Week 1 Discrete

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Language of Sets

Let S and T be sets, these are basic set operations.

$$|S| = \#S = \# \text{ of elements in S}$$
$$S \cup T = S \text{ union } T = \{\text{x: x} \in \text{S or x} \in \text{T}\}$$
$$S \cap T = S \text{ intersection } T = \{\text{x: x} \in \text{S and x} \in \text{T}\}$$
$$\emptyset = \text{empty set} = \{\ \}$$
$$S \backslash T = S \text{ minus } T = \{\text{x: x} \in \text{S and x} \notin \text{T}\}$$
$$S \sqcup T = S \text{ symmetric difference } T = \{\text{x: x} \in \text{S or x} \in \text{T but not both}\} \text{ (similar to xor)}$$
$$S \subset T = S \text{ is a subset of } T = \{\text{x: x} \in \text{S implies x} \in \text{T}\}$$
$$\bar{S} = S^{\complement} = \text{complement of } S = \{\text{x: x} \notin \text{S}\}$$

Partitioning a Set

Let S be a set. Partitioning S would create a collection $S_1, S_2, ..., S_m$ such that each subset of S is exactly one element of S. Each subset is called a *part*.

$$S = S_1 \cup S_2 \cup ... \cup S_m$$
$$S_i \cap S_j = \emptyset, \ \ (i \neq j)$$

Addition Principle

$$|S| = |S_1| + |S_2| + ... + |S_m|$$

From the principle that a whole is equal to the sum of its parts. This only applies if the whole set is correctly partitioned, or the parts don't overlap.

Example: A bag has 5 red marbles, 3 blue marbles, and 2 green marbles. How many ways can you pick a marble from the bag?

$$|S| = |S_1| + |S_2| + |S_3| = 5 + 3 + 2 = 10 \text{ ways to pick a marble from the bag}$$

Multiplication Principle

$$\text{Let } S = \{(a, b) : |a| = p, |b| = q\}$$
$$\text{Then } |S| = pq$$

This principle is a consequence of the addition principle. Let $a_1, a_2, ..., a_p$ be the partition of a. Next, partition S into parts $S_1, S_2, ..., S_p$ where S_i is the set of ordered pairs in S with a_i as the first element $(i = 1, 2, ..., p)$. For each element in a, there are q possible options to make a pair with that element in a.

Example: If there are 6 men, 4 women, 3 boys, and 2 girls, how many ways can you get one of each?

$$|S| = |S_1| \cdot |S_2| \cdot |S_3| \cdot |S_4| = 6 \cdot 4 \cdot 3 \cdot 2 = 144 \text{ ways to get one of each.}$$

Example: $N = 2^9 \cdot 5^7 \cdot 7^3$, what is the number of positive numbers that are factors of N?

Consider m such that $m|N$,

For any factor, $m = 2^a \cdot 5^b \cdot 7^c$,

Where $a = \{0 \ldots 9\}$, $b = \{0 \ldots 7\}$, $c = \{0 \ldots 3\}$

This means there are 10 options for a, 8 options for b, and 4 options for c.

Therefore, $10 \cdot 8 \cdot 4 = 320$ positive integer factors of N exist.

Subtraction Principle

Let A be a set and let U be a set larger than and containing A.
$\bar{A} = U/A = \{x \in U : x \notin A\}$ = the complement of A in U.
Therefore,
$$|A| = |U| - |\bar{A}|.$$

This principle only makes sense when working with objects in U and $\bar{A}$ is easier than working with objects in A.

Example: A computer password consists of 6 symbols from A = 0,..,9 and B = a,...,z. How many passwords have a repeated symbol?

Applying the subtraction principle, let U be the set of all possible passwords, and A be the set of all passwords with repeated symbols.

Then $|A| = |U| - |\bar{A}|$,
There are 36 total symbols, so $|U| = 36!$,
To find $|\bar{A}|$, or the number of passwords with no repeated symbols,
The first symbol has 36 possibilities, the second has 35, and so on. So $|\bar{A}| = \frac{36!}{30!}$,
Therefore, $|A| = 36! - \frac{36!}{30!}$

Permutations of Sets

Let r be a positive integer. An r-permutation of a set S of n elements, we get an ordered arrangement of r of the n elements. For example, if $S = \{$a, b, c$\}$, then

The three 1-permutations of S are:

a b c

The six 2-permutations of S are:

ab ac bc ba ca cb

The six 3-permutations of S are:

abc acb bac bca cab cba

The number of r-permutation of an n-element set is denoted by $P(n, r)$.

Theorem 1

For positive integers r and n where $r \leq n$,

$$P(n, r) = n * (n-1) * (n-2) * \ldots * (n-r+1) = \frac{n!}{(n-r)!}$$

This is used to find the number of ways to arrange r objects from a set of n objects without repetition.