Lie Groups and Lie Algebras.

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1 Introduction.

Before we go nose to grindstone and deal carefully with the details of Lie groups and Lie algebras, it merits sketching the telos of this lecture, both to motivate and to allow readers to try to anticipate significant connections throughout the presentation. We therefore begin with the most important definition: a **Lie group** is a set endowed with both group structure and **smooth manifold** structure with smooth inversion and composition maps. This might not sound much stronger than the definition of a topological group, yet Lie groups are much nicer than generic topological groups. We construct a tangent space to the identity of the Lie group, give it a bit of extra structure, and call it a Lie algebra. Then, we can do almost all our work on the Lie algebra (which conveniently, is linear), reducing the problem of operating on our group to a much simpler linear algebra problem.

2 Smooth manifolds.

2 Definitions.

A set X equipped with a function $\mathscr{N}(x)$ that assigns to each $x \in X$ a nonempty collection of subsets $\{N\}$ (called **neighborhoods** of x) is a **topological space** if it satisfies the following four axioms.

1. Each point $x \in X$ belongs to each of its neighborhoods.

2. Each superset of a neighborhood of x is also a neighborhood of x.

3. The intersection of any two neighborhoods of x is a neighborhood of x.
4. Any neighborhood N contains a subneighborhood M such that N is a neighborhood of each point in M.

A set is **open** if it contains a neighborhood around each point.

The **basis** of a topological space M is some family $\mathscr{B}$ of open subsets such that every open set in M is equal to the union of some some sub-family of $\mathscr{B}$.

A topological space is **second countable** if it has a countable basis.

A **Hausdorff space** is a topological space M where $\forall x, y \in M$, $\exists U_x, U_y$, neighborhoods of x and y, such that $U_x \cap U_y = \emptyset$.

A **neighborhood** of a point p in a topological space M is any open subset of M containing p.

A **open cover** of M is a collection of open sets in U_a whose union $\cup U_a = M$.

A topological space M is **locally Euclidean of dimension** n if $\forall p \in M$, $\exists U_p$ neighborhood of p such that there is a homeomorphism ϕ from U_p to an open set in $\mathbb{R}^n$.

The pair (U_p, ϕ) is called a **chart**, U_p is called a **coordinate neighborhood**, and ϕ is called a **coordinate map**. A chart U is **centered at** p if $\phi(p) = 0$.

2 Topological manifolds.

We simply concatenate several definitions to define a **topological manifold of dimension** n, a Hausdorff, second countable, locally Euclidean space of dimension n.

Example 0. Any open subset of the Euclidean space $S \subseteq \mathbb{R}^n$ is a topological manifold with chart (S, id).

Example 0. The graph of $y = |x|$ in $\mathbb{R}^2$ is a topological manifold of dimension 1 with the coordinate map $(x, |x|) \mapsto x$.

Example 0. The cross $M = \{(x_1, x_2) : x_1 = 0 \text{ or } x_2 = 0\}$ is not a topological manifold. Homeomorphisms preserve the number of connected components. Observe that $M \setminus \{0\}$ has 4 components, whereas $\mathbb{R}^n \setminus \{0\}$ has 2 components if $n = 1$ and 1 component otherwise. Therefore there can be no homeomorphism from M to an open subset of $\mathbb{R}^n$, so M is not a topological manifold.

Two charts (U, ϕ) and (U, ψ) are **compatible** if $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are both C^∞.

In other words, we can start in Euclidean space, go up to the manifold via the inverse of one map, then come back down via the other map, all without trouble. Obvserve that if $U \cap V = \emptyset$, then the functions $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are trivially C^∞, so this definition is only restrictive for charts on nondisjoint neighborhoods.

An **atlas** on a topological manifold M is a collection $\mathscr{U} = \{(U_a, \phi_a)\}$ of pairwise compatible charts such that $\cup\{U_a\}$ forms an open cover of M.

An atlas (U_a, ϕ_a) is **maximal** if it is not the proper subset of any atlas on M.

Finally, our main definition: a **smooth manifold** is a topological manifold M together with a maximal atlas.

Example 0. For any smooth function $f : A \to \mathbb{R}^m$ for A open in R^n, the space $M = \{(x, f(x)) : x \in A\}$ is smooth n-manifold whose atlas contains a single chart (A, ϕ), with coordinate neighborhood A, the domain of the function, and map $\phi : (x, f(x)) \mapsto x$.

Example 0. The sphere S^1 is a 1-manifold in $\mathbb{R}^2$. Define four open neighborhoods $U_+ = \{(x,y) \in S^1 : x > 0\}$, $U_- = \{(x,y) \in S^1 : x < 0\}$, $V_+ = \{(x,y) \in S^1 : y > 0\}$, and $V_- = \{(x,y) \in S^1 : y < 0\}$, with maps projecting onto the axis of the variable not restricted. Clearly, these neighborhoods cover S^1, and each is homeomorphic to an open subset of R^1.

To prove the following theorem, we need a lemma:

Lemma. Any open subset of a smooth manifold is a smooth manifold. Let M' be an open subset of M and M a smooth manifold with atlas $\{(U_a, \phi_a)\}$. Then $\{(U_a \cap M', \phi_a)\}$ is an atlas of M', so M' is itself a smooth manifold.

Definition. $M(n, \mathbb{K})$ is the set of all $n \times n$ matrices with entries in $\mathbb{K}$. (If $\mathbb{K}$ need not be distinguished, we write simply $M(n)$, and similarly for all other matrix groups.)

Definition. $GL(n)$ is the set of all $n \times n$ matrices with nonzero determinant.

Theorem. $GL(n, \mathbb{R})$ **is a smooth** n^2 **manifold.** To begin, we identify $M(n, \mathbb{R})$ with $\mathbb{R}^{n^2}$. $\mathbb{R}^{n^2}$ is trivially a smooth manifold: take atlas $\{(\mathbb{R}^{n^2}, id)\}$. Thus, $M(n, \mathbb{R})$ is a smooth manifold. The determinant $\det : GL(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is polynomial in the entries of the matrix of which it is taken, so it is continuous, and its image is an open subset of $\mathbb{R}$. The preimage of any continuous map to an open set is open, so the preimage of det, namely $GL(n, \mathbb{R})$, is an open subset of $M(n, \mathbb{R})$. Therefore, by the lemma, $GL(n, \mathbb{R})$ is a smooth manifold of dimension n^2.

Theorem. $GL(n, \mathbb{C})$ **is a smooth** $2n^2$ **manifold.** (This proof is identical to *super*, *mutatis mutandis*.) To begin, we identify $M(n, \mathbb{C})$ with $\mathbb{R}^{2n^2}$. $\mathbb{R}^{2n^2}$ is trivially a smooth manifold: take atlas $\{(\mathbb{R}^{2n^2}, id)\}$. Thus, $M(n, \mathbb{C})$ is a smooth manifold. The determinant $\det : GL(n, \mathbb{C}) \to \mathbb{R}^2 \setminus \{0\}$ is polynomial in the entries of the matrix of which it is taken, so it is continuous, and its image is an open subset of $\mathbb{R}^2$. The preimage of any continuous map to an open set is open, so the preimage of det, namely $GL(n,\mathbb{C})$, is an open subset of $M(n,\mathbb{C}$. Therefore, by the lemma, $GL(n,\mathbb{C})$ is a smooth manifold of dimension $2n^2$.

3 Lie groups.

3 Matrix groups.

A **matrix group** is just what it sounds like: a sets of matrices closed under matrix multiplication, containing an identity element, and containing an inverse of each of its elements. We will begin out treatment of Lie groups by dealing with matrix groups, which are simple to work with, then consider how our results generalize.

Theorem. $GL(n)$ **is the maximal a matrix group.**

Proof. Recall from linear algebra that $\det(AB) = \det(A)\det(B)$. Let $A, B \in GL(n)$. By the definition of $GL(n)$, $\det(A) \neq 0$ and $\det(B) \neq 0$. Then $\det(AB) = \det(A)\det(B) \neq 0$, so $AB \in GL(n)$. Also, $\det(I) \neq 0$ and $AI = IA = A$, $\forall A \in GL(n)$, so $GL(n)$ contains the identity. Finally, recall from linear algebra that a square matrix with nonzero determinant possesses an inverse. Then $\forall A \in GL(n)$, $\exists A^{-1} : AA^{-1} = A^{-1}A = I$. This implies $\det(A^{-1}) = 1/\det(A)$, so $\det(A^{-1}) \neq 0$, so $A \in GL(n)$.

Recall from linear algebra that nonsquare matrices and matrices with determinants of 0 lack inverses. Then any set with such a matrix does not have an inverse for each of its elements, so it is not a group.

$\therefore GL(n)$ is a matrix group and any set of matrices with an element not in $GL(n)$ is not a matrix group, so $GL(n)$ is the maximal matrix group. $\square$

4 Lie algebras.

5 Lie groups and Lie algebras together: the exponential and logarithm maps.