Problem Set 2

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Last updated September 25, 2024

Due: 10/8, 11:59pm.
Please typeset your solutions in LaTeX.

Problem 1 (On $\models$, borrowed from Aaron Potechin)**.** Consider the following polynomial equation in 3 variables, x, y, z.

$$(x^2 + 1)y = z^2.$$

Because it implies $y = \frac{z^2}{x^2+1}$, any solution (x, y, z) to the above must have $y \geq 0$. We will see if sum-of-squares can capture this reasoning.

1. Construct a degree 4 pseudoexpectation $\widetilde{\mathbb{E}}$ in variables x, y, z such that $\widetilde{\mathbb{E}} \models (x^2 + 1)y = z^2$ but $\widetilde{\mathbb{E}}\, y < 0$. (Computer-aided proofs are allowed.)

   By $\widetilde{\mathbb{E}} \models (x^2 + 1)y = z^2$, we mean that for any polynomial p of degree at most 1 in x, y, z, $\widetilde{\mathbb{E}}\, p(x, y, z)(x^2 + 1)y = \widetilde{\mathbb{E}}\, p(x, y, z)z^2$.

2. Despite the above, show that there exists a sum-of-squares refutation to the following system of polynomial inequalities, for any $c > 0$: $\{(x^2 + 1)y = z^2, y \leq -c\}$.

Problem 2. Suppose $\widetilde{\mathbb{E}}$ is a pseudoexpectation of degree d, with d even, and $\widetilde{\mathbb{E}} \models p \leq 0, p \geq 0$ for some polynomial p. (Informally, we have been writing $\widetilde{\mathbb{E}} \models p = 0$.) Show that for every q such that the degree of pq is at most d, we have $\widetilde{\mathbb{E}}\, pq = 0$.

Problem 3. Unreleased.