# Estimation Of Market Volatility-A Case Of Logistic Brownian Motion.

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# Editorial Board

Title

# ESTIMATION OF MARKET VOLATILITY-A CASE OF LOGISTIC BROWNIAN MOTION

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## ABSTRACT:

In this paper, we have used the Dupire's equation to derive the volatility model when the asset price follows logistic Brownian motion. We have used the analysis of Brownian motion, logistic Brownian motion, derivation of Black-Scholes Merton differential equation using $It\hat{}o$ process and $It\hat{}o$'s lemma and stochastic processes. We have also reviewed derivation of Dupire Volatility equation and used it's concept to derive a volatility model when the asset price follows logistic Brownian motion.


**Keywords:** Volatility, Modeling, Brownian motion, differential equation and Dupire's equation

## 1 Introduction:

The estimation of market volatility (or variance of returns of an asset) is a crucial issue in modern applied finance. The measure of volatility and good forecasts of future volatility are crucial for implementation, evaluation of asset and derivative pricing of asset. In particular, volatility has been used in financial markets in assessments of risk associated with short-term fluctuations in financial time-series. It is estimated as mean square deviation from trend pattern. Black-Scholes Merton Model (1973) published a landmark paper about option pricing and corporate liabilities. Not only did this specify the first successful option pricing formula, but it also described a general framework for pricing of other derivative instruments. Volatility has a key role to play in the determination of a risk and in the valuation of options and other derivative securities. The widespread of Black-Scholes model for asset prices assumes volatility is constant. Hull ( 2000) argue that volatility can be estimated by using historical data in the form of logarithms of asset returns which is referred to as historical volatility. A large number of models have been proposed to address the short comings of Black-Scholes model (Black and Scholes 1973; Merton 1973)

## 2 Preliminaries:

Wiener process is a particular type of Markov Stochastic process with mean change of zero and variance 1.0. If X (t) follows a stochastic process where $\mu$ the mean of the probability distribution is and $\sigma$ is the standard deviation. That is, X (t) ~ N ( $\mu$ , $\sigma$ ) then for Wiener process X (t) ~ N (0, 1) which means X (t) is a normal distribution with $\mu$ =0 and $\sigma$ =1. Expressed formally, a variable Z follows a Wiener process if it has the following properties:

**PROPERTY 1:** The change $\Delta$ Z during a small period of time i

$$\Delta Z = \varepsilon \sqrt{\Delta t} \tag{2.1}$$

where $\varepsilon$ has a standardized normal distribution; $\vartheta(0,1)$

**PROPERTY 2**: The values of $\Delta$ Z for any two different short time intervals of time, $\Delta$ t, are independent. That is, Var ( $\Delta Z_i$, $Z_j$ )=0, i $\neq$ j. It follows from the first property that itself has a normal distribution with Mean of $\Delta$ Z=0, Standard deviation of $\Delta$ Z= $\sqrt{\Delta t}$ and Variance of $\Delta$ Z= $\Delta t$. The second property implies that Z follows a Markov process. Consider the change in the value of Z during a relatively long period of time T. This can be denoted by Z (T)-Z (0). It can be regarded as the sum of the changes in Z in N small time intervals of length $\Delta$ (t), where

$$N=\frac{T}{\Delta t}$$

Thus

$$Z(T)-Z(0) = \sum_{i=1}^{N} \varepsilon_i \sqrt{\Delta t}, \tag{2.2}$$

where the $\varepsilon_i$ (i=1,2,3.....N) are distributed $\vartheta(0,1)$. From the second property of Weiner process, $\varepsilon_i$ are independent of each other. It follows that Z (T)-Z (0) is normally distributed with Mean =E (Z (T)-Z (0)) = 0, Variance of (Z (T)-Z (0)) = n $\Delta$ = T thus, Standard deviation of (Z (T)-Z (0)) is $\sqrt{T}$ Hence Z (T)-Z (0) ~ N (0, $\sqrt{T}$ ).

It$^\wedge$o process is a generalized Wiener process in which the parameters a and b are functions of the value of the underlying variable X and time t. An It$^\wedge$o process can be written algebraically as,

$$dX = a(X, t)\, dt + b(X, t)\, dZ. \tag{2.3}$$

It$\hat{}$o lemma is the formula used for solving stochastic differential equations. It is a treatment of wide range of Wiener-like differential process into a strict mathematical framework.

## 3 Estimation of Volatility:

Black-Scholes Merton model is derived from Black-Scholes Merton differential equation. This model has been a breakthrough as far as predictions of asset prices are concerned. One of the major assumptions of Black and Scholes model when determining price of underlying asset (depending on the time t)is that the price of underlying asset follows a geometric Brownian motion with constant drift $\mu$ and constant volatility $\sigma$ throughout the duration of a derivative constant; that is

$$d\,S_t = \mu S_t + \sigma S_t\, dZ \qquad 3.1$$

where Z is wiener process. There are several methods of estimating constant volatility; among them we have historical volatility, implied volatility. Recent studies have revealed that volatility of underlying asset is not necessarily constant during the life of an option, it may vary with time.

## 4 Derivation of the Black-Scholes-Merton differential equation:

We consider stock price process that follows a geometric Brownian motion as follows:

$$dS = \mu S\, dt + \sigma\, S\, dZ, \qquad 4.1$$

where $\mu$ (linear drift rate) and $\sigma$ (volatility) are constants and Z is a Wiener process. Suppose that G is the price of a call option or other derivative contingent twice differentiable in S and once on t. The variable G must be of some function of S and t. Hence from It$\hat{}$o process of the form;

$$dG = \left( \frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S}\sigma S dZ\,, \qquad 4.2$$

We therefore seek to eliminate the Wiener process. We choose a portfolio of stock and the derivative. The appropriate portfolio is

-1: derivative

$+\frac{\partial G}{\partial S}$ : Shares

The holder of this portfolio is short one derivative and long an amount $\frac{\partial G}{\partial S}$ of shares. This implies that portfolio holder will have a long option position and a short option position in a quantity related to shares. We define $\pi$ as the value of the portfolio. By definition,

$$\pi = -G + \frac{\partial G}{\partial S} S \qquad 4.3$$

Taking $\Delta\pi$, $\Delta S$ and $\Delta G$ as the changes in $\pi$, $S$ and $G$ in small interval $\Delta t$, equation (4.3) is given by

$$\Delta\pi = -\Delta G + \frac{\partial G}{\partial S}\Delta S \qquad 4.4$$

Substituting the discrete version of equations (4.1) and (4.2) into equation (4.4) we obtain

$$\Delta\pi = -\left( (\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2)\Delta t + \frac{\partial G}{\partial S}\sigma S\Delta Z \right) + \frac{\partial G}{\partial S}(\mu S\Delta t + \sigma S\Delta Z) \qquad 4.5$$

This simplifies to

$$\Delta\pi = \left( -\frac{\partial G}{\partial t} - \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 \right)\Delta t \qquad 4.6$$

The portfolio is risk-less during the time $\Delta t$ because (4.6) does not involve $\Delta Z$ .The assumptions of Black-Scholes-Merton differential equation imply that the portfolio must instantaneously earn the same rate of return as other short-term risk free assets. If it earned more than this return, arbitrageurs could make a risk-less profit by borrowing money to buy the portfolio. If it earned less, they could make a risk-less profit by shortening the portfolio and buying a risk-less assets. This change in the value of the portfolio must therefore be the same as the growth we could get if we put equivalent amount of cash in a risk-free interest bearing account using the non-arbitrage principle. It follows that

$$\Delta\pi = r\pi\Delta t \qquad 4.7$$

where r, is the risk-free interest rate. Substituting from equations (4.3) and (4.6) into (4.7) we obtain

$$\left(-\frac{\partial G}{\partial t}-\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(-G+\frac{\partial G}{\partial S}S\right)\Delta t$$

So that

$$\frac{\partial G}{\partial t}+rS\frac{\partial G}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 G}{\partial S^2}=rG \qquad 4.8$$

Equation (4.8) is the Black-Scholes-Merton differential equation.

## 5 Derivation of Dupire Volatility equation:

In the recent work, research by Dupire (1994), Derman and Kani(1994) and Rubinstein (1994) has concentrated on building models for price process of S(t) that can fit a certain observed pattern of volatility. In this section we summarize the work of Dupire as follows; The aim of the model is to show unique volatility function $\sigma$ (S, t) that is assumed that at a given time t, prices of European call option, for all maturity time T ≥ t and all exercise price E > 0 can be observed. The observed option price is consistent with the price of dividend yielding the following equation;

$$d\,S(t)=\mu\,(t)-y(t))S(t)\,dt+\sigma\,(S,t)S(t)d\,Z(t), \qquad 5.1$$

where $\mu$ (t) is the drift term of the underlying asset, y(t) is the dividend yield of the asset, $\sigma$ (S, t) is the volatility and Z(t) is the standard Wiener process. We then have the Black-Scholes Merton differential equation for any claim h(S, t) given by

$$\frac{\partial h(S,t)}{\partial t}+\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 h(S,t)}{\partial S^2}+(r(t)-y(t))S\frac{\partial h(S,t)}{\partial S}-r(t)h(S,t)=0 \qquad 5.2$$

where r(t) is the risk-free interest rate in the market and not $\mu$ because when it comes to price derivatives, it is the risk-free drift r, that matters and not the real drift $\mu$ . The t dependence of r(t) is the ``term structure" of interest rates. Considering a European call option at time t, priced at a discounted expectation where P (t, T) denotes the price at time t, of a risk-zero coupon bonds, then it can be shown that

$$h(S,t)=P(t,T)Exp\left\langle S(T)-K/S(T)\right\rangle \qquad 5.3$$

Hull (2000)

Defining $\vartheta(S(T)/(S(t))$ as the probability density function of condition on $S(t)$ then (5.2) can be written as

$$h(S,t) = P(t,T)\int_E^\infty \max(S(T)-K)\vartheta(S(T)/S(t))dS(T) \qquad 5.4$$

If we differentiate (5.3) twice with respect to K, we get

$$\vartheta(S(T)/S(t)) = K = \frac{\partial^2 h(S,t)}{\partial K^2}P(t,T)^{-1} \qquad 5.5$$

(Breeden and Litzenberger 1978).

Considering the use of Kolmogorov's forwarding equation for transitional probability density function $\vartheta(S(T)/(S(t))$ the Fokker-Planck equation on

$$dS(t) = \mu(S,t)S(t)dt + \sigma(S,t)S(t)dZ(t) \qquad 5.6$$

We have

$$\frac{\partial \vartheta(S(T)/S(t))}{\partial T} - \frac{1}{2}\frac{\partial^2 h(S,t)}{\partial S(T)^2}\sigma(S(T),T)^2\vartheta(S(T)/S(t)) + \frac{\partial h(S,t)}{\partial S(T)}\mu(S(T),T)\vartheta(S(T)/(t)) = 0 \qquad 5.7$$

The function $\vartheta(S(T)/(S(t))$ is the density of random variable $S(T)$ and time $T$ conditional on the initial $S(t)$ value and the process (5.6). Dupire (1993) re-stated equation (5.7) regarding $h(S,t)$ as a function of strike price $K$. With differentiation taken with respect to $K$ , and with drift and volatility functions evaluated at $K$ (because the density function in equation (5.5) is expressed in terms of $K$ ). Equation (5.7) can be re-written as

$$\frac{\partial K}{\partial T} - \frac{1}{2}\frac{\partial^2 h}{\partial K^2}[\sigma(K,T)^2 K^2\vartheta(K)] + \frac{\partial h}{\partial K}[(r(T)-y(T))K\vartheta(K)] = 0 \qquad 5.8$$

Using equation (5.5) and substituting for $\vartheta(K)$ in the first term of (5.8) we get

$$\frac{\partial}{\partial T}[P(t,T)^{-1}(\frac{\partial^2 h}{\partial K^2})] - \frac{1}{2}\frac{\partial^2 h}{\partial K^2}[\sigma(K,T)^2 K^2\vartheta(K)] + \frac{\partial h}{\partial K}[(r(T)-y(T))K\vartheta(K)] = 0 \qquad 5.9$$

Using chain rule to differentiate the first term in (5.9) with respect to T and expand we have

$$P(t,T)^{-1}\frac{\partial}{\partial T}(\frac{\partial^2 h}{\partial K^2}) + r(T)P(t,T)^{-1}\frac{\partial^2 h}{\partial K^2} + \frac{\partial h}{\partial K}[r(T)-y(T)K\vartheta(K)] - \frac{1}{2}\frac{\partial^2 h}{\partial K^2}[\sigma(K,T)^2 K^2\vartheta(K)] = 0 \qquad 5.10$$

Integrating once, substituting for $\vartheta(K)$ and multiplying by $P(t,T)$ we have

$$\frac{\partial h}{\partial K}(\frac{\partial h}{\partial T})+r(T)\frac{\partial h}{\partial K}+(r(T)-y(T)K\frac{\partial^2 h}{\partial K^2}-\frac{1}{2}\frac{\partial h}{\partial K}[\sigma(K,T)^2K^2\frac{\partial^2 h}{\partial K^2}]=\alpha T \quad 5.11$$

where $\alpha T$ is constant of integration. Integrating again with respect to $K$ we get

$$\frac{\partial h}{\partial T}r(T)h+[r(T)-y(T)]K\frac{\partial h}{\partial K}-r(T)h-y(T)h-\frac{1}{2}[\sigma(K,T)^2K^2\frac{\partial^2 h}{\partial K^2}]=\alpha T+\beta T \quad 5.12$$

where $\beta T$ is a constant of integration relating to the second integration. Rearranging and simplifying we have

$$\frac{\partial h}{\partial T}+[r(T)-y(T)]K\frac{\partial h}{\partial K}+y(T)h-\frac{1}{2}[\sigma(K,T)^2K^2\frac{\partial^2 h}{\partial K^2}]=\alpha T+\beta T \quad 5.13$$

Following Dupire (1994) it is assumed that all the terms in the left hand side of (5.13) decay when $K$ tends to $+\infty$ so that $\alpha T=\beta T=0$. Hence (5.13) becomes

$$\frac{\partial h}{\partial T}+[r(T)-y(T)]K\frac{\partial h}{\partial K}+y(T)h-\frac{1}{2}[\sigma(K,T)^2K^2\frac{\partial^2 h}{\partial K^2}]=0 \quad 5.14$$

Where $K>0$.This is the price of a European option expressed as a function of $T$ and $S$ (for fixed $t$ and $s$). Rearranging equation (5.14) we get the volatility function

$$\sigma(K,T)=\sqrt{2\left\{\frac{\frac{\partial h}{\partial T}+\left(r(T)-y(T)\right)K\frac{\partial h}{\partial K}+y(t)h}{K^2\frac{\partial^2 h}{\partial K^2}}\right\}} \quad 5.15$$

Equation (5.15) defines the value of volatility of an option at time $T$ and strike price $K$ and is referred to as Dupire volatility equation.

## 6 Estimation of Volatility when the Asset price follows Non-linear Brownian motion:

The non-linear Brownian motion is build on Walrasian price-adjustment model by introducing excess demand and applying them in the framework of Walrasian-Samuelson price adjustment mechanism to obtain a deterministic logistic equation. In this section we derive a volatility model when the asset price follows non-linear Brownian motion at any given time $t$, with the price of the asset $S(t)$ is less than equilibrium price $S^*$ giving the following equation.

$$dS(t) = \mu S(t)(S^* - S(t))dt + \sigma S(t)(S^* - S(t))dZ(t) \qquad 6.1$$

where $S^* \neq S(t)$; $S^*$ is the equilibrium price, $S(t)$ is the price of the asset or security, $\mu$ is the function of growth rate and $\sigma$ is the volatility while $dZ(t)$ is the standard weiner process. Applying the Black-Scholes Merton differential equation for any claim of asset value $V(S,t)$. From equation 4.8 we have.

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2(S,t)S(t)^2(S^* - S(t))^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS(t)(S^* - S(t))\frac{\partial V(S,t)}{\partial S} = r(t)V(S,t) \qquad 6.2$$

where $r(t)$ is the risk-free interest rate in the market and not $\mu$. Using the Black-Scholes equation, equation (6.2) can be expressed as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2(S^* - S)^2 \frac{\partial^2 V}{\partial S^2} + r(S^* - S)\frac{\partial V}{\partial S} - rV = 0 \qquad 6.3$$

where $t \in (0,T)$ , $S^* > 0$; $V(S,t) = \max((S^* - S),0)$. Where $r(t)$ risk-free interest is rate when $S^*$ and $r$ are constant then the European put option is expressed as a function of $T$ with volatility $\sigma$ being sufficiently regular. Using standard arguments, it can be shown that call option at time $t$ will be priced at the discounted expectations.

$$V(S,t) = P(t,T)Exp\langle S^* - S / S(T)\rangle \qquad 6.4$$

In which factor $P(t,T)$ denotes the price at time $t$ of a zero-risk coupon bond. Relying heavily in the transitional probability function $\vartheta(t, S(t), T, S^*)$ for a risk-neutral random walk. Then we have

$$V(S,t) = P(t,T)\int_S^\infty \max(S^* - S)\vartheta(S(T)/S(t))dS(T) \qquad 6.5$$

Differentiating (6.5) twice with respect to $S$ we get

$$\vartheta(S(T)/S(t)) = S = \frac{\partial^2 V(S,t)}{\partial S^2} P(t,T)^{-1} \qquad 6.6$$

Using Kolmogorov's forward equation for transitional probability density function, the Fokker-Planck equation on (6.1) becomes

$$\frac{\partial \vartheta(S(T)/S(t))}{\partial T}-\frac{1}{2}\frac{\partial^2 V(S,t)}{\partial S(T)^2}\sigma(S(T),T)^2 S(t)^2(S^*-S(t))^2\vartheta(S(T)/S(t)) + \frac{\partial V(S,t)}{\partial S(T)}\mu(S(T),T)S(t)(S^*-S(t))\vartheta(S(T)/(t))=0 \qquad 6.7$$

Using the spirit of Dupire (1993) regarding $V$ as a function of strike price $S$ in equation (6.7). With differentiation taken with respect to $S$, and the drift and the volatility function evaluated at $S$ (because the density function in equation (6.5) is expressed in terms of $S$). Equation (6.7) can be re-written as

$$\frac{\partial \vartheta S}{\partial T}-\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma(S,T)^2 S^2(S^*-S)^2\vartheta(S)+\frac{\partial V}{\partial S}[r(T)S(S^*-S)\vartheta(S)=0 \qquad 6.8$$

Using equation (6.6) and substituting for $\vartheta(S)$ in the first term of equation (6.8) we get

$$\frac{\partial}{\partial T}[P(t,T)^{-1}\frac{\partial^2 V}{\partial S^2}]-\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma(S,T)^2 S^2(S^*-S)^2\vartheta(S)+\frac{\partial V}{\partial S}[r(T)S(S^*-S)\vartheta(S)=0 \qquad 6.9$$

Using chain rule to differentiate the first term in (6.9) with respect to $T$ and expand we have

$$P(t,T)^{-1}\frac{\partial}{\partial T}(\frac{\partial^2 V}{\partial S^2})+r(T)P(t,T)^{-1}\frac{\partial^2 V}{\partial S^2}+\frac{\partial V}{\partial S}[r(T)S(S^*-S)\vartheta(S)] -\frac{1}{2}\frac{\partial^2 V}{\partial S^2}[\sigma(S,T)^2 S^2(S^*-S)\vartheta(S)]=0 \qquad 6.10$$

Integrating once, then substituting for $\vartheta(S)$ and multiplying by $P(t,T)$ we have

$$\frac{\partial V}{\partial S}(\frac{\partial V}{\partial T})+r(T)\frac{\partial V}{\partial S}+r(T)S(S^*-S)\frac{\partial^2 V}{\partial S^2}-\frac{1}{2}\frac{\partial V}{\partial S}[\sigma(S,T)^2 S^2(S^*-S)^2\frac{\partial^2 V}{\partial S^2}]=\alpha T \qquad 6.11$$

where $\alpha T$ is constant of integration. Integrating again with respect to $S$ we get

$$\frac{\partial V}{\partial T}r(T)V+[r(T)S(S^*-S)]\frac{\partial V}{\partial S}-r(T)V-\frac{1}{2}[\sigma(S,T)^2 S^2(S^*-S)^2\frac{\partial^2 V}{\partial S^2}]=\alpha T+\beta T \qquad 6.12$$

where $\beta T$ is the constant of integration relating to the second integration. Rearranging and simplifying we have

$$\frac{\partial V}{\partial T}+[r(T)S(S^*-S)]\frac{\partial V}{\partial S}+r(T)V-\frac{1}{2}[\sigma(S,T)^2 S^2(S^*-S)^2\frac{\partial^2 V}{\partial S^2}]=\alpha T+\beta T \qquad 6.13$$

Using the spirit of Dupire (1994) where it is assumed the all the terms in the left hand side of equation (6.13) decay when $S$ tend to $+\infty$ so that $\alpha T = \beta T = 0$. Hence (6.13) becomes

$$\frac{\partial V}{\partial T} + [r(T)S(S^* - S)]\frac{\partial V}{\partial S} + r(T)V - \frac{1}{2}\sigma(S,T)^2 S^2 (S^* - S)^2 \frac{\partial^2 V}{\partial S^2} = 0 \qquad 6.14$$

Rearranging equation (6.14) we have

$$\sigma = \sqrt{2\left\{\frac{\frac{\partial V}{\partial t} + r(T)S(S^* - S)\frac{\partial V}{\partial S} + r(T)V}{S^2 (S^* - S)^2 \frac{\partial^2 V}{\partial S^2}}\right\}} \qquad 6.15$$

## 7 Conclusion:

In this paper, we have developed a mathematical model to be used in estimating volatility when the asset price at time t follows non-linear Brownian motion rather than when the asset price at time t follows linear Brownian motion. This model has also reviewed the evidence of non-constant volatility as opposed to the wide spread Black-Sholes model for asset prices that assumes that volatility is constant.

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