Proof of Correctness of BFS

First, two kind of annoying lemmas. These help us formalize what's going on as the algorithm is running.

Lemma 1. *At end of BFS, for all $v \in V$, dist(v) is at least the distance from s to v.*

Proof. Will show by induction that at each iteration of loop, this holds for all v.

Base Case: 0th iteration.

Inductive Hypothesis: Assume true for the kth iteration.

Inductive Case: Consider the vertex v removed from queue on $k+1$st iteration.

- Only change dist(u) for a few vertices, and only if u adjacent to v.
- For verts not changed, use IH.
- Otherwise,
  * distance from s to v
  * $\leq$ distance from s to u + 1
  * $\leq$ dist(u)+1
  * $=$ dist(v).

□

Lemma 2. *For any k $(v_1, \ldots, v_r)$ be the elements of the queue at iteration k. At this iteration,*

1. $dist(v_1) \geq dist(v_r) - 1$
2. *for any* $i < j$, $dist(v_i) < dist(v_j)$

Proof. Induction

Base Case: Initially, queue is empty.

Inductive Hypothesis: Assume true for kth iteration.

Inductive Case: Consider $k+1$st iteration.

- Remove v_1 from queue. New front is v_2.
- Enqueue neighbors of v_1.
  * Let u be a neighbor.
  * dist(u) is set to $dist(v_1) + 1 \geq dist(v_r) - 1 + 1 = dist(v_r)$, so 2. still holds.
  * $dist(v_2) \geq dist(v_1) = dist(u) - 1$, so 1. still holds.

□

Corollary: Let v_k be the kth vertex to have dist(v) set. $dist(v_k)$ is increasing in k.

Finally, the proof of correctness

At the termination of BFS, if BFS explores v, then dist(v) is the distance from s to v.

Proof by contradiction.

- Assume there's some vertex with dist() not equal to the distance from s.
- Let v be such a vertex which is the smallest distance from s.
- Let u be its predecessor on shortest path from v to u.
- By lemma 16, dist(v) > distance from s to v

- = (distance from s to u) + 1 = dist(u) + 1
- Consider when u was dequeued. We'll contradict the above chain of inequalities in any of the three cases:
  - if v wasn't explored yet, v would have been enqueued with dist(v)=dist(u)+1. Contradiction to above inequalities.
  - if v on the queue, then parent(v) has lower dist than u does, by corollary to lemma 17. Then dist(v) = dist(parent(v)) + 1 $\leq$ dist(u) + 1, again contradicting the inequalities.
  - if v has already been dequeued, then $dist(v) \leq dist(u)$. Contradiction again.

□