Exam 3 Notes

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Theorems

Fubini's Theorem

Given $R := \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$

$$\iint\limits_R f(x, y)\,dy\,dx = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

General Integration

For some function f, and some region Ω:

$$\int_{\Omega} f \, d(something) = I$$

I is the average value of f on Ω times the size[1] of Ω

[1] The meaning of size differs based on the region you're integrating over

Integration Over a 2D Region

2-dimensional regions are be described by how well they can be ”sliced”

Type 1 regions are described by the following, given:

$$D_1 := \{(x, y) : a \leq x \leq b,\ f_1(x) \leq y \leq f_2(x)\}$$

$$\int_{D_1} h(x, y)\, dy\, dx$$

And Type 2 regions can be described similarly:

$$D_2 := \{(x, y) : g_1(y) \leq x \leq g_2(y),\ c \leq y \leq d\}$$

$$\int_{D_2} h(x, y)\, dx\, dy$$

Clearly, these integrals are setup with the best order of integration

Coordinate Systems

4 systems are used in this course:

1. Cartiesian: 2D & 3D
2. Polar: 2D
3. Cylindrical: 3D
4. Spherical: 3D

Conversions

$x \rightarrow rcos(\theta) \rightarrow rcos(\theta) \rightarrow \rho \sin(\theta) cos(\phi)$
$y \rightarrow rsin(\theta) \rightarrow rsin(\theta) \rightarrow \rho \sin(\theta) sin(\phi)$
$z \rightarrow z \rightarrow \rho \cos(\phi)$

$\rho \rightarrow \sqrt{x^2 + y^2 + z^2} \rightarrow \sqrt{r^2 + z^2}$
$\theta \rightarrow arctan(\frac{y}{x})$
$\phi \rightarrow arccos(\frac{z}{\rho}) \rightarrow arccos(\frac{z}{\rho})$

Jacobian constants

$dA \rightarrow r\, dr\, d\theta$
$dV \rightarrow r\, dr\, d\theta\, dz$
$dV \rightarrow \rho^2 sin(\phi)\, d\rho\, d\theta\, d\phi$

Vector Fields

Considering $\vec{F} := < x, y, z >$

Divergence of $\vec{F}$

- Denoted by $\nabla \cdot \vec{F} = F^1_x + F^2_y + F^2_z$
- Measures how much $\vec{F}$ is "spreading out" - $\nabla \cdot \vec{F} = 0$ when $\vec{F}$ is **incompressable**

Curl of $\vec{F}$

- Denoted by $\nabla \times \vec{F} = \vec{F} \times < \partial_x, \partial_y, \partial_z >$
- $\nabla \times \vec{F} = \vec{0}$ when $\vec{F}$ is **irrotational and conservative**
- Magnitude of $\nabla \times \vec{F}$ measures angular velocity
- Direction of $\nabla \times \vec{F}$ measures axis of rotation by right hand rule

Conservative Vector Fields

- $\vec{F}$ is conservative if there exists some function, f[2], where $\nabla f = \vec{F}$

- Full Theorems:

if $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as differentiable on a simply connected domain
and
$F^1_y = F^2_x$
then $\vec{F}$ is conservative.

if $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as differentiable on a simply connected domain
and
$\nabla \times \vec{F} = 0$
then $\vec{F}$ is conservative.

[2] Called the potential function

Line Integrals

Scalar line integrals

Let f be a function with a domain that includes a smooth curve C that is parametrized by $r(t) =< x(t),\, y(t),\, z(t) >$, $a \leq t \leq b$.

The scalar integral of f along C is

$$\int\limits_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t)) \cdot \|\vec{r}'(t)\|\,dt$$

Vector line integrals

The vector line integral of a vector field $\vec{F}$ along oriented smooth curve C is

$$\int\limits_C \vec{F} \cdot \vec{T}\,ds$$

Which can be simplified to

$$\vec{F} \cdot \vec{T}\,ds = \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \cdot \vec{r}\,'(t)\,dt = \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

Therefore the vector line integral of $\vec{F}$ over C is

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

Orientation of line integrals

Orientation and parametrization **do not matter for scalar line integrals**.

However, they do matter for vector line integrals. This isn't a huge problem, just remember

$$\int_a^b \vec{F}\,d\vec{r} = (-1)\int_b^a \vec{F}\,d\vec{r}$$

Fundamental theorem of line integrals

When calculating a vector line integral with a conservative vector field, the fundamental theorem of line integrals applies:

$$\int\limits_C \vec{F}\, d\vec{r} = f(b) - f(a)$$

or

$$\int\limits_C \nabla f\, d\vec{r} = f(b) - f(a)$$

Line integrals with respect to x, y, z

Assuming $\vec{r}(t) =< x(t),\, y(t),\, z(t) >$

$$\int\limits_C f\, dx = \int_a^b f(\vec{r}(t)) \cdot x'(t)\, dt$$

Integration of functions over surfaces

Parametric Surfaces

Like with line integrals, when doing a surface integral the surface has to be parametrized. Parametrized surfaces have this form:

$$r(u, v) =< x(u, v),\, y(u, v),\, z(u, v) >$$

With this form, the parameter domain is a set of 2 points on the uv plane.

Surface Integral of a Scalar Function

Assume S is a piecewise smooth surface with parametrization $r(u, v) =< x(u, v),\, y(u, v),\, z(u, v) >$ and parameter domain D. The surface integral of a scalar-valued function f over S is

$$\iint_S f(x, y, z)\, dS = \iint_D f(r(u, v)) \cdot \|r_u \times r_v\|\, dA$$

This somewhat resembles the definition for a line integral, with one key difference being the term $\|r'(t)\|$ in the line integral has become $\|r_u \times r_v\|$. $r'(t)$ is tangent to the curve whereas the vector $r_u \times r_v$ is perpendicular to the surface