Lab 5B: Numerical Integration and Differentiation

Due: Sunday, March 31st at 11:59PM

Problem 1

It can be shown that for analytic functions (i.e. those that have a derivative in a domain containing an interval $[a, b]$ in the complex plane) that

$$\mathrm{Im}(f(x+ih))/h = f'(x) + \mathcal{O}(h^2)\,.$$

The approximation $\mathrm{Im}(f(x+ih))/h$ for for the *real* derivative $f'(x)$ is called the *Squire-Trapp* formula, after its inventors. In detail, its derivation is as follows.

$$\begin{aligned} f(x+ih) &= f(x) + ihf'(x) + \frac{(ih)^2}{2}f''(x) + \frac{(ih)^3}{6}f'''(x) + \mathcal{O}(h^4) \\ \therefore\ \mathrm{Im}(f(x+ih)) &= hf'(x) - \frac{h^3}{6}f'''(x) + \mathcal{O}(h^5) \\ \text{so}\quad \frac{\mathrm{Im}(f(x))}{h} &= f'(x) - \frac{h^2}{6}f'''(x) + \mathcal{O}(h^4)\,. \end{aligned}$$

Note that that derivation assumes that x and h are real, and that $f(x)$ is real for real x. The formula doesn't work, otherwise. The similarly accurate central difference formula

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h^2}{6}f'''(x) + \mathcal{O}(h^4)$$

suffers more from cancellation error for small h. But the Squire-Trapp formula needs an implementation of f that works accurately near to the real axis (just into the complex plane), and not every mathematical function has such in MATLAB. One set of examples are the *Bessel functions* which have the integral

$$J_n(z) = \frac{1}{\pi}\int_{t=0}^{\pi} \cos(nt - z\sin(t))\,dt\,. \tag{1}$$

This translates to the single line below:

```
J = @(n, z) integral(@(t) cos(n*t - z*sin(t)), 0, pi)/pi;
```

Using this integral to evaluate the Bessel functions is surprisingly better for differentiation with the Squire-Trapp formula than the built-in `besselj` is, as you will show here. Note that the exact derivatives are given by

$$J_n'(x) = \frac{nJ_n(x)}{x} - J_{n+1}(x)$$

for $n = 0, 1, 2, \ldots$.

1. For $x = 22.3$ and `h = logspace(-14, 0, 2019);` compute $\mathrm{Im}(J_n(x+ih)/h) - J_n'(x)$ for various n and plot it on a `loglog` plot. Repeat using the built-in `besselj`. You should see that the Squire-Trapp formula works better for your integral than it does for the built-in.

2. Do the same for the central difference formula; you should see that for the integral form there isn't much difference, this time, owing to catastrophic cancellation.

3. The error terms in the Squire-Trapp formula and the central difference formula as printed above seem to be the same but with opposite sign. That suggests that by averaging the two estimates of $f'(x)$ we should get a fourth-order formula. Try it and see.

Problem 2

A calm and meditative scuba diver follows the given depth profile in a peaceful lagoon[1]. The file `profile.mat`, which can be downloaded from **OWL**, contains the depth profile measured at every two minutes for the duration of the dive. To load this data into MATLAB, use the following command

```
load profile.mat
```

This loads two variables:

- `t`: The time points at which the depth profiles were taken, in minutes.
- `profile`: The depth profile of the scuba diver, in metres.

A large man, though a moderately skilled diver, breathes at a constant rate of $r = 15$ L/minute; of course one litre of air at 10 m depth is twice as much air as one litre of air at 0 m because of the pressure of the water surrounding the diver (and the regulator and his lungs); at 20 m it's 3 times as much. This information is encoded in the formula below (valid for salt water: for fresh water, the number 10 is changed to 10.4—but this is a salt water dive).

1. How much air A does he use on the dive? Here is an integral for it, which you must evaluate numerically in some fashion.
$$A = \int_{t=0}^{40} \left(1 + \frac{d(t)}{10}\right) \cdot r\,\mathrm{d}t\,.$$

2. Diving the exact same profile dive the next day, he sees a box jellyfish at 20 m at minute 13. This increases r to $2r$ for ten minutes (box jellyfish are lethal). How much extra air gets used? [Remark: This part really happened.]

3. Ascending too fast puts a diver at risk of *decompression illness*. A common safety limitation is 9 meters per minute: does the diver exceed this limit with this profile?

You will have made some assumptions about the given depth profile in order to compute this integral and derivative. Document your assumptions. Do your assumptions impact your conclusion for part 3? **Remark**: If you can install the CurveFitting package, and learned how to use it, you could investigate different ways to fit and integrate and differentiate this data with some pretty good programs. But the intent of this question is for you to program it yourself, so we would prefer that you did it with your own programs.

Problem 3

Use MATLAB's built-in quadrature function `integral` to evaluate

$$A(x) = \frac{1}{x}\int_0^x \frac{\sqrt{-2\log\cos u^2}}{u^2}\mathrm{d}u = 1 + \frac{x^4}{60} + \frac{x^8}{480} + \cdots$$

for various values of x. Comment on your results and how you know that your results are incorrect. W. Kahan shows a work-around to numerically integrate this integral correctly in [1]. Implement this work-around and show that you obtain the correct results for various values of x.

[1] The data was made up. Sorry. We could all use some peaceful lagoons at the moment.

Problem 4

> **A real sign at a beach near Cairns, Australia reads as follows**:
> Warning! Saltwater crocodiles. Box Jellyfish. Sharks. Swim at your own risk.

A box jellyfish filament contains many tiny spring-like venom delivery packages called nematocysts. A single filament of length about 30 cm carries enough venom to kill (very painfully) an unprotected human (you can relax a bit, though: these spring-loaded spikes are very short, and even thin fabric is enough to provide protection).

A box jellyfish filament of unknown length touches the arm of our adventurous TA who is swimming off this beach near Cairns, Australia. The filament's position on her arm has the x and y data in cm, which can be found in `jelly.mat` on **OWL**. Is the length greater than 30cm? Will she live?

Answer: Of course she'll live. She's just taken the jaws of a salt-water crocodile and used them to drive off a shark, at the same beach. A mere box jellyfish won't cause our TA more than annoyance. But please do the computation of the length of the filament as if it were a normal timid tourist who had encountered the venomous creature.

Submission Requirements

All files should be submitted on OWL on the **Lab 5B** submission. Only `.m` and `.pdf` files will be accepted for grading. Submit all files needed to reproduce the results you found in the lab.

Your report should be written in a way such that someone with no programming experience can understand what the problem was, how you solved it and what the results were.

References

[1] William Kahan. Handheld calculator evaluates integrals. *Hewlett-Packard Journal*, 31(8):23–32, 1980.