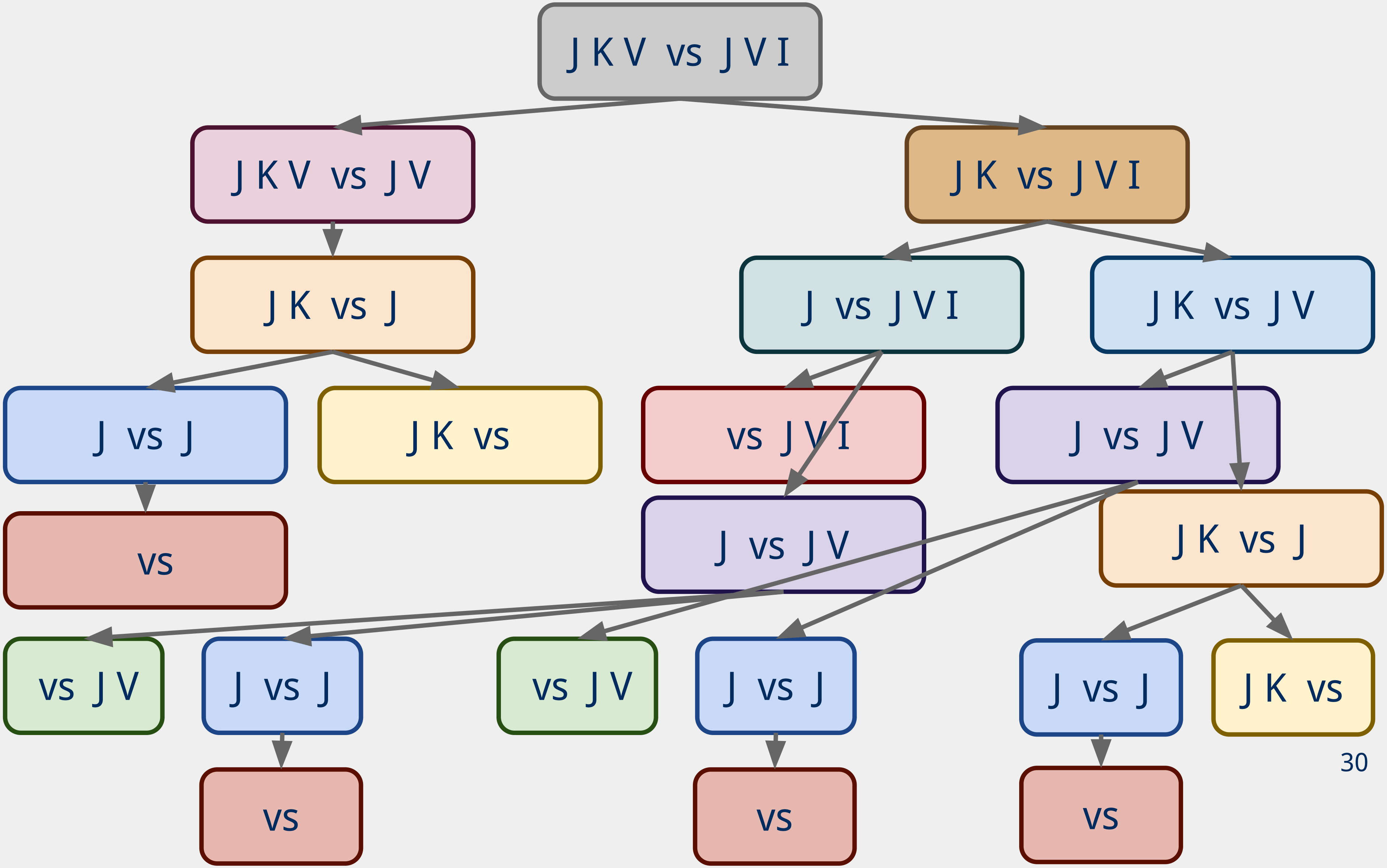
- Given two sequences, return the longest common subsequence
  - `A` **`Q`** `S R` **`J`** `K` **`V`** `B` **`I`**

    `Q` `B W F` **`J V I`** `T U`
- We'll consider a relaxation of the problem and only look for the *length* of the longest common subsequence

LCS recursive example



LCS recursive example



LCS recursive solution

```
int LCSLength(String x, String y, int m, int n) {
    if (m == 0 || n == 0)
        return 0;
    if (x.charAt(m-1) == y.charAt(n-1))
        return 1 + LCSLength(x, y, m-1, n-1);
    else
        return max(LCSLength(x, y, m, n-1),
                   LCSLength(x, y, m-1, n)
                   );
}
```

LCS dynamic programming example

`x = A Q S R J B I`

`y = Q B I J T U T`

i\j	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

LCS dynamic programming solution

```
int LCSLength(String x, String y) {
    int[][] m = new int[x.length + 1][y.length + 1];
    for (int i=0; i <= x.length; i++) {
        for (int j=0; j <= y.length; j++) {
            if (i == 0 || j == 0) m[i][j] = 0;
            if (x.charAt(i) == y.charAt(j))
                m[i][j] = m[i-1][j-1] + 1;
            else
                m[i][j] = max(m[i][j-1], m[i-1][j]);
        }
    }
    return m[x.length][y.length];
}
```

To review...

- Questions to ask in finding dynamic programming solutions:
    - Does the problem have optimal substructure?
        - Can solve the problem by splitting it into smaller problems?
        - Can you identify subproblems that build up to a solution?
    - Does the problem have overlapping subproblems?
        - Where would you find yourself recomputing values?
            - How can you save and reuse these values?

The change-making problem

- Consider a currency with n different denominations of coins $d_1, d_2, \ldots, d_n$. What is the minimum number of coins needed to make up a given value k?