

# Modeling foreign exchange rates with jumps

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## Abstract

We propose a new discrete-time model of returns in which jumps capture persistence in the conditional variance and higher-order moments. Jump arrival is governed by a heterogeneous Poisson process. The intensity is directed by a latent stochastic autoregressive process, while the jump-size distribution allows for conditional heteroskedasticity. Model evaluation focuses on the dynamics of the conditional distribution of returns using density and variance forecasts. Predictive likelihoods provide a period-by-period comparison of the performance of our heterogeneous jump model relative to conventional SV and GARCH models. Further, in contrast to previous studies on the importance of jumps, we utilize realized volatility to assess out-of-sample variance forecasts.

key words: jump clustering, jump dynamics, MCMC, predictive likelihood, realized volatility, Bayesian model average

JEL codes: C22, C11, G1

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# 1 Introduction

Measuring and forecasting the distribution of returns is important for many problems in finance. Pricing of financial securities, risk management decisions, and portfolio allocations all depend on the distributional features of returns. The most predictable component of the conditional distribution is the structure in the variance. As a result, a vast literature has sprung from the ARCH model of Engle (1982) and the stochastic volatility (SV) approach of Taylor (1986).

Current research has documented the importance of jump dynamics in combination with autoregressive volatility for modeling returns. Examples of this work include Andersen et al. (2002), Bates (2000), Chernov et al. (2003), Chib et al. (2002), Eraker et al. (2003), Jorion (1988), and Maheu & McCurdy (2004) among others. Jumps provide a useful addition to SV models by explaining occasional, large abrupt moves in financial markets, but they are generally not used to capture volatility clustering. As a result, jumps account for neglected structure, usually tail dynamics, that autoregressive SV cannot capture. Additional SV factors is another possible solution. For example, Chernov et al. (2003) find that a multifactor loglinear SV specification is equivalent to an affine class of SV with jumps for equity data.

All financial data are measured in discrete time which suggests that jumps provide a natural framework to model price moves. Nevertheless, surprisingly few applications exclude an autoregressive SV component in order to focus on the potential performance of the jump specification to capture return dynamics. Recent research, including Das (2002), Lin et al. (1999), Johannes et al. (1999), and Oomen (2006), suggests that jumps alone could provide a good specification for financial returns. As in Maheu & McCurdy (2004), those applications all feature some form of dependence in the arrival rate of jumps. For example, Johannes et al. (1999) allow jump arrival to depend on past jumps and the absolute value of returns. Indeed, the success of a jump model will depend on whether the specification can explain dynamics of the conditional distribution, in particular, volatility clustering. However, the performance of existing models in this regard is unclear. For example, can models with only jump dynamics produce good volatility forecasts? Are they competitive with standard volatility models? The purpose of this paper is to investigate these questions.

This paper proposes a new discrete-time model of returns in which jumps capture persistence in the conditional variance as well as time-variation in higher-order moments. The jump intensity is directed by a latent stochastic autoregressive process. Therefore, jump arrivals can cluster. We also allow the jump-size distribution to be conditionally

heteroskedastic. Larger jumps occur during volatile periods and smaller ones during quiet periods. In contrast to GARCH and SV models, our heterogeneous jump model allows periods of homoskedasticity and periods of heteroskedasticity.

We estimate the model using Markov chain Monte Carlo (MCMC) methods, and follow Johannes et al. (1999) in treating unobserved state variables such as jump times and jump sizes as parameters. Our MCMC estimation approach generates estimates of these quantities which incorporate parameter uncertainty.

Model evaluation focuses on the dynamics of the conditional distribution of returns. Using Bayesian simulation methods, we estimate predictive likelihoods for models as suggested by Geweke (1994). This is the relevant measure of out-of-sample predictive content of a model (Geweke & Whiteman (2006)). This allows for period-by-period comparisons of the jump model with the benchmark SV model, and is particularly useful in identifying influential observations. From these calculations, we are able report the cumulative model probability as a function of time, which provides insight into the performance of each model through time.

In addition to conditional density evaluation, we compare forecasts of volatility associated with each model, as well as the combined model average forecast. Andersen & Bollerslev (1998) and Andersen et al. (2001) show that the sum-of-squared intraday returns is an efficient and consistent estimate of ex post daily volatility. As in Andersen et al. (2001) and Maheu & McCurdy (2002), we compute this realized volatility estimate based on 5-minute bilateral exchange rate data from the yen-U.S. dollar (JPY-USD) foreign exchange market. The out-of-sample forecasts of volatility from our alternative models are assessed using these realized volatility measures.

Our empirical results show that the process governing jump arrival is highly persistent. The conditional jump probability (intensity) varies widely over time, taking on values of less than 0.05 to values in excess of 0.90. This implies that jumps will tend to cluster which can also be seen from the estimated jump times. The jump size variance is a positive function of the market's volatility as measured by last period's absolute value of daily returns. Therefore realized jumps will tend to be larger in a volatile market.

Based on the predictive Bayes factor, our jump specification is favored over both SV and GARCH alternatives. Although the evidence in favor of the jump model varies considerably over the sample, it gradually strengthens from the perspective of the cumulative model probability. However, the SV model is competitive. Therefore, when we proceed to evaluating out-of-sample variance forecasts relative to realized volatility, we also include a model average variance forecast.

The main difference in the variance forecasts from the alternative models is that

during low volatility periods the jump model produces a constant variance forecast due to a low probability of jumps. On the other hand, the SV and GARCH model forecasts tend to be too volatile during these calm periods. In other words, periods of low volatility may be better represented by the constant variance generated by the jump model. Another important feature of our model is that we can immediately move from high volatility to low volatility levels avoiding the slow decay built into SV and GARCH models. In summary, our proposed heterogeneous jump model can account for volatility persistence and is very competitive with existing models of volatility.

The paper is organized as follows. The next section presents a heterogeneous jump model for foreign exchange returns, while Section 3 briefly discusses benchmark SV and GARCH specifications used for comparison purposes. Section 4 considers Bayesian estimation of the jump model. Estimation of the predictive likelihood for model comparison is reviewed in Section 5, while model forecasts are explained in Section 6. Data sources are found in Section 7, results in Section 8, and conclusions in Section 9. An Appendix contains detailed calculations for the estimation algorithms.

## 2 Basic Jump Model

We begin with a brief review of the Press (1967) jump model. Consider the following jump diffusion,

$$\frac{dP(t)}{P(t)} = \alpha dt + \sigma dW(t) + \exp(\xi) dq(t) \quad (2.1)$$

where  $P(t)$  is the security price,  $dW(t)$  is a standard Wiener process increment,  $\exp(\xi)$  is the percent change in the share price from a jump, and  $dq(t)$  is a Poisson counter ( $dq(t) = 1$  for a jump and otherwise 0) with intensity parameter  $\lambda$ . From the solution to this differential equation the daily return evolves according to,

$$r_t = \mu dt + \sigma z_t + \sum_{k=q(t-1)}^{q(t)} \xi_k \quad (2.2)$$

where  $r_t = \log(P(t)/P(t-1))$ , and  $\mu = \alpha - \sigma^2/2$ . Although this model can produce constant conditional and unconditional skewness and kurtosis, it cannot capture volatility clustering. There is a large literature which augments this basic model with dynamics for  $\sigma^2$ . The purpose of this paper is to investigate additional dynamics in the jump intensity and the jump-size distribution. Equation 2.2 motivates our starting point for

exploring the importance of jump dynamics which we describe next.

## 2.1 Heterogeneous Jump Parameterization

This subsection proposes a new discrete-time model which can capture the autoregressive pattern in the conditional variance of returns by allowing jumps to arrive according to a heterogeneous Poisson process. Our parameterization includes a latent autoregressive structure for the jump intensity, as well as a conditionally heteroskedastic variance for the jump-size distribution. The mean of the jump-size distribution can be significantly different from zero, allowing the specification to capture a skewed distribution of returns.

The parameterization of the heterogeneous jump model is as follows:

$$r_t = \mu + \sigma z_t + J_t \xi_t, \quad z_t \sim N(0, 1) \quad (2.3)$$

$$\xi_t \sim N(\mu_\xi, \sigma_{\xi,t}^2), \quad J_t \in \{0, 1\} \quad (2.4)$$

$$P(J_t = 1|\omega_t) = \lambda_t \text{ and } P(J_t = 0|\omega_t) = 1 - \lambda_t \quad (2.5)$$

$$\lambda_t = \frac{\exp(\omega_t)}{1 + \exp(\omega_t)} \quad (2.6)$$

$$\omega_t = \gamma_0 + \gamma_1 \omega_{t-1} + u_t, \quad u_t \sim N(0, 1), \quad |\gamma_1| < 1 \quad (2.7)$$

$$\sigma_{\xi,t}^2 = \eta_0 + \eta_1 X_{t-1}, \quad (2.8)$$

where  $r_t$  denotes daily returns,  $t = 1, \dots, T$ ,  $\mu$  is the mean of the returns conditional on no jump, and  $\xi_t$  is the jump size which follows a conditional normal distribution.

$J_t$  is an indicator that identifies when jumps occur. In particular, the set  $\{\forall J_t | J_t = 1\}$  denotes jump times.  $\lambda_t$  is the time-varying jump intensity or arrival process which is directed by the latent autoregressive process  $\omega_t$ . In this paper, we allow for one jump per period, that is,  $J_t = 0, 1$ . The logistic function ensures that  $\omega_t$  is mapped into a  $(0, 1)$  interval for  $\lambda_t$ . Conditional on  $\omega_t$  the probability of a jump follows a Bernoulli distribution. This specification could be extended to allow for more than one jump per period by using an ordered probit model for  $P(J_t|\lambda_t)$ . Besides capturing occasional large moves in returns, this specification can account for volatility clustering through persistence in  $\omega_t$  and dependence in higher-order moments of the distribution. This model is a time-varying mixture-of-Normals and therefore possess time-variation in all higher-order moments. One interpretation of  $\omega_t$  is that it represents unobserved news flows into the market that cause trading activity.

The variance of the jump-size distribution,  $\sigma_{\xi,t}^2$ , is allowed to be a function of weakly exogenous regressors  $X_{t-1}$ . In this paper we consider  $X_{t-1} = |r_{t-1}|$ . This specification permits the jump-size variance to be sensitive to recent market volatility levels. For

example, if  $\eta_1 > 0$  jumps will tend to be larger (smaller) in volatile (quiet) markets.

Imposing the restrictions  $\lambda_t = \lambda, \forall t$ , and  $\eta_1 = 0$  obtains a simple pure jump model with an iid arrival of jumps and a homogeneous jump size distribution (Press (1967)). Although not considered in this paper, it is straightforward to allow additional dynamics such as a different jump-size distribution or time dependence in  $\mu_\xi$ , the mean of the jump-size distribution. In addition, other possibilities could be explored for  $X_{t-1}$  such as the range.

### 3 Benchmark Specifications

#### 3.1 SV model

Our application is to foreign exchange rates for which asymmetries and leverage effects are generally absent. Therefore, we consider a standard log-linear stochastic volatility (SV) model as a benchmark for comparison purposes. A large literature discusses estimation methods for this model, for example, Melino & Turnbull (1990), Danielsson & Richard (1993), Gouriéroux et al. (1993), Jacquier et al. (1994), Andersen & Sørensen (1996), Gallant et al. (1997), Kim et al. (1998), Alizadeh et al. (2002), and Jacquier et al. (2004). Surveys of the SV literature include Taylor (1994), Ghysels et al. (1996), and Shephard (1996).

The discrete-time SV model is parameterized as,

$$r_t = \mu + \exp(h_t/2)z_t, \quad z_t \sim N(0, 1) \quad (3.1)$$

$$h_t = \rho_0 + \rho_1 h_{t-1} + \sigma_v v_t, \quad v_t \sim N(0, 1). \quad (3.2)$$

Given  $h_t$ , the conditional variance of returns is  $\exp(h_t)$ . Conditionally, the model produces fat tails. Our estimation approach imposes stationarity by rejecting any MCMC draws that violate  $|\rho_1| < 1$ .

Compared to GARCH models, estimation is more difficult due to the latent volatility which must be integrated out of the likelihood. Bayesian methods rely on Markov Chain Monte Carlo (MCMC) sampling to estimate SV models. The properties of the estimator compare favorably with other approaches (Jacquier et al. (1994)). It is straightforward to obtain smoothed estimates of volatility from MCMC output. In addition, these estimates of volatility take parameter uncertainty into account.

### 3.2 GARCH model

The second benchmark specification that we include is a standard GARCH(1,1) model,

$$r_t = \mu + \epsilon_t, \quad \epsilon_t = \sigma_t z_t, \quad z_t \sim N(0, 1) \quad (3.3)$$

$$\sigma_t^2 = \kappa + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2. \quad (3.4)$$

We restrict the model to be covariance stationary by restricting  $\alpha + \beta < 1$  in estimation.

## 4 Posterior Inference

Johannes & Polson (2006) provide an excellent overview of Bayesian methods for financial models including a simple jump model. Eraker et al. (2003) and Johannes et al. (1999) discuss a data augmentation approach to deal with jump times and jump sizes.

From a Bayesian perspective, inference regarding parameters takes place through the posterior which incorporates both the prior and likelihood function. Let the history of data be denoted as  $\Phi_t = \{r_1, \dots, r_t\}$ . In the case of our heterogeneous Jump model, we augment the parameters  $\theta = \{\mu, \sigma^2, \mu_\xi, \eta, \gamma\}$  where  $\eta = \{\eta_0, \eta_1\}$ , and  $\gamma = \{\gamma_0, \gamma_1\}$ , with the unobserved state vectors  $\omega = \{\omega_1, \dots, \omega_T\}$ , jump times  $J = \{J_1, \dots, J_T\}$ , and jump sizes  $\xi = \{\xi_1, \dots, \xi_T\}$  and treat these as parameters. For the Jump model, Bayes rule gives us

$$p(\theta, \omega, J, \xi | \Phi_T) \propto p(r | \theta, \omega, J, \xi) p(\omega, J, \xi | \theta) p(\theta) \quad (4.1)$$

where  $r = \{r_1, \dots, r_T\}$ ,  $p(r | \theta, \omega, J, \xi)$  is the joint density of returns conditional on the state variables  $\omega, J$ , and  $\xi$ ,  $p(\omega, J, \xi | \theta)$  is the density of the state variables, and  $p(\theta)$  is the prior. By treating jump times and jump sizes as parameters to sample we avoid the difficulty of integrating them out, but more importantly we can compute smoothed estimates of them from the posterior sample. In practice, analytical results are not available and we use MCMC methods to draw samples from the posterior. Surveys of MCMC methods include Geweke (1997), Robert & Casella (1999), and Chib (2001).

MCMC theory allows valid draws from the posterior to be obtained by sampling from a series of conditional distributions. It is often much easier to work with the conditional distributions. In the limit, draws converge to samples from the posterior. A simulation consistent estimate of any function of the parameter vector can be constructed from sample averages. For instance, if we have  $\{\mu^{(i)}\}_{i=1}^N$  draws of  $\mu$  from the posterior, and assuming the integral of  $g(\mu)$  with respect to the marginal posterior exists, we can

estimate  $E[g(\mu)]$  as  $\frac{1}{N} \sum_{i=1}^N g(\mu^{(i)})$ . For example, to compute  $E\mu$ , set  $g(\mu) = \mu$ , to compute  $E\mu^2$  and the variance, set  $g(\mu) = \mu^2$ , etc. See Tierney (1994) for technical details. Further assumptions on the integrability of  $g(\mu)^2$  permit consistent estimation of the asymptotic standard error of the estimate, based on conventional time series methods.

In the following we denote the vector  $\theta$  excluding the  $k$ th element  $\theta_k$ , as  $\theta_{-\theta_k}$ , the subvector  $\{\omega_t, \dots, \omega_\tau\}$  as  $\omega_{(t,\tau)}$  and  $\omega$  excluding  $\omega_{(t,\tau)}$  as  $\omega_{-(t,\tau)}$ . Sampling is based on Gibbs and Metropolis-Hasting (MH) routines. Draws from the posterior  $\psi = \{\theta, \omega, J, \xi\}$  are obtained by cycling over the following steps.

1. sample  $\mu|\theta_{-\mu}, \omega, J, \xi, r$
2. sample  $\sigma^2|\theta_{-\sigma^2}, \omega, J, \xi, r$
3. sample  $\mu_\xi|\theta_{-\mu_\xi}, \omega, J, \xi, r$
4. sample  $\eta|\theta_{-\eta}, \omega, J, \xi, r$
5. sample blocks  $\omega_{(t,\tau)}|\theta, \omega_{-(t,\tau)}, J, \xi, r, t = 1, \dots, T$ .
6. sample  $\gamma|\theta_{-\gamma}, \omega, J, \xi, r$
7. sample  $\xi|\theta, \omega, J, r$
8. sample  $J|\theta, \omega, \xi, r$
9. goto 1

A pass through 1 - 8 provides a draw from the posterior. We repeat this several thousand times and collect these draws after an initial burn-in period. Note that the parameters  $\eta$ ,  $\gamma$ ,  $J$ ,  $\xi$ , and  $\omega$  are sampled as blocks which may contribute to better mixing of the MCMC output. Detailed steps of the algorithm are collected in the Appendix.

## 5 Model Comparison

The key ingredient in Bayesian model comparison is the marginal likelihood which can be used to form Bayes factors or model probabilities. However, a drawback of any statistical approach that summarizes a model's performance with a single number is understanding why and when a model performs well or poorly. Geweke (1994) suggests the use of a predictive likelihood decomposition of the marginal likelihood. Estimating the predictive likelihood allows us to compare models on an observation by observation



basis. This may be useful in identifying influential observations, or periods that make a large contribution to predictive Bayes factors. Applications of this idea include Gordon (1997), and Min & Zellner (1993).

Consider a model with parameter vector  $\Theta$ . The predictive density for observation  $y_{t+t}$  based on the information set  $\Phi_t$  is,

$$p(y_{t+1}|\Phi_t) = \int p(y_{t+1}|\Phi_t, \Theta)p(\Theta|\Phi_t)d\Theta \quad (5.1)$$

where  $p(\Theta|\Phi_t)$  is the posterior, and  $p(y_{t+1}|\Phi_t, \Theta)$  is the conditional distribution. When it is clear we suppress conditioning on a model for notational convenience. Evaluating (5.1) at the realized  $\tilde{y}_{t+1}$  gives the predictive likelihood,

$$\hat{p}_{t+1} = p(\tilde{y}_{t+1}|\Phi_t) = \int p(\tilde{y}_{t+1}|\Phi_t, \Theta)p(\Theta|\Phi_t)d\Theta \quad (5.2)$$

which can be estimated from MCMC output. Models that have a larger predictive likelihood are preferred to ones with a smaller value, as they are more likely to have generated the data.

Geweke (1994) shows that the predictive likelihood for observations  $\tilde{y}_u, \dots, \tilde{y}_v$ ,  $u < v$  can be decomposed as

$$p(\tilde{y}_u, \dots, \tilde{y}_v|\Phi_{u-1}) = \int p(\tilde{y}_u, \dots, \tilde{y}_v|\Phi_{u-1}, \Theta)p(\Theta|\Phi_{u-1})d\Theta = \prod_{i=u}^v \hat{p}_i. \quad (5.3)$$

Therefore we can compute the predictive likelihood for  $\tilde{y}_u, \dots, \tilde{y}_v$  observation by observation. If  $u = 1$  and  $v = T$  then (5.3) provides a full decomposition of the marginal likelihood. In practice we will use a *training sample* that we condition on for all models. This initial sample of observations  $1, 2, \dots, u-1$ , combined with the likelihood and prior, forms a new prior,  $p(\Theta|\Phi_{u-1})$ , on which all calculations are based. If  $u$  is large then  $p(\Theta|\Phi_{u-1})$  will be dominated by the likelihood function and the original prior  $p(\Theta)$  will have a minimal contribution to model comparison exercises. Note that conditional on this training sample, the log predictive Bayes factor in favor of model  $j$  versus  $k$  for the data  $\tilde{y}_u, \dots, \tilde{y}_v$  is

$$\log B_{j,k} = \sum_{i=u}^v \log \frac{\hat{p}_i^j}{\hat{p}_i^k} \quad (5.4)$$

where the predictive likelihood is now indexed by the model  $j$  and model  $k$ .

Model probabilities associated with the predictive likelihood are calculated as,

$$p(M_i|\tilde{y}_u, \dots, \tilde{y}_v, \Phi_{u-1}) = \frac{p(\tilde{y}_u, \dots, \tilde{y}_v|M_i, \Phi_{u-1})p(M_i|\Phi_{u-1})}{\sum_{k=1}^K p(\tilde{y}_u, \dots, \tilde{y}_v|M_k, \Phi_{u-1})p(M_k|\Phi_{u-1})}, \quad i = 1, \dots, K \quad (5.5)$$

where there are  $K$  models, and  $M_k$  denotes model  $k$ . In all calculations equal prior model probabilities are used. At the end of the training sample we start with an equal prior on the models  $p(M_i|\Phi_{u-1}) = P(M_i)$ , with  $P(\text{SV}) = P(\text{GARCH}) = P(\text{Jump}) = 1/3$ . Models with a high predictive likelihood will be assigned a high model probability. Calculating (5.5) for each  $v = u + 1, \dots, T$ , provides a cumulative assessment of the evidence for model  $i$  as more observations are used. The model probabilities are used to calculate the model average in the out-of-sample period. For example, the Bayesian model average of  $g(y_{v+1})$  using (5.5) is

$$E[g(y_{v+1})|\tilde{y}_u, \dots, \tilde{y}_v, \Phi_{u-1}] = \sum_{k=1}^K E[g(y_{v+1})|\tilde{y}_u, \dots, \tilde{y}_v, \Phi_{u-1}, M_k]p(M_k|\tilde{y}_u, \dots, \tilde{y}_v, \Phi_{u-1}). \quad (5.6)$$

In addition to the predictive likelihood which focuses on the whole density, we also compare models by one-step ahead out-of-sample variance forecasts. The predictive variance from (5.1) is computed by simulation methods and detailed below. The Bayesian model average for the predictive variance can be computed using (5.6) as in  $E[y_{v+1}^2|\Phi_v] - E[y_{v+1}|\Phi_v]^2$ . We report mean squared error (MSE), mean absolute error (MAE), and Mincer & Zarnowitz (1969) forecast regressions for each model's predictive variance against realized volatility. Realized volatility serves as our target ex post volatility measure and is discussed below.

## 5.1 Calculations

### 5.1.1 Jump Model

The predictive likelihood can be estimated from the MCMC output. For the Jump model, with  $y_{t+1} = r_{t+1}$ , and  $\Theta = (\theta, \omega)$  the predictive likelihood is,

$$\hat{p}_{t+1} = \int p(r_{t+1}|\theta, \omega_t, \Phi_t)p(\theta, \omega_t|\Phi_t)d\theta d\omega_t \quad (5.7)$$

$$= \int p(r_{t+1}|\theta, \omega_{t+1}, \Phi_t)p(\omega_{t+1}|\omega_t, \theta)p(\theta, \omega_t|\Phi_t)d\theta d\omega_t d\omega_{t+1} \quad (5.8)$$

$$\approx \frac{1}{N} \sum_{i=1}^N \frac{1}{R} \sum_{j=1}^R p(r_{t+1}|\theta^{(i)}, \omega_{t+1}^{(j)}, \omega_t^{(i)}, \Phi_t) \quad (5.9)$$

where  $i$  denotes the  $i$ th draw from the posterior  $p(\theta, \omega | \Phi_t)$ ,  $i = 1, \dots, N$ ,  $j$  indexes simulated values of  $\omega_{t+1}$ , and

$$p(r_{t+1} | \theta^{(i)}, \omega_{t+1}^{(j)}, \omega_t^{(i)}, \Phi_t) = \lambda_{t+1}^{(j)} \phi(r_{t+1} | \mu^{(i)} + \mu_{\xi}^{(i)}, \sigma^{2(i)} + \sigma_{J,t+1}^{2(i)}) \quad (5.10)$$

$$+ (1 - \lambda_{t+1}^{(j)}) \phi(r_{t+1} | \mu^{(i)}, \sigma^{2(i)}) \quad (5.11)$$

$$\lambda_{t+1}^{(j)} = \frac{\exp(\omega_{t+1}^{(j)})}{1 + \exp(\omega_{t+1}^{(j)})} \quad (5.12)$$

$$\omega_{t+1}^{(j)} = \gamma_0^{(i)} + \gamma_1^{(i)} \omega_t^{(i)} + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, 1). \quad (5.13)$$

$\phi(x | \mu, \sigma^2)$  denotes the normal density function evaluated at  $x$  with mean  $\mu$  and variance  $\sigma^2$ . The following steps summarize the estimation of  $\hat{p}_t^{t+1}$ .

1. Set model parameters to the  $i$ th draw from the posterior  $\{\theta^{(i)}, \omega^{(i)}\}$ .
2. Generate  $j = 1, \dots, R$ , values of  $\omega_{t+1}^{(j)}$  according to (5.13) and calculate the average of  $p(r_{t+1} | \theta^{(i)}, \omega_{t+1}^{(j)}, \omega_t^{(i)}, \Phi_t)$  for these values. Save the result.
3. If  $i < N$  then set  $i = i + 1$  and goto 1
4. Calculate the average of the  $N$  values obtained in step 2.

Standard errors for this estimate can be calculated as usual from MCMC output. Note that the measure of accuracy should account for the fact that the posterior draws are from a Markov chain and display dependence. Following Geweke (1992), we calculate the long-run variance of the terms in 2 using the Newey-West variance-covariance estimator. The square-root of this divided by the sample size is the numerical standard error. The delta method is used to convert this to a standard error for  $\log(\hat{p}_{t+1})$ . From this a numerical standard error can be derived for the log of (5.3) assuming all estimates are independent.

### 5.1.2 SV Model

Very similar calculations are used for the SV model.

$$\hat{p}_{t+1} = \int p(r_{t+1} | \theta, h_t, \Phi_t) p(\theta, h_t | \Phi_t) d\theta dh_t \quad (5.14)$$

$$= \int p(r_{t+1} | \theta, h_{t+1}, \Phi_t) p(h_{t+1} | h_t, \theta) p(\theta, h_t | \Phi_t) d\theta dh_t dh_{t+1} \quad (5.15)$$

$$\approx \frac{1}{N} \sum_{i=1}^N \frac{1}{R} \sum_{j=1}^R p(r_{t+1} | \theta^{(i)}, h_{t+1}^{(j)}, h_t^{(i)}, \Phi_t), \quad (5.16)$$

where  $\theta^{(i)} = \{\mu^{(i)}, \rho_0^{(i)}, \rho_1^{(i)}, \sigma_v^{2(i)}\}$ . To summarize,

1. Set model parameters to the  $i$ th draw from the posterior  $\{\theta^{(i)}, h^{(i)}\}$ , given  $\Phi_t$ .
2. Generate  $j = 1, \dots, R$ , values of  $h_{t+1}^{(j)}$  according to

$$h_{t+1}^{(j)} = \rho_0^{(i)} + \rho_1^{(i)} h_t^{(i)} + \sigma_v^{(i)} \nu_{t+1}, \quad \nu_{t+1} \sim N(0, 1) \quad (5.17)$$

and calculate the average of  $p(r_{t+1}|\theta^{(i)}, h_{t+1}^{(j)}, h_t^{(i)}, \Phi_t) = \phi(r_{t+1}|\mu^{(i)}, \exp(h_{t+1}^{(j)}))$  for these values. Save the result.

3. If  $i < N$  then set  $i = i + 1$  and goto 1
4. Calculate the average of the  $N$  values obtained in step 2.

### 5.1.3 GARCH Model

Unlike the the other models, the GARCH specification does not have latent variables, which simplifies calculation of the predictive likelihood. If  $\theta = \{\mu, \kappa, \alpha, \beta\}$ , then

$$p_{t+1} = \int p(r_{t+1}|\theta, \Phi_t) p(\theta|\Phi_t) d\theta \quad (5.18)$$

$$\approx \frac{1}{N} \sum_{i=1}^N p(r_{t+1}|\theta^{(i)}, \Phi_t) \quad (5.19)$$

where  $\theta^{(i)}$  is a draw from the posterior  $p(\theta|\Phi_t)$ , and  $p(r_{t+1}|\theta^{(i)}, \Phi_t) = \phi(r_{t+1}|\mu^{(i)}, \sigma_{t+1}^{2(i)})$ . Note that the full series of conditional variances must be computed starting at  $t = 1$  to obtain  $\sigma_{t+1}^{2(i)} = \kappa^{(i)} + \alpha^{(i)} \epsilon_t^2 + \beta^{(i)} \sigma_t^{2(i)}$ , with  $\epsilon_t = r_t - \mu^{(i)}$ .

## 6 Volatility Forecasts

Consider a generic model with parameter vector  $\Theta$ . Moments of  $y_{t+1}$  (assuming they exist), based on time  $t$  information, can be calculated from the predictive density as

$$E[y_{t+1}^s|\Phi_t] = \int y_{t+1}^s p(y_{t+1}|\Phi_t) dy_{t+1} \quad (6.1)$$

$$= \int y_{t+1}^s \int p(y_{t+1}|\Phi_t, \Theta) p(\Theta|\Phi_t) d\Theta dy_{t+1} \quad (6.2)$$

$$= \int E[y_{t+1}^s|\Phi_t, \Theta] p(\Theta|\Phi_t) d\Theta, \quad s = 1, 2, \dots \quad (6.3)$$

which can be approximated from MCMC output. Note that the posterior  $p(\Theta|\Phi_t)$  was used in the last section for calculating the predictive likelihood. Therefore, very little additional computation is needed to obtain out-of-sample forecasts. Conditional variance forecasts are  $E[y_{t+1}^2|\Phi_t] - E[y_{t+1}|\Phi_t]^2$ .

## 6.1 Calculations

### 6.1.1 Jump model

$$E[r_{t+1}^s|\Phi_t] = \int E[r_{t+1}^s|\Phi_t, \theta, \omega_t] p(\theta, \omega_t|\Phi_t) d\theta d\omega_t \quad (6.4)$$

$$\approx \frac{1}{N} \sum_{i=1}^N E[r_{t+1}^s|\Phi_t, \theta^{(i)}, \omega_t^{(i)}], \quad s = 1, 2, \quad (6.5)$$

where  $\{\theta^{(i)}, \omega_t^{(i)}\}$  is the  $i$ th draw from the posterior distribution,  $p(\theta, \omega|\Phi_t)$ . For the variance we require the first two moments

$$E[r_{t+1}|\Phi_t, \theta^{(i)}, \omega_t^{(i)}] = \mu^{(i)} + \mu_\xi^{(i)} E[J_{t+1}|\theta^{(i)}, \omega_t^{(i)}] \quad (6.6)$$

$$E[r_{t+1}^2|\Phi_t, \theta^{(i)}, \omega_t^{(i)}] = \mu^{(i)2} + \sigma^{(i)2} + (\mu_\xi^{(i)2} + \sigma_J^{(i)2}) E[J_{t+1}|\theta^{(i)}, \omega_t^{(i)}] \quad (6.7)$$

$$+ 2\mu^{(i)} \mu_\xi^{(i)} E[J_{t+1}|\theta^{(i)}, \omega_t^{(i)}]. \quad (6.8)$$

These moments are substituted into (6.5) for each draw of  $\{\theta^{(i)}, \omega_t^{(i)}\}$ . Since  $\lambda_{t+1}$  depends on  $\omega_{t+1}$  we approximate each conditional expectation of  $J_{t+1}$  as

$$E[J_{t+1}|\theta^{(i)}, \omega_t^{(i)}] \approx \frac{1}{R} \sum_{j=1}^R \frac{\exp(\omega_{t+1}^{(j)})}{1 + \exp(\omega_{t+1}^{(j)})} \quad (6.9)$$

$$(6.10)$$

where  $\omega_{t+1}^{(j)}$  is generated from

$$\omega_{t+1}^{(j)} = \gamma_0^{(i)} + \gamma_1^{(i)} \omega_t^{(i)} + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, 1), \quad j = 1, \dots, R. \quad (6.11)$$

### 6.1.2 SV model

For the SV model we have,

$$E[r_{t+1}^s | \Phi_t] = \int E[r_{t+1}^s | \Phi_t, \theta, h_t] p(\theta, h_t | \Phi_t) d\theta dh_t \quad (6.12)$$

$$\approx \frac{1}{N} \sum_{i=1}^N E[r_{t+1}^s | \Phi_t, \theta^{(i)}, h_t^{(i)}], \quad s = 1, 2, \quad (6.13)$$

The conditional moments are

$$E[r_{t+1} | \Phi_t, \theta^{(i)}, h_t^{(i)}] = \mu^{(i)} \quad (6.14)$$

$$E[r_{t+1}^2 | \Phi_t, \theta^{(i)}, h_t^{(i)}] = \mu^{(i)2} + \exp(\rho_0^{(i)} + \rho_1^{(i)} h_t^{(i)} + \sigma_\nu^{(i)2}/2). \quad (6.15)$$

### 6.1.3 GARCH model

The results for the SV model can be used for the GARCH model except we replace the conditional moments with,

$$E[r_{t+1} | \Phi_t, \theta^{(i)}] = \mu^{(i)} \quad (6.16)$$

$$E[r_{t+1}^2 | \Phi_t, \theta^{(i)}] = \mu^{(i)2} + \sigma_{t+1}^{2(i)}. \quad (6.17)$$

## 7 Data

Five-minute intraday FX quote data (bid and ask) were obtained from Olsen and Associates for every day from 1986/12/16 - 2002/12/31 for the Japanese yen-US dollar (JPY-USD) exchange rate. With a few exceptions, the construction of daily returns and realized volatility closely follow Andersen et al. (2001) and Maheu & McCurdy (2002).

Currencies trade 24 hours a day, 7 days a week. The raw data included a number of missing observations. Missing quotes on the 5-minute grid were linearly interpolated by Olsen and Associates from the nearest available quote. We use the midpoint of the bid and ask quote, giving us 1.8 million 5-minute price observations. A day was defined as beginning at 00:05 GMT and ending 24:00 GMT. Continuously compounded 5-minute returns (in percent, that is, scaled by 100) were constructed from the price data. Following Andersen et al. (2001), all weekends (Saturday and Sunday) were removed as well as the following slow trading days: December 24-26, 31, January 1, 2. In addition, the moving holidays: Good Friday, Easter Monday, Memorial Day, July Fourth, Labor Day, Thanksgiving and the day after, as well as any days in which more than half (144)

of the day's quotes were missing, were also removed. The remaining 5-minute returns data were linearly filtered by an MA( $q$ ) to remove high-frequency autocorrelation which may be due to the discrete nature of bid/ask quotes and market microstructure effects. We set  $q = 4$  based on the sample autocorrelations of the 5-minute returns.

From these filtered data, daily realized volatilities were constructed as

$$RV_t = \sum_{j=1}^{288} r_{t,j}^2 \quad (7.1)$$

where  $r_{t,j}$  is the  $j$ th 5-minute return in day  $t$ . Daily returns were constructed as the sum of the intraday 5-minute returns, that is,  $r_t = \sum_{j=1}^{288} r_{t,j}$ .

In a series of papers, Andersen, Bollerslev, Diebold and co-authors, Barndorff-Nielsen and Shephard, among others, have documented various features of realized volatility and derived their asymptotic properties as the sampling frequency of prices increases over a fixed-time interval. Review of this large literature include Andersen et al. (2005) and Barndorff-Nielsen et al. (2004). For our purpose, we note that  $RV_t$  serves as a natural ex post estimate of volatility and provides a superior measure to rank the forecasts of competing models. Therefore, we treat  $RV_t$  as the ideal target for conditional variance forecasts from our alternative models.

Table 1 reports our summary statistics for daily returns and realized volatility.

**Table 1 ABOUT HERE**

## 8 Results

Our daily data for JPY-USD covers 1986/12/16 to 2002/12/31 (4001 observations). Table 2 contains full-sample model estimates. For each model a total of 90000 MCMC iterations were performed. The first 10000 draws were discarded to minimize the influence of starting values. Thus,  $N = 80000$  samples from the posterior distribution are used to calculate posterior moments.

Figure 1 shows various features of the jump model while Figures 2 and 3 display characteristics of the out-of-sample performance for the SV and jump specifications for the JPY-USD data. Out-of-sample calculations condition on a training sample of the first 3000 observations. These observations provide accurate parameter estimates for all models and minimize parameter uncertainty. Therefore, predictive likelihood estimates and model forecasts, appearing in Tables 3 and 4, are based on the remaining 1001 observations. Predictive likelihood estimates are illustrated in Figure 2.

## Table 2 ABOUT HERE

Table 2 contains estimates of the standard SV and GARCH models as well as our heterogeneous jump specification. We report various features of the posterior distribution. Estimates of the SV specification in Table 2 show latent volatility to be persistent. All of the parameters are accurately estimated. The SV model implies an unconditional variance of .617 which is close to the value in Table 1. Similarly,  $\alpha + \beta$  in the GARCH model is about 0.96, and the implied unconditional variance is 0.723.

Based on the posterior mean, the unconditional expectation for the autoregressive latent variable,  $\omega_t$ , driving jump arrival is -2.62. This implies that jumps are infrequent, on average. For instance, the empirical average of  $\lambda_t$  is .07. This is consistent with previous studies which combine SV with simple jumps. However, in our case, this jump probability shows clear time dependencies. From the model estimates it is seen that the process governing jump arrival is very persistent. A 95% density interval for  $\gamma_1$  is (.937, 971).

The posterior mean of  $\eta_1$  is .4532 which indicates that lagged absolute returns are important in affecting the jump-size variance. Notice that the jump-size variance is 3-4 times larger than the normal innovation variance  $\sigma^2$ .

## Figure 1 ABOUT HERE

Features of the jump model are displayed in Figure 1. Panel A shows returns, panel B plots the inferred jump probability  $\lambda_t$ , panel C illustrates the jump size and panel D displays estimated jump times over the full sample. Clearly, this model identifies large moves in returns as jumps. Panels C and D suggest that some jumps are isolated events while others cluster and lead to more jumps and higher volatility. This is an important feature of the model.

## Table 3 ABOUT HERE

Predictive likelihood estimates appear in Table 3. The estimates were obtained by re-estimating the posterior for each observation from 3001 on. For each run, we collected  $N = 20000$  (after discarding the first 10000) samples from the posterior simulator and set  $R = 100$ . Kass & Raftery (1995) recommend considering twice the logarithm of the Bayes factor for model comparison, as it has the same scaling as the likelihood ratio statistic. They suggest the following interpretation of  $2 \log BF$ : 0 to 2 not worth more than a bare mention, 2 to 6 positive, 6 to 10 strong, and greater than 10 as very strong. Based on this, the evidence in favor of the Jump model is positive. For example,  $2 \log BF = 2(-1103.760 + 1105.4758) = 3.431$  in favor of the Jump model versus the SV model. On the other hand, the GARCH model is dominated by the other specifications. Model probabilities based on the predictive likelihood estimates are calculated as in



Equation 5.5. The final cumulative model probability in favor of the jump specification is .85 while it is .15 for the SV model.

### **Figure 2 ABOUT HERE**

Differences in the log predictive likelihood in favor of the Jump model compared to the SV model are found in panel B of Figure 2. A positive (negative) value occurs when an observation is more likely under the Jump (SV) model. There are several influential observations favoring the Jump model that appear to be high volatility episodes. These influential observations show up as spikes in panel B. The cumulative probability for the Jump model is plotted in panel C of Figure 2. It is interesting to note that there are several upward trending periods in panel C which are not from tail occurrences in returns, for instance from the middle of 2000 to 2001. The SV model appears to perform best at the start of the sample, 1999-2000.

Although Bayes factors are an assessment of all distributional features, it is interesting to also evaluate volatility forecasts. Out-of-sample forecasts for all models and the model average are assessed using realized volatility. Table 3 contains Mincer & Zarnowitz (1969) forecast regressions of realized volatility regressed on a model's conditional variance forecast. We report regression  $R^2$ , mean squared error (MSE) and mean absolute error (MAE). Although the GARCH model does poorly in accounting for the predictive density of returns, it performs very well in one period ahead forecasts of realized volatility. The jump model has the largest  $R^2$  although the differences in the models is small.

### **Figure 3 ABOUT HERE**

Panel A of Figure 3 displays realized volatility for the out-of-sample period. Panel B plots the model forecasts for SV and the Jump model. Notice that the model forecasts are much less variable than realized volatility. During high volatility periods both models produce similar forecasts, but during low periods the Jump model's variance is lower. During these times the Jump variance is essentially flat, a time-series pattern similar to that of realized volatility, while the SV forecast appears to be trending upward. This is due to the fact that the jump model can have periods of homoskedasticity while the SV model always has a time-varying variance.

The differences in the volatility forecasts become clear when we compare Figure 3B to 2C. Periods when the Jump model's variance is low are exactly when the probability for the Jump model is increasing (just before 2001 and around 2002). This suggests that state-dependent volatility clustering may be important, that is, periods of normal homoskedasticity along with periods of high volatility that clusters. The Jump model is able to capture these dynamics relatively well.

Overall the Jump model is quite competitive with a simple parameterization of SV and GARCH. Further, out-of-sample predictive likelihoods reveal periods when the heteroskedastic jump structure appears to do a better job at capture the dynamics of realized volatility.

There are several potential extensions to our heterogeneous jump model. For example, we could consider a fat-tailed distribution for return innovations and for jump-size innovations. We have allowed the conditional variance of the jump-size distribution to be a function of the absolute value of lagged returns to reflect recent volatility levels. Other conditioning variables that could be used are the range statistic or realized volatility.

## 9 Conclusions

This paper proposes a new discrete-time model of returns in which jumps capture persistence and time-variation in the conditional variance and higher-order moments of returns. Time-varying jump arrival is governed by a latent autoregressive process and the jump-size distribution is allowed to be conditionally heteroskedastic with a non-zero mean.

We discuss a Bayesian approach to estimation and advocate the use of predictive likelihoods to compare alternative models. We also compare the out-of-sample volatility forecasts against realized volatility. In particular, using JPY-USD exchange rate returns, we compare forecasts from our heterogeneous jump model, a conventional stochastic volatility and GARCH model, and the model average.

Our results indicate that the heterogeneous jump model effectively captures volatility persistence through jump clustering. In addition, we find that the jump-size variance is heteroskedastic and increasing in volatile markets. Overall this model is very competitive with the stochastic volatility specification. The state dependence in jumps allows it to perform well in both quiet periods and in volatile periods when tail realizations occur. We conclude that the modeling of jump dynamics to describe market returns is a fruitful avenue for future research.

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## 11 Appendix

Below we provide the details of the posterior simulator for the Heterogeneous Jump specification and the SV model.

### 11.1 Jump Model

For steps 1 - 3 and 6 we use standard conjugate results for the linear regression model (see Koop (2003)).

1.  $\mu|\theta_{-\mu}, \omega, J, \xi, r$ . If the prior is normal,  $p(\mu) \sim N(a, A^{-1})$

$$p(\mu|\theta_{-\mu}, \omega, J, \xi, r) \propto p(r|\theta, \omega, J, \xi)p(\mu) \quad (11.1)$$

$$\sim N(m, V^{-1}) \quad (11.2)$$

where  $V = \sigma^{-2}T + A$  and  $m = V^{-1}(\sigma^{-2}\sum_{t=1}^T(r_t - \xi_t J_t) + Aa)$ .

2.  $\sigma^2|\theta_{-\sigma^2}, \omega, J, \xi, r$ . With an inverse gamma prior,  $\sigma^2 \sim IG(\nu_{\sigma^2}/2, s_{\sigma^2}/2)$  we have

$$p(\sigma^2|\theta_{-\sigma^2}, \omega, J, \xi, r) \propto p(r|\theta, \omega, J, \xi)p(\sigma^2) \quad (11.3)$$

$$\sim IG\left(\frac{T + \nu_{\sigma^2}}{2}, \frac{\sum_{t=1}^T(r_t - \xi_t J_t - \mu)^2 + s_{\sigma^2}}{2}\right) \quad (11.4)$$

3.  $\mu_\xi|\theta_{-\mu_\xi}, \omega, J, \xi, r$ . If  $p(\mu_\xi) \sim N(b, B^{-1})$

$$p(\mu_\xi|\theta_{-\mu_\xi}, \omega, J, \xi, r) \propto p(\xi|\theta)p(\mu_\xi) \quad (11.5)$$

$$\sim N(m, V^{-1}) \quad (11.6)$$

where  $V = \sigma_J^{-2}T + B$ , and  $m = V^{-1}(\sigma_J^{-2}\sum_{t=1}^T \xi_t + Bb)$ .

4.  $\eta|\theta_{-\eta}, \omega, J, \xi, r$ . We use independent inverse gamma priors for both parameters,  $p(\eta_0) \sim IG(\nu_{\eta_0}/2, s_{\eta_0}/2)$ ,  $p(\eta_1) \sim IG(\nu_{\eta_1}/2, s_{\eta_1}/2)$ . This ensures that the jump size variance is always positive. The target distribution is

$$p(\eta|\theta_{-\eta}, \omega, J, \xi, r) \propto p(\xi|\theta)p(\eta_0)p(\eta_1) \quad (11.7)$$

$$\propto p(\eta_0)p(\eta_1) \prod_{t=1}^T (\eta_0 + \eta_1 X_{t-1})^{-1/2} \exp\left(-\frac{1}{2} \frac{(\xi_t - \mu_\xi)^2}{(\eta_0 + \eta_1 X_{t-1})}\right)$$

which is a nonstandard distribution. We use a MH algorithm to sample this parameter using a random walk proposal. The proposal distribution  $q(x|\eta^{i-1})$ , is a fat-tailed mixture of normals where the covariance matrix is calibrated so approximately 50% of candidate draws are accepted. The mixture specification is the same as what is used in step 5 below. If  $\eta^i \sim q(x|\eta^{i-1})$  is a draw from the proposal distribution, it is accepted with probability

$$\min \left\{ \frac{p(\eta^i|\theta_{-\eta}, \omega, J, \xi, r)}{p(\eta^{i-1}|\theta_{-\eta}, \omega, J, \xi, r)}, 1 \right\} \quad (11.8)$$

and otherwise  $\eta^i = \eta^{i-1}$ .

5.  $\omega_{(t,\tau)}|\theta, \omega_{-(t,\tau)}, J, \xi, r, t < \tau, t = 1, \dots, T$ . The conditional posterior is

$$p(\omega_{(t,\tau)}|\theta, \omega_{-(t,\tau)}, J) \propto p(J_{(t,\tau)}|\omega_{(t,\tau)})p(\omega_{(t,\tau)}|\omega_{-(t,\tau)}, \theta) \quad (11.9)$$

$$\propto p(\omega_{\tau+1}|\omega_{\tau}, \theta) \prod_{i=t}^{\tau} p(J_i|\omega_i)p(\omega_i|\omega_{i-1}, \theta) \quad (11.10)$$

which is a nonstandard distribution. To improve the mixing properties of the MCMC output we adapted the blocking procedure that Fleming & Kirby (2003) use for SV models. Specifically, we approximate the first-difference of  $\omega_t$  as a constant over the interval  $(t, \tau)$ , and sample a block using an independent Metropolis routine. If  $\omega_t$  is strongly autocorrelated this provides a good proposal density. The proposal distribution is a fat-tailed multivariate mixture<sup>1</sup> of normals,

$$q(x|\omega_{-(t,\tau)}) \sim \begin{cases} N(m, V) & \text{with probability } p \\ N(m, 10V) & \text{with probability } 1 - p \end{cases} \quad (11.11)$$

where  $p = .9$ . The mean vector and variance-covariance matrix are,

$$m_l = \omega_{t-1} + \frac{l}{k+1}(\omega_{t-1} + \omega_{\tau+1}) \quad l = 1, \dots, k, \quad (11.12)$$

$$V_{lm} = \min(l, m) - \frac{lm}{k+1}, \quad l = 1, \dots, k, \quad m = 1, \dots, k \quad (11.13)$$

where  $k = \tau - t + 1$  is the block length and is chosen randomly from a Poisson distribution with parameter 15.  $V^{-1}$  has a very convenient tridiagonal form in which  $V_{ii}^{-1} = 2$  and  $V_{ij}^{-1} = -1$ ,  $1 \leq i \leq k$ ,  $j = i - 1, i + 1$ , and otherwise 0. A new

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<sup>1</sup>In practice a fat-tailed proposal was found to be important when comparing the accuracy of the block version with a single move sampler.

draw  $\omega_{(t,\tau)}^{(i)}$  from  $q(x|\omega_{-(t,\tau)}^{(i-1)})$ , is accepted with probability

$$\min \left\{ \frac{p(\omega_{(t,\tau)}^{(i)}|\theta, \omega_{-(t,\tau)}^{(i-1)}, J)/q(\omega_{(t,\tau)}^{(i)}|\omega_{-(t,\tau)}^{(i-1)})}{p(\omega_{(t,\tau)}^{(i-1)}|\theta, \omega_{-(t,\tau)}^{(i-1)}, J)/q(\omega_{(t,\tau)}^{(i-1)}|\omega_{-(t,\tau)}^{(i-1)})}, 1 \right\} \quad (11.14)$$

and otherwise  $\omega_{(t,\tau)}^{(i)} = \omega_{(t,\tau)}^{(i-1)}$ .

6.  $\gamma|\theta_{-\gamma}, \omega, J, \xi, r$ . The conjugate prior for  $\gamma$  is a bivariate normal  $p(\gamma) \sim N(m, V)$  which results in a conditional distribution that is also normal. We reject any draw which does not satisfy  $|\gamma_1| < 1$ .
7.  $\xi|\theta, \omega, J, r$ . The jump size can be sampled in a block by using the conditional independence of each  $\xi_t$ . The conditional distribution of  $\xi_t$  is,

$$p(\xi_t|\theta, \omega, J, r) \propto p(r_t|\theta, \omega_t, J_t, \xi_t)p(\xi_t|\theta) \sim N(c, C^{-1}) \quad (11.15)$$

where  $C = \sigma^{-2}J_t + \sigma_{\xi,t}^{-2}$ , and  $c = C^{-1}(\sigma^{-2}J_t(r_t - \mu) + \sigma_{\xi,t}^{-2}\mu_\xi)$

8.  $J|\theta, \omega, \xi, r$ .  $J_t$  is a Bernoulli random variable with parameter  $\lambda_t$ . To find the probability of  $J_t = 0, 1$  note that,

$$\begin{aligned} p(J_t = 0|\theta, \omega, \xi, r) &\propto p(r_t|\theta, J_t, \xi_t)p(J_t = 0|\omega) \propto \exp(-.5\sigma^{-2}(r_t - \mu)^2)(1 - \lambda_t) \\ p(J_t = 1|\theta, \omega, \xi, r) &\propto p(r_t|\theta, J_t, \xi_t)p(J_t = 1|\omega) \propto \exp(-.5\sigma^{-2}(r_t - \mu - \xi_t)^2)\lambda_t \end{aligned}$$

which allows calculation of the normalizing constant and hence a draw of  $J_t$ .

9. goto 1

The priors used for this model are independent  $\mu \sim N(0, 1000)$ ,  $\mu_\xi \sim N(0, 100)$ ,  $\sigma^2 \sim IG(3, .02)$ ,  $\eta_0 \sim IG(2.5, 1)$ ,  $\eta_1 \sim IG(2.5, 1)$ ,  $\gamma_0 \sim N(0, 100)$ ,  $\gamma_1 \sim N(0, 100)I_{|\gamma_1| < 1}$ . The priors selected for this model are for the most part, non-informative. Note that  $\omega_t$  is restricted to be stationary. The priors on  $\eta_0$  and  $\eta_1$  reflect a reasonable range for the conditional variance of the jump-size distribution. For example, the probability that these parameters lie in the interval  $(.1, 2)$  is approximately .96.

## 11.2 SV Model

For the SV model we obtain draws from the full posterior iterating on the following steps:

sample  $\mu \sim p(\mu|\rho_0, \rho_1, \sigma_v^2, r, h)$

sample  $\rho_0, \rho_1 \sim p(\alpha, \delta|\mu, \sigma_v^2, r, h)$

sample  $\sigma_v^2 \sim p(\sigma_v^2|\mu, \rho_0, \rho_1, r, h)$

sample  $h_t \sim p(h_t|h_{-t}, \mu, \rho_0, \rho_1, \sigma_v^2, r), t = 1, \dots, T$ .

The priors are independent  $\mu \sim N(0, 100)$ ,  $\rho_0 \sim N(0, 100)$ ,  $\rho_1 \sim N(0, 100)I_{|\rho_1| < 1}$ , and  $\sigma_v^2 \sim IG(5/2, 0.1/2)$ .

1.  $\mu|\rho_0, \rho_1, \sigma_v, r, h$ . Note that since  $h$  is known the above model can be rewritten as

$$r_t \exp(-h_t/2) = \mu \exp(-h_t/2) + \nu_t, \quad t = 1, \dots, T. \quad (11.16)$$

This is a regression model  $y_t = x_t \mu + \nu_t$  where  $y_t = r_t \exp(-h_t/2)$ , and  $x_t = \exp(-h_t/2)$ , and the innovation term has variance 1. Hence we can apply Gibbs sampling results for the linear regression model.

2.  $\rho_0, \rho_1|\mu, \sigma_v, r, h$ . The conjugate prior for  $\rho_0, \rho_1$  is a bivariate normal which results in a conditional distribution that is also normal. We reject any draw which does not satisfy  $|\rho_1| < 1$ .
3.  $\sigma_v^2|\mu, \rho_0, \rho_1, r, h$ . The inverse Gamma prior results in an inverse Gamma posterior.
4.  $h_t|h_{-t}, \mu, \rho_0, \rho_1, \sigma_v^2, r, t = 1, \dots, T$ . We use the single move sampler of Kim et al. (1998) except that we use a MH step as opposed to accept/reject sampling. Here the proposal density is  $N(\tilde{\mu}_t, \sigma^2)$ , with  $\tilde{\mu}_t = \mu_t + \frac{\sigma^2}{2}(r_t^2 \exp(-\mu_t) - 1)$ ,  $\mu_t = \frac{\rho_0(1-\rho_1)+\rho_1(h_{t-1}+h_{t+1})}{1+\rho_1^2}$ , and  $\sigma^2 = \frac{\sigma_v^2}{1+\rho_1^2}$ . Therefore sample  $h' \sim N(\tilde{\mu}_t, \sigma^2)$  which has density  $q(\cdot|h_{t-1}, h_{t+1}, \omega)$ . and accept this with probability

$$\min \left\{ 1, \frac{p(h'_t|h_{t-1}, h_{t+1}, \omega)/q(h'_t|h_{t-1}, h_{t+1}, \omega)}{p(h_t|h_{t-1}, h_{t+1}, \omega)/q(h_t|h_{t-1}, h_{t+1}, \omega)} \right\} \quad (11.17)$$

and otherwise reject.

### 11.3 GARCH Model

If  $\theta$  denotes the GARCH model parameters then we use a multivariate random walk proposal in a MH step. Specifically, a new proposal is obtained as  $\theta' = \theta + V$ , where  $V$  is a fat-tailed mixture of two multivariate normal densities. This proposal is accepted with

probability  $\min\{1, p(\theta'|\Phi_T)/p(\theta|\Phi_T)\}$ , and otherwise the new value of the chain is set to the previous draw  $\theta$ . The first time the model is estimated we calibrate the covariance of  $V$  to be proportional to the posterior covariance matrix using output from a single-move random walk. Thereafter, we use the joint sampler with this covariance matrix. When a new observation becomes available we use the previous posterior draws to calibrate the covariance matrix in the next round of estimation. This approach provides very efficient sampling and automatic updates to the proposal density  $V$ . We collect 10000 draws for posterior inference. The priors are  $\mu \sim N(0, 100)$ ,  $\kappa \sim N(0, 100)$ ,  $\alpha \sim N(0, 100)$ ,  $\beta \sim N(0, 100)$ , and  $\kappa > 0, \alpha \geq 0, \beta \geq 0$ , and  $\alpha + \beta < 1$ .

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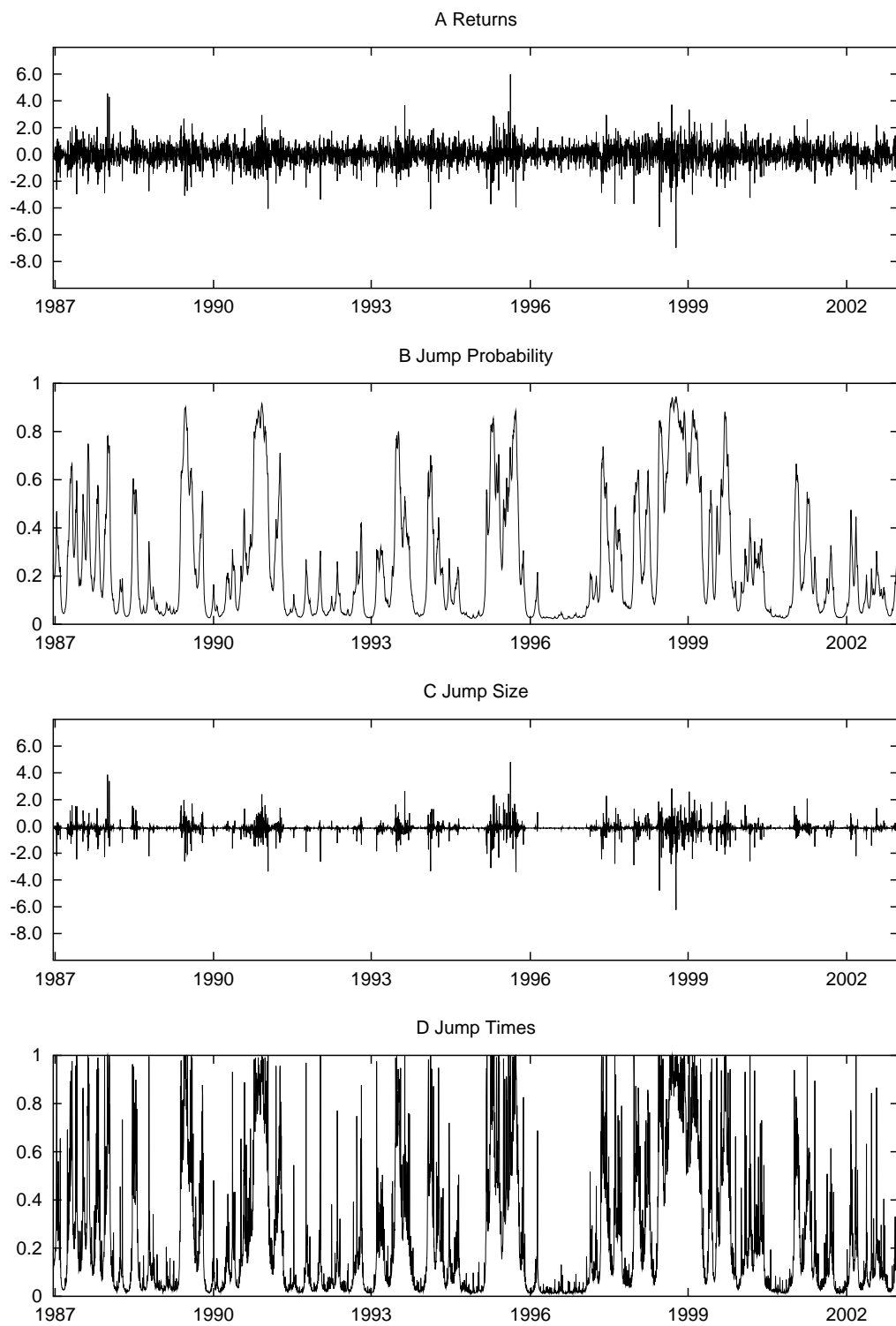


Figure 1: Jump Model

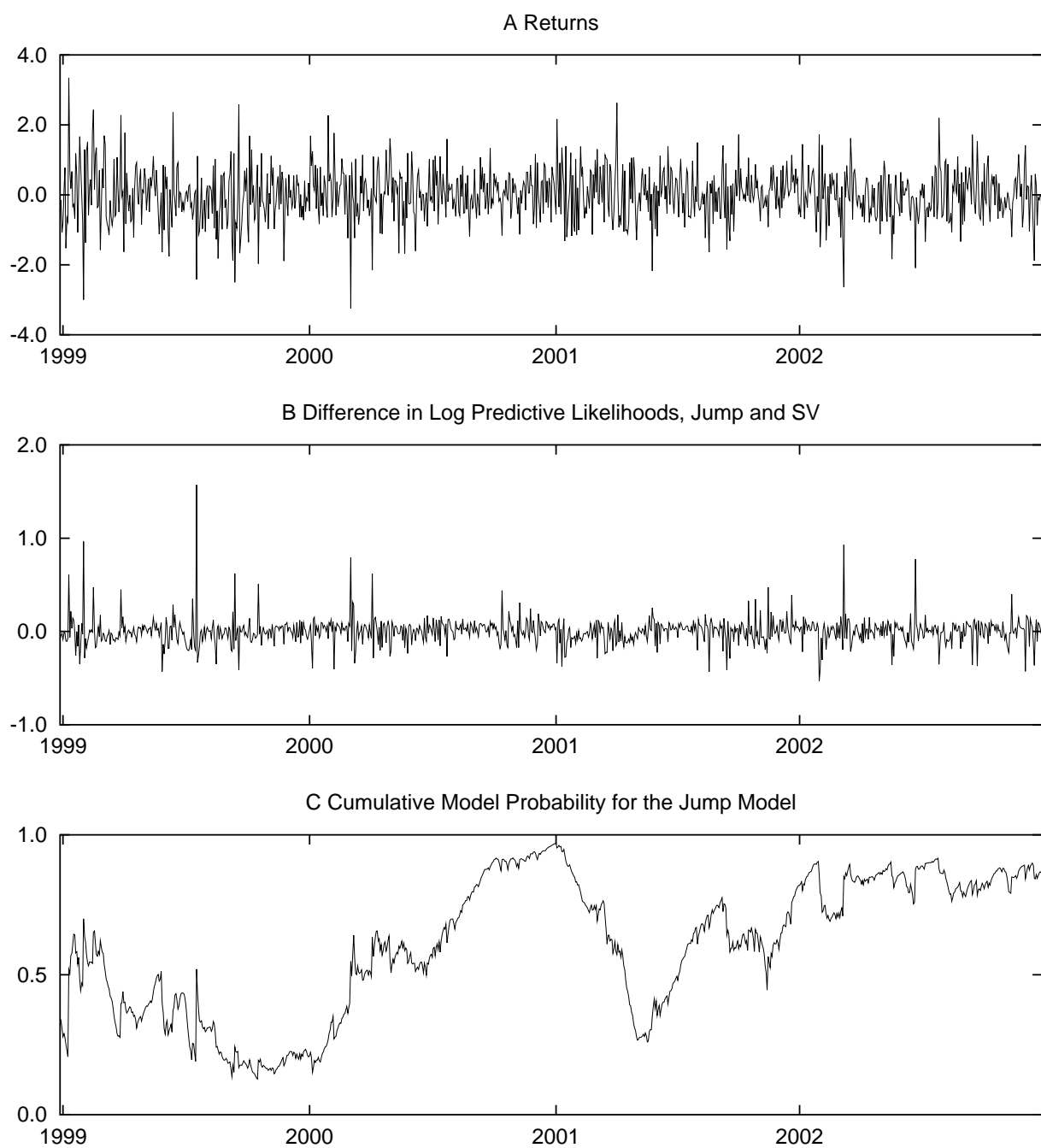


Figure 2: Returns, Predictive Likelihoods, and Cumulative Model Probabilities

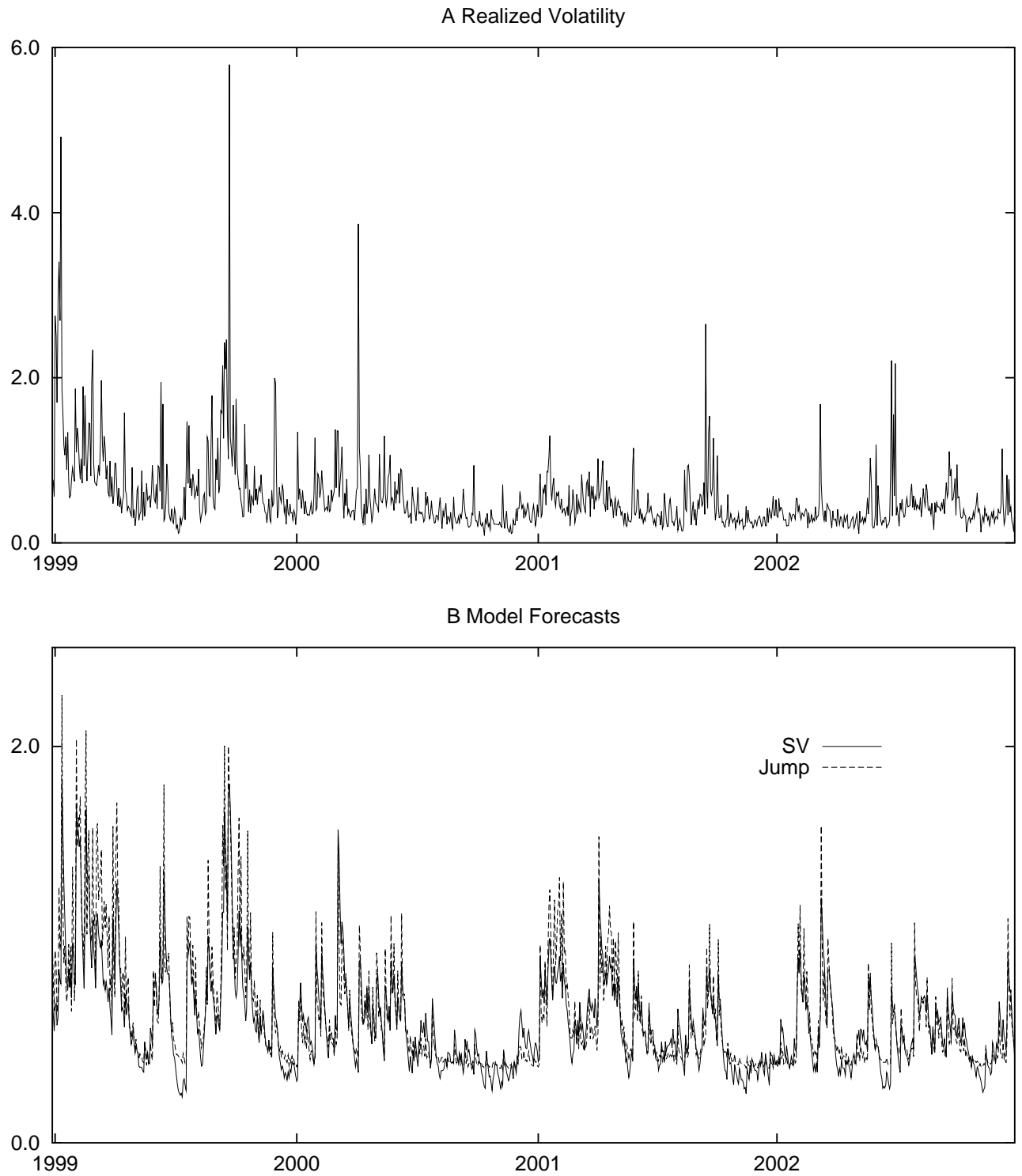


Figure 3: Volatility and Model Forecasts

Table 1: Summary Statistics, JPY-USD

Statistic	$r_t$	$RV_t$
Mean	-4.566e-6	0.5992
Variance	0.6352	0.6264
Skewness	-0.3504	21.6655
Kurtosis	8.0587	854.2149
Min	-6.9768	0.0290
Max	5.9780	34.3872
Obs	4001	4001

$r_t$  is percent log differences of daily spot exchange rates, and  $RV_t$  is realized volatility for the period 1986/12/16 - 2002/12/31.

Table 2: Model Estimates  
SV Model

$$r_t = \mu + \exp(h_t/2)z_t, \quad z_t \sim N(0, 1)$$

$$h_t = \rho_0 + \rho_1 h_{t-1} + \sigma_v v_t, \quad v_t \sim N(0, 1), \quad |\rho_1| < 1$$

	mean	stdev	median	95% DI
$\mu$	0.0165	0.0105	0.0165	(-0.0008, 0.0337)
$\rho_0$	-0.0568	0.0125	-0.0559	(-0.0786, -0.0379)
$\rho_1$	0.9234	0.0148	0.9244	( 0.8974, 0.9460)
$\sigma_v^2$	0.0763	0.0160	0.0749	( 0.0524, 0.1043)

GARCH Model

$$r_t = \mu + \epsilon_t, \quad \epsilon_t = \sigma_t z_t, \quad z_t \sim N(0, 1)$$

$$\sigma_t^2 = \kappa + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

	mean	stdev	median	95% DI
$\mu$	0.0083	0.0123	0.0083	(-0.0140, 0.0338)
$\kappa$	0.0305	0.0064	0.0302	( 0.0184, 0.0431)
$\alpha$	0.1031	0.0143	0.1022	( 0.0784, 0.1342)
$\beta$	0.8547	0.0198	0.8555	( 0.8144, 0.8919)

Jump Model

$$r_t = \mu + \sigma z_t + J_t \xi_t, \quad z_t \sim N(0, 1)$$

$$\xi_t \sim N(\mu_\xi, \sigma_{\xi,t}^2), \quad J_t \in \{0, 1\}$$

$$P(J_t = 1|\omega_t) = \lambda_t \quad \text{and} \quad P(J_t = 0|\omega_t) = 1 - \lambda_t$$

$$\lambda_t = \frac{\exp(\omega_t)}{1 + \exp(\omega_t)}$$

$$\omega_t = \gamma_0 + \gamma_1 \omega_{t-1} + u_t, \quad u_t \sim N(0, 1), \quad |\gamma_1| < 1$$

$$\sigma_{\xi,t}^2 = \eta_0 + \eta_1 X_{t-1}$$

	mean	stdev	median	95% DI
$\mu$	0.0307	0.0125	0.0307	( 0.0103, 0.0513)
$\sigma^2$	0.3226	0.0159	0.3223	( 0.2968, 0.3491)
$\mu_\xi$	-0.1272	0.0510	-0.1269	(-0.2115, -0.0440)
$\eta_0$	0.9089	0.1238	0.8961	( 0.7273, 1.1276)
$\eta_1$	0.4532	0.1153	0.4443	( 0.2793, 0.6599)
$\gamma_0$	-0.1169	0.0305	-0.1158	(-0.1688, -0.0686)
$\gamma_1$	0.9554	0.0103	0.9562	( 0.9372, 0.9709)

The data are percent log differences of daily spot exchange rates for the JPY-USD from 1986/12/16 - 2002/12/31. Tables for the alternative models report posterior mean, standard deviation, median and 95% probability density intervals (DI).



Table 3: Log Predictive Likelihoods

Model	JPY-USD
GARCH	-1117.6374 (0.0565)
SV	-1105.4758 (0.2153)
Jump	-1103.7603 (0.3982)

This table reports estimates of log predictive likelihoods for observations 3001 - 4001. Numerical standard errors appear in parentheses.

Table 4: Out-of-Sample Forecast Performance for RV

	a	b	$R^2$	MSE	MAE
GARCH	-0.0244 (0.0337)	0.9413 (0.0520)	0.2471	0.1619	0.2443
SV	0.0024 (0.0334)	0.8899 (0.0509)	0.2345	0.1657	0.2505
Jump	0.0477 (0.0298)	0.7829 (0.0428)	0.2509	0.1694	0.2582
Model Average	0.0141 (0.0318)	0.8587 (0.0475)	0.2467	0.1649	0.2511

This table reports Mincer & Zarnowitz (1969) forecast regressions of

$$RV_t = a + b\text{Var}_{t-1}(r_t) + \text{error}_t$$

where  $\text{Var}_{t-1}(r_t)$  is a model forecast of the one-period-ahead conditional variance based on time  $t - 1$  information, and  $RV_t$  is realized volatility for day  $t$ . Standard errors appear in parentheses.  $R^2$  is the coefficient of determination. MSE and MAE are the mean squared error and mean absolute error, respectively, for  $(RV_t - \text{Var}_{t-1}(r_t))$ . The number of out-of-sample forecasts are 1001.