Some Integral Inequalities for Log-Preinvex Functions

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Abstract In this paper, a new type of Hermite–Hadamard inequalities is established for log-preinvex functions. Some natural applications to special means of real numbers are also discussed.

Keywords Invex sets · Preinvex functions · Log-preinvex functions · Hermite–Hadamard inequality

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1 Introduction

Log-preinvex functions are nonconvex functions and are the generalized class of log-convex functions. Log-convex functions play an important role in the theory of special functions and mathematical statistics [9, 17, 18, 21]. Let I be a closed interval. A real valued function $f : I \to R$ is said to be convex on I if $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$ for all $x, y \in I$ and $t \in [0, 1]$. The well-known classical Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

gives us an estimate of the mean value of a convex function $f : I \to R$.

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In recent years, some refinements and several generalizations of the Hermite–Hadamard Inequality have been extensively investigated by the many authors, see [4–8, 10–13, 15, 22, 24]. Dragomir and Aggarwal [7] proved two inequalities for differentiable convex mappings.

An important and significant generalization of convexity is invexity, which was introduced by Hanson [14] in 1981. Weir and Mond [23] introduced the concept of preinvex functions while Jeyakumar [16] studied the properties of preinvex functions and their role in optimization and mathematical programming. Ahmad et al. [2] introduced the concept of geodesic η-preinvex functions on Riemannian manifolds.

The main purpose of this paper is to establish some new refined inequalities of Hermite–Hadamard's type for log-preinvex functions. Applications to special means have also been considered.

2 Preliminaries

Definition 2.1 ([14]) A set $S \subseteq R^n$ is said to be invex with respect to $\eta : S \times S \to R^n$ if for every $x, y \in S$ and $t \in [0, 1]$

$$y + t\eta(x, y) \in S \tag{1}$$

Every convex set is invex for $\eta(x, y) = x - y$ but converse need not be true. Let $S \subseteq R^n$ be an invex set with respect to $\eta : S \times S \to R^n$. For every $x, y \in S$ in the η-path P_{xv} joining the points x and $v := x + \eta(y, x)$ is defined as follows

$$P_{xv} := \{z : z = x + t\eta(y, x) : t \in [0, 1]\}.$$

Definition 2.2 ([23]) Let $S \subseteq R^n$ be an invex set with respect to $\eta : S \times S \to R^n$. Then, the function $f : S \to R$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y). \tag{2}$$

Every convex function is a preinvex function but the converse is not true. For example, the function $f(x) = - \mid x \mid$ is not a convex function but it is a preinvex function with respect to η, where

$$\eta(x, y) := \begin{cases} x - y, & \text{if } x \leq 0, y \leq 0 \text{ and } x \geq 0, y \geq 0, \\ \\ y - x, & \text{otherwise.} \end{cases}$$

The concept of invex and preinvex functions have played a very important role in the development of generalized convexities.

Definition 2.3 ([20]) A function $f : I \to (0, \infty)$ on the invex set S is said to be logarithmic preinvex with respect to η, if

$$f(y + t\eta(x, y)) \leq (f(x))^t (f(y))^{(1-t)}, \quad \text{for every } x, y \in S, \ t \in [0, 1] \tag{3}$$

Note that for $\eta(x, y) = x - y$, the invex set S reduces to convex set and consequently log-preinvex function to log-convex function.

3 Main Results

In this section, we generalize Hermite–Hadamard inequality for log-preinvex functions. Barani et al. [3] generalized the result of Dragomir and Aggarwal [7] as follows:

Lemma 3.1 ([3]) *Let $A \subset R$ be an open invex subset with respect to $\theta : AXA \to R$ and $a, b \in A$ with $\theta(a, b) \neq 0$. Suppose that $f : A \to R$ is a differentiable function. If f' is integrable on the θ path P_{bc}, $c = b + \theta(a, b)$, then the following equality holds*

$$-\frac{f(b) + f(b + \theta(a, b))}{2} + \frac{1}{\theta(a, b)} \int_b^{b+\theta(a,b)} f(x)dx = \frac{\theta(a, b)}{2} \int_0^1 (1 - 2t) f'(b + t\theta(a, b))dt$$

Using Lemma 3.1, we prove the following theorem.

Theorem 3.1 *Let $A \subset R$ be an open invex subset with respect to $\theta : AXA \to R$ and $f : A \to R$ be a differentiable function. If $|f'|$ is log-preinvex on A then for every $a, b \in A$ with $\theta(a, b) \neq 0$ the following inequality holds*

$$\left| \frac{f(b) + f(b + \theta(a, b))}{2} - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a,b)} f(x)dx \right|$$

$$\leq \frac{\theta(a, b)}{2} \left[\frac{|f'(a)| - |f'(b)|}{log|f'(a)| - log|f'(b)|} - 2 \left(\frac{\sqrt{|f'(a)|} - \sqrt{|f'(b)|}}{log|f'(a)| - log|f'(b)|} \right)^2 \right]$$

Proof Using Lemma 3.1 and log preinvexity of $|f'|$, we get

$$\left| \frac{f(b)+f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)} \int_b^{b+\theta(a,b)} f(x)dx \right| = \left| \frac{\theta(a,b)}{2} \int_0^1 (1 - 2t) f'(b + t\theta(a, b))dt \right|$$

$$\leq \frac{\theta(a,b)}{2} \int_0^1 |1 - 2t| \, |f'(b)|^{1-t} |f'(a)|^t dt$$

Now integrating by parts, the above result is obtained.

Theorem 3.2 *Let $A \subset R$ be an open invex subset with respect to $\theta : AXA \to R$ and $f : A \to R$ be a differentiable function. Suppose that $p \in R$ with $p > 1$. If $|f'|^{p/(p-1)}$ is log-preinvex on A then for every $a, b \in A$ with $\theta(a, b) \neq 0$ the following inequality holds*

$$\left|\frac{f(b)+f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx\right| \le \frac{|\theta(a,b)|}{2(p+1)^{1/p}}\left[\frac{|f'(a)^q - |f'(b)|^q}{q(log|f'(a)| - log|f'(b)|)}\right]^{1/q}$$

Proof Using Lemma 3.1 and Holders' Inequality, we get

$$\begin{aligned}
\left|\tfrac{f(b)+f(b+\theta(a,b))}{2} - \tfrac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx\right| &= \left|\tfrac{\theta(a,b)}{2}\int_0^1 (1-2t)f'(b+t\theta(a,b))dt\right| \\
&\le \tfrac{\theta(a,b)|}{2}\left[\int_0^1 |1-2t|^p dt\right]^{1/p} \\
&\quad \times\left[\int_0^1 |f'(b+t\theta(a,b)|^q dt\right]^{1/q} \\
&= \tfrac{|\theta(a,b)|}{2(p+1)^{1/p}}\left[\int_0^1 |f'(b+t\theta(a,b))|^q dt\right]^{1/q} \\
&\le \tfrac{|\theta(a,b)|}{2(p+1)^{1/p}}\left[\int_0^1 |f'(b)|^{q(1-t)}|f'(a)|^{qt} dt\right]^{1/q} \\
&= \tfrac{|\theta(a,b)|}{2(p+1)^{1/p}}\left[\tfrac{|f'(a)|^q - |f'(b)|^q}{q(log|f'(a)| - log|f'(b)|)}\right]^{1/q},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. This completes the proof.

It is to be noted that if $A = [a, b]$ and $\theta(x, y) = x - y$ for every $x, y \in A$ then, these results of log-preinvex will be converted to that of log-convex.

Mohan and Neogy [19] gives the following condition C.

Condition C The mapping $\eta : S \times S \to R^n$ is said to satisfy the condition C if for every $x, y \in S$ and $t \in [0, 1]$,

$$\eta(y, y + t\eta(x, y)) = -t\eta(x, y),$$

$$\eta(x, y + t\eta(x, y)) = (1 - t)\eta(x, y)$$

Now, from condition C, for every $x, y \in S$ and every $t_1, t_2 \in [0, 1]$, we have

$$\eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).$$

Lemma 3.2 *Let $S \subseteq R^n$ be an invex set with respect to $\eta : SXS \to R^n$ and $f : S \to R$ be a function. Suppose that η satisfies the condition C on S. Then for every $x, y \in S$*

the function f is log-preinvex with respect to η on the η path P_{xv} if and only if the function $\phi : [0, 1] \to R$ defined by $\phi(t) := f(x + t\eta(y, x))$ is log convex on [0, 1].

Proof Suppose that ϕ is log convex on [0, 1] and $z_1 := x + t_1\eta(y, x) \in P_{xv}$, $z_2 := x + t_2\eta(y, x) \in P_{xv}$. Fix $\lambda \in [0, 1]$. Since η satisfies condition C,

$$\begin{aligned} f(z_1 + \lambda\eta(z_2, z_1)) &= f(x + ((1 - \lambda)t_1 + \lambda t_2)\eta(y, x)) \\ &= \phi((1 - \lambda t_1) + \lambda t_2) \\ &= (\phi(t_1)^{1-\lambda})(\phi(t_2)^{\lambda}) \\ &= f(z_1)^{1-\lambda} f(z_2)^{\lambda}. \end{aligned}$$

Hence, f is log preinvex with respect to η on the η path P_{xv}.

Conversely, let $x, y \in S$ and the function f is log preinvex with respect to η on the η path P_{xv}. Suppose that η satisfies the condition C and $t_1, t_2 \in [0, 1]$. Then for every $\lambda \in [0, 1]$ we have,

$$\begin{aligned} \phi((1 - \lambda)t_1 + \lambda t_2) &= f(x + ((1 - \lambda)t_1 + \lambda t_2)\eta(y, x)) \\ &\le f(x + t_1\eta(y, x) + \lambda\eta(x + t_2\eta(y, x), x + t_1\eta(y, x)) \\ &\le f(x + t_1\eta(y, x))^{1-\lambda} f(x + t_2\eta(y, x))^{\lambda} \\ &= \phi(t_1)^{1-\lambda}\phi(t_2)^{\lambda}. \end{aligned}$$

Therefore, ϕ is log convex on [0, 1].

Lemma 3.3 *Assume that $a, b \in R$ with $a < b$ and $f : [a, b] \to R$ is a differentiable function on (a, b). If $|f'|$ is log convex on [a, b] then the following inequality holds true*

$$\begin{aligned} \left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2}\Bigg[&\frac{|f'(a)| - |f'(b)|}{log|f'(a)| - log|f'(b)|} \\ &- 2\left(\frac{\sqrt{|}f'(a)| - \sqrt{|}f'(b)|}{log|f'(a)| - log|f'(b)|}\right)^2\Bigg] \end{aligned}$$

Proof Since $|f'|$ is log convex, using Lemma 3.1, we get

$$\begin{aligned} \left|\tfrac{f(b)+f(a)}{2} - \tfrac{1}{b-a}\int_b^{b+\theta(a,b)} f(x)dx\right| &= \left|\tfrac{\theta(a,b)}{2}\int_0^1 (1 - 2t)f'(ta + (1 - t)b)dt\right| \\ &\le \tfrac{b-a}{2}\int_0^1 |1 - 2t|\,|f'(b)|^{1-t}|f'(a)|^t dt. \end{aligned}$$

Now integrating by parts,the above result is obtained.

Theorem 3.3 *Let $S \subseteq R^n$ be an invex set with respect to $\eta : SXS \to R^n$. Suppose that η satisfies the condition C on S. Suppose that every $x, y \in S$ the function $f : S \to R^+$ is log preinvex with respect to η on the η path P_{xv}. Then for every $(a, b) \in (0, 1)$ with $a < b$ the following inequality holds*

$$\left|1/2\int_0^a f(x+s\eta(y,x))ds + 1/2\int_0^b f(x+s\eta(y,x))ds - \frac{1}{b-a}\int_a^b\int_0^s f(x+s\eta(y,x))dtds\right|$$
$$\le (b-a)\left[\frac{f(x+b\eta(y,x)) - f(x+a\eta(y,x)) - f(x+b\eta(y,x))f(x+a\eta(y,x))}{log(f(x+b\eta(y,x))) - log(f(x+a\eta(y,x)))}\right]^{1/2}$$
$$\le \frac{b-a}{2}\left[\frac{|f(x+a\eta(y,x))| - |f(x+b\eta(y,x))|}{log|f(x+a\eta(y,x))| - log|f(x+b\eta(y,x))|}\right.$$
$$\left. - 2\left(\frac{\sqrt{|f(x+a\eta(y,x))|} - \sqrt{|f(x+b\eta(y,x))|}}{log|f(x+a\eta(y,x))| - log|f(x+b\eta(y,x))|}\right)^2\right]$$

Proof Let $x, y \in$ S and a,b $\in$ (0,1) with $a < b$. Since f is log-preinvex with respect to η on the η path P_{xv} by Lemma 3.3 the function $\phi : [0, 1] \to R^+$ defined by

$$\phi(t) := f(x + t\eta(y, x))$$

is log convex on [0, 1]. Now we define the function $\varphi : [0, 1] \to R^+$ as follows

$$\varphi(t) := \int_0^t \phi(t)ds = \int_0^t f(x + s\eta(y, x))ds$$

Obviously for every $t \in (0, 1)$ we have

$$\varphi'(t) = \phi(t) = f(x + t\eta(y, x)) \geq 0.$$

Applying Lemma 3.3 to the function φ implies that

$$\left|\frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(s)ds\right|$$

$$\le \frac{b-a}{2}\left[\frac{|\varphi'(a)| - |\varphi'(b)|}{log|\varphi'(a)| - log|\varphi'(b)|} - 2\left(\frac{\sqrt{|\varphi'(a)|} - \sqrt{|\varphi'(b)|}}{log|\varphi'(a)| - log|\varphi'(b)|}\right)^2\right]$$

and hence we deduce that the above theorem holds.

4 Inequalities for Second Order Differentiable Functions

Now, we derive some results for functions whose second order derivative absolute values are log-preinvex. The following Lemma was given by Barani et al. [3].

Lemma 4.1 ([3]) *Let $A \subset R$ be an open invex subset with respect to $\theta : AXA \to R$ and $a, b \in A$ with $\theta(a, b) \neq 0$. Suppose that $f : A \to R$ is a differentiable function. If f'' is integrable on the θ path P_{bc}, $c = b + \theta(a, b)$, then the following equality holds*

$$\frac{f(b) + f(b + \theta(a, b))}{2} - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a,b)} f(x)dx = \frac{\theta(a, b)}{2}^2 \int_0^1 t(1 - t) f''(b + t\theta(a, b))dt$$

Using Lemma 4.1, we prove the following result.

Theorem 4.1 *Let $A \subset R$ be an open invex subset with respect to $\theta : AXA \to R$ and $a, b \in A$ with $\theta(a, b) \neq 0$. Suppose that $f : A \to R$ is a twice differentiable function on A. If $|f''|$ is log-preinvex and f'' integrable on the θ path P_{bc}, $c = b + \theta(a, b)$, then the following inequality holds*

$$\left| \frac{f(b)+f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)} \int_b^{b+\theta(a,b)} f(x)dx \right|$$

$$\leq \frac{\theta(a, b)^2}{2} \left[\frac{|f''(a)| + |f''(b)|}{(log|f''(a)| - log|f''(b)|)^2} + \frac{2(|f''(b)| - |f''(a)|)}{(log|f''(a)| - log|f''(b)|)^3} \right]$$

Proof From Lemma 4.1, we have

$$\begin{aligned} \left| \frac{f(b)+f(b+\theta(a,b))}{2} - \frac{1}{\theta(a,b)} \int_b^{b+\theta(a,b)} f(x)dx \right| &= \left| \frac{\theta(a,b)}{2}^2 \int_0^1 t(1 - t) f''(b + t\theta(a, b))dt \right| \\ &\leq \frac{\theta(a,b)}{2}^2 \int_0^1 t(1 - t)|f''(b)|^{1-t}|f''(a)|^t dt \\ &= \frac{\theta(a,b)^2}{2} \left[\frac{|f''(a)|+|f''(b)|}{(log|f''(a)|-log|f''(b)|)^2} \right. \\ &\quad \left. + \frac{2(|f''(b)|-|f''(a)|)}{(log|f''(a)|-log|f''(b)|)^3} \right] \end{aligned}$$

which completes the proof.

Theorem 4.2 *Let $A \subset R$ be an open invex subset with respect to $\theta : AXA \to R$ and $a, b \in A$ with $\theta(a, b) \neq 0$. Suppose that $f : A \to R$ is a twice differentiable function on A. If $|f''|^{p/p-1}$ is log-preinvex on A for $p > 1$. If f'' integrable on the θ path P_{bc}, $c = b + \theta(a, b)$, then the following inequality holds*

$$\left| \frac{f(b) + f(b + \theta(a, b))}{2} - \frac{1}{\theta(a, b)} \int_b^{b+\theta(a,b)} f(x)dx \right|$$

$$\leq \frac{\theta(a,b)^2}{16}\sqrt{\pi}^{1/p}\,\frac{\Gamma(p+1)}{\Gamma(p+3/2)}^{1/p}\left(\frac{|f''(a)|^q-|f''(b)|^q}{q(log|f''(a)|-log|f''(b)|)}\right)^{1/q}$$

Proof By log-preinvexity of $|f''|^q$, Lemma 4.1 and Holder's inequality, we get

$$\begin{aligned}\left|\tfrac{f(b)+f(b+\theta(a,b))}{2}-\tfrac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx\right| &\leq \tfrac{\theta(a,b)}{2}^2\int_0^1 t(1-t)f''(b+t\theta(a,b))dt\\ &\leq \tfrac{\theta(a,b)^2}{2}\left(\int_0^1 (t-t^2)^p dt\right)^{1/p}\\ &\quad\times\left(\int_0^1 |f''(b+\theta(a,b)|^q dt\right)^{1/q}\\ &\leq \tfrac{\theta(a,b)}{2}^2\left[\tfrac{2^{-1-2p}\sqrt{\pi}\Gamma(1+p)}{\Gamma(3/2+p)}\right]^{1/p}\\ &\quad\times\left[\int_0^1 |f''(b)|^{q(1-t)}|f''(a)|^{qt}dt\right]^{1/q}\\ &\leq \tfrac{\theta(a,b)^2}{16}\sqrt{\pi}^{1/p}\,\tfrac{\Gamma(p+1)}{\Gamma(p+3/2)}^{1/p}\\ &\quad\times\left(\tfrac{|f''(a)|^q-|f''(b)|^q}{q(log|f''(a)|-log|f''(b)|)}\right)^{1/q}\end{aligned}$$

which completes the proof.

Theorem 4.3 *Let $A \subset R$ be an open invex subset with respect to $\theta : AXA \to R$ and $a, b \in A$ with $\theta(a,b) \neq 0$. Suppose that $f : A \to R$ is a twice differentiable function on A. If $|f''|^q$ is log-preinvex on A for $q > 1$ and f'' integrable on the θ path P_{bc}, $c = b+\theta(a,b)$, then the following inequality holds*

$$\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx$$

$$\leq \frac{\theta(a,b)^2}{12}(6)^{1/q}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(q(log|f''(a)|-log|f''(b)|))^2}+\frac{2(|f''(b)|^q-|f''(a)|^q)}{(q(log|f''(a)|-log|f''(b)|))^3}\right]^{1/q}$$

Proof By Lemma 4.1 and using the well known weighted power mean inequality, we get

$$\left|\frac{f(b)+f(b+\theta(a,b))}{2}-\frac{1}{\theta(a,b)}\int_b^{b+\theta(a,b)} f(x)dx\right|$$

$$\leq \frac{\theta(a,b)}{2}^2 \int_0^1 t(1-t)|f''(b+t\theta(a,b))|dt$$

$$\leq \frac{\theta(a,b)^2}{2}\left(\int_0^1 (t-t^2)dt\right)^{1-1/q}\left(\int_0^1 (t-t^2)|f''(b+\theta(a,b)|^q dt\right)^{1/q}$$

$$\leq \frac{\theta(a,b)^2}{2}(1/6)^{1-1/q}\left[\int_0^1 (t-t^2)|f''(b)|^{q(1-t)}|f''(a)|^{qt}dt\right]^{1/q}$$

$$\leq \frac{\theta(a,b)^2}{12}(6)^{1/q}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(q(log|f''(a)|-log|f''(b)|))^2}+\frac{2(|f''(b)|^q-|f''(a)|^q)}{(q(log|f''(a)|-log|f''(b)|))^3}\right]^{1/q}$$

which completes the proof.

5 Applications to Special Means

Let a and b be two positive numbers. We have:
Arithmetic mean

$$A(a,b)=\frac{a+b}{2},$$

Logarithmic mean

$$L_p(a,b)=\frac{a-b}{lna-lnb},$$

and generalized logarithmic mean

$$L_p(a,b)=\left[\frac{a^{p+1}-b^{p+1}}{(p+1)(a-b)}\right]^{1/p}, \quad p\neq -1,0;$$

There are several results connecting these means, see [1] for some new relations; however very few results are known for arbitrary real numbers. For this, it is clear that we can extend some of the above means as follows:

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2}, \qquad \alpha,\beta\in R$$

$$\bar{L}(\alpha,\beta)=\frac{\beta-\alpha}{ln|\beta|-ln|\alpha|}, \qquad \alpha,\beta\in R\backslash 0$$

$$L_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}} \qquad n\in N,\ \ n\geq 1,\ \ \alpha,\beta\in R,\ \ \alpha<\beta$$

Now using the results of Sects. 3 and 4, we prove the following inequalities connecting the above means for arbitrary real numbers.

Proposition 5.1 *Let* $a, b \in R$, $a < b$ *and* $n \in N$, $n \geq 2$. *Then the following inequality holds:*

$$|A(a^n, b^n) - (L_n(a,b))^n| \leq \frac{n|a-b|}{2}\left[\bar{L}(|b^{n-1}|, |a^{n-1}|) - \frac{1}{2}(\bar{L}(\sqrt{|b^{n-1}|}, \sqrt{|a^{n-1}|}))^2\right]$$

Proof The proof is immediate from Theorem 3.1 applied for $f(x) = x^n$, $x \in R$, $\theta(a,b) = a - b$.

Proposition 5.2 *Let* $a, b \in R$, $a < b$ *and* $n \in N$, $n \geq 2$. *Then, for all* $p > 1$, *the following inequality holds:*

$$|A(a^n, b^n) - L_n^n(a,b)| \leq \frac{n|a-b|}{2(p+1)^{1/p}}\left[\bar{L}(|b^{n-1}|^q, |a^{n-1}|^q)\right]^{\frac{1}{q}}$$

Proof The proof is immediate from Theorem 3.2 applied for $f(x) = x^n$, $x \in R$, $\theta(a,b) = a - b$.

Proposition 5.3 *Let* $a, b \in R\backslash 0$, $a < b$. *Then the following inequality holds:*

$$|A\left(\frac{1}{a}, \frac{1}{b}\right) - \bar{L}^{-1}(a,b)| \leq \frac{|a-b|}{2}\left[L\left(\frac{1}{b^2}, \frac{1}{a^2}\right) - \frac{1}{2}(\bar{L}(|\frac{1}{b}|, |\frac{1}{a}|))^2\right]$$

Proof The proof is obvious from Theorem 3.1 applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$, $\theta(a,b) = a - b$.

Proposition 5.4 *Let* $a, b \in R\backslash 0$, $a < b$. *Then for all* $p > 1$, *the following inequality holds:*

$$|A\left(\frac{1}{a}, \frac{1}{b}\right) - \bar{L}^{-1}(a,b)| \leq \frac{|a-b|}{2(p+1)^{1/p}}\left[L\left(\left(\frac{1}{b^2}\right)^q, \left(\frac{1}{a^2}\right)^q\right)\right]^{\frac{1}{q}}$$

Proof The proof is obvious from Theorem 3.2 applied for $f(x) = \frac{1}{x}$, $x \in [a, b]$, $\theta(a,b) = a - b$.

Proposition 5.5 *Let* $a, b \in R$, $a < b$ *and* $n \in N$, $n \geq 2$. *Then the following inequality holds*

$$\frac{1}{(n+1)(n+2)}\left|A(a^{n+2}, b^{n+2}) - L_{n+2}^{n+2}(a,b)\right| \leq \frac{1}{n^2}\bar{L}(a,b)^2\left[A(|a^n|, |b^n|) - L(|a^n| - |b^n|)\right]$$

Proof The proof is immediate from Theorem 4.1 applied for $f(x) = \frac{x^{n+2}}{(n+1)(n+2)}$, $x \in R$, $\theta(a,b) = a - b$.

Proposition 5.6 *Let $a, b \in R$, $a < b$ and $n \in N$, $n \geq 2$. Then, for all $p > 1$, the following inequality holds*

$$\frac{1}{(n+1)(n+2)}\left|A(a^{n+2}, b^{n+2}) - L_{n+2}^{n+2}(a,b)\right| \leq \frac{\theta(a,b)^2}{16}\sqrt{\pi}^{1/p}\,\frac{\Gamma(p+1)}{\Gamma(p+3/2)}^{1/p}\left(L(|a^n|^q, |b^n|^q)\right)^{\frac{1}{q}}$$

Proof The proof is immediate from Theorem 4.2 applied for $f(x) = \frac{x^{n+2}}{(n+1)(n+2)}$, $\in R$, $\theta(a,b) = a - b$.

Proposition 5.7 *Let $a, b \in R\backslash 0$, $a < b$. Then the following inequality holds*

$$\left|A(alog|a|, blog|b|) + \frac{1}{2}A(a,b) - \frac{b^2log|b| - a^2log|a|}{2(b-a)}\right| \leq L^2(a,b)\left[A(|\frac{1}{a}|, |\frac{1}{b}|) - L(|\frac{1}{a}|, |\frac{1}{b}|)\right]$$

Proof The proof is obvious from Theorem 4.1 applied for $f(x) = xlogx - x$, $x \in [a,b]$, $\theta(a,b) = a - b$.

Proposition 5.8 *Let $a, b \in R\backslash 0$, $a < b$. Then for all $p > 1$, the following inequality holds*

$$\left|A(alog|a|, blog|b|) + \frac{1}{2}A(a,b) - \frac{b^2log|b| - a^2log|a|}{2(b-a)}\right| \leq \frac{\theta(a,b)^2}{16}\sqrt{\pi}^{1/p}\,\frac{\Gamma(p+1)}{\Gamma(p+3/2)}^{1/p} \times \left[L\left(\left|\frac{1}{b^q}\right|^q, \left|\frac{1}{a^q}\right|^q\right)\right]^{\frac{1}{q}}$$

Proof The proof is obvious from Theorem 4.2 applied for $f(x) = xlogx - x$, $x \in [a,b]$, $\theta(a,b) = a - b$.

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References

1. Agarwal, R.P., Dragomir, S.S.: An application of Hayashi's inequality for differentiable functions. Comput. Math. Appl. **32**(6), 95–99 (1996)
2. Ahmad, I., Iqbal, A., Ali, S.: On properties of Geodesic η-preinvex functions. Adv. Oper. Res. Article ID 381831, 10 p. (2009)
3. Barani, A., Ghazanfari, A.G., Dragomir, S.S.: Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex. J. Inequal. Appl. **2012**, 247 (2012)
4. Bombardelli, M., Varosanec, S.: Properties of h-convex functions related to the Hermite-Hadamard-Fejer inequalities. Comput. Math. Appl. **58**, 1869–1877 (2010)
5. Dragomir, S.S.: Two mappings in connection to Hadamard's inequalities. J. Math. Anal. Appl. **167**, 49–56 (1992)
6. Dragomir, S.S.: On Hadamards inequalities for convex functions. Math. Balk. **6**, 215–222 (1992)
7. Dragomir, S.S., Agarwal, R.P.: Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl. Math. Lett. **11**, 91–95 (1998)

8. Dragomir, S.S., Pearce, C.E.M.: Quasi-convex functions and Hadamard's inequality. Bull. Aust. Math. Soc. **57**(3), 377–385 (1998)
9. Dragomir, S.S., Pearce, C.E.M.: Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA Monographs, Victoria University (2000)
10. Dragomir, S.S., Pecaric, J.E., Sandor, J.: A note on the Jensen-Hadamard inequality. Anal. Numr. Thor. Approx. **19**, 29–34 (1990)
11. Dragomir, S.S., Pecaric, J.E., Persson, L.E.: Some inequalities of Hadamard type. Soochow J. Math. **21**, 335–341 (1995)
12. Dragomir, S.S., Cho, Y.J., Kim, S.S.: Inequalities of Hadamard's type for Lipschitzian mappings and their applications. J. Math. Anal. Appl. **245**, 489–501 (2000)
13. Hadjisavvas, N.: Hadamard-type inequalities for quasi-convex functions. J. Inequal. Pure Appl. Math. **4**(1) article 13 (2003)
14. Hanson, M.A.: On sufficiency of the Kuhn-Tucker conditions. J. Math. Anal. Appl. **80**, 545–550 (1981)
15. Ion, D.A.: Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Comput. Sci. Ser. **34**, 19–22 (2007)
16. Jeyakumar, V.: Strong and weak invexity in mathematical programming. Methods Oper. Res. **55**, 109–125 (1985)
17. Manfrino, R.B., Delgado, R.V., Ortega, J.A.G.: Inequalities a Mathematical Olympiad Approach. Birkhauser, Basel (2009)
18. Mitrinovic, D.S., Pecaric, J.E., Fink, A.M.: Classical and New Inequalities in Analysis. Kluwer Academic Publishers, Dordrecht (1993)
19. Mohan, S.R., Neogy, S.K.: On invex sets and preinvex function. J. Math. Anal. Appl. **189**, 901–908 (1995)
20. Noor, M.A.: Variational-like inequalities. Optimization **30**, 323–330 (1994)
21. Peciaric, J.E., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, New York (1991)
22. Qaisar, S., Hussain, S., Chuanjiang, He.: On new inequalities of Hermite-Hadamard type for functions whose third derivatives absolute values are quasi-convex with applications. J. Egypt. Math. Soc. **22**, 137–146 (2014)
23. Weir, T., Mond, B.: Preinvex functions in multiobjective optimization. J. Math. Anal. Appl. **136**, 29–38 (1988)
24. Yang, G.S., Hwang, D.Y., Tseng, K.L.: Some inequalities for differentiable convex and concave mappings. Comput. Math. Appl. **47**, 207–216 (2004)