A GEOMETRIC MAP WITH APPLICATION TO BILLIARDS

SAMUEL EVERETT

Abstract. We study the dynamics of a geometric map where iteration is defined as a cycling composition of a finite set of piecewise maps, which act on a space composed of three or more pairwise nonparallel, nonconcurrent lines in $\mathbb{R}^2$. We first provide conditions under which this class of maps generates periodic orbits. We then use these maps to establish a result related to the area of billiard dynamics, in which we prove the existence of closed nonsmooth curves over $\mathbb{R}^2$ which satisfy particular structural constraints with respect to the intersecting lines composing the space.

1. Introduction

The theory of mathematical billiards in polygons concerns the uniform motion of a point mass (billiard) in a polygonal plane domain, with elastic reflections off the boundary according to the mirror law of reflection: the angle of incidence equals the angle of reflection. Much work on polygonal billiards has been devoted to the long-standing question of whether every polygon contains a periodic billiard orbit (see *Problem 10* in [12], and [11] for a survey); in fact, the question remains unsolved for particular obtuse triangles. Intense study on this problem has led to progress (see, e.g., [9, 16, 18, 19]), and many deep theorems have been obtained from such work. Nonetheless, the problem remains open, and many difficulties are much alive, largely due to a lack of machinery that can be applied to more than just specific cases of the problem.

This paper aims to elaborate machinery suitable for use in studying problems related to determining periodic billiard orbits in polygons. Indeed, we use the introduced system to prove a theorem closely related to this problem. We briefly introduce notation and definitions before stating this result. Let X_m denote the space of $m \geq 3$ pairwise nonparallel, nonconcurrent lines in $\mathbb{R}^2$, where each line is associated with a unique label L_i, for $i \in \{1, 2, ..., m\}$. Let $p_1, p_2, ..., p_n$, with $p_i \in X_m$, be a sequence of points such that no two consecutive points, including p_n and p_1, lie on the same line in X_m, and every line in X_m contains at least one of the points. Join consecutive pairs of such points, including the first and last, with line segments to construct a *closed curve* Γ over X_m.

We define the *positive* and *negative traversal* of a closed curve Γ to be the traversal of Γ in arbitrarily determined, albeit opposite directions; i.e. in the "clockwise" and "counterclockwise" directions when the closed curve is simple. Associated with

---

E-mail address: `samuel.everett@colorado.edu`.

2010 *Mathematics Subject Classification.* 37B99, 37E99, 51N20, 37C27.

Key words and phrases. piecewise function, contraction mapping, periodic orbits, billiards.

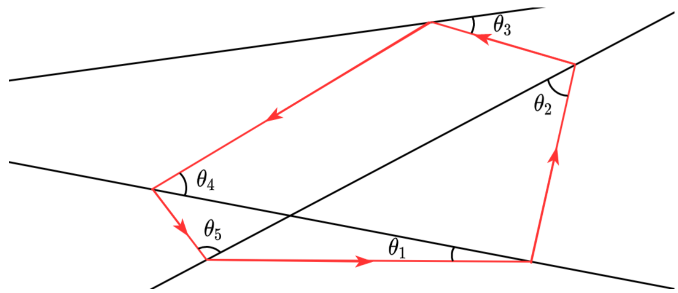

FIGURE 1. The values $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ compose an incidence sequence of the red closed curve, when traversed in a particular direction.

the positive and negative traversals of a closed curve, we obtain positive and negative *incidence sequences*, labeled $\theta_1^+, \theta_2^+, ..., \theta_n^+$ and $\theta_1^-, \theta_2^-, ..., \theta_n^-$, which are constructed by taking the acute or right angles the line segments of the closed curve make with the lines in X_m they join, with respect to the particular traversal direction. Refer to Figure 1 for visual demonstration, where the closed curve generates the incidence sequence $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ upon traversal in a particular direction. One of our primary results is the following.

Theorem 1.1. *For any space X_m, let $\theta_1, \theta_2, ..., \theta_n$, $n \geq m$, $\theta_i \in (0, \pi/2]$, be any sequence of incidence angles, and let $L_{a_1}, L_{a_2}, ..., L_{a_n}$ be a sequence of line labels such that no two consecutive labels are the same, including L_{a_n} and L_{a_1}, and each of the m possible labels occur at least once in the sequence. Then there exists a closed curve Γ over X_m such that traversal of Γ in a fixed direction generates incidence sequence $\theta_1, \theta_2, ..., \theta_n$, and visits the labeled lines in X_m with order $L_{a_1}, L_{a_2}, ..., L_{a_n}$.*

In the case where the closed curves are strictly contained within polygons, the angles of incidence implicitly define angles of reflection, and the connection with billiards in polygons is made apparent. In particular, we note that Theorem 1.1 has no condition on the pairwise line intersection angles composing the space X_m, unlike many results on polygonal billiards which assume such angles are rational multiples of pi.

Theorem 1.1 is a consequence of a dynamical system developed in this paper, which, like iterated function systems [13], is defined using a finite set of contraction mappings. We generalize the work in [8] to introduce a map we call an *n-rule map*, where iteration is defined as a cycling composition of $n \geq 1$ distinct piecewise mappings, or "rules", acting over the space X_m. In particular, we prove two theorems demonstrating how two distinct classes of n-rule maps satisfying particular geometric conditions will converge to a periodic orbit when iterated from almost any point in X_m. These results provide the structure used to prove Theorem 1.1.

n-Rule maps are defined geometrically, and provide dynamics which can subsume aspects of inner and tiling billiards [7], and also provide a strong connection to billiards with modified reflection laws [2, 15]. Furthermore, despite the emphasis of applying n-rule maps to billiards, they are interesting in their own right, providing many geometric and dynamical features of interest, and may be applicable to other areas. For example, piecewise contracting maps appear in biological networks

[4, 6], queuing systems [14], and related areas (see the introductions of [5, 17]). Additionally, we bring attention to the well studied area of piecewise isometries (see [1, 10] for review), which have similarities to the dynamical system studied in this paper [3], and have application in a variety of physical problems.

This paper is organized as follows.

In Section 2 we define n-rule maps, establishing notions for Section 3 where we begin by showing that n-rule maps satisfying an "average contraction" converge to periodic orbits. The n-rule maps defined in this section map between lines in the space X_m on a basis of the distance between points in the orbit and lines in the space. On the other hand, in Section 4 we use the results from Sections 2 and 3 to derive a different class of n-rule maps which map between the lines in X_m on a basis of unique symbolic labels associated with each line in the space. This "symbolic" class of n-rule maps provides less robust although more controllable dynamics, and upon showing how these maps converge to periodic orbits, we use them to prove Theorem 1.1.

2. Preliminaries

If L_i, L_j are two distinct lines in X_m where $x \in L_i$, we define

$$d(x, L_j) \coloneqq \inf\{d(x, y) | y \in L_j\}$$

to be the distance between point x and line L_j, where $d(x, y)$ is the Euclidean metric and $d(x, L_i) = 0$.

For every point $x \in X_m$, we construct the set

$$D(x) = \{d(x, L_i) | 1 \leq i \leq m\}$$

and define the partially ordered set $\mathcal{D}(x) = (D(x), \leq)$. We let l denote an index on $\mathcal{D}(x)$, so that $l = i$, $1 \leq i \leq m$ corresponds with the ith farthest line from point x. Note that if there exist m' lines in X_m, $m' < m$, that are all the same distance from point x, then there are m' values of l that do not correspond with a unique distance value in $\mathcal{D}(x)$, and thus do not correspond with a unique line in X_m.

For each $x \in L_i$ with $L_i, L_j \subset X_m$, $i \neq j$, we may define two lines, $\mathscr{L}_0, \mathscr{L}_1$ such that $\{x\} = \mathscr{L}_0 \cap \mathscr{L}_1$ and $\mathscr{L}_0, \mathscr{L}_1$ intersect with line L_j at angle $\theta \in (0, \pi/2)$, with intersection points z_0 and z_1, respectively. We call z_0, z_1 the *orientation 0 and 1, angle θ-projections of x onto L_j*. If $\theta = \pi/2$, then we call the line intersection point z the *perpendicular projection of x onto L_j*. In the case when $x \in L_i \cap L_j$, the projection of x is simply x itself.

Definition 1 (Rule)**.** *We call the mapping $r : X_m \to X_m$ a* rule*, denoted by a triple of form $r \coloneqq (\theta, o, l)$, where $\theta \in (0, \pi/2]$, $o \in \{0, 1\}$, $l \in \{2, 3, ..., m\}$. Then $r(x)$ is an orientation o, angle θ-projection of x onto the lth farthest line from x. If the index l corresponds with a distance value in $\mathcal{D}(x)$ that is not unique in $\mathcal{D}(x)$, put $r(x) = x$.*

In the case when the index l corresponds to a distance value in $\mathcal{D}(x)$ that is not unique so that $r(x) = x$, then we say x is an *invariant* point under rule r. When the rule projection angle θ is equal to $\pi/2$, we simply write the rule as $(\frac{\pi}{2}, l)$ as there is only one possible orientation. Figure 2 provides visual demonstration of the composition of two rules,

$$r_1(x) \coloneqq (\theta_1, 1, 2) \text{ and } r_2(x) \coloneqq (\theta_2, 0, 3)$$

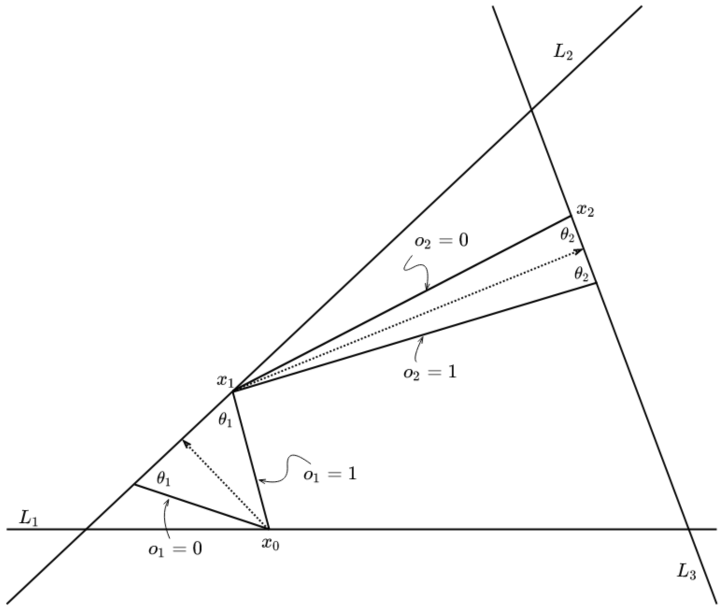

FIGURE 2. An illustration of the mapping action of two rules composed over a initial point x_0 in a space X_3.

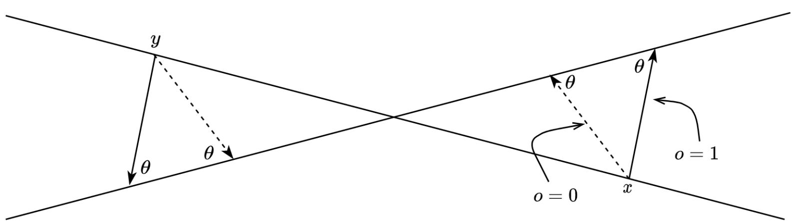

FIGURE 3. How orientation is preserved under point translation (from x to y) under a rule with projection angle θ. Note how the projection lines corresponding with a fixed orientation are antiparallel across line intersection points.

over a point $x_0 \in X_3$, so that $r_1(x_0) = x_1$ and $r_2(r_1(x_0)) = x_2$. Intuitively, rule r_1 maps to the *closest* line from a point x, and rule r_2 maps to the *farthest* line from a point x when applied to a space X_3.

We require rule orientation to be defined in a predictable and consistent way, so that it is never ambiguous which projection points correspond to which orientation value. For this paper, we choose a natural and mathematically convenient convention where the orientation 0 and 1 projection points under a rule $r(x)$ are the "left" and "right" points, "from the perspective of x". Figures 2 and 3 demonstrate this convention under point translation.

Definition 2 (Rule sequence). *A* rule sequence *is an iterative composition of n rules, denoted by*

$$r_1 r_2 ... r_n \coloneqq (\theta_1, o_1, l_1)(\theta_2, o_2, l_2)...(\theta_n, o_n, l_n)$$

so that for $x_0 \in X_m$, application of the rule sequence to point x_0 is defined as

$$r_n(...(r_2(r_1(x_0))))$$

and generates the sequence of points $x_0, x_1, x_2, ..., x_n$ contained in X_m.

Definition 3 (n-Rule Map)**.** *Let $K_n : X_m \to X_m$ denote an n-*rule map*. We define K_n as a length n rule sequence*

$$K_n \coloneqq r_1 r_2 ... r_n$$

so that iteration of an n-rule map over a point $x \in X_m$ is defined as a cyclic composition of the defining rule sequence. That is,

$$K_n(K_n^{n+1}(x)) = K_n^{n+2}(x) = r_2(r_1(r_n(...r_2(r_1(x)))))$$

For any n-rule map K_n, it is required that at least one of the rules in the associated rule sequence has l index value greater than $l = 2$; such a restriction ensures the dynamics of an n-rule map in X_m are nontrivial. If all rules of the rule sequence have l index value of $l = 2$, then each iteration maps to the "closest" line, and iteration approaches a line intersection point of X_m, failing to exhibit behavior of interest.

Unless otherwise stated, the pair (X_m, K_n) denotes a dynamical system. Referring back to Figure 2, we can define the 2-rule map

$$K_2 \coloneqq (\theta_1, 1, 2)(\theta_2, 0, 3)$$

where Figure 2 shows iterations $K_2(x_0) = x_1$ and $K_2^2(x_0) = x_2$ of K_2 over x_0.

For point $x \in X_m$, we let $\mathcal{O}(x)$ denote the *orbit of* x under n-rule map K_n, so that

$$\mathcal{O}(x) = \{x, K_n(x), K_n^2(x), ...\}$$

For a point $x \in X_m$, let $\omega(x)$ label the *limit set of point x under n-rule map K_n*, formally defined as

$$\omega(x) \coloneqq \bigcap_{j=0}^{\infty} \overline{\mathcal{O}(K_n^j(x))}$$

If a point $x^* \in X_m$ is invariant for $n' < n$ of the rules in the n-rule sequence defining K_n, then we say x^* is *sometimes invariant* under K_n. If x^* is invariant under all rules defining K_n, we say x^* is *strictly invariant* under K_n. As such, any 1-rule map has only strictly invariant points.

Remark 1. *For any (X_m, K_n) dynamical system, the set of strictly invariant and sometimes invariant points is finite.*

We call an n-rule map *collapsible* if there exists a length n' rule sequence with $1 \leq n' < n$, such that for all $x \in X_m$, the orbit of x under the n'-rule map is equal to the orbit of x under the n-rule map. An example of a collapsible map is that given by the 4-rule sequence $r_1 r_2 r_1 r_2$, which may be collapsed into the 2-rule sequence $r_1 r_2$, so we say $K_4 \coloneqq r_1 r_2 r_1 r_2$ is a collapsible 4-rule map. If an n-rule sequence cannot be collapsed, we say the corresponding n-rule map is *non-collapsible.* For the purpose of this paper we assume all n-rule maps are non-collapsible.

2.1. **Convergence and contraction of n-rule maps.** The space X_m is composed of m pairwise nonparallel, nonconcurrent lines, so any given space X_m has $\binom{m}{2}$ pairwise line intersection points. Then, for each pairwise intersection point, let η_i label the ith pairwise line intersection angle, where $0 < \eta_i \leq \pi/2$. Let

$$\delta = \min\left\{\eta_i | 1 \leq i \leq \binom{m}{2}\right\}$$

label the least pairwise intersection angle between any two lines in X_m. Note δ must be acute by definition of X_m.

Definition 4 (Average Contraction Condition)**.** *For n-rule map K_n, let $\bar{\theta}$ label the average of all projection angles in the n-rule sequence defining K_n. Then if*

$$\frac{\pi - \delta}{2} < \bar{\theta} \leq \frac{\pi}{2} \tag{1}$$

for least angle δ in X_m, we say K_n satisfies the average contraction condition *in X_m.*

We motivate the introduction of the average contraction condition through the following observations. Let L_1, L_2 denote lines in $\mathbb{R}^2$, intersecting at point z with acute angle δ. Without loss of generality, let $x, y \in L_1$, and take a rule r, such that $r(x), r(y) \in L_2$, and $x, y, r(x), r(y)$ are on the same side of intersection point z. Further, let the orientation value of r be the choice that maps farthest from z. For example, in Figure 3 rule orientation value $o = 1$ maps farther away from the line intersection point when mapping from the particular line.

Assume $d(x, y) = \epsilon > 0$, and $a = d(z, x)$, $a + \epsilon = d(z, y)$. Let θ denote the projection angle of rule r, and let $\gamma = \pi - \delta - \theta$, so that $\gamma = \angle zxr(x) = \angle zyr(y)$. Then if

$$0 < \theta < \frac{\pi - \delta}{2}$$

it follows by use of the law of sines that

$$d(r(x), r(y)) = \left|\frac{a\sin(\gamma)}{\sin(\theta)} - \frac{(a+\epsilon)\sin(\gamma)}{\sin(\theta)}\right| = \frac{\epsilon\sin(\gamma)}{\sin(\theta)}$$

but $\theta < (\pi - \delta)/2$ and δ is acute, so under our choice of rule orientation value

$$\frac{\sin(\gamma)}{\sin(\theta)} > 1$$

and then $d(r(x), r(y)) > d(x, y)$: iteration of r over L_1 and L_2 in such a way is then expansive. By similar argument, we see that if $\theta = (\pi - \delta)/2$, then the rule defines an isometry and $d(r(x), r(y)) = d(x, y)$, and when

$$\frac{\pi - \delta}{2} < \theta \leq \frac{\pi}{2}$$

then $d(r(x), r(y)) \leq cd(x, y)$, $0 \leq c < 1$. Further, δ is acute, so in the case when $r(x)$ maps opposite the angle $\pi - \delta$, we have $\pi - \delta > \delta$, so if r is contractive when mapping opposite δ, it must also be contractive when mapping opposite $\pi - \delta$.

From the above example, we see that for any rule r, with colinear x, y and colinear $r(x), r(y)$ all on the same side of the line intersection point, we have

$$d(r(x), r(y)) = cd(x, y),\ c \geq 0$$

where c can be computed directly via the law of sines, as a function of the rule projection angle and the opposite line intersection angle. We call such values c, *rule separation constants.*

Lemma 2.1. *Let lines $L_1, L_2 \subset X_m$ intersect at a point z with acute angle δ, and let $x, y \in L_1$ lie on the same side of z. Let r_1, r_2 label rules with distinct orientation values, which are chosen so that the rules map farthest from z, and let the points $r_i(x), r_i(y) \in L_2$ and $r_i(r_j(x)), r_i(r_j(y)) \in L_1$, $i, j = 1, 2$, $i \neq j$ all lie on the same side of z. Then for corresponding rule separation constants c_1 and c_2, $0 \leq c_1c_2 < 1$ if and only if*

$$\frac{\pi - \delta}{2} < \frac{\theta_1 + \theta_2}{2} \leq \frac{\pi}{2} \tag{2}$$

for rule projection angles θ_1, θ_2 corresponding with rules r_1, r_2.

Remark 2. *Note that $c_1c_2 < 1$ implies composition of the two rules defines a contraction:*

$$d(r_i(r_j(x)), r_i(r_j(y))) \leq c_1c_2 d(x, y),\ \ c_1c_2 < 1$$

For distinct i, j. Further, the orientation values of the rules are chosen so that the rules map farthest from the line intersection point in each case, and thus the corresponding separation constants are maximized.

Proof of Lemma 2.1. Let $\gamma_1 = \pi - \theta_1 - \delta$ and $\gamma_2 = \pi - \theta_2 - \delta$. We assume that

$$\frac{\pi - \delta}{2} < \theta_1 \leq \frac{\pi}{2}$$

so by Equation 2, we require that

$$\pi - \theta_1 - \delta < \theta_2 \leq \pi - \theta_1 \tag{3}$$

By Equation 3 we see that $\sin(\theta_2) > \sin(\pi - \theta_1 - \delta)$, and that

$$\begin{aligned} \sin(\gamma_2) &= \sin(\pi - \theta_2 - \delta) \\ &= \sin(\theta_2 + \delta) \\ &\leq \sin(\pi - \theta_1 + \delta) \end{aligned}$$

Then, through substitution we obtain

$$c_1c_2 = \frac{\sin(\gamma_1)}{\sin(\theta_1)} \frac{\sin(\gamma_2)}{\sin(\theta_2)} < \frac{\sin(\pi - \theta_1 - \delta)}{\sin(\theta_1)} \frac{\sin(\pi - \theta_1 + \delta)}{\sin(\pi - \theta_1 - \delta)}$$

but $\sin(\pi - \theta_1 + \delta) = \sin(\theta_1 - \delta) < \sin(\theta_1)$, so

$$\frac{\sin(\pi - \theta_1 - \delta)}{\sin(\theta_1)} \frac{\sin(\pi - \theta_1 + \delta)}{\sin(\pi - \theta_1 - \delta)} = \frac{\sin(\pi - \theta_1 + \delta)}{\sin(\theta_1)} < 1$$

Going the other direction, let $\gamma_1 = \pi - \theta_1 - \delta$ and $\gamma_2 = \pi - \theta_2 - \delta$. Then from

$$c_1c_2 = \frac{\sin(\gamma_1)\sin(\gamma_2)}{\sin(\theta_1)\sin(\theta_2)} < 1$$

with substitution we obtain

$$\sin(\pi - \theta_1 - \delta)\sin(\pi - \theta_2 - \delta) < \sin(\theta_1)\sin(\theta_2)$$

By the product identity for sine, we have

$$\frac{\cos(-\theta_1 + \theta_2) - \cos(2\pi - \theta_1 - \theta_2 - 2\delta)}{2} < \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2}$$

but $\cos(-1(\theta_1 - \theta_2)) = \cos(\theta_1 - \theta_2)$ so upon simplifying we have

$$\cos(2\pi - \theta_1 - \theta_2 - 2\delta) > \cos(\theta_1 + \theta_2)$$

and by removing cosine we obtain

$$2\pi - \theta_1 - \theta_2 - 2\delta < \theta_1 + \theta_2$$

Note that although cosine is not monotone, by the restrictions on the angles we can remove cosine in such a way. This gives us

$$\pi - \delta < \theta_1 + \theta_2 \implies \frac{\pi - \delta}{2} < \frac{\theta_1 + \theta_2}{2}$$

The case for the upper bound $(\theta_1 + \theta_2)/2 \le \pi/2$ is clear. □

In the above lemma, we took the rule orientation values to be chosen in a way that ensures the rules map farthest from the line intersection point. By instead using rule orientation values that force the rules to map closer to the line intersection points, then so long as the two projection angles θ_1, θ_2 satisfy Equation 2, *both rules* must provide contraction (i.e. $c_1 < 1$ and $c_2 < 1$). That is, if $\gamma_2 = \pi - \theta_2 - \delta$, and

$$\frac{\sin(\gamma_2)}{\sin(\theta_2)} > 1$$

under a rule orientation value mapping farther from a line intersection point, then under opposite rule orientation value, we have $\gamma_2' = \pi - (\pi - \theta_2) - \delta$, so

$$\frac{\sin(\gamma_2')}{\sin(\pi - \theta_2)} = \frac{\sin(\theta_2 - \delta)}{\sin(\theta_2)} < 1$$

As an immediate consequence of the above remark and Lemma 2.1, we obtain the following corollary.

Corollary 2.1.1. *Let $L_1, L_2 \subset X_m$ intersect at point z with acute angle δ. Further, let K_n be an n-rule map so that iterates of K_n map between L_1 and L_2, opposite angle δ, and for initial points $x, y \in L_1$, let the first n points of $\mathcal{O}(x), \mathcal{O}(y)$ remain on the same side of z. Then if K_n satisfies the average contraction condition for least angle δ, then*

$$d(K_n^n(x), K_n^n(y)) \le Cd(x,y),\ \ C \in [0,1)$$

We remark that here $C = c_1 c_2 \cdots c_n$, the product of the n separation constants for the rule sequence.

Lemma 2.2. *For all lines $L_i \subset X_m$ and $x, y \in L_i$, if each closed interval*

$$[K_n^i(x), K_n^i(y)] \subset X_m,\ \ 0 \le i \le n$$

contains no line intersection points or invariant points, then if K_n satisfies the average contraction condition in X_m, then

$$d(K_n^n(x), K_n^n(y)) \le Cd(x,y),\ \ C \in [0,1)$$

Proof. Let δ label the least pairwise intersection angle in X_m. Then by Corollary 2.1.1, if K_n satisfies the average contraction condition over X_m, and iteration of K_n is strictly opposite angle δ, then $d(K_n^n(x), K_n^n(y)) \le Cd(x,y)$ for Lipschitz constant C. But δ is the least angle in X_m, so if iteration of K_n contracts opposite angle δ

on average, then it must also contract opposite every other angle in X_m on average: if η_i is a distinct line intersection angle, then $\eta_i \geq \delta$, and

$$\frac{\pi - \eta_i}{2} \leq \frac{\pi - \delta}{2}$$

As such, assuming the conditions of the statement, it follows that

$$d(K_n^n(x)K_n^n(y)) \leq Cd(x, y)$$

for $C \in [0, 1)$. □

We note the average contraction condition ensures contraction regardless of rule orientation. The average contraction condition provides sufficient but not necessary conditions for an n-rule map to define a contraction on average.

Lemma 2.3. *If K_n satisfies the average contraction condition over X_m, then there exists bounded regions $R, R' \subset X_m$ such that for all $x \in R$, $\overline{\mathcal{O}(x)} \subset R'$.*

Proof. By definition of n-rule maps and the average contraction condition, iteration of an n-rule map K_n in X_m must, on average, map closer to line intersection points. The lines composing X_m are pairwise nonparallel, so all lines must intersect, and there must exist a bounded region R containing all such line intersection points. As such, if iteration of K_n maps closer to line intersection points on average, then iteration of the map must remain in a bounded region R'. □

Immediate from Lemma 2.3 we obtain the following corollary.

Corollary 2.3.1. *If K_n satisfies the average contraction condition over X_m, then the sequence of points obtained from taking successive preimages of K_n over non-invariant points $x \in X_m$ diverges in X_m.*

3. Asymptotic behavior of n-rule maps

In this section, we study the asymptotic properties of n-rule maps satisfying the average contraction condition over X_m. For n-rule map K_n and point $x \in X_m$, we call a *cycle* of K_n over x the application of K_n to x, n times; the cycle of $x_0 \in X_m$ under K_n is the sequence of points $x_0, x_1, ..., x_n$, where $K_n^n(x_0) = x_n$. We let $\mathcal{K}_n \coloneqq K_n^n$ label the *cycle map* of K_n, so that for $x_0 \in X_m$, $\mathcal{K}_n(x_0) = x_n$, and $\mathcal{K}_n^t(x_0) = K_n^{tn}(x_0) = x_{tn}$.

If rule r_i in the rule sequence of n-rule map K_n has invariant point q in X_m, then for $h \in X_m$, if $K_n^i(h) = q$, then h is a *pre-invariant* point of rule r_i. Associated with the (X_m, K_n) dynamical system, we let Ω denote the set of invariant points of all types, as well as preimages of the cycle map $\mathcal{K}_n$ from all pre-invariant points. Further, if p labels a strictly invariant point under K_n, then all points $a \in X_m$ such that $\mathcal{K}_n^j(a) = p$ for some $j \in \mathbb{Z}^+$, are also contained in Ω.

For example, if q is a sometimes invariant point of K_n in X_m, and h labels the corresponding pre-invariant point of q, then

$$\{q, h, \mathcal{K}_n^{-1}(h), \mathcal{K}_n^{-2}(h), ...\} \subset \Omega$$

Where $\mathcal{K}_n^{-1}(h)$ is the point obtained by inverting the map K_n over point h, n times.

Put $X'_m = X_m \setminus \Omega$. We call the dynamical system (X_m, K_n) *degenerate* when iteration of K_n eventually maps to an invariant point of any type; it follows that for the dynamical system $(X'_m, \mathcal{K}_n)$ to be well defined, (X_m, K_n) must be a non-degenerate dynamical system. Such degenerate systems occur at bifurcation points, and the remainder of this section focuses on the study of non-degenerate systems. The main result of this section is as follows.

Theorem 3.1. *Let (X_m, K_n) be a non-degenerate system, with n-rule map K_n satisfying the average contraction condition over X_m. Then for all $x \in X'_m$, the orbit $\mathcal{O}(x)$ converges to a periodic orbit of period kn, $k \in \mathbb{Z}^+$.*

We need some preparatory lemmas to prove Theorem 3.1. If K_n satisfies the average contraction conditions in X_m, then let U_m denote the set of open intervals $I_a \subset X_m$ such that the boundary values of each I_a are given by elements in Ω; no element in Ω is contained within an open interval I_a. Note that as consequence of Corollary 2.3.1, if K_n satisfies the average contraction condition then preimages diverge in X_m, so Ω is guaranteed not to be dense in X_m, and thus each $I_a \in U_m$ must exist. If however K_n fails to satisfy the condition then such a guarantee may not be made. Let

$$\hat{\mathcal{O}}(x) = \{x, \mathcal{K}_n(x), \mathcal{K}_n^2(x), ...\}$$

denote the orbit of x under cycle map $\mathcal{K}_n$.

Lemma 3.2. *For non-degenerate (X_m, K_n) and K_n satisfying the average contraction condition over X_m, if $I_a \in U_m$, then there exists an $I_b \in U_m$ such that $\mathcal{K}_n[I_a] \subset I_b$, where I_a, I_b need not be distinct.*

Proof. We proceed by contradiction and assume $\mathcal{K}_n[I_a] \subset I_b \cup I_c$. By definition, the boundary values of each $I_a \in U_m$ are invariant points or preimages of invariant points under $\mathcal{K}_n$. It follows that if $\mathcal{K}_n[I_a] \subset I_b \cup I_c$, then $I_a = I_d \cup I_e$, as I_a would contain preimages of such boundary values, a contradiction. □

Lemma 3.3. *If (X_m, K_n) is non-degenerate with K_n satisfying the average contraction condition and $x \in X'_m$, then $\hat{\mathcal{O}}(x) \subset \bigcup_{i=1}^{s} I_i$ for finite s and $I_i \in U_m$.*

Proof. We first note that taking $x \in X'_m$ is, by definition, equivalent to taking $x \in I_a$, for I_a in U_m. For any (X_m, K_n) dynamical system, there may only be a finite number of invariant points of any type under K_n, and by Corollary 2.3.1, preimages of n-rule maps satisfying the average contraction condition diverge from points in X_m. It then follows by definition of the set Ω and corresponding construction of intervals in U_m, that for any bounded region $R \subset X_m$, there may only be a finite number of such intervals I_a in R. Further, by Lemma 2.3, orbits of n-rule maps satisfying the average contraction condition must remain in a bounded region. Finally, by Lemma 3.2, for every $I_a \in U_m$, $\mathcal{K}_n[I_a] \subset I_b$, and it thus follows that the orbit of x under $\mathcal{K}_n$ is contained in a finite number of intervals. □

We call an interval $I_c \in U_m$ *confining* if there is a $t \in \mathbb{Z}^+$, $t = t(I_c, K_n)$, such that $\mathcal{K}_n^t[I_c] \subset I_c$.

Lemma 3.4. *If (X_m, K_n) is a non-degenerate dynamical system with K_n satisfying the average contraction conditions over X_m, then there exists a confining interval I_c in X_m, and iteration of $\mathcal{K}_n$ over any $x \in X'_m$ maps into a confining interval in a finite number of iterations.*

Proof. By Lemma 3.3, the orbit of any $x \in X'_m$ under $\mathcal{K}_n$ is restricted to a finite number of intervals. Thus, by the pigeon hole principle, iteration of $\mathcal{K}_n$ is forced to map to an interval it has already visited in a finite number of iterations. Then there exists at least one confining interval, and iteration of $\mathcal{K}_n$ over $x \in X'_m$ maps into such an interval in a finite number of iterations. □

Definition 5. *Let $I_c \in U_m$ be a confining interval in X_m, and let $\hat{\mathcal{K}}_n : I_c \to I_c$ be the* induced map *of $\mathcal{K}_n$ over the interval of continuity I_c, defined so that if $x \in I_c$ and $\mathcal{K}_n^k(x) \in I_c$ for minimal k, we put $\hat{\mathcal{K}}_n(x) = \mathcal{K}_n^k(x) = K_n^{kn}(x)$.*

Lemma 3.5. *If (X_m, K_n) is a non-degenerate dynamical system with K_n satisfying the average contraction condition over X_m and I_c is a confining interval in X_m, then the induced map $\hat{\mathcal{K}}_n$ has a unique fixed point on I_c.*

Proof. By hypothesis, K_n satisfies the average contraction condition over X_m, so $\hat{\mathcal{K}}_n$ is a contraction mapping over confining interval I_c as consequence of Lemma 2.2. Further, we take the system (X_m, K_n) to be non-degenerate, so as consequence of Lemma 3.2, $\hat{\mathcal{K}}_n[I_c] \subset I_c$. As such, for any $x \in I_c$, the sequence of points $x, \hat{\mathcal{K}}_n(x), \hat{\mathcal{K}}_n^2(x), ...$ is a Cauchy sequence, and must converge to a unique point on the interval of continuity I_c. It follows that there is a point $x^* \in I_c$ such that $\hat{\mathcal{K}}_n(x^*) = x^*$. □

We now prove Theorem 3.1.

Proof of Theorem 3.1. By hypothesis, (X_m, K_n) is a non-degenerate dynamical system and K_n satisfies the average contraction condition, and we take $x \in X'_m$ so iteration of K_n over x does not map to an invariant point of any type. It then follows as a consequence of Lemma 3.4 that iteration of K_n over $x \in X'_m$ maps into a confining interval $I_c \in U_m$ in a finite number of iterations. And by consequence of Lemma 3.5 and Definition 5, iteration of K_n in a confining interval must converge to a periodic orbit of period kn, $k \in \mathbb{Z}^+$. □

Figure 4 illustrates the kind of dynamics Theorem 3.1 provides, showing the periodic orbit a 4-rule map converged to in a space X_5. We remark that for particular periodic orbits generated by an n-rule map in X_m, we cannot claim that the corresponding basin of attraction is all of X', as the periodic orbit is also dependent on initial condition $x_0 \in X'$. Indeed, work established in [17] for piecewise contractions of the interval motivates questions regarding upper bounds for the number of distinct periodic orbits a fixed (X_m, K_n) dynamical system can admit.

4. Symbolic n-rule maps and closed curves

4.1. **A symbolic redefinition of n-rule maps.** In the original definition of n-rule maps, the index value l of each rule was used to determine the lth farthest line from a point. We now seek to use the rule index value l in a different way to obtain more controllable asymptotic behavior from n-rule maps.

Consider a labelling of the lines composing X_m, so that each line in X_m is associated with a unique label L_i, $1 \le i \le m$. We define a *symbolic rule* denoted $\tilde{r} \coloneqq (\theta, o, L_i)$ so for $x \in X_m$, $r(x)$ defines an orientation o, angle θ-projection of x onto the line in X_m with label L_i. In such symbolic rules, we refer to L_i as the *symbol value* of the rule. We then call n-rule maps defined by a rule sequence of symbolic rules *symbolic n-rule maps*, denoted $\tilde{K}_n$. We leave all other pertinent properties corresponding to n-rule maps to apply to symbolic n-rule maps in the same way as before, such as the average contraction condition.

We require that no two consecutive rules in a rule sequence defining a symbolic n-rule map have the same symbol value L_i (including the first and last rules in the rule sequence). Further, we require each line in X_m to be mapped to by at least one of the symbolic rules in the defining rule sequence of the symbolic n-rule map.

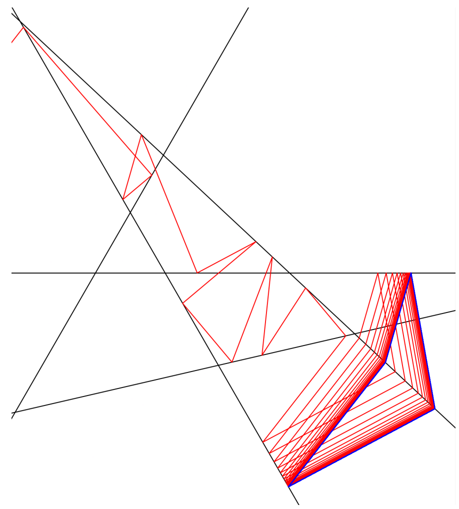

FIGURE 4. The black lines are the lines in X_5, the red curve indicates iterates of the map, and the blue curve traces the periodic orbit.

Note that symbolic n-rule maps have no invariant points as defined previously. It is possible that two or more iterations of a symbolic n-rule map map to the same line intersection point, but by definition of the space X_m, the lines are nonconcurrent so we cannot obtain a point which always remains invariant under iteration of symbolic n-rule maps. Further, such line intersection points are not discontinuities; symbolic n-rule maps are continuous everywhere on X_m, unlike the previous definition.

Theorem 4.1. *Let $(X_m, \tilde{K}_n)$ be a dynamical system with symbolic n-rule map $\tilde{K}_n$ satisfying the average contraction condition over X_m. Then for all $x \in X_m$, the orbit $\mathcal{O}(x)$ converges to a periodic orbit of period n.*

Remark 3. *We note two key differences between Theorem 3.1 and Theorem 4.1. First, Theorem 4.1 does not require the non-degeneracy assumption of the dynamical system $(X_m, \tilde{K}_n)$, as a consequence of lack of discontinuities (invariant points) in the system. Further, symbolic n-rule maps generate periodic orbits of primitive period n, instead of kn for positive integer k; reasoning for this is made clear in the proof of the theorem.*

Proof of Theorem 4.1. Let $\tilde{\mathcal{K}}_n := \tilde{K}_n^n$ label the cycle map of symbolic n-rule map $\tilde{K}_n$, and let $\hat{\mathcal{O}}(x)$ denote the orbit of $x \in X_m$ under the symbolic cycle map $\tilde{\mathcal{K}}_n$. It follows from definition of symbolic n-rule maps, that for any $x \in X_m$, the orbit $\hat{\mathcal{O}}(x) \setminus \{x\}$ of x under $\tilde{\mathcal{K}}_n$, must be a subset of some line $L_i \subset X_m$.

By hypothesis, $\tilde{K}_n$ satisfies the average contraction condition over X_m. Then by consequence of Lemma 2.2, if $\tilde{\mathcal{K}}_n[L_i] \subset L_i$ for line L_i in X_m, then for distinct $x, y \in L_i$,

$$d(\tilde{\mathcal{K}}_n(x), \tilde{\mathcal{K}}_n(y)) \leq Cd(x,y),\ 0 \leq C < 1$$

But L_i is a closed subset of $\mathbb{R}^2$ and necessarily a complete metric space. Hence, by the contraction mapping theorem there exists a unique $x^* \in L_i$ such that $\tilde{\mathcal{K}}_n(x^*) = x^*$, and for any $x \in L_i$, $\tilde{\mathcal{K}}_n^k(x) \to x^*$ as $k \to \infty$.

Finally, by definition of cycle map $\tilde{\mathcal{K}}_n$, if $\tilde{\mathcal{K}}_n$ admits a fixed point, then $\tilde{K}_n$ admits a periodic orbit of period n. As such, for any $x \in X_m$, iteration of $\tilde{K}_n$ over x converges to a periodic orbit of period n. □

Although the above theorem used the average contraction condition, the symbolic redefinition of n-rule maps allows for a significant relaxation of this condition. Given a space X_m and symbolic n-rule map $\tilde{K}_n$, it can be known exactly which rules map opposite which pairwise line intersection angles in X_m, and thus restrictions on the projection angles of each rule need not be determined by the least intersection angle in the whole space. Hence we obtain the following corollary.

Corollary 4.1.1. *Let $\tilde{\mathcal{K}}_n$ be a cycle map of a symbolic n-rule map $\tilde{K}_n$, with separation constant $C = c_1 c_2 \cdots c_n$. Then if $C < 1$, for all $x \in X_m$, the orbit $\mathcal{O}(x)$ converges to a periodic orbit of period n.*

Proof. Proof is equivalent to that of Theorem 4.1. □

Note that the inverse of symbolic n-rule map is itself a symbolic n-rule map, and is unique. Such an inverse map is simply obtained by determining the inverse of each rule, and then rearranging the rules in the rule sequence appropriately. In particular, the inverse map is defined so that for $x \in X_m$, and points $x, \tilde{K}_n(x), ..., \tilde{K}_n^n(x)$, then

$$\tilde{K}_n^{-1}(\tilde{K}_n^n(x)) = \tilde{K}_n^{n-1}(x), ..., \tilde{K}_n^{-n}(\tilde{K}_n^n(x)) = x$$

Finally, we give the following trivial, but important remark.

Remark 4. *Let L_1, L_2 be lines in X_m with intersection angles angles $\eta \in (0, \pi/2]$ and $\pi - \eta$ and intersection point z. If $x \in L_1$ and $\tilde{K}_n^{i+1}(x) \in L_2$ for some symbolic n-rule map $\tilde{K}_n$ and nonnegative integer i, and if the line segment $\overline{x\tilde{K}_n^{i+1}(x)}$ is opposite angle η or $\pi - \eta$, then the line segment $\overline{y\tilde{K}_n^{i+1}(y)}$ lies opposite the same line intersection angle for all $y \in L_1$, $y \neq z$.*

4.2. **Existence of closed curves over X_m.** We are now in a position to prove Theorem 1.1

Proof of Theorem 1.1. Consider the symbolic n-rule map given by

$$\tilde{K}_n := (\theta_1, o_1, L_{a_1})(\theta_2, o_2, L_{a_2})...(\theta_n, o_n, L_{a_n}) \tag{4}$$

Then, using the identification $L_{a_{n+1}} = L_{a_1}$ and $r_{n+1} = r_1$, we define separation constants c_i for the ith rule in the rule sequence in $\tilde{K}_n$ inductively by

$$d(r_{i+1}(x), r_{i+1}(y)) = c_i d(r_i(x), r_i(y))$$

for $x, y, \in L_{a_n}$. We define $C = c_1 c_2 \cdots c_n$ to be the separation constant for the cycle map of $\tilde{K}_n$.

As such, if $C < 1$, the symbolic n-rule map $\tilde{K}_n$ is a contraction mapping on average, and by Corollary 4.1.1, iteration of this map converges to a periodic orbit of period n in X_m. But then by the definition of $\tilde{K}_n$ given by Equation 4, joining the consecutive periodic points of the symbolic n-rule map with line segments must generate a closed curve Γ with incidence sequence $\theta_1, \theta_2, ..., \theta_n$, visiting the lines in X_m with order $L_{a_1}, L_{a_2}, ..., L_{a_n}$.

We now consider the case where the incidence sequence $\theta_1, \theta_2, ..., \theta_n$ and orientation values are such that $C > 1$. This would imply $\tilde{K}_n$ is an expansive mapping on average, but then its inverse map, which is unique and must exist by definition of symbolic n-rule maps, is a contraction mapping on average. Then iteration of the inverse map converges to a periodic orbit by Corollary 4.1.1, and the periodic points generate a closed curve Γ.

But immediate from the definition and geometry of symbolic n-rule maps (i.e. see Remark 4), we know angles are preserved regardless of the order in which points in the orbit are traversed, or if they are translated across lines and their intersection points. It follows that traversal of the closed curve produced by the inverse map, in the opposite direction, generates the incidence sequence $\theta_1, \theta_2, ..., \theta_n$, and visit the lines in X_m with order $L_{a_1}, L_{a_2}, ..., L_{a_n}$.

Consider the case where the incidence sequence $\theta_1, \theta_2, ..., \theta_n$ and orientation values are such that $C = 1$. Then we define two approximating maps for $\tilde{K}_n$ by

$$\tilde{K}_n^{(+)} \coloneqq ((\theta_1 + \epsilon), o_1, L_{a_1})(\theta_2, o_2, L_{a_2})...(\theta_n, o_n, L_{a_n})$$

and

$$\tilde{K}_n^{(-)} \coloneqq ((\theta_1 - \epsilon), o_1, L_{a_1})(\theta_2, o_2, L_{a_2})...(\theta_n, o_n, L_{a_n})$$

for real $\epsilon > 0$. But these maps must then have cycle maps with separation constants $C^{(-)} < 1$ and $C^{(+)} > 1$, so by the above, there must exist associated closed curves with incidence sequences $(\theta_1 + \epsilon), \theta_2, ..., \theta_n$ and $(\theta_1 - \epsilon), \theta_2, ..., \theta_n$, visiting the lines in X_m with order $L_{a_1}, L_{a_2}, ..., L_{a_n}$. But as $\epsilon \to 0$, these closed curves approach one another, and the limiting curve must exist: by definition symbolic n-rule maps do not have discontinuities.

Finally, consider the case in which two or more consecutive rules of the symbolic n-rule map $\tilde{K}_n$ map to the same line intersection point, which is also a periodic point. In such a situation the corresponding closed curve does not admit every incidence angle in the set $\theta_1, \theta_2, ..., \theta_n$, as there are strictly less than n unique periodic points generated by the n-rule map.

By definition of X_m, the lines are not all concurrent, so for any legal sequence of line labels $L_{a_1}, L_{a_2}, ..., L_{a_n}$, there cannot be more than $n - 1$ rules that all map onto the same line intersection point(s) in a periodic orbit. In particular, for any space X_m and corresponding symbolic n-rule map $\tilde{K}_n$, there are at most

$$\sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r+1}{r} < 2^{n-1}$$

different ways two or more consecutive rules map to the same line intersection point in a periodic orbit. But there are 2^n distinct symbolic n-rule maps of form

$$\tilde{K}_n \coloneqq (\theta_1, o_1, L_{a_1})(\theta_2, o_2, L_{a_2})...(\theta_n, o_n, L_{a_n})$$

for the 2^n distinct rule orientation combinations, so there must exist at least one such symbolic n-rule map that has no consecutive rules mapping onto the same line intersection point. □

Acknowledgements. The author is grateful for the time and feedback provided by the reviewers of this article, to Nikhil Krishnan (Princeton University) for the helpful feedback and valuable discussions, and to Nicolai Haydn (USC) for the helpful suggestions.

References

[1] R. Adler, B. Kitchens, and C. Tresser, *Dynamics of non-ergodic piecewise affine maps of the torus*, Ergod. Theory Dyn. Syst. **21** (2001), 959–999.

[2] A. Arroyo, R. Markarian, and D. P. Sanders, *Bifurcations of periodic and chaotic attractors in pinball billiards with focusing boundaries*, Nonlinearity **22** (2009), 1499–1522.

[3] H. Bruin and J. H. B. Deane, *Piecewise contractions are asymptotically periodic*, Proc. Amer. Math. Soc. **137** (2009), 1389–1395.

[4] E. Catsigeras and P. Guiraud, *Integrate and fire neural networks, piecewise contractive maps and limit cycles*, J. Math. Biol. **67** (2013), 609–655.

[5] E. Catsigeras, P. Guiraud, A. Meyroneinc, and E. Ugalde, *On the asymptotic properties of piecewise contracting maps*, Dyn. Syst. **31** (2016), 107–135.

[6] R. Coutinho, B. Fernandez, R. Lima, and A. Meyroneinc, *Discrete time piecewise affine models of genetic regulatory networks*, J. Math. Biol. **52** (2006), 524–570.

[7] D. Davis, K. DiPietro, J. Rustad, and A. St Laurent, *Negative refraction and tiling billiards*, Adv. Geom. **18** (2018), 133–159.

[8] S. Everett, *A piecewise contractive map on triangles*, J. Dyn. Syst. Geom. Theor., To Appear.

[9] G. Galperin, A. Stepin, and Ya. Vorobets, *Periodic billiard trajectories in polygons: generating mechanisms*, Russian Math. Surveys **47** (1992), 5–80.

[10] A. Goetz, *Dynamics of piecewise isometries*, Illinois J. Math. **44** (2000), 465–478.

[11] E. Gutkin, *Billiard in polygons: survey of recent results*, J. Stat. Phys. **83** (1996), 7–26.

[12] E. Gutkin, *Billiard dynamics: an updated survey with the emphasis on open problems*, Chaos **22** (2012), 1–13.

[13] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. J. Math. **30** (2001), 713–747.

[14] I. M. MacPhee, M. V. Menshikov, S. Popov, and S. Volkov, *Periodicity in the transient regime of exhaustive polling systems*, Ann. App. Probab. **16** (2006), 1816–1850.

[15] G. D. Magno, J. L. Dias, P. Duarte, J. P. Gaivão, and D. Pinheiro, *Srb measures for polygonal billiards with contracting reflection laws*, Commun. Math. Phys. **329** (2014), 687–723.

[16] H. Masur, *Closed trajectories for quadratic differentials with an application to billiards*, Duke Math. J. **53** (1986), 307–314.

[17] A. Nogueira and B. Pires, *Dynamics of piecewise contractions of the interval*, Ergod. Theory Dyn. Syst. **35** (2015), 2198–2215.

[18] R. E. Schwartz, *Obtuse triangular billiards ii: one hundred degrees worth of periodic trajectories*, Exp. Math. **18** (2009), 137–171.

[19] S. Troubetzkoy, *Periodic billiard orbits in right triangles*, Ann. Inst. Fourier **55** (2005), 29–46.