Potential Future Exposure Calculations Using the BGM Model

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Abstract
We investigate the use of the Brace–Gatarek–Musiela (BGM) model in potential future exposure (PFE) calculations for various exotic interest rate products. Monte Carlo simulation is required to generate PFE scenarios. If the pricing model is also implemented using Monte Carlo simulation, PFE calculations are exceptionally demanding, involving *nested* Monte Carlo simulations. To reduce computation time, we look at two approximate pricing methods—a Longstaff-Schwartz regression-based approach, and a finite difference approach using a drift approximated BGM model. Numerical results are presented for several different exotic trade types. PFE results delivered by the two approximate methods are compared with results obtained by full resimulation, i.e. nested Monte Carlo. In the test cases we looked at, both approaches produced good results and were several orders of magnitude faster than full resimulation. However, care must be taken when using the regression-based approach. In particular, results can vary significantly depending on the order of the polynomial basis functions used in the regressions.

Keywords
counterparty credit risk, potential future exposure, PFE, state space sampling, interest rate derivatives, BGM model

1 Introduction
Counterparty credit risk is the risk that a counterparty in a financial contract will fail to meet its contractual obligations. The counterparty exposure is the value, floored at zero, of the entire portfolio of trades entered with a given counterparty. This corresponds to the net loss incurred if the counterparty were to default and no amount recovered.

Potential future exposure (PFE) calculations are used to set counterparty credit limits and to determine economic and regulatory capital requirements. A PFE calculation requires the simulation of counterparty exposure at a number of future time points in many risk factor scenarios. The PFE at a future exposure date t is obtained from the distribution of simulated counterparty exposures at time t. PFE(t) is a user-specified percentile of this distribution; typically the 95$^{\text{th}}$ percentile. The function $t \to$ PFE(t) is called the *PFE profile*.[1]

PFE calculations must account for all trades with a given counterparty regardless of the underlying risk factors. Therefore the PFE simulation model must capture correlations between all risk factors relevant to the portfolio. For portfolios of single currency interest rate products, the Brace–Gatarek–Musiela (BGM) model is one such model as it is able to capture a wide variety of correlation structures. See Appendix A for more details regarding the BGM model. Except for a handful of simple products admitting closed form prices, Monte Carlo simulation is typically required for BGM pricing. Since PFE calculations involve pricing in many different scenarios at many future time points, a Monte Carlo pricing approach mandates the use of nested Monte Carlo simulations. This would be computationally intensive, perhaps infeasibly so, for reasonable accuracy levels and realistic portfolio sizes. However, an alternative approximate approach is to generate a state space of prices at all future exposure dates. Instead of repricing the counterparty portfolio at every time step of every scenario, we instead sample the populated state space to obtain the counterparty exposure for each PFE scenario.

Note that the steps involved in performing PFE calculations are very similar to those required for credit value adjustment (CVA) calculations. Thus, the techniques discussed here could

1. Here we sidestep details related to the treatment of collateral and netting rules. For more on PFE calculations in general, see Duffie and Canabarro (2004), Pykhtin and Zhu (2007), and Lomibao and Zhu (2006).

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also applied to CVA calculations. For more on CVA calculations, see Pykhtin and Zhu (2007).

In this article, we investigate the use of state space sampling techniques to accelerate PFE calculations based on the BGM model. We look at two methods for generating a pricing state space: a Longstaff–Schwartz regression-based approach and a finite difference-based approach using a drift approximated BGM model. Numerical results are presented for several different interest rate product types, including trades having both path dependency and cancellation features. PFE profiles obtained using the two approximate methods are compared with results obtained using nested Monte Carlo full resimulation.

2 Longstaff–Schwartz Regression Approach

Longstaff and Schwartz (2001) present a simple, elegant Monte Carlo method for pricing options having early exercise features. Here, least-squares regression is used to approximate the continuation value at every exercise date. We briefly outline the details of the method following the treatment by Glasserman (2004). The continuation value C_t at some future time t is the expectation of all future cashflows $V_{\tau>t}$:

$$C_t = E\left[V_{\tau>t} | X_t = x\right] \tag{1}$$

where X_t are the state variables of the model at time t. The method approximates the conditional expectation as:

$$E\left[V_{\tau>t} | X_t = x\right] \approx \sum_{r=1}^{M} \beta_r \psi_r(x) \tag{2}$$

for some basis functions ψ_r and constants β_r. The constants β_r are obtained by performing a least squares regression on a set of pre-simulated paths. The basis functions that are used are typically polynomials over a set of explanatory variables that can be calculated from the state variables. The explanatory variables are chosen to be quantities (such as a set of Libor rates or swap rates in the case of interest rate product) that represent the value of the trade. The efficacy of a particular selection of basis functions and explanatory variables is typically trade dependent; there is no universal "best choice" suitable for all trade types.[2] The exercise decision is then based on the comparison of this continuation value with the exercise value. If the exercise value cannot be calculated analytically, another separate regression can be performed to provide an estimate. The exercise decision information is then stored and used in any required subsequent regressions.

The regressions used to estimate conditional expectations in the Longstaff–Schwartz method can also be used for state space sampling in PFE calculations. A regression can be performed at every PFE date, whether it is an exercise date or not. Product prices for an arbitrary PFE scenario can then be obtained from the resulting regression surface. All the caveats regarding the choice of basis functions and explanatory variables apply here, doubly so in the context of a PFE calculation. Although a specific choice of basis functions and explanatory variables may give good pricing results, it need not give good results for PFE calculations. In pricing, the regressions are only used for making exercise decisions. Thus, good approximations to the continuation value are only really necessary near the exercise boundary. However, when using regressions for PFE calculations, a good approximation to the entire state space is important, particularly in the tails of the underlying risk factor distribution. Figure 1 shows a simple example using the Black—Scholes model of how well a regression function fits the value of a call option on a future simulation date. The example shows the value of a call option expiring in 2 years observed 1 year into the future. The parameters used in the Black-Scholes model were $r = 0.05$, $\sigma = 0.15$, $S(0) = 100$ and $K = 100$. The regression was performed using 20000 paths with cubic basis functions on the stock price.

3 Finite Difference Approach

For pricing methods based on finite difference or tree methods, a state space of state-contingent prices is automatically generated as part of the backward iterative pricing procedure. This state space can evidently be used for state space sampling in PFE calculations. However, a general BGM model is Markovian only in the underlying forward rates, not in the stochastic driver W. Thus, finite difference methods cannot be applied directly to the BGM model because the number of forward rates usually far exceeds the number of dimensions where finite difference methods are computationally feasible. However, in the special case when the BGM model has a separable volatility structure, Pietersz, Pelsser, and van Regenmortel (2004) show how the model can be approximated by a low-dimensional Markov process, via a judicious drift approximation using the longest dated forward measure. Additional details on separable BGM models are provided in Appendix B.

One major drawback of the separable BGM model is that it is less flexible than a general BGM model in its ability to calibrate to different volatility and correlation structures. A more general BGM model can be calibrated to an entire at-the-money swaption matrix and a matrix of instantaneous correlations. A single calibration can be used to price many different trades. Following Piterbarg (2004a), we refer to such models as *global models*. In contrast, models that have to be recalibrated for each trade are referred to as *local models*. Local models are calibrated to a set of trade-specific quantities; for example, Bermudan swaptions are calibrated to a suitable swaption diagonal. Piterbarg (2004a) describes a local projection calibration method where results obtained from a more flexible global model are used to calibrate a local model so that the local model approximates the global model well for a specific trade. Ng (2009) studies the local projection calibration method with a separable BGM model as

2. A good overview of the Longstaff–Schwartz method in the context of exotic interest rate product pricing is presented in Piterbarg (2004b).

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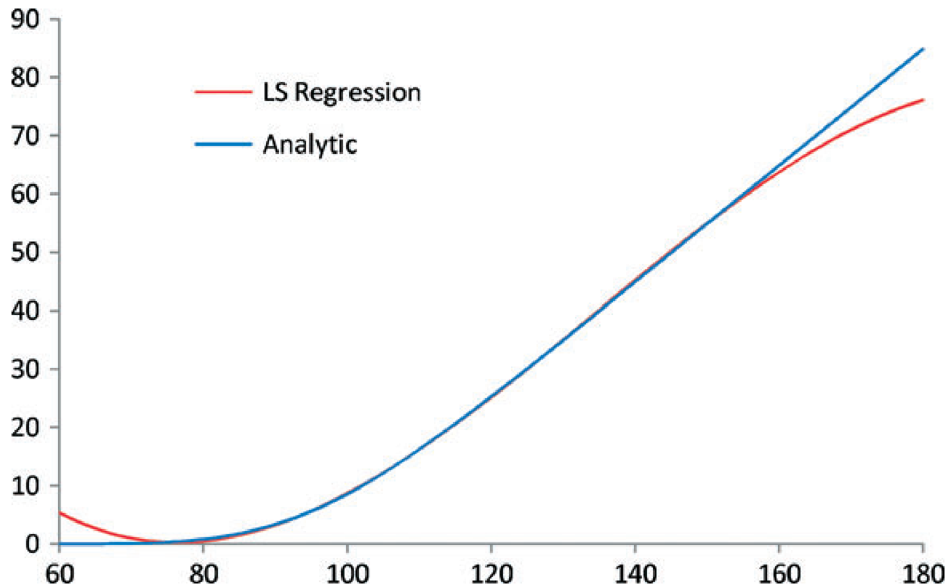

Figure 1: European Call Option Longstaff–Schwartz Regression Example using the Black–Scholes Model.

the local model and a general BGM model as the global model. Here, pricing results are presented for a variety of trades using both a general BGM model calibrated with the popular Pedersen (1998) method and a separable BGM model calibrated using local projection.

As previously discussed, it is important that the model used in PFE calculations capture correlations between all risk factors relevant to the portfolio. Thus, to generate PFE simulations, we use a global BGM model. In order to achieve scenario consistency across trades, thereby capturing exposure netting effects, these scenarios are common to all trades in the counterparty portfolio.[3] At the individual trade level, a local separable BGM model, calibrated by the projection method, is used to generate a state space of values for state space sampling. The calibration procedure and the resulting state space are specific to each trade in the portfolio.[4]

3.1 State Space Sampling Using BGM Finite Difference

Suppose that we have populated trade–specific finite difference state spaces for every trade in a counterparty portfolio. To obtain the PFE profile, we must sample every state space, at every exposure date t, for every PFE scenario ω, to obtain each trade's contribution to counterparty exposure. Focusing on one particular trade and one particular exposure date t, let $S = S(\omega, t)$ be the vector of simulated time t forward rates corresponding to scenario ω. Let $V(S)$ be the corresponding trade value. The trade's finite difference state space can be viewed as a discrete function $\hat{S} \to \hat{V}(\hat{S})$, where $\hat{\mathcal{S}} = \{\hat{S}\}$ are the solver state variables at the node points of the t^{th} time slice of the finite difference solver. In an idealized situation where the PFE scenario S maps unambiguously to a point $\hat{S} \in \hat{\mathcal{S}}$, we could simply set $V(S)$ equal to $\hat{V}(\hat{S})$. However, since the state space $\hat{\mathcal{S}}$ is finite, this is not generally possible. We must instead base the definition of $V(S)$ primarily on points $\hat{S}$ that are, in some sense, "close to" S.

To this end, let $\Delta(S, \hat{S})$ be a measure of the distance between the simulation S and a point $\hat{S}$ in the state space $\hat{\mathcal{S}}$. This distance function will be trade–specific; specific choices of Δ are discussed in the following subsections. Define $V(S)$ to be a weighted sum of all the values $\hat{V}(\hat{S})$ in the state space, i.e.

$$V(S) = \sum_{\hat{\mathcal{S}}} w(S, \hat{S}) \times \hat{V}(\hat{S}).$$

The weight function $w(S, \hat{S})$ is defined as

$$w(S, \hat{S}) = v(S, \hat{S}) \;/\; \sum_{\hat{\mathcal{S}}} v(S, \hat{S}),$$

where

$$v(S, \hat{S}) = \exp\left(-\frac{\Delta(S, \hat{S})^2}{\delta(S)^2}\right), \quad \delta(S) = \min_{\hat{\mathcal{S}}} \Delta(S, \hat{S}).$$

This choice of weights has several desirable features. First, points in the state space that are closer to the simulation S are more heavily weighted. Also, the weights are similar to a Gaussian distribution with the standard deviation proportional to the minimum distance $\delta(S)$. By doing this, the closer the nearest point in the state space is to S, the more sharply peaked the

3. More generally the simulation measure may differ from the pricing measure, in which case a non-risk neutral model may be used for simulation purposes. However, the Basel II recommendations permit either approach.

4. For more details on the calibration of separable BGM models, see Ng (2009).

weighting function will be. In addition, if there are several points in the state space that are equally close to S, they will be equally weighted.

3.1.1 Δ_{Libor}

The choice of the distance function Δ used in the weight calculation depends on the trade being priced. For Libor-based products, we use the Euclidean distance between the logs of surviving forward rates specific to the trade, i.e.

$$\Delta_{\text{Libor}}(S,\hat{S}) = \sqrt{\sum_i |\ln(F_i) - \ln(\hat{F}_i)|^2},$$

where F_i and $\hat{F}_i$ denote the respective forward rates obtained from the global BGM model (the PFE scenario) and the separable BGM model (the trade–level state space). The sum is taken over all surviving forward rates, i.e. all forward rates that have not yet expired and which are required for pricing that specific trade.[5] There will generally be more unexpired forward rates in the general model as it is used to generate scenarios for the entire portfolio, not just the specific trade.

3.1.2 Δ_{CMS}

For CMS spread-based products, where the coupon is based on the spread between two swap rates R_1 and R_2, we use a different distance function. Instead of using the remaining forward rates, we use the Euclidean distance between the logs of the remaining forward swap rates relevant to the trade, i.e.

$$\Delta_{\text{CMS}}(S,\hat{S}) = \sqrt{\sum_i \left(|\ln(R_{1,i}) - \ln(\hat{R}_{1,i})|^2 + |\ln(R_{2,i}) - \ln(\hat{R}_{2,i})|^2\right)},$$

where $R_{1,i}, R_{2,i}$ and $\hat{R}_{1,i}, \hat{R}_{2,i}$ denote the respective forward swap rates from the general BGM model and the separable BGM model. As an example, suppose we have a 5-year trade with quarterly payments based on the 2-year and 30 year CMS rates set at the start of the coupon periods. If we are performing our PFE calculation 1 year into the future, the remaining forward swap rates relevant to the trade would be 2 and 30 year CMS rates starting at times $T_i = 0, 0.25, 0.5, \ldots, 3.5, 3.75$ years from the PFE observation date.

We use this distance function for CMS spread-based products because it is more closely related to the local projection calibration procedure used for CMS spread products. As discussed in Ng (2009), these products are best calibrated to the relevant trade-dependent swap rates. Accordingly, a distance function based on these swap rates is likely a better choice than one based on forward rates.[6]

3.2 State Space Sampling Using Interpolation Methods

Another approach commonly used to perform state space sampling in PFE calculations is to use interpolation methods to obtain values from the generated state space of prices. When the state space is represented with a regular grid (as in the case with finite difference methods), interpolation can be performed using standard techniques (such as linear or cubic). However, when the state space is represented by an unstructured mesh, interpolation becomes more difficult. In addition, there has to be a well defined transformation that maps a PFE scenario to a point in the state space for interpolation to be used.

In the case of the separable BGM model, the state space is discretized using the state variables $X(t)$, not the underlying forward rates $F_i(t)$. Using drift approximations, one can obtain values of $F_i(t)$ from $X(t)$ but the inverse transformation is not available. The PFE scenarios generated using Monte Carlo simulation are represented using the forward rates $F_i(t)$ as the state variables. Thus, even if the state space is a regular grid in $X(t)$, the interpolation-based approach cannot be applied here since we do not have a direct mapping that transforms a PFE scenario represented by $F_i(t)$ to a set of values for $X(t)$.

The weight-based state space sampling method we proposed in section 3.1 is quite general and can be used in a variety of situations where interpolation cannot easily be performed. One example is with the stochastic mesh method of Broadie and Glasserman (2004) which can be applied to higher dimensional problems than tree or finite difference methods. The stochastic mesh method generates a state space with an irregular mesh making state space sampling using interpolation methods difficult. Another example is with a state space generated using a Markov functional model. As with separable BGM models, Markov functional models generate a state space where the state variable is a driftless Brownian motion. Thus, the same difficulties encountered when performing state space sampling with separable BGM models are also present for Markov functional models.

4 Numerical Results

In this section, numerical results are presented for several different exotic interest rate products. The test trades considered here comprise a subset of the test trades considered in Ng (2009). However, for the sake of brevity, only a subset of these results is included here. A description of the test market data and details relating to the calibration procedure and the implementation of the finite difference pricing can be found in Ng (2009).

4.1 Test Trades

All the trades are swaps starting in 3 months with maturities of 5, 10, or 15 years. The swaps have quarterly payments based

5. These rates correspond to all the unexpired forward rates remaining in the separable BGM model; it is a local model calibrated specifically for that trade and is therefore only aware of this subset of all forward rates.

6. Numerical tests comparing a forward rate-based distance function to its swap rate-based counterpart show that the swap rate-based approach gives better results for CMS products.

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on a \$100 notional. Floating payments based on 3-month Libor are received and a structured coupon is paid. Our approach to daycounts is somewhat simplified; we assume that all coupon periods have a year fraction of 0.25. We analyze both cancelable and non-cancelable trades. For cancelable trades, the exercise dates are quarterly with the first cancellation date occuring one coupon period into the swap.

The following coupon types are considered:[7]

1. Inverse Floaters

$$\text{InverseFloaterCpn} = \max\left[k_l \times 3\text{mLibor} + k_s, 0\right]$$

where 3mLibor is the 3-month Libor rate.

2. Ratchets

$$\text{RatchetCpn} = \max\left[k_l \times 3\text{mLibor} + k_p \times \text{PrevCpn} + \text{FixedRate}, 0\right] \tag{3}$$

where PrevCpn is the ratchet coupon for the previous coupon period, an initial rate of k_f is used for the first period, FixedRate $= k_r$ for the first ratchet coupon (second coupon period) and FixedRate increases by an amount k_s for each subsequent coupon.

3. Targets

$$\text{TargetCpn} = \min\left[\text{InverseFloaterCpn}, \frac{k_t - \text{SumPrevPayments}}{0.25}\right] \tag{4}$$

where SumPrevPayments is the sum of all previous target coupon payments ($\sum 0.25 \times \text{TargetCpn}_i$). Also, if the target amount k_t is met, the trade terminates and if the trade goes to maturity, the last structured coupon payment is $k_t - \text{SumPrevPayments}$.

4. CMS Spreads

$$\text{CMSSpreadCpn} = \max\left[k_\ell \times (\text{CMS2} - \text{CMS1}), 0\right] \tag{5}$$

where CMS1 and CMS2 are CMS rates are based on quarterly payments. CMS2 has a longer maturity than CMS1. k_ℓ is a leverage factor.

4.2 PFE Results

PFE profiles are calculated for the 95$^{\text{th}}$, 85$^{\text{th}}$, and 75$^{\text{th}}$ percentiles using 1000 scenarios and 10 day time steps. Results are generated for Longstaff-Schwartz regression–based state space sampling (LS), finite difference-based state space sampling (FD), and full Monte Carlo resimulation (MC). The full resimulation profile should be viewed as the ''correct'' (or closest to correct) PFE profile.

In the regressions, the following choices for explanatory variables are used:

1. Inverse Floaters—3-month Libor and coterminal swap rate.
2. Ratchets—3-month Libor, coterminal swap rate and previous coupon rate.
3. Targets—3-month Libor, coterminal swap rate and the sum of all previous coupon payments.
4. CMS Spreads—3-month Libor, coterminal swap rate and the 2 swap rates of the spread.

When pricing cancelable trades using full MC resimulation, cubic basis functions were used. For regression–based state space sampling (LS), the order of polynomial used for basis functions can have a significant effect on the PFE profiles generated.[8] Thus, LS results were generated using 3$^{\text{rd}}$, 4$^{\text{th}}$, 5$^{\text{th}}$, 6$^{\text{th}}$, and 7$^{\text{th}}$ order polynomial basis functions.

In the case of MC, 1000 simulation paths are used for repricing at every time step of every ''outer'' PFE scenario. The PFE scenarios are simulated using a log-Euler discretization under the spot Libor measure using a 5-factor BGM model calibrated with the Pedersen (1998) method. For LS, 20,000 paths are used to generate regression coefficients. In the case of FD, a 2-factor separable BGM model with a 35×35 grid is used for Libor–based trades. For path-dependent ratchets and targets, an auxiliary variable with 35 grid points is used. For CMS spread trades, a 3-factor separable BGM model with a $20 \times 20 \times 20$ grid is used. Again, more details are provided in Ng (2009) on the finite difference implementation.

Table 1 summarizes the trades and execution times required for the PFE calculations. These execution times account for both the cancelable and non-cancelable versions of the trade. For FD and LS results, PFE times include the time required to generate the state space and the state space sampling time involved in the exposure calculation. These timings are presented only to illustrate roughly how long the computations take using the 3 different methods. They are indicative only—they should not be interpreted as strict benchmarks since computations were performed on different machines having various loadings.

Figures 2 and 3 illustrate the effect of the order of the polynomial basis functions used with the LS method. Figure 2 shows the 95$^{\text{th}}$ percentile PFE profile for a non-cancelable 10 year ratchet trade for basis functions of different orders. For this trade, we see that using lower order basis functions (3$^{\text{rd}}$, 4$^{\text{th}}$ and 5$^{\text{th}}$ order) can result in large discrepancies from the correct value. Figure 3 shows the 75$^{\text{th}}$ percentile PFE profile for a cancelable 5-year inverse floater trade. In this example, we find that a significant bias exists using the LS method even when using fairly high order basis functions. Although not presented in Figure 3, results using 8$^{\text{th}}$ and 9$^{\text{th}}$ order basis functions were also generated and were found to be quite similar to 7$^{\text{th}}$ order results. In general, we find that the LS method can produce poor results when lower order basis functions are used.[9] When using higher order basis functions, the agreement is often quite good but there are still cases, as shown in Figure 3, where the

7. Note that the parameter values used for the test trades are listed in Table 1.
8. We would like to thank an anonymous reviewer of this article for suggesting we try using higher order basis functions.
9. This is interesting as Piterbarg (2004b) advocates the use lower-order polynomial basis functions when using the Longstaff–Schwartz method for pricing.

Table 1: Test Trades and PFE Times.								
Trade Type (Length)	Trade Parameters	PFE Time in Seconds						
		MC	FD	LS				
				Polynomial Order				
				3	4	5	6	7
Inverse Floater (5Y)	$k_l = -2, k_s = 0.15$	6993	29	17	21	26	34	46
Ratchet (10Y)	$k_l = -1, k_p = 1, k_f = 0.12, k_r = 0.012, k_s = 0.00125$	47226	177	57	96	147	234	367
Target (15Y)	$k_l = -2, k_s = 0.1, k_t = 0.15$	116451	268	101	151	221	368	564
2-10 CMS Spread (5Y)	$k_l = 5$	22955	343	65	118	222	449	1296
2-20 CMS Spread (10Y)	$k_l = 5$	127900	1042	201	319	505	955	2673
5-30 CMS Spread (15Y)	$k_l = 5$	474481	2016	462	615	856	1533	4067

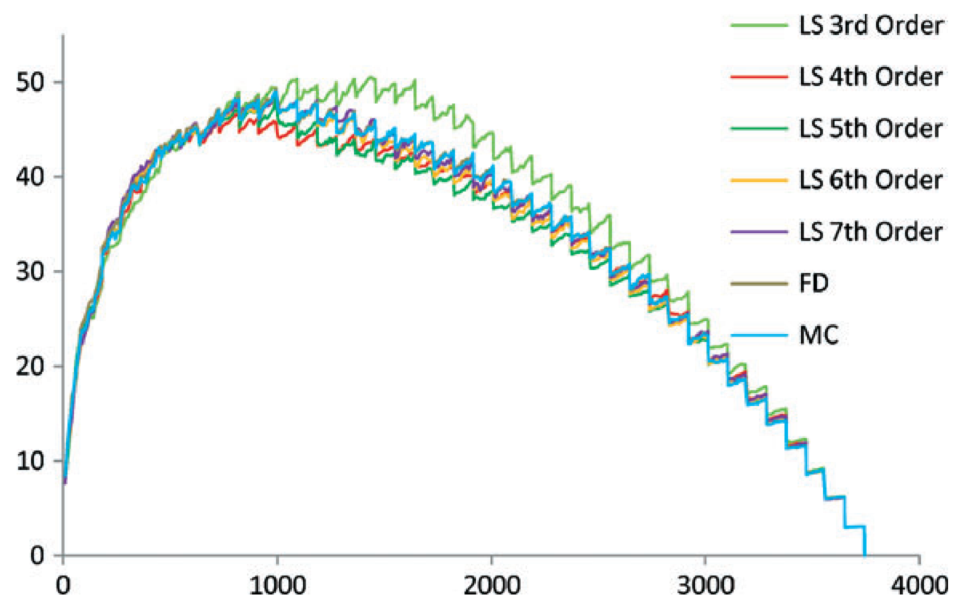

Figure 2: 95th Percentile PFE for 10 Year Ratchet (LS with different order polynomials).

differences can be significant. In these such cases, a better choice of explanatory variables is probably required to achieve more accurate results.

Figures 4–15 show FD, LS (7th order only) and MC results for the various test trades summarized in Table 1. For Libor–based products (Figures 4–9), we see good agreement between FD with MC. Of particular note, observe the difference in execution speed; the FD approach is several order of magnitude faster than the full resimulation approach. We also see decent results using the LS method with 7th order polynomials except for the case that was highlighted in Figure 3. In general, the results using LS are not as good as results from FD. Note that in our implementation, the calculation times for the LS method, although still much faster than MC, starts to exceed that of FD at about 6th order for the basis functions.

For CMS spread based trades (Figures 10–15), we also find that, with the exception of some longer non–cancelable trades, FD gives better agreement with MC than LS does. For these longer maturity trades, the agreement between LS and MC tends to be better than the agreement between FD and MC at shorter exposure dates. This is likely an artefact of the drift approximations used in the local separable BGM model. As discussed by Pietersz et al. (2004) and Joshi and Stacey (2006), drift approximations perform worse when the number of Libors is large and the step date is further away from the forward measure date. This is exactly the situation arising here. For drift approximations, in our tests we use the non-iterative

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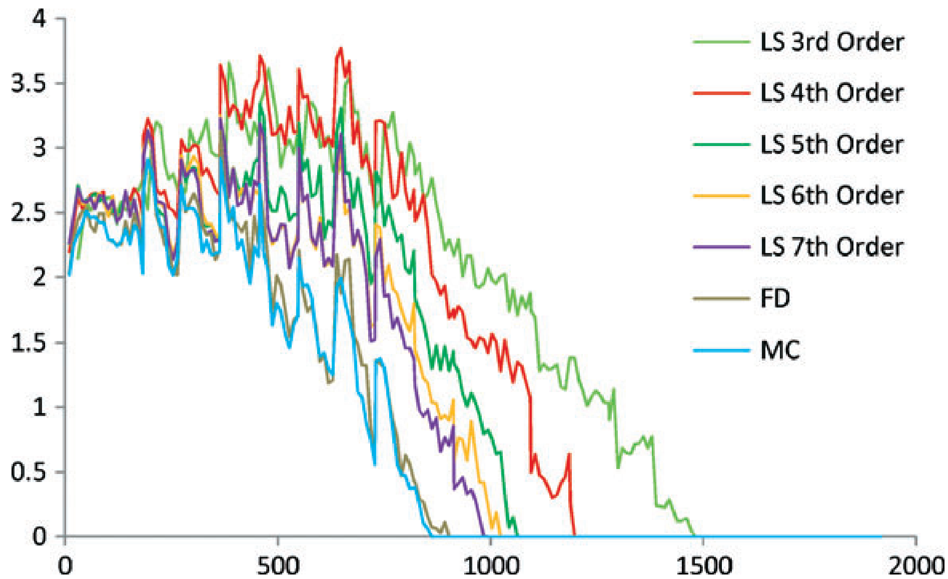

Figure 3: 75th Percentile PFE for Cancelable 5 Year Inverse Floater (LS with different order polynomials).

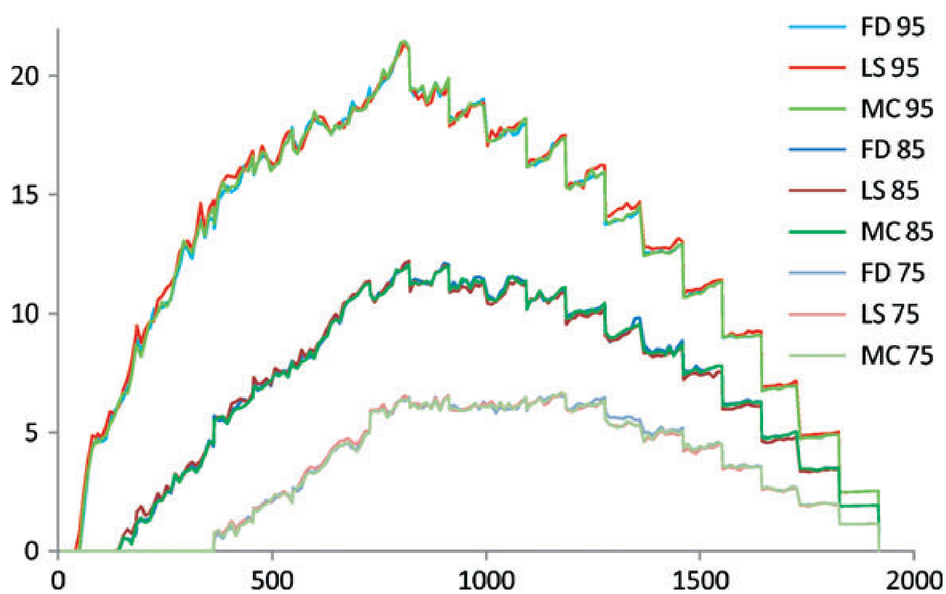

Figure 4: PFE for 5 Year Inverse Floater.

predictor-corrector method as described in Joshi and Stacey (2006). A more accurate drift approximation formula may give better results.

5 Conclusion

We investigate the use of the BGM model for Potential Future Exposure calculations. We find through numerical examples that PFE profiles obtained via state space sampling using a drift approximated BGM model agree well with those obtained by full Monte Carlo resimulation. However, the drift approximated model is orders of magnitude faster than its full resimulation counterpart. Regression–based state space sampling is also fast, but somewhat more erratic, occasionally having large discrepancies from full resimulation results. In particular, we find that using low order polynomial basis functions can lead to poor results.

Although we only consider interest rate products in a single currency, the finite difference sampling method can be applied to portfolios involving multiple currencies as long the payoff for each trade is only in one currency. Here, a multi–currency version of the BGM model can be used for scenario generation, while state space sampling can be performed using single currency local BGM models in the currency specific to each trade. It is likely also possible to apply the finite difference sampling method to other asset classes, e.g. equity, FX, cross-currency, and hybrid. Further research is required to determine the extent to

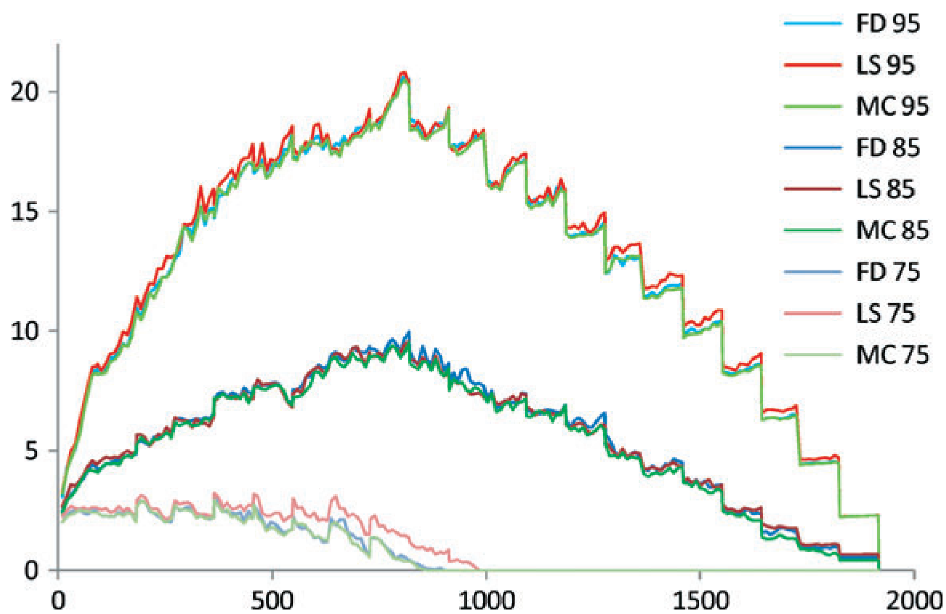

Figure 5: PFE for 5 Year Cancelable Inverse Floater.

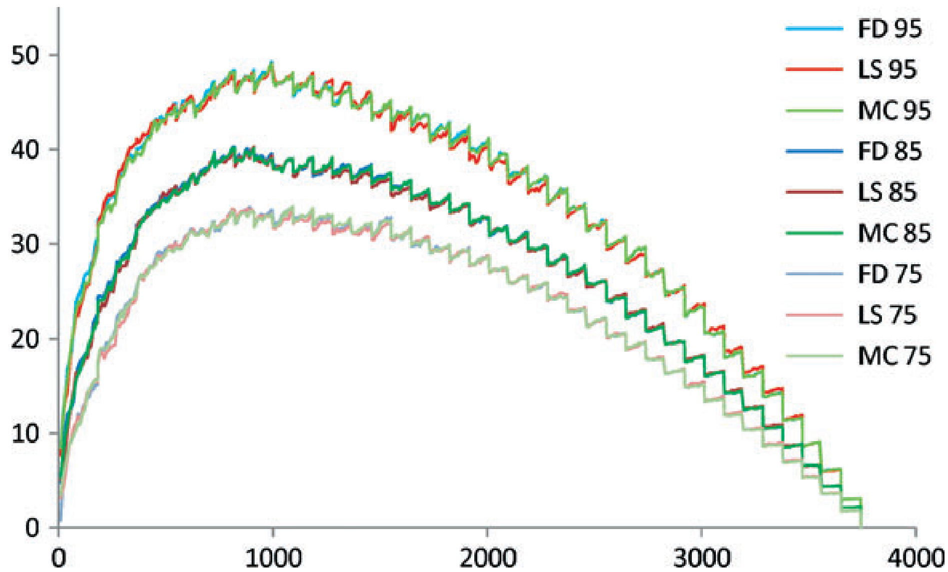

Figure 6: PFE for 10 Year Ratchet.

which BGM finite difference approximation and local projection calibration procedures can be extended to additional product types.

Appendix A: BGM Model

In the BGM model, the model primitives are forward rates. Specifically, let $F_i(t)$ be the forward rate from time T_i to T_{i+1}. Forward rates can be expressed in terms of zero coupon bond prices $P(t, T)$ as:

$$F_i(t) = \frac{1}{\delta_i}\left(\frac{P(t, T_i)}{P(t, T_{i+1})} - 1\right) \tag{6}$$

where δ_i is the year fraction between T_i and T_{i+1}. In the BGM model, the dynamics of the forward rates $\mathcal{F}(t) = \{F_0(t), F_1(t), \cdots, F_{n-1}(t)\}$ spanning the node dates $0 = T_0 < T_1 < \cdots < T_n$ take the form:

$$\frac{dF_i(t)}{F_i(t)} = \mu_i(t, \mathcal{F}(t))dt + \gamma_i(t) \cdot dW \tag{7}$$

Here W is a d-dimensional Wiener process (relative to a chosen pricing measure), $\mu_i(t, \mathcal{F}(t))$ is a state–dependent drift rate, and $\gamma_i(t)$ is a d–dimensional volatility vector for $F_i(t)$. The dimension d of the driving process is typically small, e.g. d is typically between three and ten, while the number of forward rates may number

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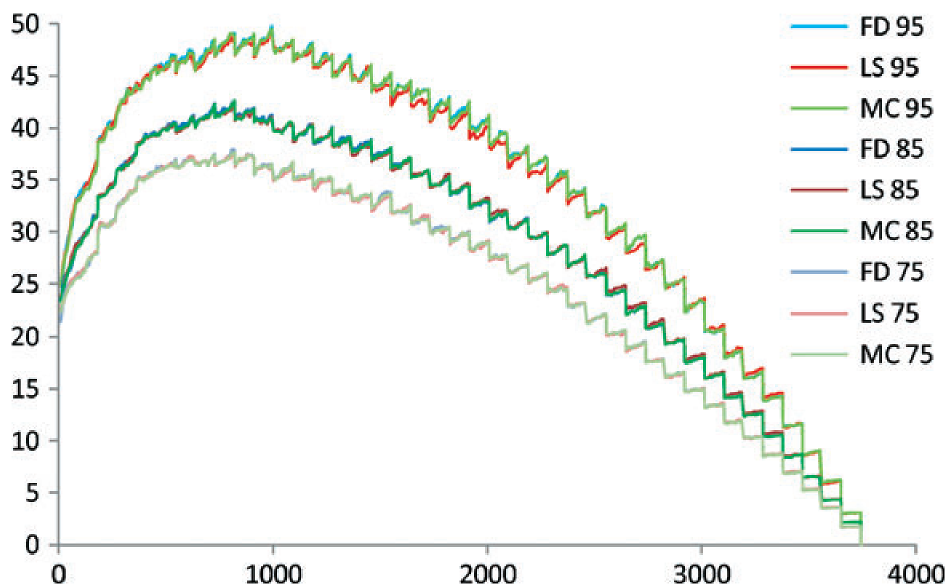

Figure 7: PFE for 10 Year Cancelable Ratchet.

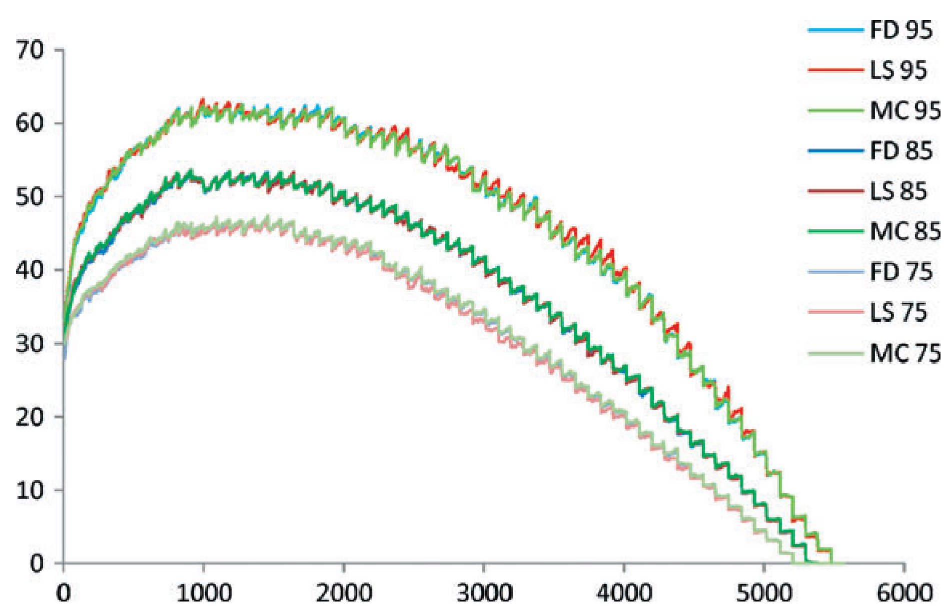

Figure 8: PFE for 15 Year Target.

in the low hundreds.[10] The forward rates F_i can be evolved from time t to T by

$$F_i(T) = F_i(t)\exp\left[\int_t^T \mu_i(s)ds - \frac{1}{2}\int_t^T |\gamma_i(s)|^2 ds + \int_t^T \gamma_i(s)\cdot dW\right] \quad (8)$$

Appendix B - Separable BGM Model

Pietersz et al. (2004) show that finite difference methods can be applied to the BGM model if:

1. The forward measure is used for pricing.
2. The volatility structure is separable (i.e. can be expressed in the form):

$$\gamma_j(t) = \nu_j * \sigma(t) \quad (9)$$

For a d factor model, $\nu_j = \{\nu_{j,1}, \nu_{j,2}, \ldots, \nu_{j,d}\}$ is a constant d dimensional vector (one for each forward rate F_k) and $\sigma(t) = \{\sigma_1(t), \sigma_2(t), \ldots, \sigma_d(t)\}$ is time varying d dimensional vector. The $*$ operator denotes element by element multiplication.

10. For more details on the BGM model, see Brace et al. (1997) or Brigo and Mercurio (2001).

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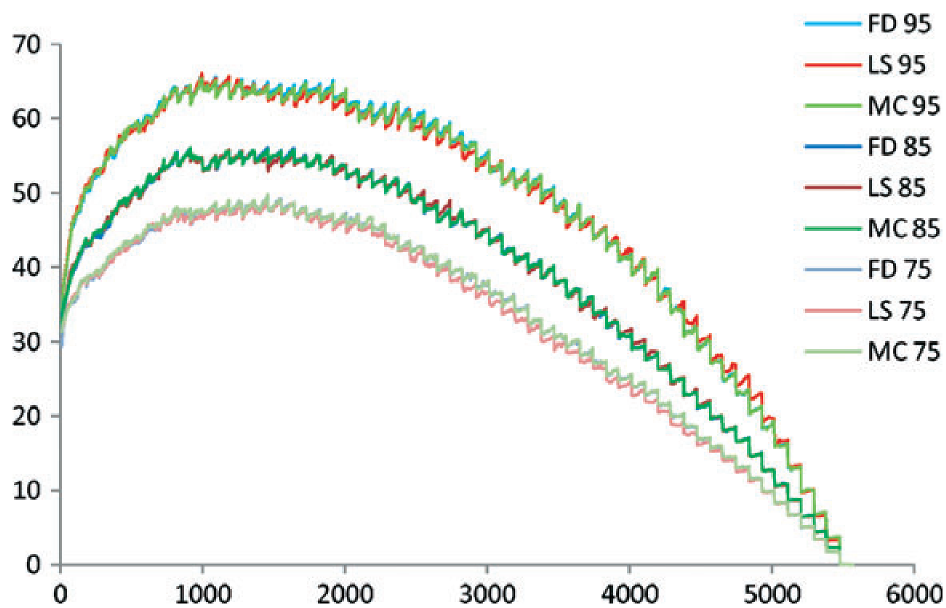

Figure 9: PFE for 15 Year Cancelable Target.

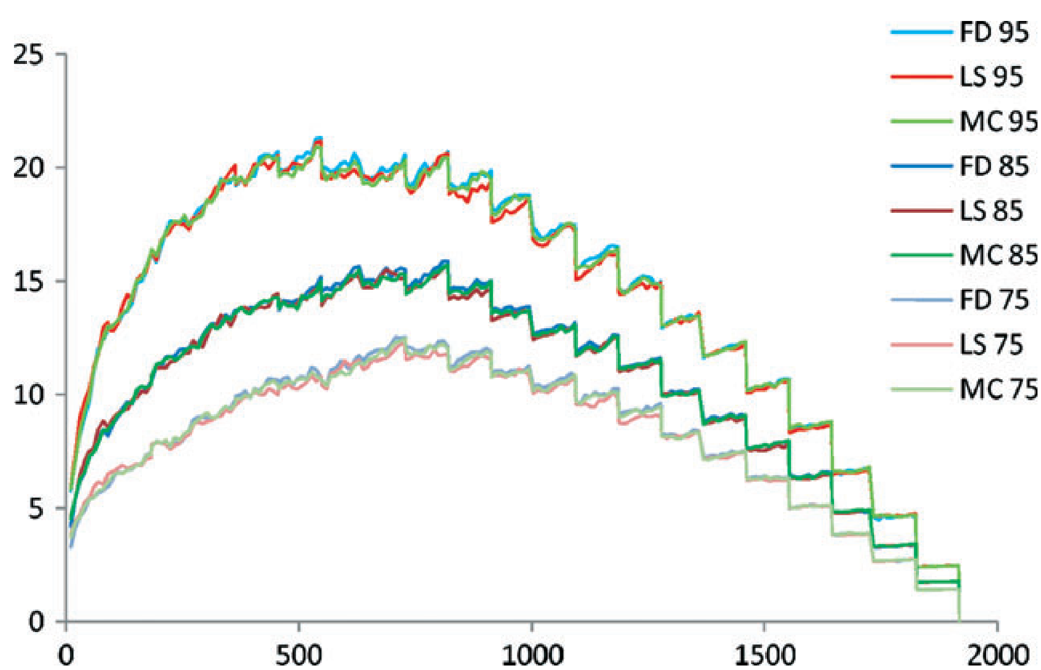

Figure 10: PFE for 5 Year 2-10 CMS Spread.

3. Drift approximations are used for the state dependent drift integral term in Eq. (8):

$$I^{DA}_{\mu_k}(t,T) \approx I_{\mu_k}(t,T) = \int_t^T \mu_k(s)ds \tag{10}$$

Various methods have been proposed to approximate the drift integral term in Eq. (10). For more details see Pietersz et al. (2004) or Joshi and Stacey (2006).

Using the separability condition, drift approximations and Eq. (8) for long stepping from time 0 to T, we obtain the following formula for Libors at time T:

$$F_k(T) \approx F_k(0)\exp\left[I^{DA}_{\mu_k}(0,T) - \frac{1}{2}\int_0^T |\gamma_k(t)|^2 dt + \nu_k \cdot X(T)\right] \tag{11}$$

where

$$X(T) = \int_0^T \sigma(t) * dW_{T_N} = \{X_1(T), X_2(T), \ldots, X_d(T)\} \tag{12}$$

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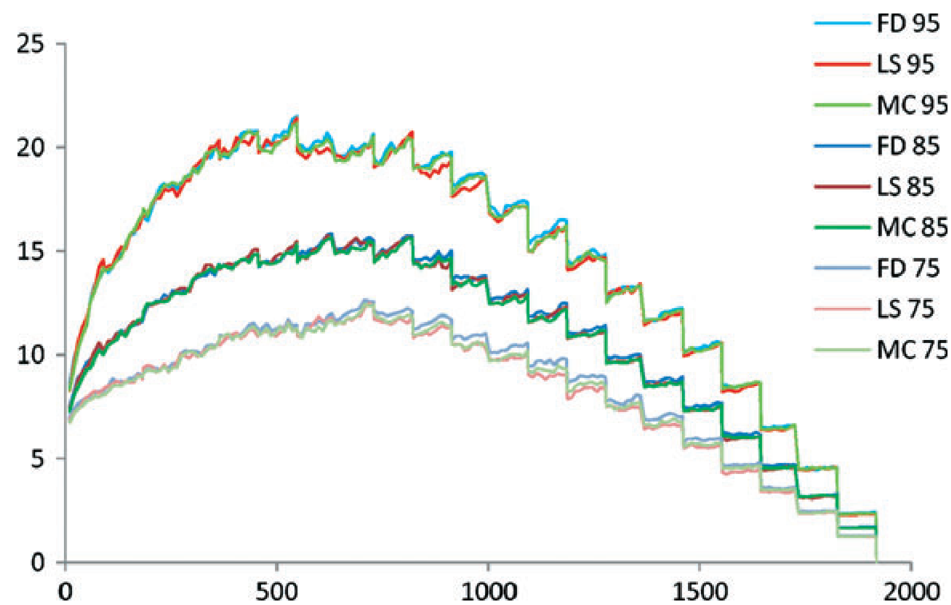

Figure 11: PFE for 5 Year Cancelable 2-10 CMS Spread.

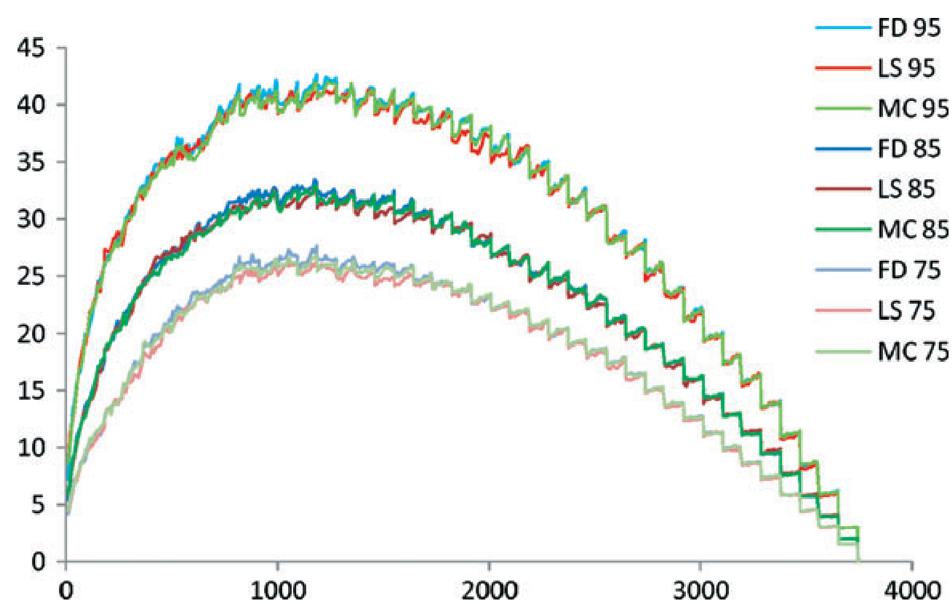

Figure 12: PFE for 10 Year 2-20 CMS Spread.

The drift approximated integral term $I^{DA}_{\mu_k}(t,T)$ can be a function of the state variable term $X(t)$ as well. Eq. (11) allows $X(t)$ to be used as a state variable. Instruments can be priced using the following PDE:

$$\frac{\partial V(X,t)}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sigma_i^2(t)\frac{\partial^2 V(X,t)}{\partial X_i^2} = 0 \tag{13}$$

At each point $X(t)$ in the state space, the forward rates can be determined using the drift approximated equation Eq. (11) and the value of the numeraire can be obtained from the forward rates.

In our implementation, we discretize the function $\sigma(t)$ so it is piecewise constant on the intervals $\mathcal{I} = \{(0,T_1),(T_1,T_2),\ldots,(T_{n-1},T_n)\}$ corresponding to the tenor dates of the underlying BGM model:

$$\sigma(t) = \{\sigma_1(t),\sigma_2(t),\ldots,\sigma_d(t)\} = \{\sigma_{\beta(t),1},\sigma_{\beta(t),2},\ldots,\sigma_{\beta(t),d}\} \tag{14}$$

where, $\beta(t)$ is the index:

$$\beta(t) = \{i : T_{i-1} \le t < T_i\} \tag{15}$$

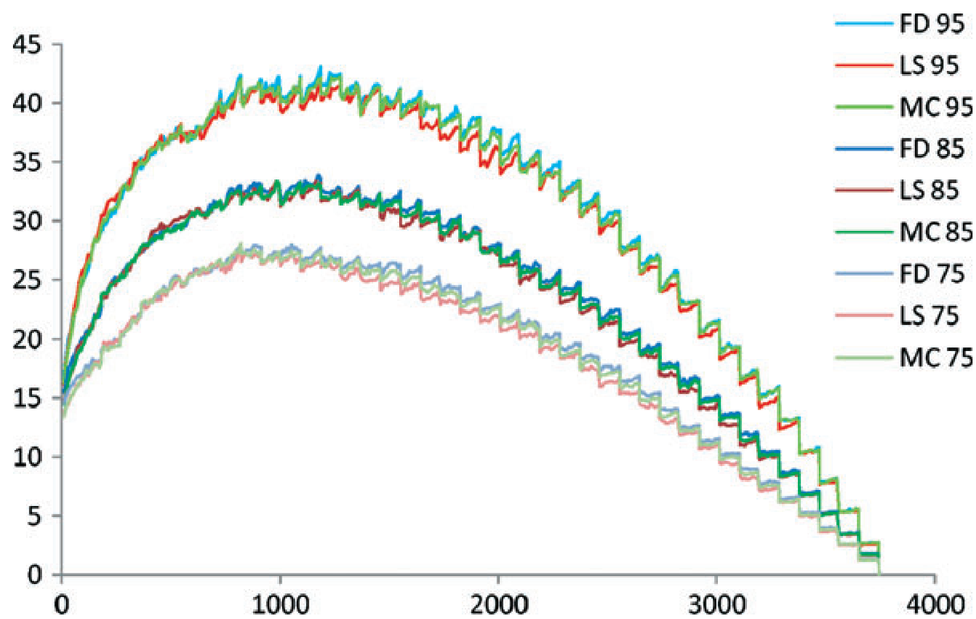

Figure 13: PFE for 10 Year Cancelable 2-20 CMS Spread.

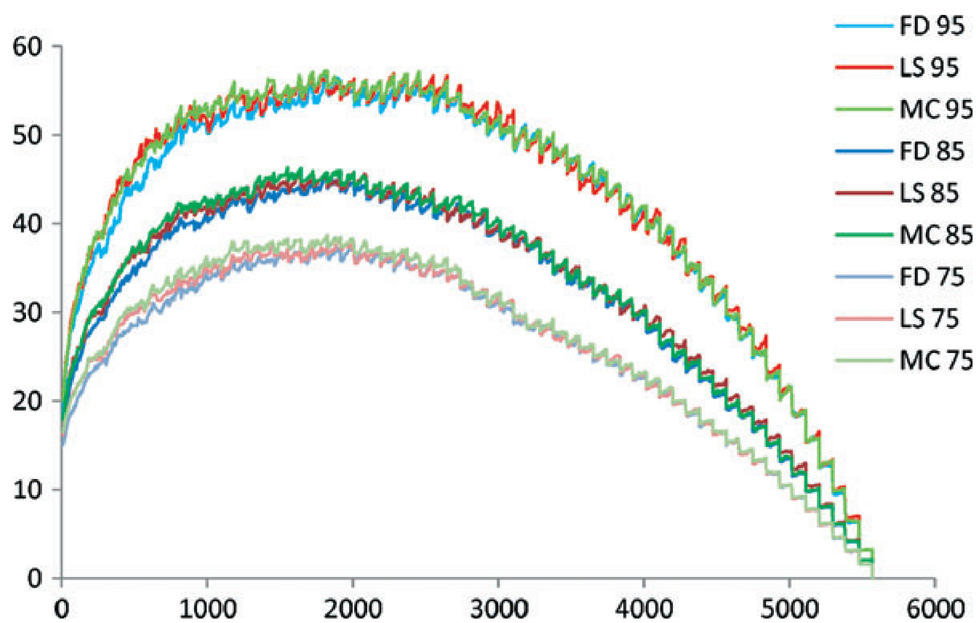

Figure 14: PFE for 15 Year 5-30 CMS Spread.

Thus, the volatility vector $\gamma_k(t)$ for the forward rate F_k at time t is given by:

$$\gamma_k(t) = \{\nu_{k,1}\sigma_{\beta(t),1}, \nu_{k,2}\sigma_{\beta(t),2}, \ldots, \nu_{k,d}\sigma_{\beta(t),d}\} \tag{16}$$

With this discretization, for a d factor separable BGM model with $M+1$ underlying forward rates going out in time until T_N, there are $d \times M$ entries for $\nu_{j,k}$ and $d \times N$ entries for $\sigma_{i,k}$ giving a total of $d \times (M+N)$ parameters in the model.

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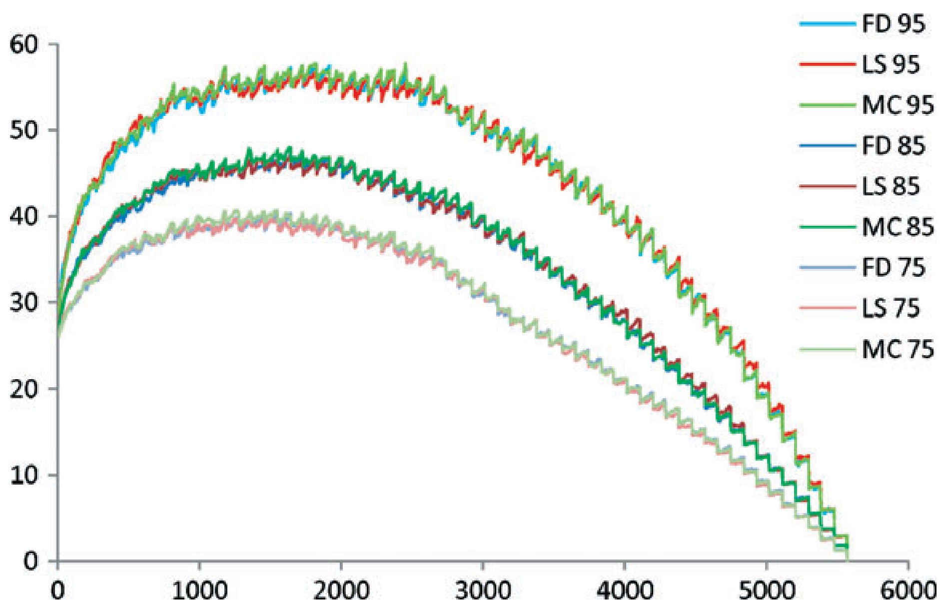

Figure 15: PFE for 15 Year Cancelable 5-30 CMS Spread.

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