Q3

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> *If $T(n)$ is defined by a standard recurrence, with parameters $a \geq 1$, $b > 1$, and $d \geq 0$, then*
>
> $$T(n) = \begin{cases} O(n^d \log n) & \text{if } a = b^d \quad \text{[Case 1]} \\ O(n^d) & \text{if } a < b^d \quad \text{[Case 2]} \\ O(n^{\log_b a}) & \text{if } a > b^d \quad \text{[Case 3]} \end{cases}$$

a) $T(n) = 2T(n/4) + \sqrt{n}$

$a = 2 \quad b = 4 \quad d = \frac{1}{2}$

$b^d = 4^{\frac{1}{2}} = 2$

$2 = 2$

Therefore case 1 because $a = b^d$

$T(n) = O(\sqrt{n} \log n)$

b)

Conjecture:

we will guess that $T(n) = O(\sqrt{n} \lg n)$

meaning that there exists $n_0 > 0$ and a constant $c > 0$ such that for all $n \geq n_0$:

$T(n) \leq c \cdot \sqrt{n} \lg n$

Basis:

We know that $T(n)$ is constant when $n \leq 2$, therefore the base case trivially holds and we can choose $n_0 = 2$.

Inductive hypothesis : Suppose for every $m < n$:

$$T(m) \leq c\sqrt{m} \lg m$$

Inductive step:

$T(n) = 2T(n/4) + \sqrt{n}$ by def. of $T(n)$

$T(n) \leq 2(c\sqrt{n/4}\lg(n/4)) + \sqrt{n}$ by inductive hypothesis

$T(n) \leq (c\frac{\sqrt{n}}{2}) \cdot (2\lg n - 2) + \sqrt{n}$

$T(n) \leq (c\sqrt{n}) \cdot (\lg n - 1) + \sqrt{n}$

$T(n) \leq (c\sqrt{n}) \cdot \lg n - c\sqrt{n} + \sqrt{n}$

$T(n) \leq c\sqrt{n}\lg n$ if we choose $c \geq 1$

Conclusion:

- we chose n_0 in the basis
- The induction step works if $c \geq 1$
- So we pick $n_0 = 2$ and $c = 1$