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Numerical Investigation of the Two-Body Problem: Mass Ratio and Eccentricity Effects

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ABSTRACT

This paper presents a numerical investigation of the Newtonian two-body problem for varying mass ratios and orbital eccentricities. We reduce the two-body system to an equivalent one-body problem in the center-of-mass frame and integrate the resulting nondimensional equations using an adaptive fourth-order Runge–Kutta method implemented through SciPy's `solve_ivp` solver. Orbits are computed for four mass ratios ($m_1 : m_2 = 1{:}1, 1{:}2, 1{:}4, 1{:}16$) and for four eccentricities ($e = 0, 0.25, 0.50, 0.75$), producing a total of sixteen configurations. The relative trajectories are then used to reconstruct the individual motions of both bodies about the center of mass. The results show that increasing eccentricity produces progressively elongated elliptical orbits, while increasing mass imbalance causes the heavier body's motion to become increasingly localized near the center of mass and the lighter body to trace larger orbits. These findings confirm key theoretical predictions from celestial mechanics and demonstrate the effectiveness of numerical methods for studying gravitationally bound systems such as planetary orbits.

Keywords: Two-Body Problem, Numerical Integration, Eccentricity, Gravitational Dynamics

INTRODUCTION

The two-body problem represents one of the most fundamental problems in classical and celestial mechanics. It describes the motion of two point masses under their mutual gravitational attraction as described by Newton's law of universal gravitation. This seemingly simple system has profound implications across astrophysics and astronomy.

This problem arises naturally in a wide range of physical systems, including planet–star systems, binary stars, planet–moon systems, artificial satellite motion, and spacecraft navigation. The exact solution of the two-body problem provides the mathematical foundation for Kepler's laws of planetary motion and for modern orbital mechanics. Even in contemporary astrophysics, two-body dynamics remains essential for modeling exoplanetary systems, compact binaries, and gravitational interactions in star clusters (Murray & Dermott 1999).

The two-body problem is fundamentally important not only for its direct physical relevance but also for its critical role in the historical development of physics. Newton's successful explanation of planetary orbits using inverse-square gravity served as one of the earliest validations of classical mechanics. Furthermore, the two-body system provides the only exactly solvable case of the more general N-body problem, which governs the complex dynamics of star clusters, galaxies, and large-scale cosmic structure. Consequently, as noted by Aarseth (2003), computational gravitational simulations ultimately reduce locally to the principles of two-body physics.

This problem also serves as a benchmark system for evaluating the accuracy and stability of numerical integration techniques. Although an analytical solution exists, numerical solvers become essential when the system is embedded in a larger dynamical environment or subject to perturbations. High-precision numerical integration of the two-body equations is fundamental to fields such as astrodynamics, long-term orbital prediction, and mission planning (Battin 1999). This project employs the classical two-body framework to investigate how orbital shape and center-of-mass motion depend on eccentricity and mass ratio, thereby demonstrating both the physical behavior of gravitational systems and the effectiveness of numerical solution methods.

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REDUCTION OF THE TWO-BODY PROBLEM TO A ONE-BODY PROBLEM

To reduce the original two-body problem to a one-body problem, we introduce the relative position vector

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{1}$$

and the center-of-mass coordinate

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}. \tag{2}$$

Substituting these variables into Newton's equations of motion decouples the dynamics into two independent equations: one describing the motion of the center of mass and the other describing the relative motion between the two bodies. The center-of-mass motion satisfies

$$\ddot{\mathbf{R}} = 0, \tag{3}$$

indicating uniform motion. By choosing a reference frame that moves with the center of mass, we set

$$\mathbf{R} = \mathbf{0}. \tag{4}$$

The relative motion then satisfies

$$\ddot{\mathbf{r}} = -\frac{G(m_1 + m_2)}{r^3}\mathbf{r}, \tag{5}$$

which is mathematically equivalent to the motion of a single particle in a central gravitational field. This transformation converts the original coupled two-body system into an effective one-body problem while preserving the full orbital dynamics of the system.

INITIAL CONDITIONS AND NUMERICAL SOLUTION

We solved the equations of motion in nondimensional form, choosing initial conditions at the point of closest approach (pericenter). The coordinate system was oriented so that the orbit is symmetric about the x-axis, with the initial position set to

$$x(0) = -1, \quad y(0) = 0, \tag{6}$$

corresponding to unit minimum separation. The initial velocity was chosen to be purely tangential,

$$\dot{x}(0) = 0, \quad \dot{y}(0) = \sqrt{1 + e}, \tag{7}$$

where e is the orbital eccentricity. We investigated four eccentricities,

$$e = 0,\ 0.25,\ 0.50,\ 0.75,$$

and four mass ratios,

$$m_1 : m_2 = 1{:}1,\ 1{:}2,\ 1{:}4,\ 1{:}16.$$

For each eccentricity, the nondimensional orbital period was computed as

$$T = \frac{2\pi}{(1 - e)^{3/2}}, \tag{8}$$

and the equations were integrated over one full orbital period.

The second-order differential equations were rewritten as a system of four first-order ordinary differential equations and solved numerically using SciPy's `solve_ivp` solver with an adaptive Runge–Kutta (RK45) method. Each solution was evaluated at 1000 evenly spaced time points, with tight relative and absolute error tolerances of 10^{-9} and 10^{-12} to ensure numerical accuracy. After solving for the relative motion, the individual trajectories of both masses were reconstructed in the center-of-mass frame using mass-weighted coordinate transformations, allowing visualization of the motion of each body for all sixteen configurations.

ORBIT VISUALIZATIONS

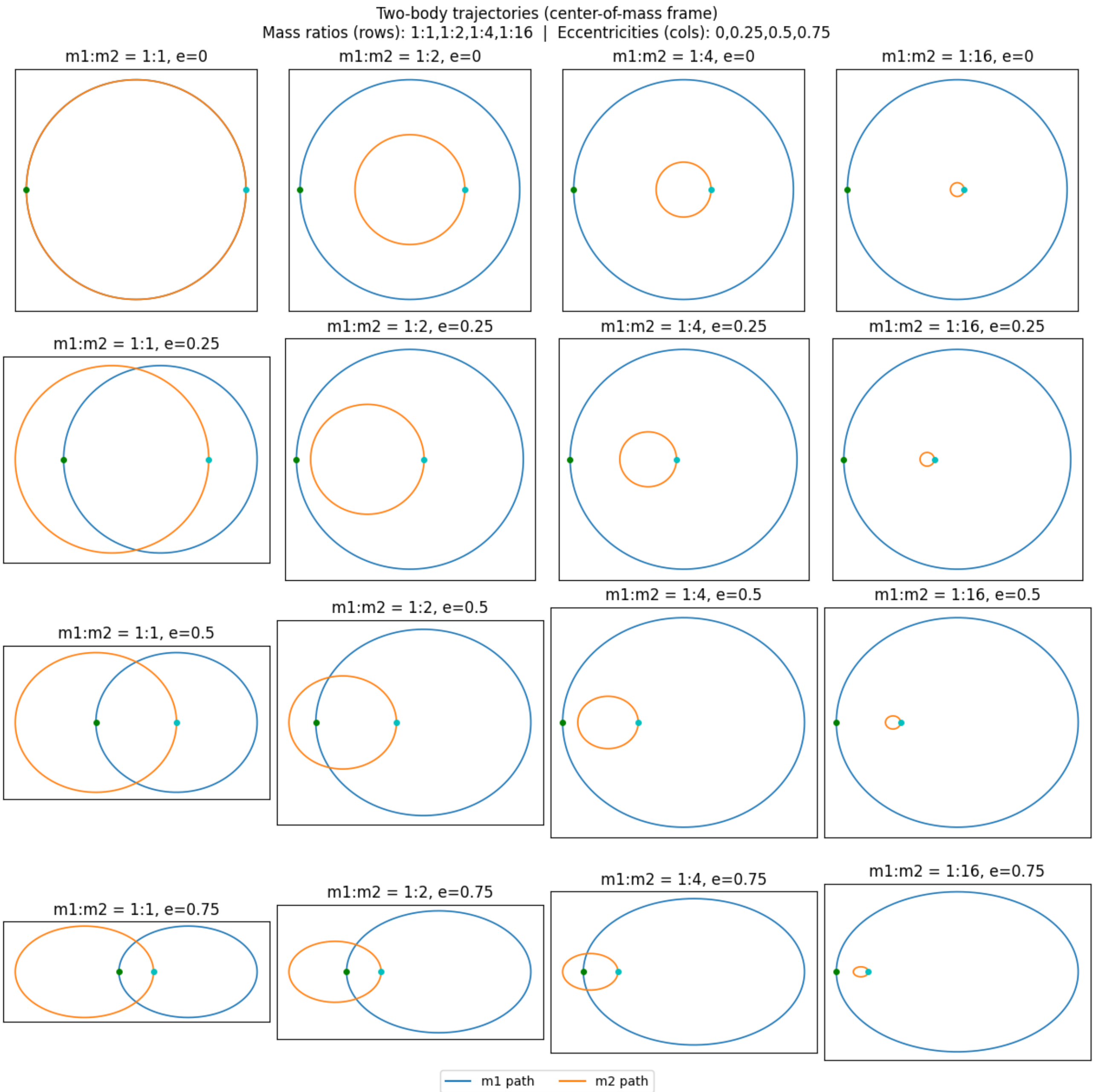

Figure 1. Center-of-mass trajectories of the two bodies for sixteen configurations spanning four mass ratios and four eccentricities.

RESULTS AND SUMMARY OF FINDINGS

Our numerical simulations produced orbits for sixteen configurations spanning four mass ratios and four eccentricities. In the center-of-mass frame, the equal-mass case ($m_1 : m_2 = 1{:}1$) shows both bodies following identical circular or elliptical paths on opposite sides of the origin, reflecting the symmetry of the system. As the mass ratio becomes more unequal ($1{:}2, 1{:}4, 1{:}16$), the lighter body traces a progressively larger orbit while the heavier body remains confined to an increasingly smaller region near the center of mass. In the most extreme case (1:16), the heavier mass moves only slightly, and the lighter mass sweeps out a wide orbit, closely resembling a planet orbiting an almost fixed star.

Across all mass ratios, increasing eccentricity from $e = 0$ to $e = 0.75$ transforms the trajectories from circles into increasingly elongated ellipses, with the bodies moving fastest near *pericenter* (the point of closest approach between the two masses) and slowest near *apocenter* (the point of maximum separation), consistent with Keplerian motion.

These results confirm two key trends: eccentricity primarily controls the shape and radial variation of the orbit, while the mass ratio determines how that orbital motion is partitioned between the two bodies. The center of mass remains fixed at the origin in every configuration, and the orbits close neatly after one period, indicating that the numerical scheme preserves the conservative nature of the gravitational system. Physically, the equal-mass, moderately eccentric cases resemble symmetric binary star systems, while the high mass-ratio cases mimic star–planet systems in which only the lighter body exhibits large-scale motion. Overall, the simulations reproduce classical expectations from celestial mechanics and provide a clear visual link between mass ratio, eccentricity, and the resulting orbital dynamics.

FUTURE WORK

Several extensions of this work could further contribute to our understanding of the two-body problem. First, this study was limited to bound elliptical orbits with $0 \leq e < 1$; extending the numerical analysis to parabolic and hyperbolic trajectories ($e \geq 1$) would enable investigation of gravitational flybys and scattering events relevant to comet dynamics and stellar encounters (Danby 1992).

Second, although an adaptive Runge–Kutta method was sufficient for single-orbit simulations, long-term integrations would benefit from the use of symplectic integrators, which are known to preserve energy and angular momentum more accurately over many orbital periods (Yoshida 1990; Hairer et al. 2006).

Third, small perturbations could be added to the gravitational force law, such as relativistic corrections or weak third-body interactions, to study orbital precession and resonance phenomena that occur in realistic astrophysical systems (Aarseth 2003).

Finally, performing a systematic parameter sweep across continuous ranges of eccentricity and mass ratio, paired with quantitative diagnostics such as energy error, angular momentum conservation, and orbital stability, would yield a more comprehensive and statistically robust characterization of two-body dynamics and its fundamental role in many-body gravitational systems.

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