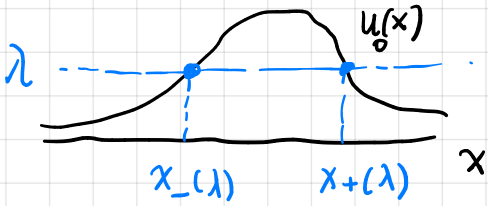
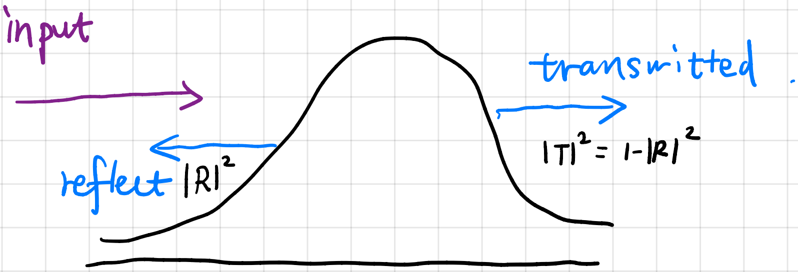
$$\varepsilon^2 y''(x) = (u_0(x) - \lambda)\, y(x), \quad \boxed{0<\lambda<1} \qquad (1)$$



$u_0(x) \to 0, \quad |x| \to \infty.$

$$\begin{cases} x_-(\lambda) < x_+(\lambda) \\ u_0(x_\pm(\lambda)) = \lambda \end{cases}$$

WKB to compute Reflection coefficient:



So set boundary condition

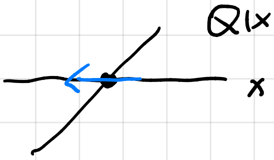
$$(2). \quad y(x) \sim \begin{cases} \overrightarrow{\exp(-ix\lambda^{1/2}/\varepsilon)} + \overleftarrow{R(\lambda)\exp(ix\lambda^{1/2}/\varepsilon)}, & x \to -\infty \\ \overrightarrow{T(\lambda)\exp(-ix\lambda^{1/2}/\varepsilon)}, & x \to +\infty. \end{cases}$$

Lemma

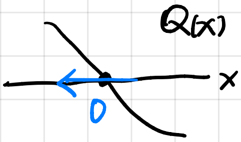
Simple One-Turning-Point connection formula:

$$\begin{cases} Q(0)=0 \\ y(+\infty)=0 \\ \varepsilon^2 y'' = Q\, y \end{cases}$$

Type 1: $Q(x) \sim ax$, $a>0$



Type 2: $Q(x) \sim ax$, $a<0$



$$2[-Q(x)]^{-\frac14}\sin\left[\frac{1}{\varepsilon}\int_x^0 \sqrt{-Q(t)}\,dt + \frac14\pi\right]$$

$$\leftarrow [Q(x)]^{-\frac14}\exp\left[-\frac{1}{\varepsilon}\int_0^x \sqrt{Q(t)}\,dt\right]$$

$$[Q(x)]^{-\frac14}\exp\left[\frac{i\pi}{4} + \frac{1}{\varepsilon}\int_x^0 \sqrt{Q(t)}\,dt\right], \quad x<0$$

$$\leftarrow [-Q(x)]^{-\frac14}\exp\left[-\frac{i}{\varepsilon}\int_0^x \sqrt{-Q(t)}\,dt\right], \quad x>0.$$

Based on Lemma, apply to (1), $Q = U_0(x) - \lambda$.

Assume $x U_0(x) \to 0$ as $|x| \to \infty$, then.

$$\int_{x_+}^{x} \sqrt{\lambda - U_0(t)}\, dt \sim \sqrt{\lambda}\, x + \rho_+(\lambda), \quad x \to \infty$$

$$\int_{x}^{x_-} \sqrt{\lambda - U_0(t)}\, dt \sim -\sqrt{\lambda}\, x + \rho_-(\lambda), \quad x \to -\infty.$$

$$\rho_\pm(\lambda) = \mp x_\pm \lambda^{\frac{1}{2}} \pm \int_{x_\pm}^{\pm\infty} \left(\sqrt{\lambda - U_0(t)} - \lambda^{\frac{1}{2}}\right) dt$$

Then together (2) with connection formulas:

we have.

$$y(x) \sim T(\lambda) \exp(-i\lambda^{\frac{1}{2}} x/\varepsilon) \qquad x \to +\infty$$

$$\sim [-Q(x)]^{-\frac{1}{4}} \exp\left[-\frac{i}{\varepsilon}\int_0^x \sqrt{-Q(t)}\, dt\right], x>0. \qquad 0 \to x_+$$

$$\sim C\, [\lambda - U_0(x)]^{-\frac{1}{4}} \exp\left(-\frac{i}{\varepsilon}\int_{x_+}^{x} \sqrt{\lambda - U_0(t)}\, dt\right), \quad x \to +\infty.$$

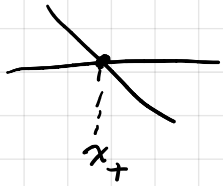
need to be determined.

$$\sim C \lambda^{-\frac{1}{4}} \exp\left[-\frac{i}{\varepsilon}\left(\lambda^{\frac{1}{2}} x + \rho_+(\lambda)\right)\right]$$

$$\therefore \quad C \lambda^{-\frac{1}{4}} \exp\left(-\frac{i}{\varepsilon}\rho_+(\lambda)\right) = T(\lambda).$$

$\boxed{x \to +\infty}$.

$$\therefore \quad y(x) \sim T(\lambda) \left[\frac{\lambda - U_0(x)}{\lambda}\right]^{-\frac{1}{4}} e^{i\rho_+/\varepsilon} \exp\left(-\frac{i}{\varepsilon}\int_{x_+}^{x} \sqrt{\lambda - U_0(t)}\, dt\right)$$

Now apply the connection formula, Type 2: 

$$y(x) \sim T(\lambda)\, e^{i\rho_+/\varepsilon}\left[\frac{u_0(x)-\lambda}{\lambda}\right]^{-\frac14}\exp\left(\frac{i\pi}{4}+\frac1\varepsilon\int_x^{x_+}\sqrt{u_0(t)-\lambda}\,dt\right).$$

$\boxed{x_- < x < x_+}$

$$\sim T(\lambda)\,e^{i\rho_+/\varepsilon}\exp\left(\frac1\varepsilon\int_{x_-}^{x_+}\sqrt{u_0(t)-\lambda}\,dt + \frac{i\pi}{4}\right)\left(\frac{u_0(x)-\lambda}{\lambda}\right)^{-\frac14}$$

$$\times\exp\left(-\frac1\varepsilon\int_{x_-}^{x}\sqrt{u_0(t)-\lambda}\,dt\right)$$

apply type 1 connection formula:

$$\sim T e^{i\rho_+/\varepsilon} e^{\tau/\varepsilon} e^{\frac{i\pi}{4}} \left(\frac{\lambda-u_0(x)}{\lambda}\right)^{-\frac14} 2\sin\left(\frac1\varepsilon\int_x^{x_-}\sqrt{\lambda-u_0(t)}\,dt+\frac\pi4\right).$$

$\boxed{x<x_-}$

$$= T e^{i\rho_+/\varepsilon}e^{\tau/\varepsilon}\left(\frac{\lambda-u_0(x)}{\lambda}\right)^{-\frac14}\left\{\exp\left(\frac{i}{\varepsilon}\int_x^{x_-}\sqrt{\lambda-u_0}\,dt\right)+i\exp\left(-\frac{i}{\varepsilon}\int_x^{x_-}\sqrt{\lambda-u_0}\,dt\right)\right\}$$

use $\int_x^{x_-}\sqrt{\lambda-u_0}\,dt \sim -\lambda^{\frac12}x+\rho_-$ and $u_0(x)\to 0$ as $x\to-\infty$

$$\sim Te^{i\rho_+/\varepsilon}e^{\tau/\varepsilon}\left\{e^{i\rho_-/\varepsilon}\exp\left(-\frac{i}{\varepsilon}\lambda^{\frac12}x\right)+ie^{-i\rho_-/\varepsilon}\exp\left(\frac{i}{\varepsilon}\lambda^{\frac12}x\right)\right\}.$$

compare with (2), we have, $\begin{pmatrix} + & \leftarrow \\ - & \rightarrow\end{pmatrix}$.

$$\begin{cases} T e^{i\rho_+/\varepsilon} e^{\tau/\varepsilon} e^{i\rho_-/\varepsilon} = 1 \\ i T e^{i\rho_+/\varepsilon} e^{\tau/\varepsilon} e^{-i\rho_-/\varepsilon} = R \end{cases}$$

i.e.
$$R(\lambda) = i e^{-2i\rho_-(\lambda)/\varepsilon}$$

$$T(\lambda) = e^{-\tau/\varepsilon} e^{-i(\rho_- + \rho_+)/\varepsilon}$$

Now let $\varepsilon \to 0^+$, we have

$$|T(\lambda)|^2 = 1 - |R(\lambda)|^2 \sim e^{-2\tau/\varepsilon}$$

where $\tau = \frac{1}{\varepsilon} \int_{x_-}^{x_+} \sqrt{u_0(t) - \lambda}\, dt$

$\blacksquare$