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Logistic Maps and Chaos Theory

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ABSTRACT

The logistic map $x_{n+1} = \mu x_n(1 - x_n)$ provides an exceptionally simple framework for exploring nonlinear dynamics and deterministic chaos. In this study we numerically iterate the map for $2.4 \leq \mu \leq 4.0$ and generate a high-resolution bifurcation diagram, a plot showing long-term behavior vs. a control parameter. The results reproduce the classic period-doubling route to chaos: a single stable fixed point gives way to cycles of periods $2, 4, 8$, and so on until a chaotic attractor emerges. Our computation clearly captures key features predicted by May (1976) and later visualized by Boeing (2016), demonstrating that even a one-dimensional deterministic equation can produce unpredictable, fractal-like behavior central to modern chaos theory.

Keywords: Logistic map, Deterministic chaos, Bifurcation diagram, Discrete dynamical systems, Fixed Point

INTRODUCTION

The study of population growth and self-limiting processes began with Pierre-François Verhulst's work on the logistic curve (Pastijn 2005). In 1844 the Belgian mathematician presented *Recherches mathématiques sur la loi d'accroissement de la population* to the Belgian Academy of Sciences, introducing the now-famous logistic growth law to describe a closed population constrained by finite resources (Pastijn 2005). Verhulst's continuous-time model,

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right),$$

captures the way growth slows as the population approaches a carrying capacity K.

Nearly a century later, the **discrete-time analogue** of Verhulst's equation became a keystone of modern **chaos theory**. When May (1976) analyzed the iterated form $x_{n+1} = \mu x_n(1-x_n)$, he showed that as the parameter μ increases the system undergoes a sequence of **period-doubling bifurcations** (successive splits where a stable cycle of period p becomes a cycle of period $2p$ as a control parameter changes) that accumulate at a finite limit, beyond which the dynamics are chaotic (aperiodic but deterministic behavior). Subsequent expositions have illustrated and extended these results, from Bubolo's educational treatment of the logistic map (the discrete iteration $x_{n+1} = \mu x_n(1 - x_n)$ describing normalized population growth) and difference equations (Bubolo 2019) to Boeing's detailed visual analysis of nonlinear dynamical systems and their fractal structure (Boeing 2016). These works emphasize how **deterministic** rules can generate aperiodic, unpredictable behavior which is an insight that transformed fields ranging from physics and chemistry to economics and ecology.

Our analysis thus bridges the gap between Verhulst's historical attempt to model population limits, through the twentieth-century rediscovery of the discrete map, to a contemporary visualization of chaos using modern computational tools. We investigated the discrete logistic map defined by $x_{n+1} = \mu * x_n(1 - x_n)$, where $0 < x_n < 1$ represents a normalized population and μ is in [2.4, 4] is the growth-rate parameter. The objectives are as following:

1. Determining long-term behavior of the sequence $x_{n+1} = \mu x_n(1-x_n)$ for each μ, identifying whether it converges to a fixed point (a value x^* such that $f(x^*) = x^*$), a periodic orbit (a repeating cycle of distinct points), or

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a chaotic attractor (a set of states toward which trajectories evolve that shows sensitive dependence on initial conditions).

2. Constructing a **bifurcation diagram** that plots the asymptotic values of x_n versus μ, exposing successive period-doubling bifurcations and the onset of deterministic chaos.

3. Assessing stability of **fixed points** using the derivative $f'(x) = \mu(1 - 2x)$, classifying parameter ranges where $|f'(x^*)| < 1|f'(x)|$ (stable) or $|f'(x^*)| > 1$ (unstable).

DATA GENERATION

To examine the logistic equation $x_{n+1} = \mu x_n(1 - x_n)$ and how its fixed points interact within the range $[2.4, 4]$ for the value of μ, a coding program was developed to simulate the fixed points. This program was created in Jupyter Notebook.

To achieve this simulation, the program first broke up the range of μ values, $[2.4, 4]$, into a NumPy array. For this project, the number of μ values used was 1000, with each point equally spaced from the next. The program then took each μ value and substituted it into the equation $x_{n+1} = \mu x_n(1 - x_n)$. After substitution, the equation was iterated through a set number of iterations. This initial iteration is called the *transient*. During this initial iteration, the values of $x_{n+1} = \mu x_n(1 - x_n)$ were not saved by the program, because these values served only to approach the fixed point rather than to estimate it.

Once the initial iteration was complete, the program ran a second round of iterations. Unlike the first iteration, the values from this stage were saved to estimate the fixed point for each given μ. The amount of iterations saved is known as the *resolution*, which determines how accurately the final estimate of the fixed point is captured. These saved iteration values were stored in a two-dimensional array, where each row represents a μ value and the columns represent the iterated values from the second iteration set. The 2D array was then used to generate a bifurcation diagram that shows how μ affects the fixed points of the equation $x_{n+1} = \mu x_n(1 - x_n)$.

For this paper, the data for $x_{n+1} = \mu x_n(1-x_n)$ were generated with μ ranging from $[2.4, 4]$, a starting value $x_0 = 0.5$, 1000 transient intervals, and 400 resolution intervals.

RESULTS

When examining the equation $x_{n+1} = \mu * x_n(1 - x_n)$, an interesting pattern is discovered. For range [2.4, 3] there is a single fixed point for each μ value. However at $\mu = 3$, something odd happens. When the fixed point is approached at $\mu >= 3$, the fixed point begins to oscillate between two points. This process repeats at $\mu = 3.45$, where the number of fixed points doubles from two to four. This repeats again and again, with the number of fixed points at a given μ doubling, until around $\mu = 3.57$ where the number of fixed points is so large it becomes chaotic. This is most apparent in the bifurcation diagram, where an almost wave like motion is examined past $\mu = 3.57$.

However, it's important to mention how there might be errors in this graph compared to the actual visualization of the logistic map. Due to the finite amount of μ values examined, the bifurcation diagram may be inaccurate, especially when $\mu > 3.57$ where the number of fixed points rapidly increases. This is due to the fact any difference between two points of μ could have extremely different numbers of fixed points. There may also be errors in how the program stored these numbers due to the limited amount of bits stored per float number, however these errors should be minimal.

DISCUSSION

Chaos theory examines how deterministic systems, those governed by exact mathematical rules, can exhibit long-term behavior that appears random and is highly sensitive to initial conditions. A well-known illustration is the *butterfly effect*, in which tiny differences in starting points grow exponentially, making long-range prediction impossible even though the underlying equations are fully deterministic.

For the logistic map, the key transition occurs when the fixed point loses stability at $\mu = 3$. Stability is determined using the criterion $|2 - \mu| < 1$, which indicates that the fixed point is stable whenever $1 < \mu < 3$. This stable regime is evident in the bifurcation diagram for $2.4 < \mu < 3$, where $|2 - \mu| < 1$.

Once μ reaches 3, the inequality is no longer satisfied ($|2-\mu| \geq 1$), and the fixed point becomes unstable. The system first oscillates between two values, a phenomenon known as a *period-doubling bifurcation*. Additional period doublings follow rapidly, with another clear bifurcation at $\mu \approx 3.45$. Beyond $\mu \approx 3.57$, these successive doublings accumulate and the orbit enters a regime of deterministic chaos. This cascade of period doublings highlights the classic route to chaos in the logistic map.

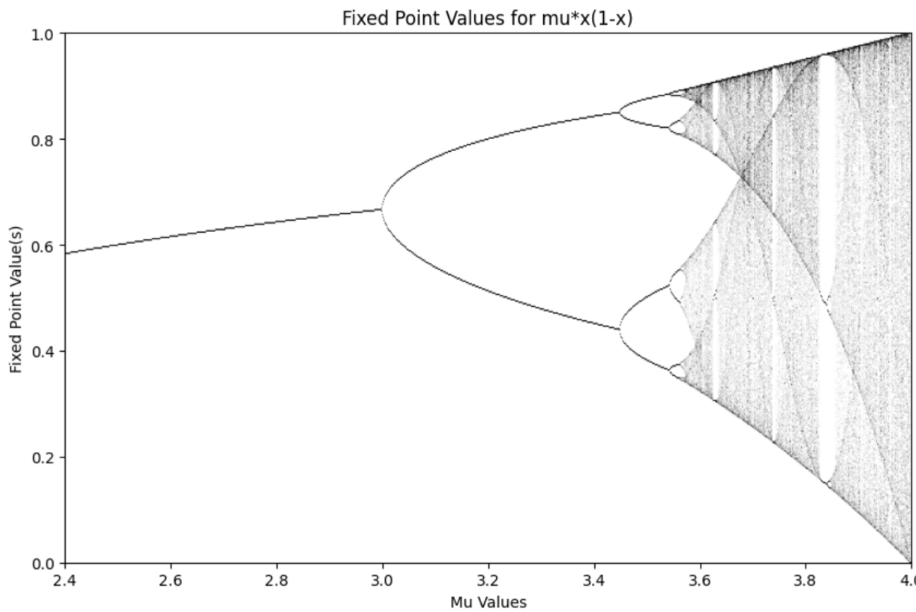

Figure 1. High-resolution bifurcation diagram of the logistic map $x_{n+1} = \mu * x_n(1 - x_n)$.

SUMMARY / CONCLUSION

Our investigation successfully demonstrated the onset of chaos in the logistic map using only elementary numerical methods. The bifurcation diagram we produced displays each stage of the period-doubling cascade and the transition to a chaotic attractor with striking clarity, confirming the theoretical predictions of May (1976) and others. By iterating a simple quadratic equation, we showed that deterministic systems can produce complex, aperiodic behavior which is an essential lesson of chaos theory. Future work could enrich these findings in several directions. Calculating Lyapunov exponents across the parameter range would provide a quantitative measure of sensitivity to initial conditions and definitively locate the boundary between regular and chaotic regimes. Estimating the Feigenbaum constant from successive bifurcation intervals would connect our computations to the universal scaling laws of nonlinear dynamics. Extending the analysis to coupled logistic maps or comparing results with other nonlinear systems, such as the tent map or Hénon map, could illuminate how chaos emerges in higher dimensions. Such extensions would not only refine the present results but also contribute to the broader understanding of universality and complexity in chaos theory.

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