A Technique for Calibrating Derivative Security Pricing Models: Numerical Solution of an Inverse Problem

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A technique for calibrating derivative security pricing models: numerical solution of an inverse problem

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A technique is presented for calibrating derivative security pricing models with respect to observed market prices. This technique can be applied in a very general multifactor setting where model parameters such as volatilities and correlations are allowed to be functions of the underlying state variables. These functions are estimated from price observations by solving the inverse problem associated with the parabolic partial differential equation governing arbitrage-free derivative security prices. A detailed exposition is given for consistent pricing of equity index options under a stochastic model that treats index volatility as a deterministic function of index level and time.

1. INTRODUCTION

Arbitrage-free pricing of derivative securities begins with the specification of a stochastic process that characterizes the underlying asset prices, indexes, or rates. Such processes typically incorporate one or more parameters (volatilities, correlations, etc.) that may be constants or deterministic functions. Assuming that the chosen process is a 'realistic' model for the underlying dynamics, the successful application in hedging and other trading activities will depend critically on how the parameters are quantified.

The approach used to estimate parameters may vary depending upon the intended purpose for the pricing model. Frequently, the choice of parameters is based upon a forecast of the future behavior of the underlying, perhaps guided by statistical inference from past behavior. Another approach is to estimate the parameters by calibrating the pricing model with respect to observed market prices of actively traded derivative securities. This market calibration is typically employed when the purpose of the model is to price and hedge exotic derivatives within a framework that is consistent with the prices of more liquid exchange-traded instruments.

In this paper, we present a technique for calibrating derivative pricing models with respect to observed market prices. This technique can be applied in a very general multifactor setting where model parameters such as volatilities and correlations are allowed to be functions of the underlying state variables. We estimate these functions from a discrete and finite set of price observations by solving the inverse problem associated with the parabolic partial differential equation (PDE) governing arbitrage-free derivative security prices.

A detailed exposition of the technique is given in the context of equity index options. Here we consider a single-factor stochastic model for the underlying index, incorporating a volatility that is both time and index-level dependent. This volatility function appears as a coefficient of the second-order partial derivative in the pricing PDE and, in principle, can be uniquely determined given enough information about option price solutions. In practice, however, this inverse problem is underdetermined because the market provides only a limited amount of information: prices of exchange-traded options corresponding to the current index level for a finite number of strikes and maturities. In addition, the problem is more generally ill-posed in the sense that the estimated volatility function will typically not depend continuously on the market data. Both of these problems are alleviated by a regularization technique. Here we minimize the L^2 norm of the gradient of local volatility over an appropriate space of smooth functions subject to a constraint that ensures that solutions of the pricing PDE match observed market prices. This minimization is carried out numerically using a gradient descent procedure implemented in a finite-difference framework. The procedure requires values of the variational derivative of the option pricing function with respect to the volatility function. We present a novel numerical approach for evaluating this variational derivative.

2. FORMULATION

2.1 Market Calibration of Exchange-Traded Index Options

Consider a class of exchange-traded options on an equity index such as the S&P 500, composed of calls and puts with a variety of maturities and strike prices. A calibrated pricing model should match the market prices of all the options in that class (at least to within the bid–ask spread) given a single assignment of model parameters. The traditional Black–Scholes model, based upon the assumption of constant index volatility, usually fails to accommodate this calibration in the sense that no single value of volatility will generate the market prices of options across the entire range of expiration dates and strike prices. This shortcoming of the Black–Scholes model is viewed by practitioners as a 'market phenomenon', where dependence of implied volatility on strike price is termed the 'smile' and dependence on time to expiration is called the 'term structure'.

Recently, Derman and Kani (1994), Dupire (1994), and Rubinstein (1994) have independently shown how a class of index options that exhibits an implied volatility smile and/or term structure can be consistently priced using a complete single-factor no-arbitrage model. All three approaches assume that the underlying index level S follows a general diffusion process of the form

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma(S, t)\,\mathrm{d}Z, \tag{2.1}$$

where μ is the drift, Z is a standard Brownian motion, and the local volatility σ is a deterministic function that may depend on both the index level and time t. Dupire (1994) has shown that this local volatility and the associated risk-neutral process for S are uniquely determined by complete knowledge of the arbitrage-free prices of European calls (or puts) for all conceivable strikes $K > 0$ and expiration dates $T > 0$. Unfortunately, the problem of estimating σ is underspecified. Exchange-traded index options are quoted only for a discrete set of strikes and maturities, and, consequently, the market alone does not provide enough information to make a unique determination of the 'implied process'.

Derman and Kani, Dupire, and Rubinstein all show how to construct a discrete approximation to the continuous risk-neutral process for the underlying index in the form of a binomial or trinomial tree. These 'implied' trees are devised to accommodate the implied volatility smile and term structure. In all cases, some *ad hoc* prescription or interpolation and extrapolation from the sparsely distributed set of market prices is needed to permit the unique determination of the local volatility function. By and large, implied trees are effective for consistently pricing an entire class of standard European options. Nevertheless, some limitations are apparent; see Lagnado and Osher (1996) for some elaboration on this point.

The focus of this paper is a general approach for model calibration that is implemented via numerical solution of the PDE governing derivative prices. In the single-factor setting of equation (2.1), this approach is an alternative to implied trees for the determination of the local volatility function, with the disadvantage of being more computationally intensive. However, some important advantages over implied trees include straightforward extension to more complex multifactor settings and the fact that any instrument (standard or exotic, European or American) may be used in the calibration process.

2.2 Solving the Inverse Problem

Suppose we are given the market prices of options (calls, puts, or some combination thereof) spanning a set of expiration dates $T_1, \ldots, T_N$. Assume that, for each expiration date T_i, options are traded with strike prices $K_{i1}, \ldots, K_{iM_i}$, where the number of strikes M_i and their values may be different for each expiration date. Let V_{ij}^{b} and V_{ij}^{a} denote the bid and ask prices, respectively, at the present time $t = 0$ for an option with expiration date T_i and strike price K_{ij}.

Assume that the underlying index follows the general stochastic process specified in equation (2.1). Let $V(S, t; K, T, \sigma)$ denote the theoretical price of an option at time t when the index level is S and for which the strike price is K and the expiration date is T. The dependence of option price on the particular choice for the local volatility function is also indicated explicitly in this notation. The pricing function V must satisfy the following generalization of the Black–Scholes PDE:

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} = rV, \tag{2.2}$$

where r is the riskless continuously compounded interest rate and q is the continuous dividend yield on the index. We assume here that the interest rate and dividend yield are deterministic and constant to simplify the discussion, but a less restrictive treatment may also be made without changing the essential details of our approach.

If the functional form of σ is specified, then the option price $V(S_0, 0; K, T, \sigma)$ corresponding to a spot index level of S_0 at the present time $t = 0$ can be uniquely determined by solving equation (2.2) subject to the appropriate initial and boundary conditions. The conditions for a standard European call are, for example,

$$\left.\begin{aligned} V(S,T;K,T,\sigma) &= \max(S-K,0) & &\text{for } S \geqslant 0,\\ V(0,t;K,T,\sigma) &= 0 & &\text{for } 0 \leqslant t \leqslant T,\\ \frac{\partial V}{\partial S}(S,t;K,T,\sigma) &\to \mathrm{e}^{-q(T-t)} \quad \text{as } S \to \infty & &\text{for } 0 \leqslant t \leqslant T. \end{aligned}\right\} \tag{2.3}$$

In this general context, market calibration involves finding a local volatility function σ such that solutions of (2.2) fall between the corresponding bid and ask market quotes, viz.,

$$V_{ij}^{\mathrm{b}} \leqslant V(S_0, 0; K_{ij}, T_i, \sigma) \leqslant V_{ij}^{\mathrm{a}}, \tag{2.4}$$

for $i = 1, \ldots, N$ and $j = 1, \ldots, M_i$. We would like to satisfy these inequality constraints by minimizing and possibly driving to zero a functional of the form

$$G(\sigma) = \sum_{i=1}^{N} \sum_{j=1}^{M_i} [V(S_0, 0; K_{ij}, T_i, \sigma) - \bar{V}_{ij}]^2, \tag{2.5}$$

where $\bar{V}_{ij} = \frac{1}{2}(V_{ij}^{\mathrm{a}} + V_{ij}^{\mathrm{b}})$ is the arithmetic mean of the bid and ask prices.

At this point, the problem of minimizing G over some general space of admissible functions is ill-posed, essentially because the set of price observations is discrete and finite. The function σ cannot be uniquely determined with guaranteed continuous dependence on the market price observations. This lack of continuous dependence means that small perturbations in the price data can result in large changes in the minimizing function. To make the problem well-posed we must introduce some type of regularization (see Tykhonov 1963), which will also lead to a robust numerical procedure for finding the optimal solution.

The regularization technique we propose involves a functional of the form

$$F(\sigma) = \| \, |\nabla\sigma| \, \|_2^2 + \lambda G(\sigma), \tag{2.6}$$

where $\lambda > 0$ is a constant[1] and $\| \cdot \|_2$ denotes the usual L^2 norm in (S, t) space. By minimizing $F(\sigma)$ over a more restricted space of admissible functions, we are finding the 'smoothest' σ that satisfies the constraint $G(\sigma) = 0$.

We use a gradient descent procedure to minimize $F(\sigma)$. First introduce a 'false' time parameter θ and a function $\hat{\sigma}(S, t, \theta)$. Starting with some initial guess $\hat{\sigma}(S, t, 0)$, we solve a perturbed parabolic equation of the form

$$\frac{\partial\hat{\sigma}}{\partial\theta} = \frac{\partial^2\hat{\sigma}}{\partial S^2} + \frac{\partial^2\hat{\sigma}}{\partial t^2} - \lambda \sum_{i=1}^{N} \sum_{j=1}^{M_i} \frac{\delta V^{ij}}{\delta\sigma}(S_0, 0; S, t)[V(S_0, 0; K_{ij}, T_i, \hat{\sigma}) - \bar{V}_{ij}]. \tag{2.7}$$

If $\hat{\sigma}(S, t, \theta)$ tends to a steady-state solution as $\theta \to \infty$, then this limit will satisfy the Euler–Lagrange equation for the functional F—a necessary condition for the existence of an extremum.

In order to solve equation (2.7), we must be able to evaluate the variational derivative $\delta V^{ij}(S_0, 0; S, t)/\delta\sigma$. This term represents the local variation of an option value at $t = 0$ with expiration date T_i and strike price K_{ij} with respect to a variation of σ at a point (S, t) and is defined for general arguments by

$$\frac{\delta V^{ij}}{\delta\sigma}(S, t; \xi, \tau) = \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon} V(S, t; K_{ij}, T_i, \sigma + \epsilon h)\right]_{\epsilon=0}, \tag{2.8}$$

where the perturbation h is a product of Dirac delta functions

$$h(S, t) = \delta(S - \xi)\delta(t - \tau). \tag{2.9}$$

This variational derivative can be evaluated by solving PDE (2.2) with an additional source term. To show this, we first note that the pricing function associated with a perturbed volatility function satisfies a PDE similar to (2.2) of the form

$$\mathcal{L}(\sigma + \epsilon h)V(S, t; K_{ij}, T_i, \sigma + \epsilon h) = 0, \tag{2.10}$$

[1] The constant λ can be interpreted as a Lagrange multiplier. In practice, the value of λ is set by experimenting to optimize the rate of convergence of a numerical minimization procedure.

where the operator $\mathcal{L}(\sigma)$ is given by

$$\mathcal{L}(\sigma) \equiv \frac{\partial}{\partial t} + (r-q)S\frac{\partial}{\partial S} + \tfrac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2}{\partial S^2} - rI, \tag{2.11}$$

where I denotes the identity operator. Differentiating (2.10) with respect to ϵ and evaluating for $\epsilon = 0$, we obtain

$$\mathcal{L}(\sigma)\frac{\delta V^{ij}}{\delta \sigma}(S,t;\xi,\tau) = -\delta(S-\xi)\delta(t-\tau)\sigma(S,t)S^2\frac{\partial^2}{\partial S^2}V(S,t;K_{ij},T_i,\sigma). \tag{2.12}$$

It can also be shown easily that the variational derivative must satisfy homogeneous boundary and initial conditions.

The calibration procedure requires the numerical solution of PDEs (2.2), (2.7), and (2.12). This can be accomplished conveniently using a finite-difference technique; see Wilmott *et al.* (1993) for an extensive discussion of the finite-difference method applied to option pricing. Although the PDEs are posed for $(S,t) \in [0,\infty) \times [0,T]$, numerical solution with finite differences requires that the semi-infinite domain be truncated or mapped into a bounded domain $[0, S_{\max}] \times [0, T]$. We can obtain sufficiently accurate solutions for our purpose simply by truncating the domain and applying the boundary condition for $S \to \infty$ at $S = S_{\max} = 2S_0$.

The numerical procedure to minimize (2.6) and determine σ can now be summarized by the following algorithmic description:

Step 1. Select a finite-difference grid $\{(S_m, t_n) : m = 0, \ldots, N_S \text{ and } n = 0, \ldots, N_t\}$ covering the domain $[0, S_{\max}] \times [0, T_{\max}]$, where $T_{\max}$ is the longest time-to-expiration of the observed option prices.

Step 2. Choose an initial guess for the values of $\hat{\sigma}$ at these grid points (e.g. $\hat{\sigma}(S_m, t_n, 0) =$ constant).

Step 3. Solve PDE (2.2) using a finite-difference method for the values $V(S_m, t_n; K_{ij}, T_i, \hat{\sigma})$ at every grid point for each $i = 1, \ldots, N$ and $j = 1, \ldots, M_i$. Evaluate the pricing constraint functional (2.5). If the value is smaller than a desired tolerance, then stop the procedure. Otherwise, proceed to Step 4.

Step 4. Solve PDE (2.12) using a finite-difference method and an appropriate discretization of the source term for the variational derivatives $\delta V^{ij}(S_0, 0; S_m, t_n)/\delta\sigma$ for each $i = 1, \ldots, N$, $j = 1, \ldots, M_i$, $m = 1, \ldots, N_S - 1$, and $n = 1, \ldots, N_t - 1$.

Step 5. Solve a discrete form of (2.7) for a small increment $\Delta\theta$ to find updated values of $\hat{\sigma}$ at all interior grid points. Return to Step 3.

This iterative procedure is clearly computationally demanding in that a large number of direct solutions to the PDE (2.12) must be effected. However, the amount of work can be reduced by exploiting the fact that each solution involves the same PDE with a different point of application for the source term. Consequently, an implementation based on an implicit finite-difference scheme only requires a single matrix inversion per time step for each iteration. It may also be possible to improve efficiency using fast wavelet-based methods as an alternative to the finite-difference method (see Engquist *et al.* 1994 and Jiang and Osher 1996).

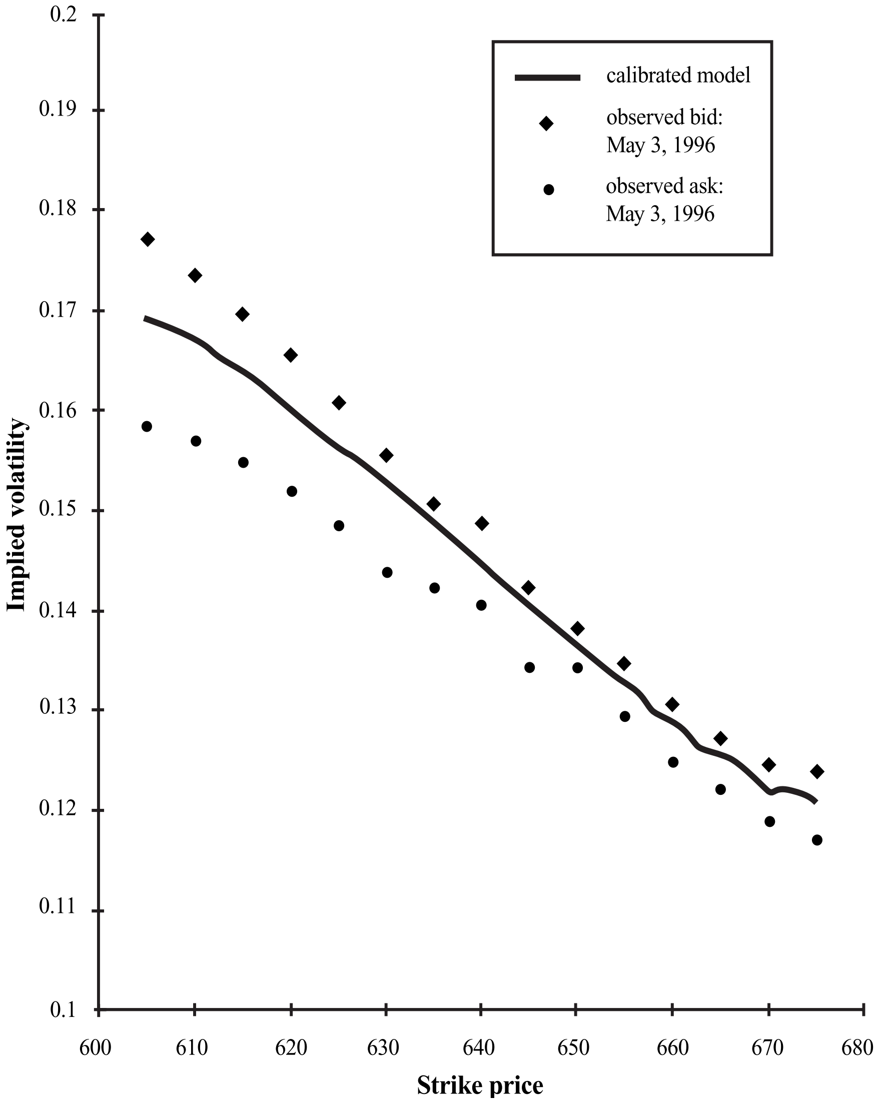

FIGURE 1. S&P 500 June calls (49 days to expiration).

3. NUMERICAL TESTS

The viability of our technique can be demonstrated by carrying out the calibration using index option prices observed in the market. We selected the closing prices of some S&P 500 index options as of 3 May 1996 for this test. In particular, we chose the call options expiring in June (49 days to expiration) and July (77 days to expiration) with strikes ranging from 605 to 675 in increments of 5. The closing index price on this day was $S_0 = 641.63$. Assuming an interest rate of $r = 0.055$ and a dividend yield of $q = 0.022$, implied volatilities ranged from roughly 12 to 18%.

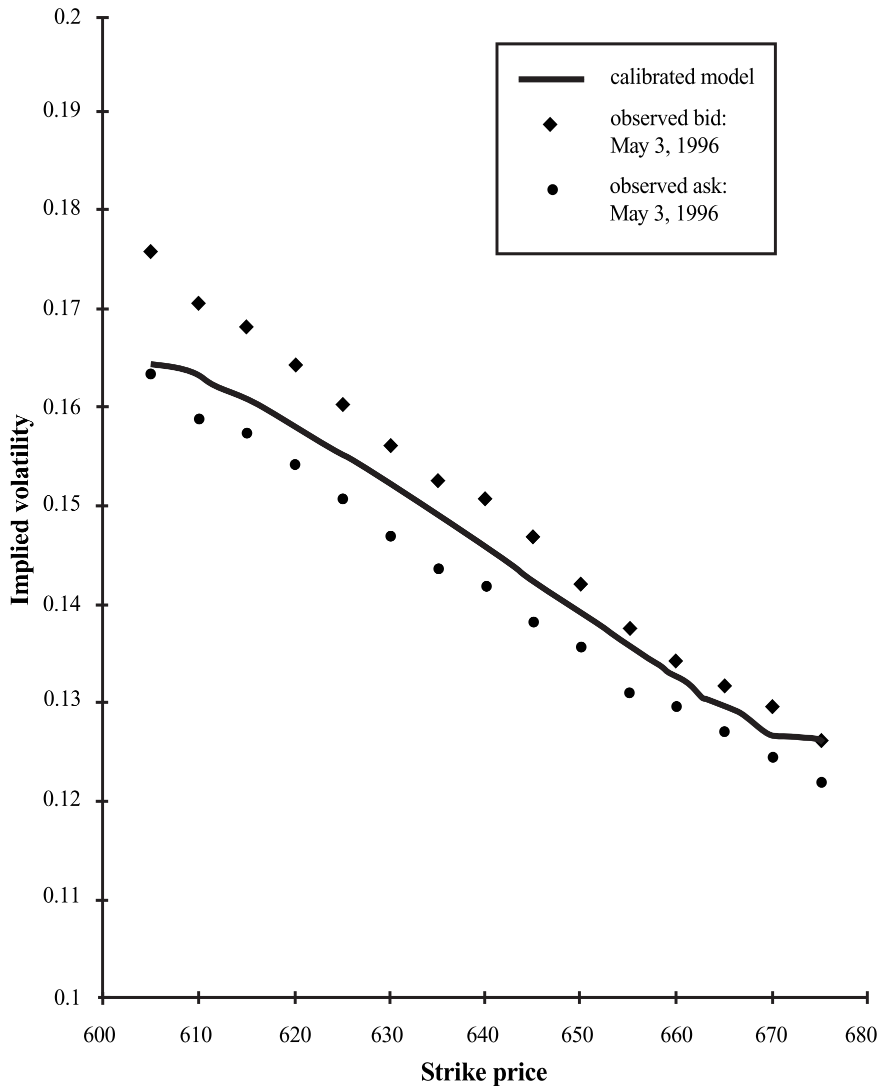

FIGURE 2. S&P 500 July calls (77 days to expiration).

For the calibration, we solved PDEs (2.2) and (2.12) with an implicit finite-difference algorithm using a backward difference approximation for the t-derivative and central difference approximations for the S-derivatives. The computational domain was discretized by a grid with 51 uniformly spaced grid lines for the t-dimension and 51 nonuniformly spaced grid lines for the S-dimension, with refinement around the center grid line at $S = 641.63$. The minimum grid spacing in the S-direction was approximately $\Delta S \approx 1$ and the maximum spacing was approximately $\Delta S \approx 75$. The gradient descent procedure was implemented by an explicit finite-difference solution of (2.7), starting with an initial guess of $\hat{\sigma}(S, t, 0) = 0.15$, an iteration step size of $\Delta\theta = 0.00001$, and a regularization parameter $\lambda = 50$.

The gradient descent procedure was terminated after 30 iterations, and call prices were calculated with the calibrated model for strikes ranging from 605 to 675. The results for

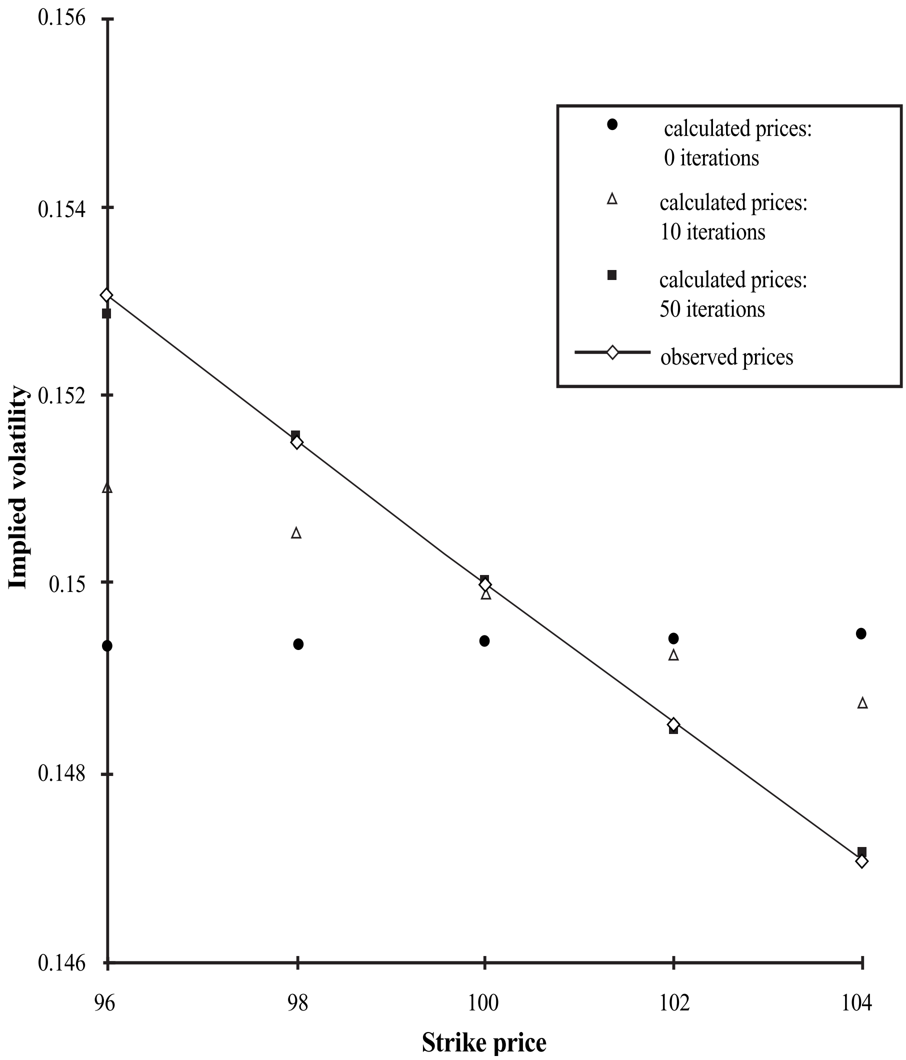

FIGURE 3. Convergence to one-year option prices (5 observations).

options with June and July expiration are plotted in Figures 1 and 2, respectively. Note that the calibration was successful since the calculated option prices all fall between the observed bid and ask market quotes.

Another interesting test of our technique was conducted using price observations that correspond to a known form of the local volatility function. This permitted an investigation of how the range and density of the price observations affects the recovery of the 'true' functional parameter σ. For this test, we selected a volatility function of the form

$$\sigma(S, t) = C/S,$$

where C is a constant. In this case, the underlying index follows an absolute diffusion process, with option prices given by the closed-form solution of Cox and Ross (1976). European call prices were generated with this formula for a variety of strikes and maturities of 0.5 and 1 year, assuming $C = 15$, a spot index level of $S_0 = 100$, an interest rate of

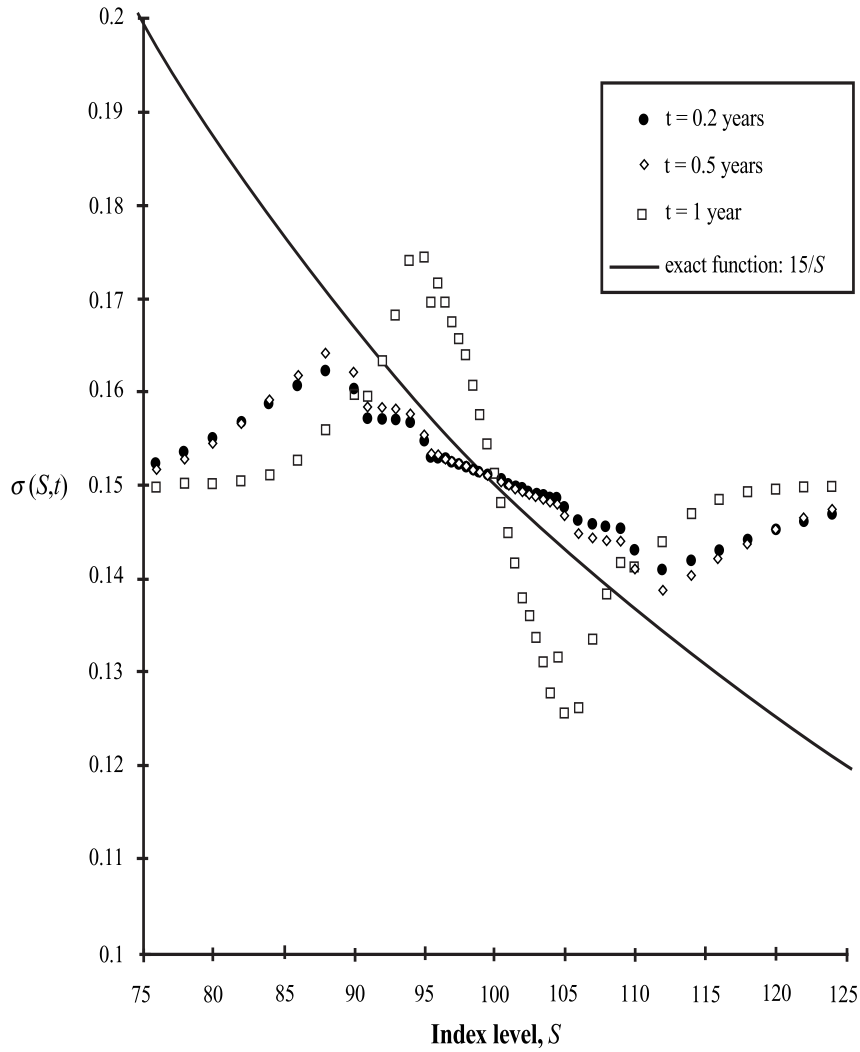

FIGURE 4. Reconstruction of local volatility (5 observations).

$r = 0.05$, and a continuous dividend yield of $q = 0.02$. Using these prices to represent market observations, we implemented our calibration technique starting with an initial guess of $\hat{\sigma}(S, t, 0) = 0.15$. The same approach described above was used to solve the PDEs. In this case, the computational domain $\{(S, t) : 0 \leqslant S \leqslant 200, 0 \leqslant t \leqslant 1\}$ was covered by a 101×51 grid with uniform spacing in the t-direction and refinement in the S-direction around the center grid line $S = 100$. The minimum grid spacing in the S-direction was $\Delta S = 0.5$ and the maximum spacing was $\Delta S = 10$.

In a first experiment, we used only five price observations corresponding to a single expiration date of $T_1 = 1$ year and the strike prices $K_{1j} \in \{96, 98, 100, 102, 104\}$ for $j = 1, \ldots, 5$. The calibration procedure was then run for a total of 50 iterations. Figure 3 compares the target observations with the numerically calculated option prices at different stages of the iterative procedure. The prices are represented in terms of their Black–Scholes implied volatilities. (Note that the absolute diffusion model produces a negatively sloped

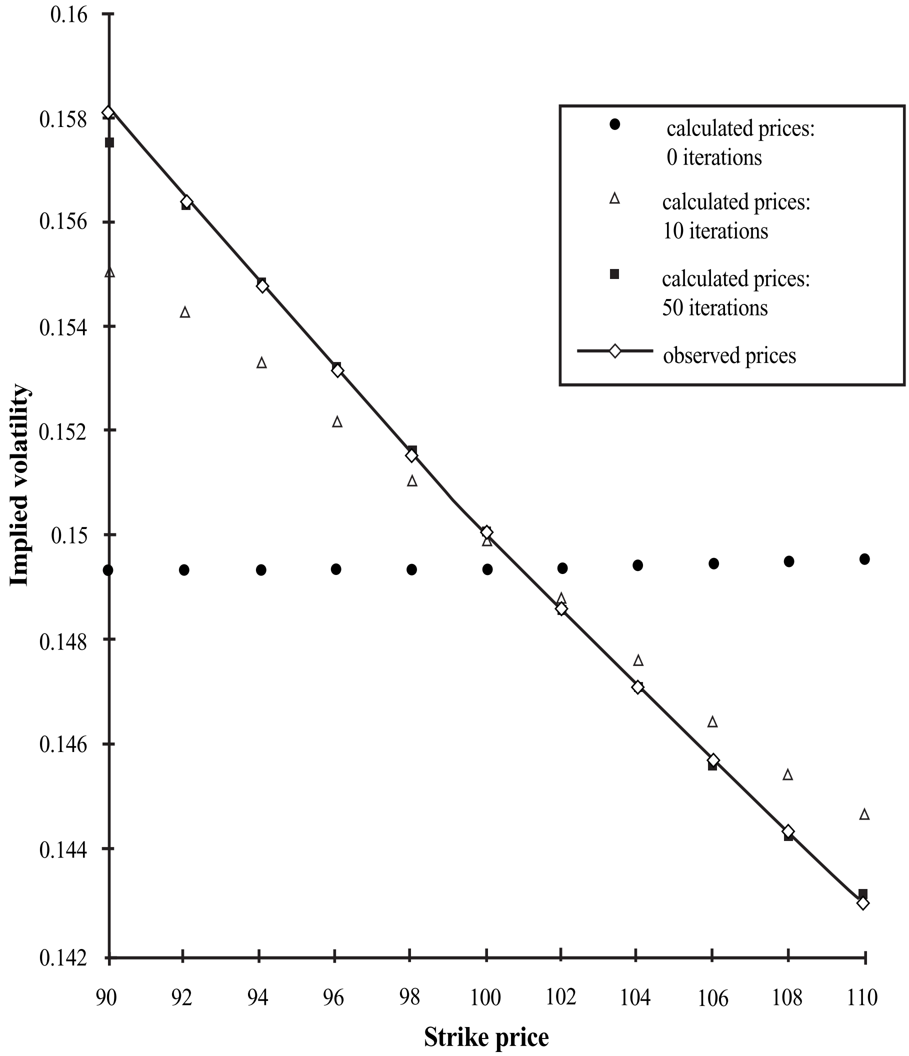

FIGURE 5. Convergence to one-year option prices (22 observations).

smile with a very small positive curvature.) After 50 iterations, the calculated prices match the corresponding target observations to within a root mean square error of 0.00005 in terms of implied volatility.

In Figure 4 we display numerically determined values of $\sigma(S, t)$ at some representative grid points obtained after 50 iterations and compare to the graph of the exact function $\sigma(S, t) = 15/S$. These values are indicated by the symbols in Figure 4, and they correspond to grid points lying on the three lines t = 0.2, 0.5, and 1 year. Note that, for sufficiently small and large values of S, the numerically determined values of $\sigma(S, t)$ remain essentially unchanged from the initially guessed value of 0.15. This effect may be attributed to the fact that values of $\sigma(S, t)$ in these peripheral regions have little or no influence on an option price evaluated at the point $(S, t) = (100, 0)$, particularly when the initial condition is applied for a strike price close to 100. For S in an interval bounded roughly by the lower strike 96 and the upper strike 104, the numerically determined values conform somewhat more closely to the

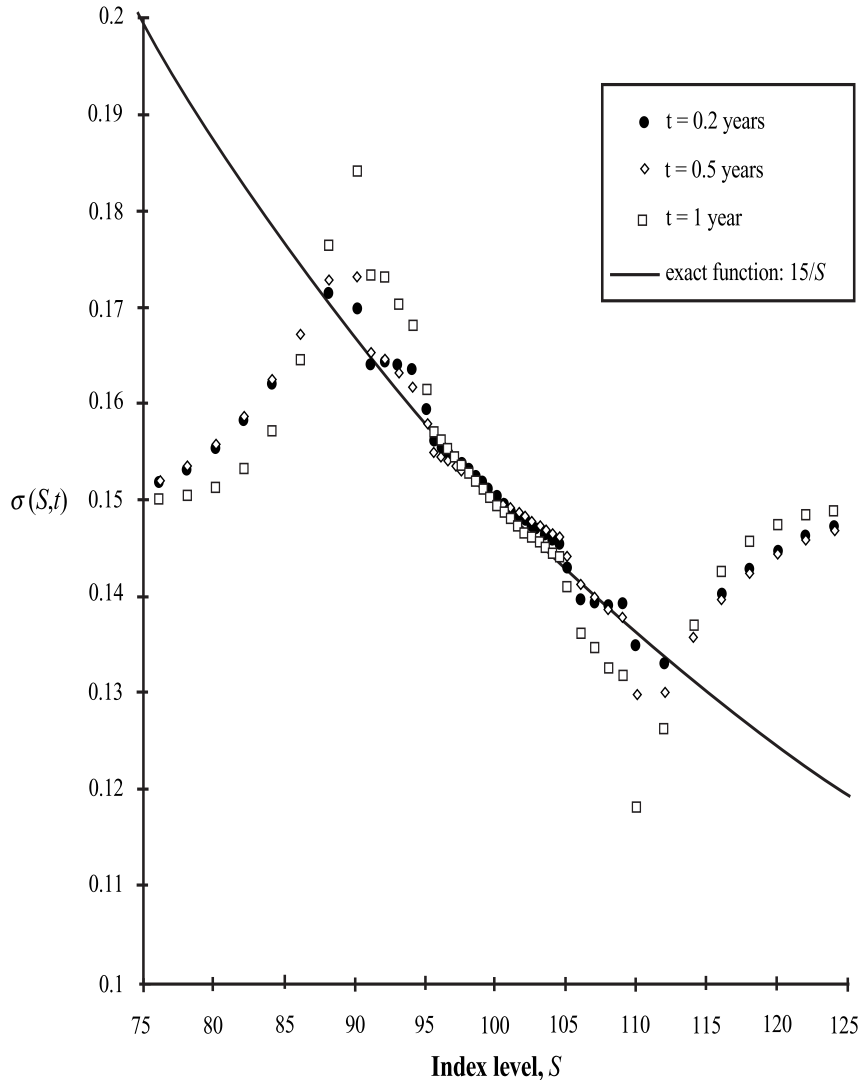

FIGURE 6. Reconstruction of local volatility (22 observations).

exact function. There is, however, a noticeable discrepancy, particular for values along the line $t = 1$. Nevertheless, this is the best 'smooth' approximation of σ that can be obtained on this particular grid using nothing more than the five price observations.

In a second experiment, we added more price observations with a wider distribution of strike prices and expiration dates. We used a total of 22 option prices corresponding to the expiration dates $T_1 = 0.5$ and $T_2 = 1$ year and the strike prices $K_{ij} \in \{90, 92, \ldots, 108, 110\}$ for $i = 1, 2$ and $j = 1, \ldots, 11$. Figure 5 shows the convergence history for the 11 one-year option prices. After 50 iterations, the pricing constraint is again satisfied to within a very small tolerance. Figure 6 then compares the numerically determined values of $\sigma(S, t)$ at some selected grid points with the graph of the exact function $\sigma(S, t) = 15/S$. The agreement is clearly much better than in the first experiment. The numerical reconstruction agrees more closely with the exact function over a wider range of S-values and shows considerably less dependence on time. As expected, the addition of more price observations covering a wider range of strikes and expiration dates improves the recovery of σ.

4. CONCLUDING REMARKS

We have not attempted to prove that the gradient descent procedure will always converge to the optimal solution of the inverse problem. However, our numerical tests provide strong evidence that the procedure is robust and will typically converge given a reasonably good initializing estimate. This question of convergence and related issues concerning existence, uniqueness, and continuous dependence of solutions will be addressed in a more theoretically oriented sequel to this paper.

We have also implied that our calibration technique extends in a straightforward manner beyond a single-factor setting to more complex pricing models. For example, consider the pricing of derivatives that are contingent upon two underlying assets, where the asset prices S_1 and S_2 each follow a general diffusion process of the form (2.1), and where we allow the instantaneous correlation between prices to be a function of S_1, S_2, and time t. This stochastic model should presumably provide enough degrees of freedom to achieve a good match with observed prices of options on the individual assets as well as more exotic dual-asset derivatives.

For such a general model, derivative prices must satisfy a PDE of the form

$$\begin{aligned}\frac{\partial V}{\partial t} + (r - q_1)S_1\frac{\partial V}{\partial S_1} + (r - q_2)S_2\frac{\partial V}{\partial S_2} + \tfrac{1}{2}\sigma_1^2(S_1, t)S_1^2\frac{\partial^2 V}{\partial S_1^2}\\ + \tfrac{1}{2}\sigma_2^2(S_2, t)S_2^2\frac{\partial^2 V}{\partial S_2^2} + \rho(S_1, S_2, t)\sigma_1(S_1, t)\sigma_2(S_2, t)S_1S_2\frac{\partial^2 V}{\partial S_1 \partial S_2} = rV,\end{aligned}$$

where σ_1 and σ_2 are the local volatility functions for the respective asset prices and ρ is the instantaneous correlation function. Calibration to market prices now amounts to a constrained minimization problem involving a more general functional of the form

$$F(\sigma_1, \sigma_2, \rho) = \| \, |\nabla\sigma_1| \, \|_2^2 + \| \, |\nabla\sigma_2| \, \|_2^2 + \| \, |\nabla\rho| \, \|_2^2 + \lambda G(\sigma_1, \sigma_2, \rho),$$

where G is the appropriate functional defining a price constraint similar to (2.5). The gradient descent procedure can be applied to this problem with minimal modification. However, the computational demands will be much greater since we must now estimate three functional parameters.

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