Graphs and complexes of lattices

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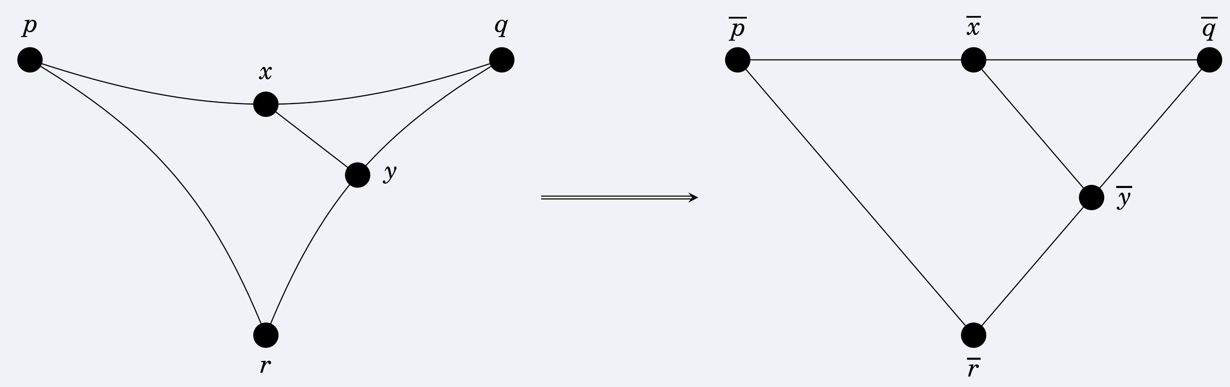
$\mathbf{CAT}(0)$ spaces

A geodesic metric space X is $\mathbf{CAT}(0)$ if for every geodesic triangle $P = \triangle(p, q, r) \subseteq X$ there exists a comparison triangle in $\mathbb{E}^2$ with the same side lengths as P such that for each pair of points $x, y \in \partial P$ we have:

$$d_X(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$



A group is $\mathbf{CAT}(0)$ if it acts properly cocompactly by isometries on a $\mathbf{CAT}(0)$ space.

Examples

- $\mathbb{E}^n$;
- Trees;
- Non-compact symmetric spaces (e.g. $\mathbb{R}\mathrm{H}^2$);
- Infinite buildings.

Lattices

Let $H = \mathbf{Isom}(X)$ be a locally compact group with Haar measure μ. A discrete subgroup $\Gamma \leq H$ is:

- a *lattice* if X/Γ has finite covolume;
- a *uniform lattice* if X/Γ is compact.

For a lattice Γ in a product $\prod_{i=1}^n H_i$ we say Γ is:

- *irreducible* if the projection to each subproduct of the H_i is non-discrete;
- *reducible* otherwise.

Examples

- Crystallograhic groups in $\mathbf{Isom}(\mathbb{E}^n)$;
- Free groups acting on trees in $\mathbf{Aut}(\mathcal{T})$;
- Arithmetic subgroups of Lie groups e.g. $\mathbf{SL}_2(\mathbb{Z}[\sqrt{2}])$ in $\mathbf{SL}_2(\mathbb{R})^2$;
- Graph products of finite groups acting on right-angled buildings.

Questions

Let X be a metric space and $H = \mathbf{Isom}(X)$ with Haar measure μ.

1. *Does H have lattices?*
2. *What are properties of a generic lattice in H?*
3. *Do the properties of lattices in H reflect properties of H?*
4. *Can we classify lattices in H up to isomorphism, commensurability, or isometry?*

Trees

A *tree* $\mathcal{T}$ is a connected graph with no loops.

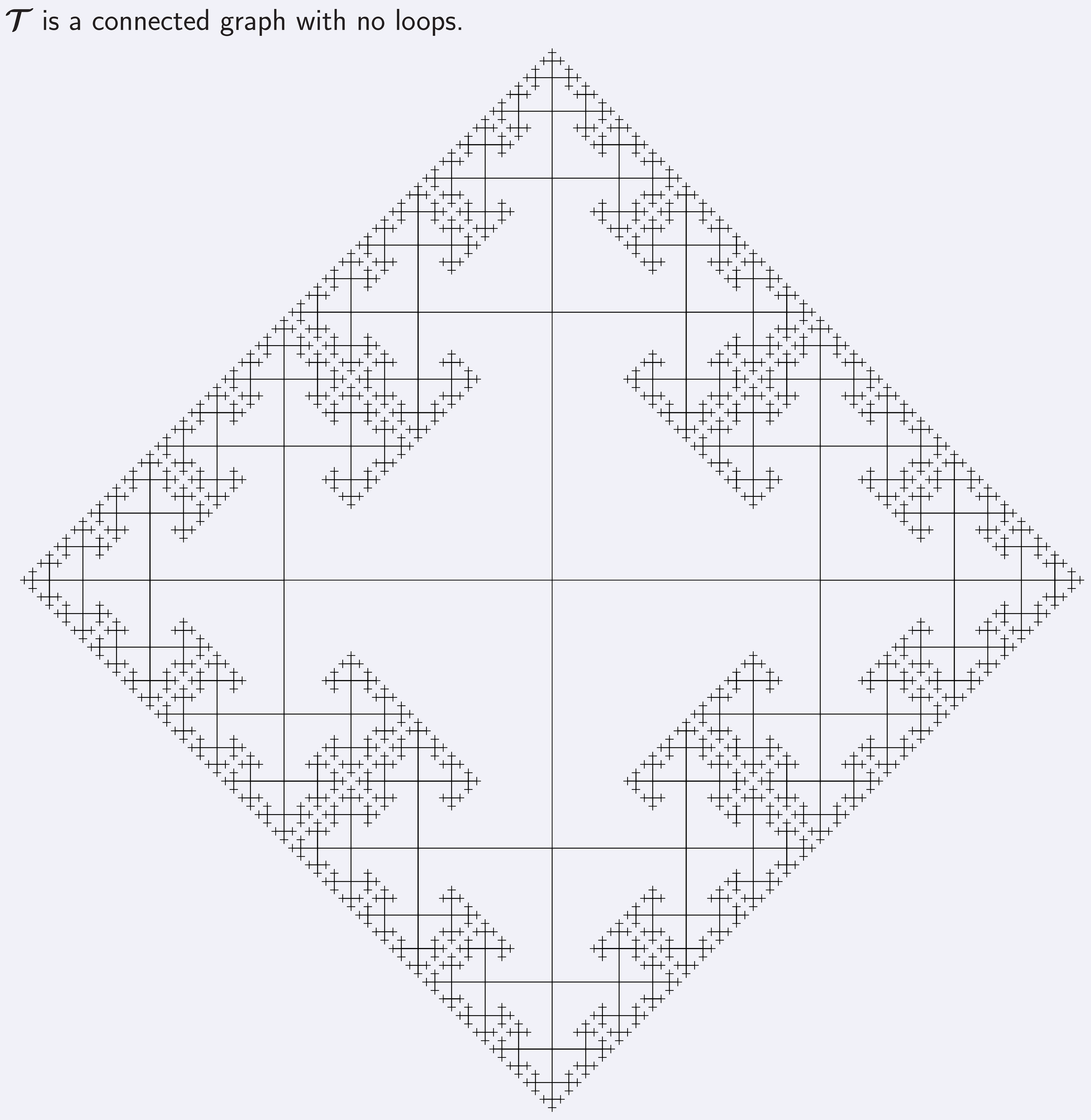


Figure 1: A 4-regular tree.

Theorem (Bass-Kulkarni 1990)

1. $\mathbf{Aut}(\mathcal{T})$ *admits a uniform lattice if and only if $\mathcal{T}$ is the universal cover of a finite connected graph.*
2. *Lattices in $\mathbf{Aut}(\mathcal{T})$ are fundamental groups of graphs of groups acting faithfully on their Bass-Serre tree. In particular, any uniform lattice is virtually free.*

Biautomatic groups

- An *automatic group* is a finitely generated group equipped with several finite-state automata. These automata represent the Cayley graph of the group. That is, they can tell if a given word representation of a group element is in a "canonical form" and can tell if two elements given in canonical words differ by a generator.
- A group is *biautomatic* if it has two multiplier automata, for left and right multiplication by elements of the generating set, respectively.

Until 2019 it was not known if every $\mathbf{CAT}(0)$ group is biautomatic.

Leary-Minasyan groups

Let $A \in \mathbf{O}(2)$, let L_1 be a finite index subgroup of $\mathbb{Z}^2$ and let $L_2 = A(L_1)$. Consider the following graph of groups:

$$\mathbb{Z}^2 \circlearrowleft L_1^t = L_2$$

We call the fundamental group a *Leary-Minasyan group*. Such a group has the following presentation:

$$\mathbf{LM}(A) = \langle a, b, t \mid [a, b],\ tL_1t^{-1} = L_2 \rangle$$

The group is equipped with a representation into $\mathbf{Isom}(\mathbb{E}^2)$ which can be described as follows:

$$\phi : \mathbf{LM}(A) \to \mathbf{Isom}(\mathbb{E}^2) \quad \text{by} \quad \begin{cases} a \mapsto [1, 0]^T \\ b \mapsto [0, 1]^T \\ t \mapsto \quad A. \end{cases}$$

The group acts freely cocompactly on $\mathbb{E}^2 \times \mathcal{T}$ where $\mathcal{T}$ is the Bass-Serre tree. In particular $\mathbf{LM}(A)$ is a $\mathbf{CAT}(0)$ group. Note that the construction generalises to $\mathbb{E}^n \times \mathcal{T}$.

Example

Concretely we can take

$$A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}, \quad L_1 = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle \quad \text{and} \quad L_2 = \left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\rangle$$

in this case

$$\mathbf{LM}(A) = \langle a, b, t \mid [a, b],\ ta^2bt^{-1} = a^2b^{-1},\ tab^2t^{-1} = a^{-1}b^2 \rangle.$$

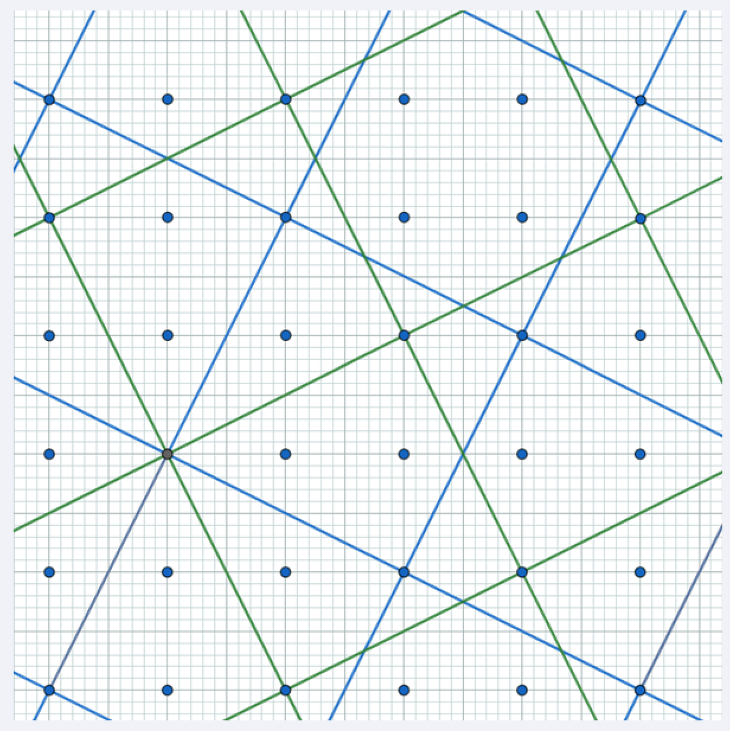


Figure 2: The action of a Leary-Minasyan group on $\mathbb{E}^n$. The orthogonal matrix maps the green squares to the blue squares.

Theorem (Leary-Minasyan 2019)

Let $\Gamma = \mathbf{LM}(A)$, $\mathcal{T}$ be the Bass-Serre tree of Γ and $T = \mathbf{Aut}(\mathcal{T})$. Then Γ is virtually biautomatic if and only if A has finite order if and only if Γ is reducible as an $(\mathbf{Isom}(\mathbb{E}^n) \times T)$-lattice.

Uniform $\mathrm{Isom}(\mathbb{E}^n) \times T$)-lattices

The following lemma gives a rough classification of $\mathbf{Isom}(\mathbb{E}^n) \times T$-lattices.

Lemma (H. 2021)

Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{E}$ and let $T = \mathbf{Aut}(\mathcal{T})$. Every uniform lattice in $\mathbf{Isom}(\mathbb{E}^n) \times T$ splits as a finite graph of virtually abelian groups.

Using the lemma we can prove a number of generic properties for irreducible lattices in $\mathbf{Isom}(\mathbb{E}^n) \times T$.

Theorem (H. 2021)

Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{E}$ and let $T = \mathbf{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathbf{Isom}(\mathbb{E}^n) \times T)$-lattice. The following are equivalent:

1. *Γ is an irreducible $(\mathbf{Isom}(\mathbb{E}^n) \times T)$-lattice;*
2. *Γ is irreducible as an abstract group;*
3. *Γ acts on $\mathcal{T}$ faithfully;*
4. *Γ does not virtually fibre;*
5. *Γ is C^*-simple;*
6. *and if $n = 2$, Γ is non-residually finite and not virtually biautomatic.*

The theorem is optimal in the sense that we can show for $n \geq 3$ all irreducible lattices are non-residually finite and not virtually biautomatic. However, there are also reducible lattices with these properties (consider $\mathbb{Z} \times \mathbf{LM}(A)$).

Theorem (H. 2021)

Let $n \geq 2$ and let X be a pentagonal building of thickness $10n$. There exist uniform lattices acting on $\mathbb{E}^n \times X$ which are not virtually biautomatic.

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