Problem Set 1 – Solutions

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Problem 1 (SoS proofs beyond eigenvalues)**.** We saw in lecture that if $M \in \mathbb{R}^{n \times n}$ is a symmetric matrix with maximum eigenvalue λ_{max}, then there is always a degree-2 SoS proof that $x^\top M x \leq \lambda_{max} \cdot \|x\|^2$ – that is, the polynomial $\lambda_{max} \cdot \|x\|^2 - x^\top M x$ is a sum of squares.

1. Show that this bound is tight, in the sense that if c is such that $c\|x\|^2 - x^\top M x$ is a sum of squares, then $c \geq \lambda_{max}$.
2. Construct a symmetric matrix M such that there exists $c < \lambda_{max}(M)$ and linear functions $f_1, \ldots, f_m$ such that $c \cdot \|x\|^2 - x^\top M x = \sum_{i \leq m} f_i(x)^2$ for every $x \in \{\pm 1\}^n$. This shows that the flexibility in a quadratic proof to use a sum of squares polynomial which is equal to $c\|x\|^2 - x^\top M x$ only for certain xs (namely, $x \in \{\pm 1\}^n$) makes the definition more powerful.

Solution, Part 1 We begin with the first item. Suppose that $c\|x\|^2 - x^\top M x = \sum_{i \leq m} f_i(x)^2$ for some linear functions $f_1, \ldots, f_m$. Let L be the matrix whose rows are the vectors $\ell_1, \ldots, \ell_m$ such that $f_i(x) = \langle \ell_i, x \rangle$. Then the quadratic forms of the matrices $cI - M$ and $L^\top L$ are identical. If the quadratic forms of two symmetric matrices are the same, then so are the matrices, so we have $cI - M = L^\top L$, which is to say that all eigenvalues of $cI - M$ are nonnegative. Note that $c - \lambda_{\max}$ is an eigenvalue of $cI - M$, so $c - \lambda_{\max} \geq 0$, hence $c \geq \lambda_{\max}$.

Solution, Part 2 Consider the matrix $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The maximum eigenvalue of M is 1. We claim that $\frac{1}{2}\|x\|^2 - x^\top M x = 0$ for every $x \in \{\pm 1\}^2$. Since 0 is a (trivial) sum of squares, this will finish the proof. Indeed, if $x = (x_1, x_2)$, then $\frac{1}{2}\|x\|^2 - x^\top M x = \frac{1}{2}(x_1^2 + x_2^2) - x_1^2 = 1 - 1 = 0$.

Problem 2 (Max cut in almost-bipartite graphs)**.** Show that there is a polynomial-time algorithm with the following guarantee: given a graph $G = (V, E)$ such that there is a cut which cuts $(1 - \varepsilon)|E|$ edges, the algorithm outputs a cut which cuts $(1 - \tilde{O}(\sqrt{\varepsilon}))|E|$ edges. ($\tilde{O}$ can hide factors of $\log(1/\varepsilon)$, though this is not strictly necessary.)

You may use the following basic anticoncentration fact for Gaussians: if $Z \sim N(0, 1)$, then $Pr(|Z| \leq \delta) = O(\delta)$.

Solution The algorithm is simply the Goemans-Williamson (GW) Max-Cut algorithm from lecture. Let $G = (V, E)$ be a graph with a cut which cuts $(1 - \varepsilon)|E|$ edges. Let $\widetilde{\mathbb{E}}$ be a pseudoexpectation such that

$$\widetilde{\mathbb{E}} \sum_{(i,j) \in E} \frac{1}{2} - \frac{x_i x_j}{2} \geq (1 - \varepsilon)|E| \,.$$

Since the GW algorithm will find such a pseudoexpectation and round it, it will be enough to show that the GW rounding scheme, applied to such a pseudoexpectation, will in expectation produce a cut which cuts $(1 - O(\sqrt{\varepsilon}))|E|$ edges.

We will use the following lemma, whose proof we defer.

Lemma. Let g, h be multivariate Gaussians distributed as $\mathcal{N}\left(0, \begin{pmatrix} 1 & -(1-\rho) \\ -(1-\rho) & 1 \end{pmatrix}\right)$, where $\rho \geq 0$. Then $\Pr(\operatorname{sign}(g) \neq \operatorname{sign}(h)) \geq 1 - O(\sqrt{\rho})$.

Recall that in the rounding phase of the GW algorithm, the probability that the edge (i, j) is cut is the probability that a pair of Gaussians g, h with $\mathbb{E}\, g = \mathbb{E}\, h = 0$ and $\mathbb{E}\, g^2 = \mathbb{E}\, h^2 = 1$ and $\mathbb{E}\, gh = \widetilde{\mathbb{E}}\, x_i x_j$ have opposite signs. Let S be the cut produced by the GW algorithm. By the lemma, then,

$$\mathbb{E}\, |E(S, \overline{S})| \geq \sum_{(i,j)\in E} 1 - O\left(\sqrt{1 + \widetilde{\mathbb{E}}\, x_i x_j}\right) .$$

By Cauchy-Schwarz,

$$\sum_{(i,j)\in E} \sqrt{1 + \widetilde{\mathbb{E}}\, x_i x_j} \leq \sqrt{|E|} \cdot \sqrt{\sum_{(i,j)\in E} 1 + \widetilde{\mathbb{E}}\, x_i x_j} = \sqrt{|E|} \cdot \sqrt{|E| + \widetilde{\mathbb{E}} \sum_{(i,j)\in E} x_i x_j} \leq O(\sqrt{\varepsilon}|E|) .$$

Proof of Lemma. We can write

$$h = -(1-\rho)g + \sqrt{1 - (1-\rho)^2} z = -(1-\rho)g + \sqrt{2\rho - \rho^2} z .$$

where $z \sim \mathcal{N}(0,1)$ is independent of g. To have sign(g) = sign(h), we have to have $|z| \geq \frac{1-\rho}{\sqrt{2\rho}}|g| = \Omega(|g|/\sqrt{\rho})$. For each fixed g, we have $\Pr(|z| \geq \Omega(|g|/\sqrt{\rho})) = \exp(-\Omega(|g|^2/\rho))$. So,

$$\mathbb{E}_g \Pr(|z| \geq \Omega(|g|/\sqrt{\rho})) \leq \mathbb{E}_g \exp(-\Omega(|g|^2/\rho)) = O(\sqrt{\rho}) .$$

where for the last step we used the standard formula

$$\mathbb{E}_g \exp(-cg^2) = \frac{1}{\sqrt{1 + 2c}} .$$

□

Problem 3 (Cauchy-Schwarz for Pseudoexpectations)**.** An important fact about any probability distribution μ is that for any real-valued f and g, $\mathbb{E}_{x\sim\mu} f(x)g(x) \leq \sqrt{\mathbb{E} f(x)^2} \cdot \sqrt{\mathbb{E} g(x)^2}$. Show that if $\tilde{\mathbb{E}}$ is a (degree 2) pseudoexpectation and f, g are linear functions, one has $\tilde{\mathbb{E}} f(x)g(x) \leq \sqrt{\tilde{\mathbb{E}} f(x)^2} \cdot \sqrt{\tilde{\mathbb{E}} g(x)^2}$.

Solution Fix a degree-2 pseudoexpectation $\widetilde{\mathbb{E}}$ and linear functions f, g. For any real numbers α, β, we have

$$0 \leq \widetilde{\mathbb{E}}(\alpha f + \beta g)^2 = \alpha^2\, \widetilde{\mathbb{E}}\, f^2 + 2\alpha\beta\, \widetilde{\mathbb{E}}\, fg + \beta^2\, \widetilde{\mathbb{E}}\, g^2 ,$$

and hence

$$-2\alpha\beta\, \widetilde{\mathbb{E}}\, fg \leq \alpha^2\, \widetilde{\mathbb{E}}\, f^2 + \beta^2\, \widetilde{\mathbb{E}}\, g^2 .$$

Now take $\alpha^2 = \sqrt{\widetilde{\mathbb{E}}\, g^2} / \sqrt{\widetilde{\mathbb{E}}\, f^2}$ and $\beta^2 = \sqrt{\widetilde{\mathbb{E}}\, f^2} / \sqrt{\widetilde{\mathbb{E}}\, g^2}$ to get the conclusion.

Problem 4 (Max cut on the triangle)**.** The three-edge triangle graph has a max-cut value of 2. We saw in class that there is a quadratic proof that the maximum cut is at most 2.9. Is there a quadratic proof that the max-cut value is at most 2.00001? (Prove the correctness of your answer.)

Solution There is no such quadratic proof. It will suffice to exhibit a degree-2 pseudoexpectation $\widetilde{\mathbb{E}}$ such that $\widetilde{\mathbb{E}} \sum_{(i,j)\in E} \frac{1}{2} - \frac{x_i x_j}{2} \geq 2.01$, where E is the set of edges of the triangle graph. We will let $\widetilde{\mathbb{E}} x_1 = \widetilde{\mathbb{E}} x_2 = \widetilde{\mathbb{E}} x_3 = 0$, and $\widetilde{\mathbb{E}} x_i x_j = -0.5$ for $i \neq j$. We need to check that this is a valid pseudoexpectation, for which it will suffice to show that the following matrix is positive semidefinite:

$$\begin{pmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{pmatrix}.$$

This matrix is diagonal-dominant, so it is positive semidefinite.

For the $\widetilde{\mathbb{E}}$ we have defined, $\widetilde{\mathbb{E}} \sum_{(i,j)\in E} \frac{1}{2} - \frac{x_i x_j}{2} = \frac{3}{2} + \frac{3}{4} = 2.25$. This finishes the proof.