Week 2 Discrete

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Combinations of Sets

Let S be a set of n elements, A *combination* of set S is an unordered selection of r elements in S. The result is a subset, A, of S. The number of r-subset or r-combination of S with n elements is denoted by $\binom{n}{r}$ and the formula is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

For example, if $S = \{a, b, c, d\}$, there are 3-subsets of S:

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$$

Combinations are different from permuations because when finding combinations, the results are sets, and the order of a elements in a set do not matter. This is why $\{a, b, c\}$ exists but $\{b, a, c\}$ doesn't.

Obvious rules for combinations:

$$\binom{n}{r} = 0 \text{ if } r > n,$$
$$\binom{0}{r} = 0 \text{ if } r > 0$$
And,
$$\binom{n}{0} = 1,\ \binom{n}{1} = n,\ \binom{n}{n} = 1$$

Properties of Combinations

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

Multi-Sets

Multisets are sets that hold multiple copies of the same element. For example, the multiset $\{a, a, b, c, c, c\}$, also written as $\{2a, b, 3c\}$. The 'coefficients' attached to each element is sometimes called a repetition number.

Permuations of Multi-Sets

An r-permutations of a multiset S is an ordered arrangment of r elements of S. There are 2 major theorems for finding the number of permuations of a multiset, one gives the number of permuations when the repetition numbers of *all* elements are infinite, and the other gives the number of permuations when the repetition numbers of all elements are finite.

Theorem Let S be a multiset with k types of objects, where each object has an infinite repetition number. The number of r-permuations of S is k^r.

Theorem Let S be a multiset with k different types of objects, each with finite repetition numbers, $n_1, n_2, \ldots, n_k$ respectively. Let the size of S be $n = n_1 + n_2 + \cdots + n_k$. The number of permuations of S equals

$$\frac{n!}{n_1! n_2! \ldots n_k!}$$

From this theorem, another interpretation occurs when partitioning a set of objects into parts of prescribed size and where the parts have labels to them. This is best shown with an example.

Example Consider a set of four objects $\{a, b, c, d\}$ that needs to be partitioned into two sets each of size 2. If each part is not labeled, then there are 3 different ways of partitioning.

$$\{a, b\}, \{c, d\}; \{a, c\}, \{b, d\}; \{a, d\}, \{b, c\};$$

Now suppose that each part is labeled, maybe where each part is a box and the label is the color of the box. In this case, The number of permutations is now 6. This can be generalized into the following theorem.

Theorem Let n be a positive integer and let $n_1, n_2, \ldots n_k$ be positive integers with $n = n_1 + n_2 + \cdots + n_k$. The number of ways to partition a set of n objects into k *labeled* boxes where box 1 contains n_1 objects, box 2 contains n_2 objects, ... box k contains n_k objects equals.

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

If the boxes are not labeled and $n_1 = n_2 = \cdots = n_k$, then the number of partitions equals

$$\frac{n!}{k!n_1!n_2!\dots n_k!}$$

Combinations of Multisets

If S is a multiset, then an r-combination of S is an unordered selection of r of the objects of S. Thus, an r-combination of S is itself a multiset, or for short, an r-submultiset. If S has n objects, then there is only one n-combination of S, namely S itself. Similarly to permuations of multisets, we first count the number of r-combinations of a multiset whose repetition numbers are all infinte.

Theorem Let S be a multiset with objects of k types, each with an infinite repetition number. Then the number of r-combinations of S equals

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1} = \frac{(r+k-1)!}{r!(k-1)!}$$