

The Black-Scholes Model

In these notes we will use Itô's Lemma and a replicating argument to derive the famous Black-Scholes formula for European options. We will also discuss the weaknesses of the Black-Scholes model and geometric Brownian motion, and this leads us directly to the concept of the volatility surface which we will discuss in some detail. We will also derive and study the Black-Scholes Greeks and discuss how they are used in practice to hedge option portfolios.

1 The Black-Scholes Model

We are now able to derive the Black-Scholes PDE for a call-option on a non-dividend paying stock with strike K and maturity T. We assume that the stock price follows a geometric Brownian motion so that

$$dS_t = \mu S_t \, dt \; + \; \sigma S_t \, dW_t \tag{1}$$

where W_t is a standard Brownian motion. We also assume that interest rates are constant so that 1 unit of currency invested in the cash account at time 0 will be worth $B_t := \exp(rt)$ at time t. We will denote by $C(S,t)$ the value of the call option at time t. By Itô's lemma we know that

$$dC(S,t) = \left(\mu S_t \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt \; + \; \sigma S_t \frac{\partial C}{\partial S} \, dW_t \tag{2}$$

Let us now consider a *self-financing* trading strategy where at each time t we hold x_t units of the cash account and y_t units of the stock. Then P_t, the time t value of this strategy satisfies

$$P_t = x_t B_t + y_t S_t. \tag{3}$$

We will choose x_t and y_t in such a way that the strategy replicates the value of the option. The self-financing assumption implies that

$$\begin{aligned} dP_t &= x_t \, dB_t \; + \; y_t \, dS_t & (4) \\ &= r x_t B_t \, dt + y_t \left(\mu S_t \, dt \; + \; \sigma S_t \, dW_t\right) \\ &= \left(r x_t B_t + y_t \mu S_t\right) \, dt \; + \; y_t \sigma S_t \, dW_t. & (5) \end{aligned}$$

Note that (4) is consistent with our earlier definition of self-financing. In particular, any gains or losses on the portfolio are due entirely to gains or losses in the underlying securities, i.e. the cash-account and stock, and **not** due to changes in the holdings x_t and y_t.

Returning to our derivation, we can equate terms in (2) with the corresponding terms in (5) to obtain

$$\begin{aligned} y_t &= \frac{\partial C}{\partial S} & (6) \\ r x_t B_t &= \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}. & (7) \end{aligned}$$

If we set $C_0 = P_0$, the initial value of our self-financing strategy, then it must be the case that $C_t = P_t$ for all t since C and P have the same dynamics. This is true by construction after we equated terms in (2) with the corresponding terms in (5). Substituting (6) and (7) into (3) we obtain

$$r S_t \frac{\partial C}{\partial S} \; + \; \frac{\partial C}{\partial t} \; + \; \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \; - \; rC \; = \; 0, \tag{8}$$

the **Black-Scholes** PDE. In order to solve (8) boundary conditions must also be provided. In the case of our call option those conditions are: $C(S,T) = \max(S-K,0)$, $C(0,t) = 0$ for all t and $C(S,t) \to S$ as $S \to \infty$.

The solution to (8) in the case of a call option is

$$\begin{aligned} C(S,t) &= S_t\Phi(d_1) \; - \; e^{-r(T-t)}K\Phi(d_2) \qquad (9) \\ \text{where } \; d_1 &= \frac{\log\left(\frac{S_t}{K}\right) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ \text{and } \; d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

and $\Phi(\cdot)$ is the CDF of the standard normal distribution. One way to confirm (9) is to compute the various partial derivatives using (9), then substitute them into (8) and check that (8) holds. The price of a European put-option can also now be easily computed from put-call parity and (9).

The most interesting feature of the Black-Scholes PDE (8) is that μ does not appear[1] anywhere. Note that the Black-Scholes PDE would also hold if we had assumed that $\mu = r$. However, if $\mu = r$ then investors would not demand a premium for holding the stock. Since this would generally only hold if investors were risk-neutral, this method of derivatives pricing came to be known as *risk-neutral pricing*.

1.1 Martingale Pricing

It can be shown[2] that the Black-Scholes PDE in (8) is consistent with martingale pricing. In particular, if we deflate by the cash account then the deflated stock price process, $Y_t := S_t/B_t$, must be a $\mathcal{Q}$-martingale where Q is the EMM corresponding to taking the cash account as numeraire. It can be shown that the $\mathcal{Q}$-dynamics of S_t satisfy[3]

$$dS_t \;\; = \;\; rS_t\,dt \; + \; \sigma S_t\,dW_t^{\mathcal{Q}} \qquad (10)$$

where $W_t^{\mathcal{Q}}$ is a Q-Brownian motion. Note that (10) implies

$$S_T \; = \; S_t e^{(r-\sigma^2/2)(T-t)+\sigma(W_T^{\mathcal{Q}}-W_t^{\mathcal{Q}})}$$

so that S_T is log-normally distributed under $\mathcal{Q}$. It is now easily confirmed that the call option price in (9) also satisfies

$$C(S_t,t) \; = \; \mathrm{E}_t^{\mathcal{Q}}\left[e^{-r(T-t)}\;\max(S_T-K,\;0)\right] \qquad (11)$$

which is of course consistent with martingale pricing.

1.2 Dividends

If we assume that the stock pays a continuous dividend yield of q, i.e. the dividend paid over the interval $(t,\; t+dt]$ equals $qS_t dt$, then the dynamics of the stock price can be shown to satisfy

$$dS_t \; = \; (r-q)S_t\,dt \; + \; \sigma S_t\,dW_t^{\mathcal{Q}}. \qquad (12)$$

In this case the *total gain process*, i.e. the capital gain or loss from holding the security plus accumulated dividends, is a $\mathcal{Q}$-martingale. The call option price is still given by (11) but now with

$$S_T \; = \; S_t e^{(r-q-\sigma^2/2)(T-t)+\sigma\left(W_T^{\mathcal{Q}}-W_t^{\mathcal{Q}}\right)}.$$

[1]The discrete-time counterpart to this observation was when we observed that the true probabilities of up-moves and down-moves did not have an impact on option prices.

[2]We would need to use stochastic calculus tools that we have not discussed in these notes to show exactly why the Black-Scholes call option price is consistent with martingale pricing. It can also be shown that the Black-Scholes model is complete so that there is a unique EMM corresponding to any numeraire.

[3]You can check using Itô's Lemma that if S_t satisfies (10) then Y_t will indeed be a $\mathcal{Q}$-martingale.

In this case the call option price is given by

$$C(S,t) = e^{-q(T-t)}S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2) \tag{13}$$

$$\text{where } d_1 = \frac{\log\left(\frac{S_t}{K}\right) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\text{and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

Exercise 1 *Follow the replicating argument given above to derive the Black-Scholes PDE when the stock pays a continuous dividend yield of q.*

---

2 The Volatility Surface

The Black-Scholes model is an elegant model but it does not perform very well in practice. For example, it is well known that stock prices jump on occasions and do not always move in the continuous manner predicted by the GBM motion model. Stock prices also tend to have fatter tails than those predicted by GBM. Finally, if the Black-Scholes model were correct then we should have a flat *implied volatility surface*. The volatility surface is a function of strike, K, and time-to-maturity, T, and is defined implicitly

$$C(S,K,T) := \mathsf{BS}\left(S,T,r,q,K,\sigma(K,T)\right) \tag{14}$$

where $C(S,K,T)$ denotes the current market price of a call option with time-to-maturity T and strike K, and $\mathsf{BS}(\cdot)$ is the Black-Scholes formula for pricing a call option. In other words, $\sigma(K,T)$ is the volatility that, when substituted into the Black-Scholes formula, gives the market price, $C(S,K,T)$. Because the Black-Scholes formula is continuous and increasing in σ, there will always[4] be a unique solution, $\sigma(K,T)$. If the Black-Scholes model were correct then the volatility surface would be flat with $\sigma(K,T) = \sigma$ for all K and T. In practice, however, not only is the volatility surface not flat but it actually varies, often significantly, with time.

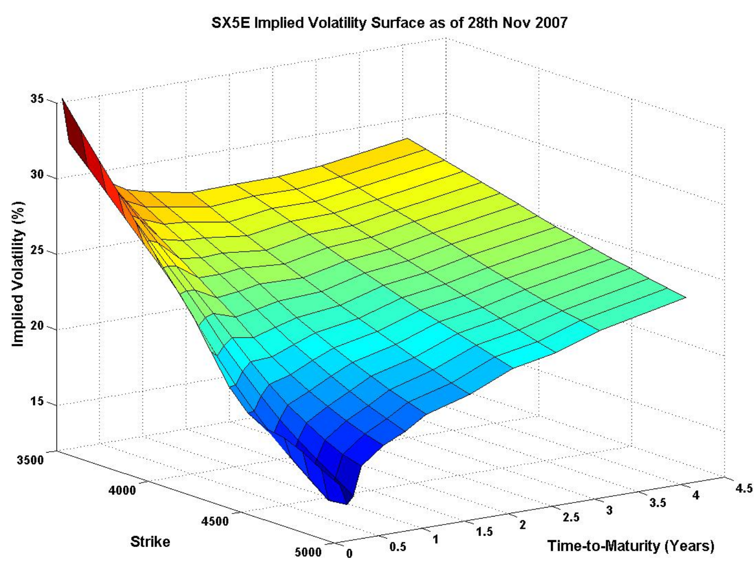


Figure 1: The Volatility Surface

[4] Assuming there is no arbitrage in the market-place.

In Figure 1 above we see a snapshot of the[5] volatility surface for the Eurostoxx 50 index on November 28^{th}, 2007. The principal features of the volatility surface is that options with lower strikes tend to have higher implied volatilities. For a given maturity, T, this feature is typically referred to as the volatility **skew** or **smile**. For a given strike, K, the implied volatility can be either increasing or decreasing with time-to-maturity. In general, however, $\sigma(K,T)$ tends to converge to a constant as $T \rightarrow \infty$. For T small, however, we often observe an inverted volatility surface with short-term options having much higher volatilities than longer-term options. This is particularly true in times of market stress.

It is worth pointing out that different implementations[6] of Black-Scholes will result in different implied volatility surfaces. If the implementations are correct, however, then we would expect the volatility surfaces to be very similar in shape. Single-stock options are generally American and in this case, put and call options will typically give rise to different surfaces. Note that put-call parity does not apply for American options.

Clearly then the Black-Scholes model is far from accurate and market participants are well aware of this. However, the language of Black-Scholes is pervasive. Every trading desk computes the Black-Scholes implied volatility surface and the Greeks they compute and use are Black-Scholes Greeks.

Arbitrage Constraints on the Volatility Surface

The shape of the implied volatility surface is constrained by the absence of arbitrage. In particular:

1. We must have $\sigma(K,T) \geq 0$ for all strikes K and expirations T.
2. At any given maturity, T, the skew cannot be too steep. Otherwise butterfly arbitrages will exist. For example fix a maturity, T and consider put two options with strikes $K_1 < K_2$. If there is no arbitrage then it must be the case (why?) that $P(K_1) < P(K_2)$ where $P(K_i)$ is the price of the put option with strike K_i. However, if the skew is too steep then we would obtain (why?) $P(K_1) > P(K_2)$.
3. Likewise the *term structure* of implied volatility cannot be too inverted. Otherwise calendar spread arbitrages will exist. This is most easily seen in the case where $r = q = 0$. Then, fixing a strike K, we can let $C_t(T)$ denote the time t price of a call option with strike K and maturity T. Martingale pricing implies that $C_t(T) = \mathsf{E}_t[(S_T - K)^+]$. We have seen before that $(S_T - K)^+$ is a Q-submartingale and now standard martingale results can be used to show that $C_t(T)$ must be non-decreasing in T. This would be violated (why?) if the term structure of implied volatility was too inverted.

In practice the implied volatility surface will not violate any of these restrictions as otherwise there would be an arbitrage in the market. These restrictions can be difficult to enforce, however, when we are "bumping" or "stressing" the volatility surface, a task that is commonly performed for risk management purposes.

Why is there a Skew?

For stocks and stock indices the shape of the volatility surface is always changing. There is generally a skew, however, so that for any fixed maturity, T, the implied volatility *decreases* with the strike, K. It is most pronounced at shorter expirations. There are two principal explanations for the skew.

1. **Risk aversion** which can appear as an explanation in many guises:
   (a) Stocks do not follow GBM with a fixed volatility. Instead they often jump and jumps to the downside tend to be larger and more frequent than jumps to the upside.
   (b) As markets go down, fear sets in and volatility goes up.
   (c) Supply and demand. Investors like to protect their portfolio by purchasing out-of-the-money puts and so there is more demand for options with lower strikes.

[5]Note that by put-call parity the implied volatility $\sigma(K,T)$ for a given European call option will be also be the implied volatility for a European put option of the same strike and maturity. Hence we can talk about "the" implied volatility surface.

[6]For example different methods of handling dividends would result in different implementations.

2. The **leverage effect** which is due to the fact that the total value of company assets, i.e. debt + equity, is a more natural candidate to follow GBM. If so, then equity volatility should increase as the equity value decreases. To see this consider the following:

   Let V, E and D denote the total value of a company, the company's equity and the company's debt, respectively. Then the fundamental accounting equations states that

$$V = D + E. \tag{15}$$

   (Equation (15) is the basis for the classical structural models that are used to price risky debt and credit default swaps. Merton (1970's) recognized that the equity value could be viewed as the value of a call option on V with strike equal to D.)

   Let ΔV, ΔE and ΔD be the change in values of V, E and D, respectively. Then $V + \Delta V = (E + \Delta E) + (D + \Delta D)$ so that

$$\begin{aligned} \frac{V + \Delta V}{V} &= \frac{E + \Delta E}{V} + \frac{D + \Delta D}{V} \\ &= \frac{E}{V}\left(\frac{E + \Delta E}{E}\right) + \frac{D}{V}\left(\frac{D + \Delta D}{D}\right) \end{aligned} \tag{16}$$

   If the equity component is substantial so that the debt is not too risky, then (16) implies

$$\sigma_V \approx \frac{E}{V}\,\sigma_E$$

   where σ_V and σ_E are the firm value and equity volatilities, respectively. We therefore have

$$\sigma_E \approx \frac{V}{E}\,\sigma_V. \tag{17}$$

Example 1 (The Leverage Effect)
Suppose, for example, that $V = 1$, $E = .5$ and $\sigma_V = 20\%$. Then (17) implies $\sigma_E \approx 40\%$. Suppose σ_V remains unchanged but that over time the firm loses 20% of its value. Almost all of this loss is borne by equity so that now (17) implies $\sigma_E \approx 53\%$. σ_E has therefore increased despite the fact that σ_V has remained constant. ∎

It is interesting to note that there was little or no skew in the market before the Wall street crash of 1987. So it appears to be the case that it took the market the best part of two decades before it understood that it was pricing options incorrectly.

What the Volatility Surface Tells Us

To be clear, we continue to assume that the volatility surface has been constructed from European option prices. Consider a **butterfly** strategy centered at K where you are:

1. long a call option with strike $K - \Delta K$
2. long a call with strike $K + \Delta K$
3. short 2 call options with strike K

The value of the butterfly, B_0, at time $t = 0$, satisfies

$$\begin{aligned} B_0 &= C(K - \Delta K, T) - 2C(K, T) + C(K + \Delta K, T) \\ &\approx e^{-rT}\ \text{Prob}(K - \Delta K \leq S_T \leq K + \Delta K) \times \Delta K/2 \\ &\approx e^{-rT}\ f(K, T) \times 2\Delta K \times \Delta K/2 \\ &= e^{-rT}\ f(K, T) \times (\Delta K)^2 \end{aligned}$$

where $f(K,T)$ is the (risk-neutral) probability density function (PDF) of S_T evaluated at K. We therefore have

$$f(K,T) \quad \approx \quad e^{rT}\ \frac{C(K-\Delta K,T) - 2C(K,T) + C(K+\Delta K,T)}{(\Delta K)^2}. \tag{18}$$

Letting $\Delta K \to 0$ in (18), we obtain

$$f(K,T) = e^{rT}\ \frac{\partial^2 C}{\partial K^2}.$$

The volatility surface therefore gives the **marginal** risk-neutral distribution of the stock price, S_T, for any time, T. It tells us **nothing** about the **joint** distribution of the stock price at multiple times, $T_1, \ldots, T_n$.

This should not be surprising since the volatility surface is constructed from European option prices and the latter only depend on the marginal distributions of S_T.

Example 2 (Same marginals, different joint distributions)
Suppose there are two time periods, T_1 and T_2, of interest and that a non-dividend paying security has risk-neutral distributions given by

$$S_{T_1} \quad = \quad e^{(r-\sigma^2/2)T_1 + \sigma\sqrt{T_1}\,Z_1} \tag{19}$$

$$S_{T_2} \quad = \quad e^{(r-\sigma^2/2)T_2 + \sigma\sqrt{T_2}\,\left(\rho Z_1 + \sqrt{1-\rho^2}\,Z_2\right)} \tag{20}$$

where Z_1 and Z_2 are independent $\mathsf{N}(0,1)$ random variables. Note that a value of $\rho > 0$ can capture a *momentum* effect and a value of $\rho < 0$ can capture a mean-reversion effect. We are also implicitly assuming that $S_0 = 1$.

Suppose now that there are two securities, A and B say, with prices $S_t^{(A)}$ and $S_t^{(B)}$ given by (19) and (20) at times $t = T_1$ and $t = T_2$, and with parameters $\rho = \rho_A$ and $\rho = \rho_B$, respectively. Note that the *marginal* distribution of $S_t^{(A)}$ is identical to the marginal distribution of $S_t^{(B)}$ for $t \in \{T_1, T_2\}$. It therefore follows that options on A and B with the same strike and maturity must have the same price. A and B therefore have identical volatility surfaces.

But now consider a knock-in put option with strike 1 and expiration T_2. In order to knock-in, the stock price at time T_1 must exceed the barrier price of 1.2. The payoff function is then given by

$$\text{Payoff} \ = \ \max\left(1 - S_{T_2},\ 0\right)\ 1_{\{S_{T_1} \geq 1.2\}}.$$

Question: Would the knock-in put option on A have the same price as the knock-in put option on B?

Question: How does your answer depend on ρ_A and ρ_B?

Question: What does this say about the ability of the volatility surface to price barrier options? ▮

---

3 The Greeks

We now turn to the sensitivities of the option prices to the various parameters. These sensitivities, or the **Greeks** are usually computed using the Black-Scholes formula, despite the fact that the Black-Scholes model is known to be a poor approximation to reality. But first we return to put-call parity.

Put-Call Parity

Consider a European call option and a European put option, respectively, each with the same strike, K, and maturity T. Assuming a continuous dividend yield, q, then put-call parity states

$$e^{-rT}\,K + \text{ Call Price } \ = \ e^{-qT}\,S + \text{ Put Price.} \tag{21}$$

This of course follows from a simple arbitrage argument and the fact that both sides of (21) equal $\max(S_T, K)$ at time T. Put-call parity is useful for calculating Greeks. For example[7], it implies that Vega(Call) = Vega(Put) and that Gamma(Call) = Gamma(Put). It is also extremely useful for **calibrating dividends** and constructing the volatility surface.

The Greeks

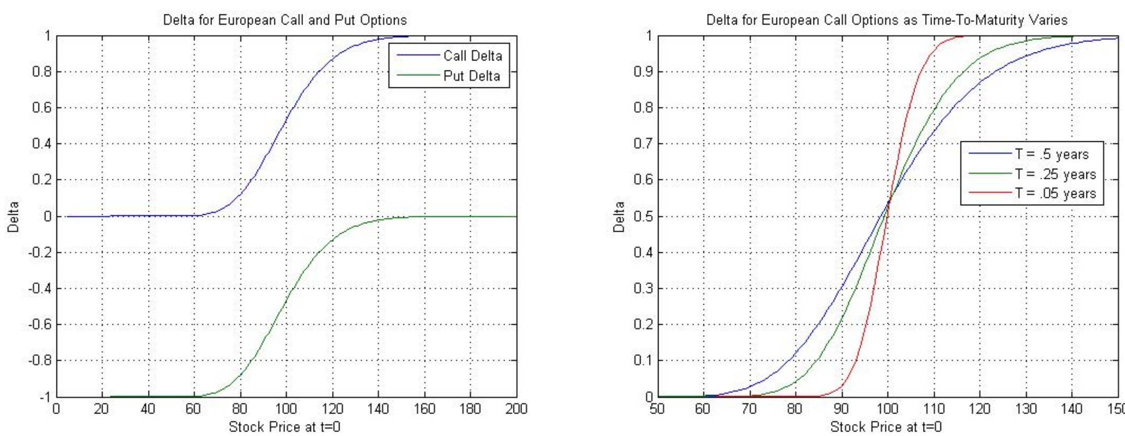
The principal Greeks for European call options are described below. The Greeks for put options can be calculated in the same manner or via put-call parity.

Definition: The **delta** of an option is the sensitivity of the option price to a change in the price of the underlying security.

The delta of a European call option satisfies

$$\text{delta} = \frac{\partial C}{\partial S} = e^{-qT}\,\Phi(d_1).$$

This is the usual delta corresponding to a volatility surface that is *sticky-by-strike*. It assumes that as the underlying security moves, the volatility of the option does **not** move. If the volatility of the option did move then the delta would have an additional term of the form vega $\times\, \partial\sigma(K,T)/\partial S$. In this case we would say that the volatility surface was *sticky-by-delta*. Equity markets typically use the sticky-by-strike approach when computing deltas. Foreign exchange markets, on the other hand, tend to use the sticky-by-delta approach. Similar comments apply to gamma as defined below.



(a) Delta for European Call and Put Options (b) Delta for Call Options as Time-To-Maturity Varies

Figure 2: Delta for European Options

By put-call parity, we have $\text{delta}_{\text{put}} = \text{delta}_{\text{call}} - e^{-qT}$. Figure 2(a) shows the delta for a call and put option, respectively, as a function of the underlying stock price. In Figure 2(b) we show the delta for a call option as a function of the underlying stock price for three different times-to-maturity. It was assumed $r = q = 0$. What is the strike K? Note that the delta becomes steeper around K when time-to-maturity decreases. Note also that $\text{delta}_{\text{call}} = \Phi(d_1)$ is often loosely interpreted as the (risk-neutral) probability that the option will expire in the money. (This risk-neutral probability is in fact $\Phi(d_2)$.)

In Figure 3 we show the delta of a call option as a function of time-to-maturity for three options of different

[7]See below for definitions of vega and gamma.

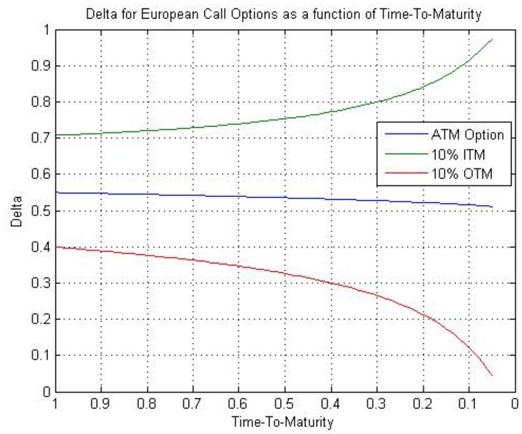


Figure 3: Delta for European Call Options as a Function of Time-To-Maturity

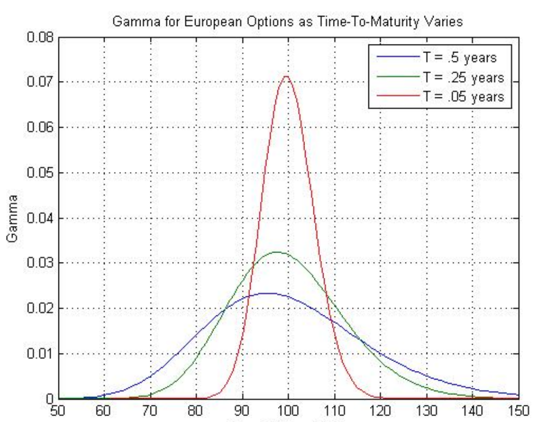
money-ness. Are there any surprises here? What would the corresponding plot for put options look like?

Definition: The **gamma** of an option is the sensitivity of the option's delta to a change in the price of the underlying security.

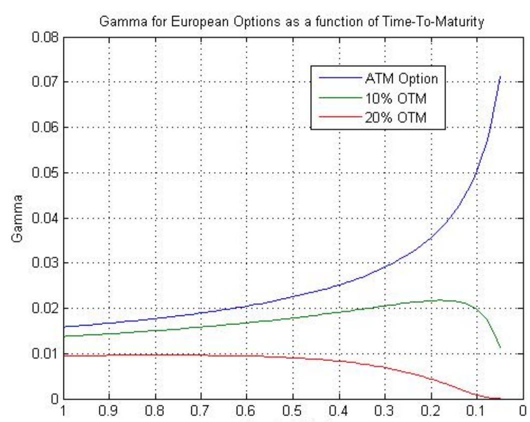
The gamma of a call option satisfies

$$\text{gamma} \;=\; \frac{\partial^2 C}{\partial S^2} \;=\; e^{-qT}\,\frac{\phi(d_1)}{\sigma S\sqrt{T}}$$

where $\phi(\cdot)$ is the standard normal PDF.



(a) Gamma as a Function of Stock Price



(b) Gamma as a Function of Time-to-Maturity

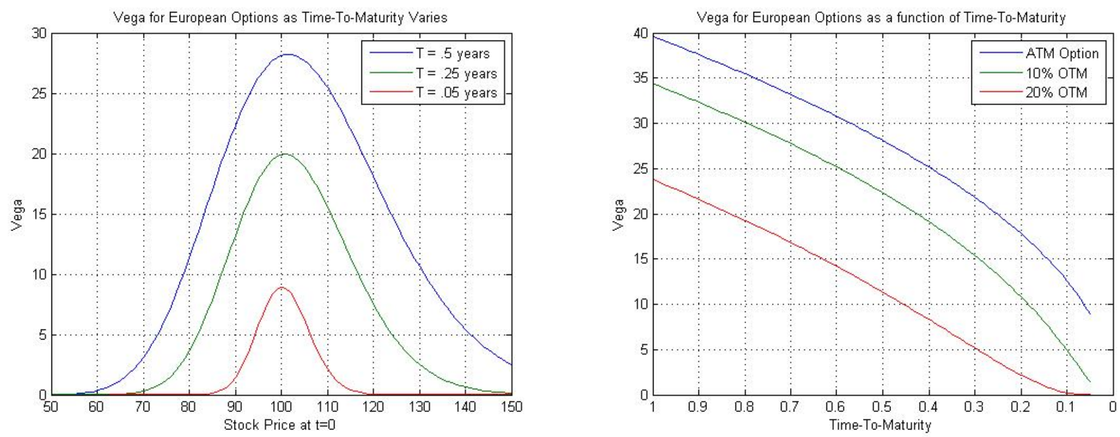
Figure 4: Gamma for European Options

In Figure 4(a) we show the gamma of a European option as a function of stock price for three different time-to-maturities. Note that by put-call parity, the gamma for European call and put options with the same strike are equal. Gamma is always positive due to option **convexity**. Traders who are long gamma can make money by gamma *scalping*. Gamma scalping is the process of regularly re-balancing your options portfolio to be delta-neutral. However, you must pay for this long gamma position up front with the option premium. In Figure 4(b), we display gamma as a function of time-to-maturity. Can you explain the behavior of the three curves in Figure 4(b)?

Definition: The **vega** of an option is the sensitivity of the option price to a change in volatility.

The vega of a call option satisfies

$$\text{vega} \;=\; \frac{\partial C}{\partial \sigma} \;=\; e^{-qT} S\sqrt{T}\;\phi(d_1).$$



(a) Vega as a Function of Stock Price

(b) Vega as a Function of Time-to-Maturity

Figure 5: Vega for European Options

In Figure 5(b) we plot vega as a function of the underlying stock price. We assumed $K = 100$ and that $r = q = 0$. Note again that by put-call parity, the vega of a call option equals the vega of a put option with the same strike. Why does vega increase with time-to-maturity? For a given time-to-maturity, why is vega peaked near the strike? Turning to Figure 5(b), note that the vega decreases to 0 as time-to-maturity goes to 0. This is consistent with Figure 5(a). It is also clear from the expression for vega.

Question: Is there any "inconsistency" to talk about vega when we use the Black-Scholes model?

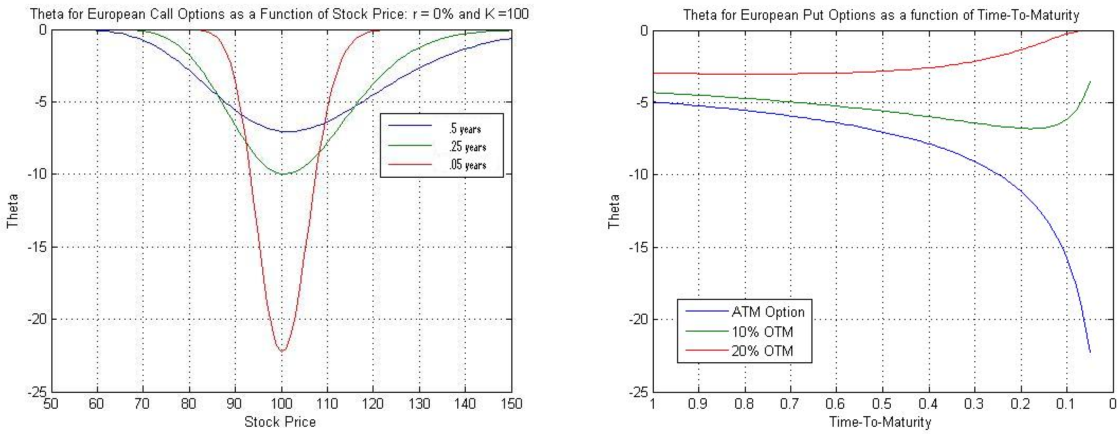
Definition: The **theta** of an option is the sensitivity of the option price to a *negative* change in time-to-maturity.

The theta of a call option satisfies

$$\text{theta} \;=\; -\frac{\partial C}{\partial T} \;\;=\;\; -e^{-qT} S\phi(d_1)\frac{\sigma}{2\sqrt{T}} \;+\; qe^{-qT} S\mathsf{N}(d_1) \;-\; rKe^{-rT}\mathsf{N}(d_2).$$

In Figure 6(a) we plot theta for three call options of different times-to-maturity as a function of the underlying stock price. We have assumed that $r = q = 0\%$. Note that the call option's theta is always negative. Can you explain why this is the case? Why does theta become more negatively peaked as time-to-maturity decreases to 0?

In Figure 6(b) we again plot theta for three call options of different money-ness, but this time as a function of time-to-maturity. Note that the ATM option has the most negative theta and this gets more negative as



(a) Theta as a Function of Stock Price (b) Theta as a Function of Time-to-Maturity

Figure 6: Theta for European Options

time-to-maturity goes to 0. Can you explain why?

Options Can Have Positive Theta: In Figure 7 we plot theta for three put options of different money-ness as a function of time-to-maturity. We assume here that $q = 0$ and $r = 10\%$. Note that theta can be positive for in-the-money put options. Why? We can also obtain positive theta for call options when q is large. In typical scenarios, however, theta for both call and put options will be negative.

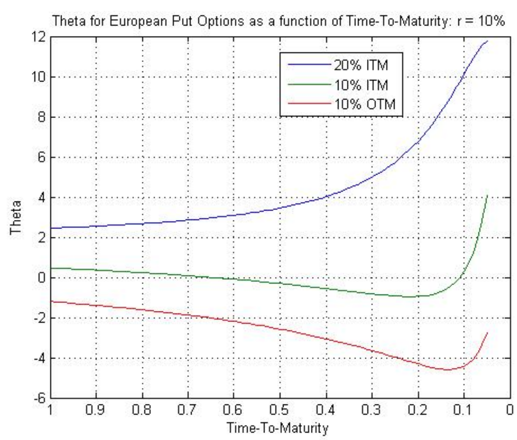


Figure 7: Positive Theta is Possible

Delta-Gamma-Vega Approximations to Option Prices

A simple application of Taylor's Theorem says

$$
\begin{aligned}
C(S+\Delta S, \sigma+\Delta\sigma) &\approx C(S,\sigma) + \Delta S \frac{\partial C}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 C}{\partial S^2} + \Delta\sigma \frac{\partial C}{\partial \sigma} \\
= C(S,\sigma) &+ \Delta S \times \delta + \frac{1}{2}(\Delta S)^2 \times \Gamma + \Delta\sigma \times \text{vega}
\end{aligned}
$$

where $C(S,\sigma)$ is the price of a derivative security as a function[8] of the current stock price, S, and the implied volatility, σ. We therefore obtain

$$\begin{aligned}\text{P\&L} &= \delta\Delta S + \frac{\Gamma}{2}(\Delta S)^2 + \text{vega } \Delta\sigma \\ &= \text{delta P\&L } + \text{ gamma P\&L } + \text{ vega P\&L}\end{aligned}$$

When $\Delta\sigma = 0$, we obtain the well-known *delta-gamma* approximation. This approximation is often used, for example, in historical **Value-at-Risk** (VaR) calculations for portfolios that include options. We can also write

$$\begin{aligned}\text{P\&L} &= \delta S\left(\frac{\Delta S}{S}\right) + \frac{\Gamma S^2}{2}\left(\frac{\Delta S}{S}\right)^2 + \text{vega } \Delta\sigma \\ &= \text{ESP} \times \text{Return} + \text{\$Gamma} \times \text{Return}^2 + \text{vega } \Delta\sigma\end{aligned}$$

where ESP denotes the **equivalent stock position** or **"dollar" delta**.

---

4 Delta Hedging

In the Black-Scholes model with GBM, an option can be replicated exactly by **delta-hedging** the option. In fact the Black-Scholes PDE we derived earlier was obtained by a delta-hedging / replication argument. The idea behind delta-hedging is to re-balance the portfolio of the option and stock continuously so that you always have a total delta of zero after re-balancing. Of course it is not practical in to hedge continuously and so instead we hedge periodically. Periodic or discrete hedging then results in some *replication error*.

Let P_t denote the time t value of the discrete-time self-financing strategy that attempts to replicate the option payoff and let C_0 denote the initial value of the option. The replicating strategy is then given by

$$P_0 := C_0 \tag{22}$$

$$P_{t_{i+1}} = P_{t_i} + (P_{t_i} - \delta_{t_i} S_{t_i})\, r\Delta t + \delta_{t_i}\left(S_{t_{i+1}} - S_{t_i} + qS_{t_i}\Delta t\right) \tag{23}$$

where $\Delta t := t_{i+1} - t_i$ is the length of time between re-balancing (assumed constant for all i), r is the annual risk-free interest rate (assuming per-period compounding), q is the dividend yield and δ_{t_i} is the Black-Scholes delta at time t_i. This delta is a function of S_{t_i} and some assumed implied volatility, σ_{imp} say. Note that (22) and (23) respect the self-financing condition. Stock prices are simulated assuming $S_t \sim \text{GBM}(\mu,\sigma)$ so that

$$S_{t+\Delta t} = S_t e^{(\mu-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} Z}$$

where $Z \sim \text{N}(0,1)$. Note the option implied volatility, σ_{imp}, need not equal σ which in turn need not equal the *realized* volatility (when we hedge periodically as opposed to continuously). This has interesting implications for the trading P&L which we may define as

$$\text{P\&L} := P_T - (S_T - K)^+$$

in the case of a short position in a call option with strike K and maturity T. Note that P_T is the terminal value of the replicating strategy in (23). Many interesting questions now arise:

Question: If you sell options, what typically happens the total P&L if $\sigma < \sigma_{imp}$?

Question: If you sell options, what typically happens the total P&L if $\sigma > \sigma_{imp}$?

Question: If $\sigma = \sigma_{imp}$ what typically happens the total P&L as the number of re-balances increases?

Some Answers to Delta-Hedging Questions

Recall that the price of an option increases as the volatility increases. Therefore if realized volatility is higher than expected, i.e. the level at which it was sold, we expect to lose money on average when we delta-hedge an

[8]The price may also depend on other parameters, in particular time-to-maturity, but we suppress that dependence here.

option that we sold. Similarly, we expect to make money when we delta-hedge if the realized volatility is lower than the level at which it was sold.

In general, however, the payoff from delta-hedging an option is **path-dependent**, i.e. it depends on the price path taken by the stock over the entire time interval. In fact, we can show that the payoff from continuously delta-hedging an option satisfies

$$\text{P\&L} \;=\; \int_0^T \frac{S_t^2}{2}\frac{\partial^2 V_t}{\partial S^2}\;\left(\sigma_{imp}^2 - \sigma_t^2\right)\; dt \tag{24}$$

where V_t is the time t value of the option and σ_t is the realized instantaneous volatility at time t.

The term $\frac{S_t^2}{2}\frac{\partial^2 V_t}{\partial S^2}$ is often called the **dollar gamma**, as discussed earlier. It is always positive for a call or put option, but it goes to zero as the option moves significantly into or out of the money.

Returning to self-financing trading strategy of (22) and (23), note that we can choose any model we like for the security price dynamics. In particular, we are not restricted to choosing geometric Brownian motion and other diffusion or jump-diffusion models could be used instead. It is interesting to simulate these alternative models and to then observe what happens to the replication error in (24) where the δ_{t_i}'s are computed assuming (incorrectly) a geometric Brownian motion price dynamics.