BNSR INVARIANTS AND ℓ^2-HOMOLOGY

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Dedicated to the memory of Peter A. Linnell.

ABSTRACT. We prove that if the nth ℓ^2-Betti number of a group is non-zero then its nth BNSR invariant over $\mathbb{Q}$ is empty, under suitable finiteness conditions. We apply this to answer questions of Friedl–Vidussi and Llosa Isenrich–Py about aspherical Kähler manifolds, to verify some cases of the Singer Conjecture, and to compute certain BNSR invariants of poly-free and poly-surface groups.

1. INTRODUCTION

Given a compact CW complex X, how can we tell if X, or perhaps a complex homotopy-equivalent to one of its finite covers, fibres over the circle? Or, to say the same thing in a different way, does there exists a compact CW complex Y and a homeomorphism f thereof such that a finite cover of X is homotopic to the mapping torus of f, that is, the quotient of $Y \times [0,1]$ by the relation $(y,0) \sim (f(y),1)$?

Answering this question is not easy, but homology gives us tools that obstruct the existence of such mapping tori.

It was shown by Lück [Lüc94] that non-vanishing ℓ^2-Betti numbers $b_n^{(2)}$ obstruct finite CW complexes being homotopic to mapping tori. If we change focus from spaces to groups (passing via classifying spaces), the problem is reformulated to asking about the existence of maps from G to the integers with kernels with good finiteness properties.

Theorem 1.1 (Lück)**.** *Let G be a group of type F_n, that is admitting a classifying space with finite n-skeleton, and let $\varphi\colon G \twoheadrightarrow \mathbb{Z}$ be an epimorphism. If $b_n^{(2)}(G) \neq 0$, then $\ker\varphi$ is not of type F_n.*

For a similar statement for connected CW complexes, see [Lüc02, Theorem 6.63].

The ℓ^2-Betti numbers can be computed for CW complexes with a cocompact group action. They are analytic in flavour, and their definition makes heavy use of the Murray–von Neumann theory. We will discuss them in detail in the main text. A wealth of information about them can be found in Lück's book [Lüc02].

Bieri, Neumann, Renz, and Strebel [BNS87, BR88] introduced a family of group invariants which are intimately related to the finiteness properties of kernels of homomorphisms $G \to \mathbb{R}$. The definition is a little technical,

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and we postpone it until Section 2.G; on an intuitive level, given a non-trivial homomorphism $\varphi\colon G \to \mathbb{R}$, the BNSR invariants focus on finiteness properties of the monoid $\varphi^{-1}([0,\infty)) \subset G$, rather than that of $\ker\varphi$. Also, instead of looking for type F_n as above, we are interested in its homological counterpart, type FP_n.

Even though the connection between the BNSR invariants and existence of epimorphisms to $\mathbb{Z}$ with kernels of type FP_n (algebraic fibring) is well established, the precise relation between the nth ℓ^2-Betti number and the nth BNSR invariants Σ^n has remained mysterious. Indeed, the question is rather subtle: for $n = 1$, Brown [Bro87] gave a characterisation of the integral characters $G \twoheadrightarrow \mathbb{Z}$ in $\Sigma^1(G)$ as those that are induced by G splitting as an ascending HNN extension over a finitely generated group; such a splitting constitutes the structure of an *algebraic mapping torus*. There is however no such characterisation for $n > 1$; the best result in this direction seems to be a theorem of Renz for $n = 2$ [Ren89] which gives combinatorial conditions for a character to be in $\Sigma^2(G)$ coming from the relations of G.

Our main theorems clarify the link between the BNSR invariants and ℓ^2-Betti numbers.

Theorem A. *Let G be a group of type $\mathsf{FP}_n(\mathbb{Q})$. If $b_n^{(2)}(G) \neq 0$, then $\Sigma^n(G;\mathbb{Q}) = \varnothing$.*

Recall that type FP_n is a homological version of type F_n; we give a formal definition in Section 2.G, where we also introduce BNSR invariants over the rationals.

The theorem was already known in a special case, namely for RFRS groups (see [HK25, Theorem 5.10] building on [Kie20] and [Fis24]). For finitely generated groups, being RFRS is the same as being residually {virtually abelian and locally indicable}, see [OS25].

It is also significantly easier to establish Theorem A when G is torsion free and known to satisfy the Atiyah conjecture. We will discuss this in detail in Section 6.

BNSR invariants are homological in nature, and they are defined using a resolution of a trivial G-module. Instead of the resolution, one can use any chain complex, in particular the cellular chain complex of the universal cover of a connected CW complex. This allows one to define BNSR invariants of connected CW complexes. In this context, we offer the following.

Theorem B. *Let X be a connected CW complex with finite n-skeleton. If $b_n^{(2)}(\widetilde{X};\pi_1 X) \neq 0$, then $\Sigma^n(X;\mathbb{Q}) = \varnothing$.*

The special case of the above result with X being a closed smooth manifold can be deduced from work of Farber [Far00, Corollary 2] (based on [NS86]). The methods used here are of a very different nature: they are ring-theoretic rather than analytic, as is the case of Farber.

The motivation for the authors to begin investigating the connection between vanishing of the nth BNSR invariants (over $\mathbb{Q}$) and non-vanishing of the nth ℓ^2-Betti number was provided by the following question which appeared in [FV21, remark after Proposition 3.4] and [LIP24, Question 7].

Question 1.2 (Friedl–Vidussi, Llosa Isenrich–Py)**.** *Let M be a closed aspherical Kähler $2n$-manifold. If $\chi(M) \neq 0$, is $\Sigma^n(\pi_1 M)$ empty?*

We answer the question in the affirmative; in fact, we do a little more. Since the techniques used here are algebraic, it is more convenient (and more general) to work with Poincaré duality complexes and groups.

For a commutative ring R, recall that an *n-dimensional Poincaré duality complex M over R* (or a $\mathsf{PD}_n(R)$*-complex*) is a finitely dominated connected CW complex with a distinguished class $[M]$ in the nth homology group $H_n(M; D)$, where D is an $R\pi_1 M$ module isomorphic to R as an R-module, such that the cap product

$$[M] \frown -\colon H^k(M; A) \to H_{n-k}(M; A \otimes_R D)$$

is an isomorphism for all $R\pi_1 M$-modules A, where the action on $A \otimes_R D$ is diagonal. Similarly, a $\mathsf{PD}_n(R)$*-group* is a group G of type $\mathsf{FP}(R)$ with cohomological dimension $\operatorname{cd}_R(G) = n$ for which $D = H^n(G; RG)$ is isomorphic to R as an R-module and $H^i(G; A) \cong H_{n-i}(G; A \otimes_R D)$ for all RG-modules A (here $A \otimes_R D$ is equipped with the diagonal action).

Corollary C. *Let M be a closed connected $2n$-manifold or (more generally) a finite $\mathsf{PD}_{2n}(\mathbb{Q})$-complex. If $\chi(M) \neq 0$, then $\Sigma^n(M) = \Sigma^n(M; \mathbb{Q}) = \varnothing$. In particular, if M is additionally aspherical, then $\Sigma^n(\pi_1 M) = \Sigma^n(\pi_1 M; \mathbb{Q}) = \varnothing$.*

Remark 1.3. A near-identical proof yields the following result: If G is a $\mathsf{PD}_{2n}(\mathbb{Q})$-group such that $\chi(G) \neq 0$, then $\Sigma^n(G; \mathbb{Q}) = \varnothing$.

The fundamental group of an aspherical $\mathsf{PD}_n(R)$-complex is a finitely presented $\mathsf{PD}_n(R)$-group, but for $n \geqslant 4$ there exist examples of $\mathsf{PD}_n(R)$-groups which are not finitely presented [Dav98a, Dav98b] (in fact there are uncountably many such groups [Lea18, KLS20]). Hence, Remark 1.3 is not a consequence of Corollary C.

The *Singer Conjecture* is one of the major unresolved problems regarding ℓ^2-Betti numbers and the topology of manifolds. For more information the reader is directed to [Lüc02, Chapter 11].

The Singer Conjecture. *If M is a closed aspherical n-manifold, then $b_p^{(2)}(\widetilde{M}; \pi_1 M) = 0$ for all $p \neq n/2$.*

The Singer Conjecture can clearly be generalised to $\mathsf{PD}_n(\mathbb{Q})$-groups; in this setting we have the following corollary.

Corollary D. *Let G be a $\mathsf{PD}_n(\mathbb{Q})$-group and let $k = \lceil n/2 \rceil - 1$. If $\Sigma^k(G; \mathbb{Q}) \neq \varnothing$, then the Singer Conjecture holds for G.*

The idea behind the proofs. The essence of the proof is unfortunately rather technical. We give here an outline which we hope to be helpful to at least some readers. All of the terminology used below will be introduced in the text.

The main technical result is Proposition 5.1, in which we prove that if a chain complex over a group ring $\mathbb{Q}G$ has vanishing Novikov homology up to some dimension, then its ℓ^2-homology vanishes as well, up to the same dimension. Novikov homology has been shown by Sikorav to encode the BNSR invariants, and ℓ^2-homology can also be understood algebraically, as homology with coefficients in the algebra $\mathcal{U}G$ of operators affiliated to the group von Neumann algebra $\mathcal{N}G$ of G. Hence, we are trying to show that if we have a partial chain contraction defined over the Novikov ring, then we also have one over the algebra $\mathcal{U}G$. We do this by constructing the ring of *weakly rational sequences*, that maps to $\mathcal{U}G$, and that houses a subring mapping to the Novikov ring. The last map has particular structure that allows us to lift invertibility of certain matrices over Novikov rings to the ring of weakly rational sequences.

The construction of the ring of weakly rational sequences is where the blood, sweat, tears, and toil went. In an ideal world, for example when G is torsion free and satisfies the Atiyah conjecture, one argues as follows: the Novikov ring corresponding to $\varphi\colon G \to \mathbb{Z}$ consists naturally of twisted Laurent power series with coefficients in $\mathbb{Q}\ker\varphi$, and so in particular in the Linnell skew field $\mathcal{D}\ker\varphi$, a particularly useful subring of $\mathcal{U}\ker\varphi$. Since twisted Laurent polynomials over $\mathcal{D}\ker\varphi$ satisfy the Ore condition, one can look at the rational functions, and it is not hard to show that, on the one hand, such rational functions contain the division closure of $\mathbb{Q}G$ inside the Novikov ring, and on the other, are naturally contained in $\mathcal{U}G$, even though the ring of Laurent power series is not. Unfortunately, the authors were unable to show that the twisted Laurent polynomial ring over $\mathcal{U}\ker\varphi$ satisfies the Ore condition – this remains an open problem. Instead, we show that this ring satisfies an approximate version of the Ore condition, which allows us to form a ring of 'approximate rational functions', that is precisely the ring of weakly rational sequences.

After finishing the work on this paper, the authors were informed by Andrei Jaikin-Zapirain that his paper with López-Álvarez [JZLÁ20, Section 3.2] contains a construction of a ring $\mathcal{P}^{\mathcal{U}_N}_{\omega,\tau}$ that contains both the Novikov ring of G with respect to φ and $\mathcal{U}G$. The intersection of these latter rings in $\mathcal{P}^{\mathcal{U}_N}_{\omega,\tau}$ can then play the same role as the ring of weakly rational sequences does. The construction of $\mathcal{P}^{\mathcal{U}_N}_{\omega,\tau}$ is very different to the one provided here: it is more general, and can be applied to a greater variety of rings than the Novikov ring and $\mathcal{U}G$; on the other hand, it relies on non-principal ultrafilters, and the weakly rational sequences we define are arguably more explicit in nature.

Outline of the paper. In Section 2, we give the relevant background on ℓ^2-homology, BNSR theory, some ring theoretic tools, and several rings and algebras related to a countable discrete group that we will need in our proofs, with particular emphasis on group von Neumann algebras.

Section 3 is the technical heart of the paper. Here we introduce the ring of weakly rational sequences in $\mathcal{U}G$.

In Section 4, we prove that vanishing of Novikov homology can be passed to other rings. We highlight two results of independent interest: The first is Proposition 4.3 that states that for a discrete group G the division closure of RG in a Novikov ring $\widehat{RG}^{\varphi}$ is equal to its rational closure. The second is Proposition 4.4 that gives a method for taking chain contractions over $\widehat{RG}^{\varphi}$ and rebuilding them over other suitable rings.

In Section 5, we prove each of the results A - D.

In Section 6, we prove a characteristic p version of Theorem A and Theorem B (see Theorem 6.4) for groups whose group rings admit embeddings into certain universal skew fields. We also give a simplified proof of A - D when G is torsion free and satisfies the Atiyah Conjecture.

In Section 7, we detail a number of example applications of our results. In Theorem 7.2 we prove that $\Sigma^n(G;\mathbb{Q}) = \varnothing$ when G is a poly-elementarily-free group of length n. This class includes poly-free and poly-surface groups. The result generalises [KV23, Proposition 1.5] where the authors compute $\Sigma^2(G)$ for groups isomorphic to $F_n \rtimes F_m$ or $\pi_1\Sigma_g \rtimes \pi_1\Sigma_h$ with $n, m, g, h > 1$ (see the first part of [KW19, Theorem 6.1] for the case $F_2 \rtimes F_m$); here Σ_g is the closed orientable connected surface of genus g. We also highlight how our Theorem A applies to real and complex hyperbolic lattices.

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2. Preliminaries

Throughout, all rings are assumed to be associative and unital. All groups will be discrete, and modules will be right modules, unless stated otherwise.

2.A. Group von Neumann algebras.

Let G be a countable group and let ℓ^2G denote the Hilbert space of square summable formal sums of elements of G with complex coefficients, that is the space of expressions

$$\sum_{g\in G} \lambda_g g \text{ such that } \sum_{g\in G} |\lambda_g|^2 < \infty \text{ where } \lambda_g \in \mathbb{C}.$$

The group G acts on ℓ^2G by right multiplication. The inner product on ℓ^2G is defined by

$$\langle \sum_{g\in G} \lambda_g g, \sum_{g\in G} \mu_g g \rangle = \sum_{g\in G} \lambda_g \overline{\mu}_g$$

with $\overline{\mu}$ being the complex conjugate of μ.

Definition 2.1 (Group von Neumann algebra)**.** We define the *group von Neumann algebra* $\mathcal{N}G$ of G to be the algebra of G-equivariant bounded operators $\ell^2G \to \ell^2G$.

Since G acts on ℓ^2G from the right, it is natural to view $\mathcal{N}G$ as operating from the left on ℓ^2G. In particular, the copy of $\mathbb{C}G$ in $\mathcal{N}G$ acts on ℓ^2G from the left. Since $\mathcal{N}G$ contains a copy of $\mathbb{C}G$, multiplication turns it into a $\mathbb{C}G$-bimodule.

One of the key features of group von Neumann algebras is that for every closed G-invariant subspace V of ℓ^2G, there is an associated projection $\pi_V \in \mathcal{N}G$; it is a self-adjoint idempotent with $\operatorname{im}\pi_V = V$ and $\ker\pi_V = V^\perp$.

Another important class of operators in $\mathcal{N}G$ are *partial isometries*, that is operators u such that $u^*u = \pi_{(\ker u)^\perp}$. This implies that $\operatorname{im} u$ is closed and that $uu^* = \pi_{\operatorname{im} u}$, see [Kad, Proposition 6.1.1] and the discussion preceding it for details.

Definition 2.2. An operator p on an inner-product space is *non-negative* if for every x in the domain of p we have $\langle px, x\rangle \geqslant 0$.

The *polar decomposition* of $s \in \mathcal{N}G$ is a canonical factorization $s = vp$ where v is a partial isometry and p is a non-negative operator, and both lie in $\mathcal{N}G$. Furthermore, v^*v is the projection onto $\ker s^\perp$, and vv^* is the projection onto the closure of $\operatorname{im} s$, whence it follows that $\ker v = \ker s$ and $\operatorname{im} v$ is the closure of $\operatorname{im} s$.

The existence of v and p as above is proved in [Kad, Theorem 6.1.2 and Proposition 6.1.3].

The group von Neumann algebra comes equipped with the *von Neumann trace* $\operatorname{tr}\colon \mathcal{N}G \to \mathbb{R}$ given by $a \mapsto \langle a(1), 1\rangle$. There is a canonical dimension function $\dim_{\mathcal{N}G}$, taking values in $[0, \infty]$, defined on $\mathcal{N}G$-modules called the *von Neumann dimension* - the precise definition and basic properties we use can be found in [Lüc02, Chapter 6]. One can also define such a dimension for closed G-invariant subspaces of ℓ^2G, in which case it is equal to the von Neumann trace of the corresponding projection. Since the only projection with zero trace is 0, the only G-invariant subspace of ℓ^2G of von Neumann dimension zero is the trivial subspace.

We will also use the following theorem.

Theorem 2.3 (Linnell [Lin92, Theorem 4])**.** *Let $H \trianglelefteq G$ be such that G/H is right orderable with total order $\leqslant$ and let $\varphi\colon G \twoheadrightarrow G/H$ be the natural epimorphism. Let T be a transveral for H in G, let $x \in \ell^2G$, and write $x = \sum_{t\in T} x_t t$ where $x_t \in \ell^2H$ for all $t \in T$. Suppose that there exists $t_0 \in T$ such that $x_t = 0$ for t with $\varphi(t) < \varphi(t_0)$. If $x_{t_0}y' \neq 0$ for all non-zero $y' \in \mathcal{N}H$, then $xy \neq 0$ for all non-zero $y \in \ell^2G$.*

The notation needs a word of explanation: the products $x_{t_0}y'$ and xy above are instances of the natural products $\ell^2H \times \ell^2H \to \ell^\infty H$ and $\ell^2G \times \ell^2G \to \ell^\infty G$.

2.B. ℓ^2-homology and Betti numbers.

Definition 2.4 (ℓ^2-homology and Betti numbers)**.** Let X be a G-CW complex. We define the ℓ^2-*homology* of X with respect to G as

$$H_p^G(X; \mathcal{N}G) := H_p\big(C_\bullet(X;\mathbb{Q}) \otimes_{\mathbb{Q}G} \mathcal{N}G\big),$$

where $C_\bullet(X;\mathbb{Q})$ is the cellular chain complex of X with rational coefficients considered as a complex of free $\mathbb{Q}G$-modules.

We define the ℓ^2-*Betti numbers* of X with respect to G to be

$$b_p^{(2)}(X; G) := \dim_{\mathcal{N}G} H_p^G(X; \mathcal{N}G).$$

The ℓ^2-Betti numbers of a group G are defined to be the ℓ^2-Betti numbers of the universal free G-space EG. A number of properties of ℓ^2-Betti numbers can be found in [Lüc02, Theorem 6.54].

The $\mathcal{N}G$-dimension does not change when one divides an $\mathcal{N}G$-module M by its torsion submodule $\mathbf{T}M$. Here

$$\mathbf{T}M = \{x \in M : f(x) = 0 \text{ for all } f \in \operatorname{Hom}_{\mathcal{N}G}(M; \mathcal{N}G)\}.$$

There are two other ways of defining ℓ^2-homology; for details, see [Lüc98]. One can look at the reduced homology of the complex

$$C_\bullet(X; \mathbb{Q}) \otimes_{\mathbb{Q}G} \ell^2 G,$$

where *reduced* means that we divide kernels by closures of images. This way the homology groups are actually closed subspaces of powers of $\ell^2 G$, and one can look at the von Neumann dimension of such a subspace. The dimension coincides with the ℓ^2-Betti number, as defined above.

The third way of defining ℓ^2-homology involves the algebra of affiliated operators $\mathcal{U}G$, which we will define in a moment. One can look at the homology of $C_\bullet(X; \mathbb{Q}) \otimes_{\mathbb{Q}G} \mathcal{U}G$, which ends up being equal to the homology of $C_\bullet(X; \mathbb{Q}) \otimes_{\mathbb{Q}G} \mathcal{N}G$ tensored with $\mathcal{U}G$. This is essentially the ℓ^2-homology equivalent of passing from integral homology to homology with $\mathbb{Q}$, that is, we lose the torsion submodule information but often computations are easier. There is again a notion of a dimension for $\mathcal{U}G$-modules, and again one obtains the same Betti numbers.

2.C. **Ore localisation.** In this section we will describe an analogue of localisation for non-commutative rings.

Definition 2.5. Let R be a ring. An element $x \in R$ is a *zero-divisor* if $x \neq 0$, and $xy = 0$ or $yx = 0$ for some non-zero $y \in R$. A non-zero element that is not a zero-divisor will be called *regular*.

Definition 2.6 (Right Ore condition)**.** Let R be a ring and $S \subseteq R$ a multiplicatively closed subset consisting of regular elements. The pair (R, S) satisfies the *right Ore condition* if for every $r \in R$ and $s \in S$ there are elements $r' \in R$ and $s' \in S$ satisfying $rs' = sr'$.

Definition 2.7 (Right Ore localisation)**.** If (R, S) satisfies the right Ore condition we may define the *right Ore localisation*, denoted RS^{-1}, to be the following ring. Elements are represented by pairs $(r, s) \in R \times S$ up to the following equivalence relation: $(r, s) \sim (r', s')$ if and only if there exists $u, u' \in R$ such that the equations $ru = r'u'$ and $su = s'u'$ hold, and $su = s'u'$ belongs to S. The addition is given by

$$(r, s) + (r', s') = (rc + r'd, t), \text{ where } t = sc = s'd \in S,$$

and the multiplication is given by

$$(r, s)(r', s') = (rc, s't), \text{ where } sc = r't \text{ with } t \in S.$$

There is a natural ring homomorphism $R \to RS^{-1}$ defined by $r \mapsto (r, 1)$. For more information on this construction the reader is referred to [Pas85, Section 4.4]. Note that there is also an analogously defined *left* Ore condition.

2.D. **The algebra of affiliated operators.**

Definition 2.8 (Affiliated operators)**.** We say that an operator

$$f\colon \operatorname{dom}(f) \to \ell^2 G$$

with $\operatorname{dom}(f) \subseteq \ell^2 G$ is *affiliated* (to $\mathcal{N}G$) if the domain $\operatorname{dom}(f)$ of f is dense in $\ell^2 G$, the operator f is closed, and it is G-equivariant, that is, $\operatorname{dom}(f)$ is a linear G-invariant subspace and $f(x).g = f(x.g)$ for all $g \in G$ (recall that G acts on $\ell^2 G$ on the right).

The set of all operators affiliated to $\mathcal{N}G$ forms the *algebra of affiliated operators* $\mathcal{U}G$ *of* G.

We define affiliated operators and the algebra they form following Lück, see [Lüc02, Definitions 8.1 and 8.9]. The product in $\mathcal{U}G$ is obtained from function composition – this is explained in detail in the paragraph before [Lüc02, Lemma 8.8].

Since an adjoint of a densely defined closed operator is densely defined and closed, every $x \in \mathcal{U}G$ has a well-defined adjoint $x^* \in \mathcal{U}G$. The map $x \mapsto x^*$ is a ring anti-automorphism of $\mathcal{U}G$.

Note that we have inclusions of $\mathbb{Q}G$-modules

$$\mathbb{Q}G \rightarrowtail \mathbb{C}G \rightarrowtail \mathcal{N}G \rightarrowtail \mathcal{U}G.$$

For more information on these constructions the reader is referred to [Lüc02] – specifically Theorem 8.22 and more generally Chapter 8. We highlight one theorem of special importance to us.

Theorem 2.9. [Lüc02, Theorem 8.22(1)] *The set S of regular elements of $\mathcal{N}G$ forms a right Ore set. Moreover, $\mathcal{U}G$ is canonically isomorphic to $(\mathcal{N}G)S^{-1}$.*

Since we have the anti-automorphism $*$, we immediately see that S satisfies also the left Ore condition, and $\mathcal{U}G$ is canonically isomorphic to $S^{-1}(\mathcal{N}G)$.

A ring R is *von Neumann regular* if for every non-zero $a \in R$, there exists $x \in R$ such that $axa = a$. We say that x is a *partial inverse* of a.

The algebra $\mathcal{U}G$ is von Neumann regular, with explicit control on what the partial inverses look like. In particular, for every $x \in \mathcal{U}G$ there exists a canonical $x^\dagger \in \mathcal{U}G$ such that $xx^\dagger = \pi_{\overline{\operatorname{im} x}}$ as an affiliated operator. Note that in reality the composition $x \circ x^\dagger$ is defined only on $\operatorname{im} x \oplus (\operatorname{im} x)^\perp$, and coincides with $\pi_{\overline{\operatorname{im} x}}$ on this subspace. However, the graph of this composition is not closed, and hence the composition is not an affiliated operator. It can be extended to one, and this extension is precisely $\pi_{\overline{\operatorname{im} x}}$. Similarly, we have $x^\dagger x = \pi_{(\ker x)^\perp}$. We also have $\ker x^\dagger = (\operatorname{im} x)^\perp$ and $\overline{\operatorname{im} x^\dagger} = (\ker x)^\perp$.

The partial inverse can be constructed directly, as in the proof of [Lüc94, Lemma 8.22(3)], or its existence can be deduced algebraically, as in [JZ19, Proposition 3.2].

2.E. **Division and rational closures.**

Definition 2.10 (Division and rational closure)**.** We say that an element of a ring is *invertible* if it admits a left-inverse and a right-inverse; such inverses are then necessarily equal and unique.

Let R be a ring and S a subring. We say that S is *division closed* if every element of S that is invertible as an element of R is already invertible over S. We say that S is *rationally closed* if every finite square matrix over S invertible over R is invertible over S.

Define the *division closure of S in R*, denoted by $\mathcal{D}(S \subset R)$, to be the smallest division-closed subring of R containing S. Define the *rational closure of S in R*, denoted by $\mathcal{R}(S \subset R)$, to be the smallest rationally closed subring of R containing S.

2.F. **Twisted polynomial rings.**

Definition 2.11 (Twisted polynomial ring)**.** Let R be a ring and let $R[t^{\pm 1}]$ be the abelian group of Laurent polynomials over R. Given a homomorphism $\nu\colon \mathbb{Z} \to \mathrm{Aut}(R)$ we may endow $R[t^{\pm 1}]$ with a (not necessarily commutative) multiplication given by

$$t^m x \cdot t^n y = t^{m+n} \nu(t^n)(x) y,$$

and extended linearly. We call this new ring the *ring of twisted Laurent polynomials over R with respect to ν* or simply a *twisted Laurent polynomial ring.*

Remark 2.12. Suppose that G splits as $H \rtimes \mathbb{Z}$. The group ring RG is isomorphic to a ring of twisted Laurent polynomials $RH[t^{\pm 1}]$ in a natural way.

Using analogous multiplication we can define the ring of twisted Laurent power series that we will denote by $R[\![t^{\pm 1}]\!]$.

2.G. **BNSR invariants.** In this section we follow the treatment of Farber–Geoghegan–Schütz [FGS10, Section 2]. Note that the authors only give statements and proofs for BNSR invariants over the ring $\mathbb{Z}$, however the arguments readily generalise to arbitrary rings R.

2.G.i. *Homological BNSR invariants.* Let R be a ring. A monoid M is of *type* $\mathsf{FP}_n(R)$ if there exists a projective resolution $P_\bullet \to R$ of the trivial RM-module R with P_i finitely generated for $i \leqslant n$. Let G be a finitely generated group. Define $S(G) = \mathrm{Hom}(G;\mathbb{R}) \smallsetminus \{0\}$. Note that we are always considering $\mathrm{Hom}(G,\mathbb{R})$ with the usual topology (in fact, since our G is always finitely generated, this is homeomorphic to $\mathbb{R}^n$ for some n). Given $\varphi \in S(G)$, define a submonoid of G by

$$G_\varphi := \{g \in G : \varphi(g) \geqslant 0\}.$$

Definition 2.13 (Homological BNSR invariants of groups)**.** Let R be a ring and G be a group of type $\mathsf{FP_n}(R)$ for $n \in \mathbb{N} \cup \{\infty\}$. We define $\Sigma^n(G;R)$, the *nth homological BNSR-invariant over R*, to be the subset of $S(G) = \mathrm{Hom}(G;\mathbb{R}) \smallsetminus \{0\}$ consisting of those $\varphi\colon G \to \mathbb{R}$ for which G_φ is of type $\mathsf{FP}_n(R)$.

For R a ring, $n \in \mathbb{N}$, and $C_\bullet$ a chain complex over R, we say that $C_\bullet$ has *finite n-type* if there exists a chain complex $P_\bullet$ of finitely generated projective R-modules and a chain map $f\colon C_\bullet \to P_\bullet$ such that $f_i\colon H_i(C_\bullet) \to H_i(P_\bullet)$ is an isomorphism for $i < n$ and an epimorphism for $i = n$. In this case we call f an *n-equivalence.*

Let $C_\bullet$ be a chain complex of RG-modules and let $k \in \mathbb{N}$. We define

$$\Sigma^k(C_\bullet; R) := \{\varphi \in S(G) : C_\bullet \otimes_{RG} RG_\varphi \text{ is of finite } k\text{-type}\}.$$

Definition 2.14 (Homological BNSR invariants of spaces)**.** Let X be a connected pointed CW complex with finite n-skeleton. For $k \leqslant n$ the *k-th (homological) BNSR invariant of X over R* is defined to be

$$\Sigma^k(X; R) := \Sigma^k(C_\bullet(\widetilde{X}; R); R),$$

where $\widetilde{X}$ is the universal cover of X, and $C_\bullet(-; R)$ denotes the cellular chain complex with coefficients in R.

2.G.ii. *Homotopic BNSR invariants.* The following definitions are taken from [FGS10, Section 2]. For more information about the topology of closed 1-forms the reader is referred to [FS08, Section 3].

A *closed 1-form* ω on a pointed topological space X is defined to be a collection $\{f_U\}_{U \in \mathcal{U}}$ of continuous real valued functions $f_U : U \to \mathbb{R}$, where $\mathcal{U}$ is an open covering of X such that for any pair $U, V \in \mathcal{U}$ the difference

$$f_U|_{U \cap V} - f_V|_{U \cap V} : U \cap V \to \mathbb{R}$$

is a locally constant function.

A closed 1-form ω on X in the above sense behaves similarly to a smooth closed 1-form on a manifold; for example, ω can be integrated along paths and integration on loops gives a homomorphism $\xi_\omega : \pi_1 X \to \mathbb{R}$.

A continuous function $f : X \to S^1$ determines a closed 1-form as follows: write $S^1 = \mathbb{R}/\mathbb{Z}$ and let $p : \mathbb{R} \to S^1$ be the projection. If $I = (a, a+1) \subset \mathbb{R}$ is an open interval, then I is homeomorphic to $p(I)$. The collection

$$\omega = \{(p|_I)^{-1} \circ f|_{f^{-1}(p(I))} : f^{-1}(p(I)) \to I \mid I = (a, a+1), a \in \mathbb{R}\}$$

then defines a closed 1-form. The map $\xi_\omega : \pi_1 X \to \mathbb{R}$ can be identified with the map $\pi_1(f) : \pi_1 X \to \pi_1 S^1$ if one views $\pi_1 S^1$ as the subgroup of $\mathbb{R}$ generated by 1.

Definition 2.15 (Homotopic BNSR invariants of groups and spaces via closed 1-forms, [FGS10, Definition 5])**.** Let X be a connected pointed CW complex with finite n-skeleton and with $\pi_1 X = G$. Let $0 \leqslant k \leqslant n$. The *$k$-th homotopic BNSR invariant of* X is the set $\Sigma^k(X) \subseteq S(G)$ defined as follows: $\varphi \in \Sigma^k(X)$ if given ω, a closed 1-form on X representing φ, there exists an $\epsilon > 0$ and a cellular homotopy $H : X^{(k)} \times I \to X$ such that $H(x, 0) = x$ and

$$\int_{\gamma_x} \omega \geqslant \epsilon$$

for all $x \in X^{(k)}$, where $\gamma_x : [0,1] \to X$ is given by $\gamma_x(t) = H(x, t)$.

For a group G we take $\Sigma^n(G) := \Sigma^n(BG)$.

2.G.iii. *Properties of BNSR invariants.* The following theorem essentially combines Theorem 4, Proposition 3, and Corollaries 1 and 2 of [FGS10]; we have taken the liberty to state these results over a general ring R.

Theorem 2.16 (Basic properties of BNSR invariants)**.** *Let X be a pointed connected CW complex with finite n-skeleton and with $\pi_1 X = G$, and let R be a ring. Then,*

(1) $\Sigma^k(X)$ *and* $\Sigma^k(X;R)$ *are open subsets of* $S(G)$*;*
(2) $\Sigma^k(X) \subseteq \Sigma^k(X;R)$*;*
(3) if $\widetilde{X}$ *is* k*-connected then*

$$\Sigma^k(X) = \Sigma^k(G) \text{ and } \Sigma^k(X;R) = \Sigma^k(G;R)$$

and

$$\Sigma^{k+1}(X) \subseteq \Sigma^{k+1}(G) \text{ and } \Sigma^{k+1}(X;R) \subseteq \Sigma^{k+1}(G;R);$$

(4) if X *is finite, then for all* $k \geqslant \dim X$ *we have*

$$\Sigma^k(X) = \Sigma^{\dim X}(X) \text{ and } \Sigma^k(X;R) = \Sigma^{\dim X}(X;R).$$

2.H. **Novikov–Sikorav homology.**

Definition 2.17 (Novikov–Sikorav ring)**.** Let G be a finitely generated group, let $\varphi \in S(G)$, and define the *Novikov–Sikorav ring* of G with respect to φ to be

$$\widehat{RG}^{\varphi} := \left\{ \sum_{g \in G} n_g g \ : \ \left|\{g : n_g \neq 0 \text{ and } \varphi(g) < t\}\right| < \infty \text{ for all } t \in \mathbb{R} \right\}.$$

Given an element $x = \sum_{g\in G} n_g g \in \widehat{RG}^{\varphi}$, the set of group elements g with $n_g \neq 0$ is the *support* of x. We say that x is of *positive support* if $x = 0$ or if its support lies in $\phi^{-1}((0,\infty))$.

Definition 2.18 (Truncation)**.** Given an element $x = \sum_{g\in G} n_g g \in \widehat{RG}^{\varphi}$ and a real number r, we define the *truncation* of x at r to be the element $\sum_{g\in G} n'_g g$ where

$$n'_g = \begin{cases} n_g & \text{if} \quad \varphi(g) \leqslant r \\ 0 & \text{if} \quad \varphi(g) > r. \end{cases}$$

The truncation lies in RG.

Definition 2.19 (Novikov–Sikorav homology)**.** Let X be a pointed CW complex with $\pi_1 X = G$ and let $\varphi \in S(G)$. The *Novikov–Sikorav homology*

$$H^G_\bullet(\widetilde{X}; \widehat{RG}^{\varphi})$$

of X is the G-equivariant homology of $\widetilde{X}$, the universal cover of X, with non-trivial coefficients $\widehat{RG}^{\varphi}$, that is, the homology of the chain complex $C_\bullet(\widetilde{X};R) \otimes_{RG} \widehat{RG}^{\varphi}$.

Remark 2.20. Straight from the definition, it follows that if H is a finite-index subgroup of G and $H^G_i(\widetilde{X}; \widehat{RG}^{\varphi}) = 0$, then $H^H_i(\widetilde{X}; \widehat{RH}^{\varphi|_H}) = 0$ as well, since $\widehat{RG}^{\varphi} = RG \otimes_{RH} \widehat{RH}^{\varphi|_H}$.

Theorem 2.21 (Sikorav's Theorem for groups [Fis24, Theorem 5.3])**.** *Let* G *be a group of type* $\mathsf{FP}_n(R)$*. Let* $\varphi \in S(G)$ *and let* $k \leqslant n$*. The following are equivalent:*

(1) $\varphi \in \Sigma^k(G;R)$*;*
(2) $H_i(G; \widehat{RG}^{\varphi}) = 0$ *for* $i \leqslant k$*.*

The following result has an essentially identical proof as [Fis24, Theorem 5.3] (see also [FGS10, Proposition 5], [Bie07, Appendix], [Kie20, Theorem 3.11], and [Sik87]).

Theorem 2.22 (Sikorav's Theorem for spaces)**.** *Let X be a connected pointed CW complex with finite n-skeleton and $\pi_1 X = G$. Let $\varphi \in S(G)$ and let $k \leqslant n$. The following are equivalent:*

(1) $\varphi \in \Sigma^k(X; R)$;

(2) $H_i^G(\widetilde{X}; \widehat{RG}^{\varphi}) = 0$ for $i \leqslant k$.

2.I. **Chain contractions.** We record here a standard observation.

Lemma 2.23. *Let R be a ring. Let $C_{n+1} \to C_n \to \cdots \to C_0 \to 0$ be a finite chain complex of projective R-modules, and let $H_i \colon C_i \to C_{i+1}$ be homomorphisms of R-modules for all $i < n$ such that*

$$H_{i-1}\partial_i + \partial_{i+1} H_i = \mathrm{id}_{C_i}$$

for every $i < n$. The chain complex is exact if and only if there exists $H_n \colon C_n \to C_{n+1}$ with $H_{n-1}\partial_n + \partial_{n+1} H_n = \mathrm{id}_{C_n}$.

Using induction, one immediately sees that the chain complex

$$C_n \to C_{n-1} \to \cdots \to C_0 \to 0$$

is exact if and only if one can build the maps H_i as above for all $i < n$; such maps are known as (partial) *chain contractions*.

3. Approximate Ore condition

Throughout this section, G denotes a group endowed with an epimorphism $\varphi \colon G \to \mathbb{Z}$, with $K = \ker \varphi$ and $t \in G$ such that $\varphi(t)$ generates $\operatorname{im} \varphi$. Various polynomials and power series in t are always twisted, and the action of t is always the conjugation action inside $\mathbb{Z}G$, $\mathcal{N}G$, and $\mathcal{U}G$, depending on context.

Definition 3.1. Given a non-zero $x = \sum_{i=k}^{\infty} t^i x_i \in \mathcal{U}K[\![t^{\pm 1}]\!]$ with $x_i \in \mathcal{U}K$ for all i, and with $x_k \neq 0$, we define its *initial term* $\operatorname{init} x$ to be $t^k x_k$. Note that k might be negative. We will refer to x_k as the *pure part* of $\operatorname{init} x$, and to k as the *associated power*. We also set $\operatorname{init} 0 = 0$, with pure part 0 and associated power 0. The *nullity* $\operatorname{null} x$ is defined to be the $\mathcal{N}K$-dimension of the kernel of the pure part of $\operatorname{init} x$. Note that the kernel of an affiliated operator in $\mathcal{U}K$ is always a closed subspace of $\ell^2 K$, since the operator is closed, and hence it makes sense to talk about the $\mathcal{N}K$-dimension of the kernel. The definitions in particular apply to Laurent polynomials in $\mathcal{U}K[t^{\pm 1}]$ and in $\mathcal{N}K[t^{\pm 1}]$.

Definition 3.2. We say that a sequence $(q_n)_n$ in $\mathcal{U}K[\![t^{\pm 1}]\!]$ is *admissible* if the sequence of powers associated to $\operatorname{init} q_n$ is bounded from below.

The sequence is *asymptotically injective* if

$$\sum_n \operatorname{null} q_n < \infty$$

and the powers associated to $\operatorname{init} q_n$ are eventually zero.

For operators a, b defined on dense subsets of $\ell^2 G$ we will make frequent use of the estimate $\dim_{\mathcal{N}G} \ker(ab) \leqslant \dim_{\mathcal{N}G} \ker a + \dim_{\mathcal{N}G} \ker b$.

Lemma 3.3. *A term-wise product of asymptotically injective sequences is also asymptotically injective.*

Proof. Let $(p_n)_n$ and $(q_n)_n$ be asymptotically injective sequences in $\mathcal{U}K[t^{\pm 1}]\!]$. The definition tells us that there exists N such that for all $n \geqslant N$ we have $\operatorname{null} p_n < 1/2$ and $\operatorname{null} q_n < 1/2$. For such n we then have $\operatorname{init} p_n \operatorname{init} q_n \neq 0$, and so $\operatorname{init}(p_n q_n) = \operatorname{init} p_n \operatorname{init} q_n$. Therefore

$$\begin{aligned}\operatorname{null} p_n q_n &= \dim_{\mathcal{N}K} \ker(\operatorname{init} p_n \operatorname{init} q_n)\\ &\leqslant \dim_{\mathcal{N}K} \ker \operatorname{init} p_n + \dim_{\mathcal{N}K} \ker \operatorname{init} q_n\\ &= \operatorname{null} p_n + \operatorname{null} q_n.\end{aligned}$$

Summing over n, we obtain

$$\begin{aligned}\sum_n \operatorname{null} p_n q_n &= \sum_{n<N} \operatorname{null} p_n q_n + \sum_{n \geqslant N} \operatorname{null} p_n q_n\\ &\leqslant \sum_{n<N} \operatorname{null} p_n q_n + \sum_{n \geqslant N} \operatorname{null} p_n + \sum_{n \geqslant N} \operatorname{null} q_n\\ &< \infty.\end{aligned}$$

Now, observe that for $n \geqslant N$ the power associated to $\operatorname{init}(p_n q_n)$ is the sum of the powers associated to $\operatorname{init} p_n$ and $\operatorname{init} q_n$, and hence eventually zero. $\square$

Recall that $\mathcal{U}K[t^{\pm 1}] \subseteq \mathcal{U}G$ and so consists of operators defined on dense subsets of $\ell^2 G$. In particular, for $x \in \mathcal{U}K[t^{\pm 1}]$ the von Neumann dimensions $\dim_{\mathcal{N}G} \ker x$ and $\dim_{\mathcal{N}G} \overline{\operatorname{im} x}$ are well defined.

Lemma 3.4. *For every $x \in \mathcal{U}K[t^{\pm 1}]$ we have*

$$\dim_{\mathcal{N}G} \ker x \leqslant \operatorname{null} x.$$

Proof. The result is clear when $x = 0$. Let us assume that $x \neq 0$.

Without loss of generality we may assume that the power associated to $\operatorname{init} x$ is 0. Since $\mathcal{U}K$ is the Ore localisation of $\mathcal{N}K$, there exists an injective operator $z \in \mathcal{N}K$ such that $zx \in \mathcal{N}G$ and $\ker zx = \ker x$. Let $V = \ker x$.

The polar decomposition gives us a partial isometry $v \in \mathcal{N}K$ such that $\ker v = \ker \operatorname{init}(zx)$ and $\operatorname{im} v = \overline{\operatorname{im} \operatorname{init}(zx)}$. In other words, the projections $v^* v = \pi_{\ker \operatorname{init}(zx)^\perp} = 1 - \pi_{\ker \operatorname{init}(zx)}$ and $vv^* = \pi_{\overline{\operatorname{im} \operatorname{init}(zx)}}$ are *Murray–von Neumann equivalent*.

Every projection in $\mathcal{N}K$ is *finite*, meaning that it cannot be Murray–von Neumann equivalent to a projection onto a proper closed subspace of its image. To see this, note that if w is a partial isometry then $\operatorname{tr} ww^* = \operatorname{tr} w^* w$, and hence the von Neumann dimensions of images of ww^* and $w^* w$ must be equal. When considering closed K-invariant subspaces of $\ell^2(K)$, passing to a proper subspace always strictly decreases the von Neumann dimension, and so $\operatorname{im} ww^*$ cannot be a proper subspace of $\operatorname{im} w^* w$.

Now, by [Tak02, Proposition V.1.38], the projections

$$\pi_{\ker \operatorname{init}(zx)} \quad \text{and} \quad 1 - \pi_{\overline{\operatorname{im} \operatorname{init}(zx)}} = \pi_{\operatorname{im} \operatorname{init}(zx)^\perp}$$

are also Murray–von Neumann equivalent. Again by the definition of the equivalence, there exists a partial isometry $u \in \mathcal{N}K$ with $\ker u = (\ker \operatorname{init} zx)^\perp$ and $\operatorname{im} u = (\operatorname{im} \operatorname{init}(zx))^\perp$. Then

$$\operatorname{init}(u + zx) = u + \operatorname{init} zx$$

is injective, and therefore $u + zx$ is injective by Theorem 2.3. This means that $u|_V$ is injective as well, and hence $\dim_{\mathcal{N}G} V \leqslant \dim_{\mathcal{N}G} \operatorname{im} u$. Now, the latter dimension is equal to the $\mathcal{N}G$-trace of uu^*. Since $uu^* \in \mathcal{N}K$, this is equal to the $\mathcal{N}K$-trace, and hence to $\dim_{\mathcal{N}K} \operatorname{im} u = \operatorname{null}(zx) = \operatorname{null} x$. ☐

Remark 3.5. The proof above also shows that if $x \in \mathcal{U}K$ then $\dim_{\mathcal{N}G} \ker x = \dim_{\mathcal{N}K} \ker x$.

3.A. **Approximate Ore condition.** We now introduce the main technical tool of this section.

Proposition 3.6 (Approximate Ore condition)**.** *For every $q, q' \in \mathcal{N}K[t^{\pm 1}]$ and every $\epsilon > 0$ there exist $r, r' \in \mathcal{N}K[t^{\pm 1}]$ such that* $\operatorname{null} r < \operatorname{null} q' + \epsilon$*,* $\operatorname{null} r' < \operatorname{null} q + \operatorname{null} q' + \epsilon$*,*

$$qt^i r = q' t^{i'} r',$$

and the power associated to $\operatorname{init} r$ *is zero and that associated to* $\operatorname{init} r'$ *is at least zero, where i is minus the power associated to* $\operatorname{init} q$ *and i' is minus the power associated to* $\operatorname{init} q'$*.*

Proof. The proof is inspired by Tamari's argument [Tam57]. Let q, q' and ϵ as above be given.

If $\operatorname{null} q' + \epsilon \geqslant 1$ we take $r = r' = 0$. Henceforth we will assume that $\operatorname{null} q' + \epsilon < 1$.

We may assume that $i = i' = 0$, and ignore the terms t^i and $t^{i'}$ above. Let N denote the maximum of the degrees of q and q'. We write

$$q = \sum_{i=0}^{N} t^i q_i, \qquad q' = \sum_{i=0}^{N} t^i q'_i$$

with $q_i, q'_i \in \mathcal{N}K$. For a natural number k, consider the right $\mathcal{N}K$-linear map

$$\lambda_k \colon \mathcal{N}K^{2k} \to \mathcal{N}K^{k+N}$$
$$(x_0, y_0, \ldots, x_{k-1}, y_{k-1}) \mapsto \left(\sum_{i+j=l} t^{-j}(q_i t^j x_j - q'_i t^j y_j) \right)_l .$$

(Secretly, we think of $(x_0, y_0, \ldots, x_{k-1}, y_{k-1})$ as representing two twisted polynomials $r = \sum_{i=0}^{k-1} t^i x_i$ and $r' = \sum_{i=0}^{k-1} t^i y_i$, and

$$(x_0, y_0, \ldots, x_{k-1}, y_{k-1}) \in \ker \lambda_k$$

is equivalent to $qr = q'r'$, since the right-hand side above collects the terms of $qr - q'r'$ according to the power of t.)

Let $d_k = \dim_{\mathcal{N}K} \ker \lambda_k$, and note that $d_k \geqslant k - N$. We may embed $\mathcal{N}K^{2k} \to \mathcal{N}K^{2+2k}$ and $\mathcal{N}K^{k+N} \to \mathcal{N}K^{1+k+N}$ by augmenting vectors with zeroes in initial positions. These embeddings form commutative squares with the maps λ_k and λ_{k+1}. The image of $\ker \lambda_k$ under the first map will be denoted by $t \ker \lambda_k$; it is a subspace of $\ker \lambda_{k+1}$ of dimension d_k.

Let $p, p' \colon \mathcal{N}K^{2k} \to \mathcal{N}K$ denote the projections onto, respectively, the first and the second factor. We will use the notation p and p' for every k.

Claim 3.7. *For some k we have*

$$\dim_{\mathcal{N}K} p(\ker \lambda_k) > 1 - \operatorname{null} q' - \epsilon/2.$$

Proof of claim. Consider $(x_0, y_0, \ldots, x_{k-1}, y_{k-1}) \in \ker p|_{\ker \lambda_k}$. We immediately see that $x_0 = 0$, and $q_0' y_0 = 0$. Hence p' sends $\ker p|_{\ker \lambda_k}$ to $\ker q_0'$. Therefore, we have the short exact sequence of $\mathcal{N}K$-modules

$$0 \to \ker p|_{\ker \lambda_k} \cap \ker p'|_{\ker \lambda_k} \to \ker p|_{\ker \lambda_k} \xrightarrow{p'} \ker q_0'$$

yielding

$$\dim_{\mathcal{N}K} (\ker p|_{\ker \lambda_k}) \leqslant \dim_{\mathcal{N}K} \left(\ker p|_{\ker \lambda_k} \cap \ker p'|_{\ker \lambda_k}\right) + \operatorname{null} q'.$$

The intersection of the kernels on the right-hand side is precisely $t \ker \lambda_{k-1}$. Hence,

$$d_k - \dim_{\mathcal{N}K} p(\ker \lambda_k) \leqslant d_{k-1} + \operatorname{null} q'.$$

Rearranging, we obtain

$$\dim_{\mathcal{N}K} p(\ker \lambda_k) \geqslant d_k - d_{k-1} - \operatorname{null} q'.$$

If the left-hand side is bounded above by $1 - \operatorname{null} q' - \epsilon/2$ for all k, then

$$1 - \epsilon/2 \geqslant d_k - d_{k-1}$$

and so adding such terms together gives

$$(1 - \epsilon/2)k \geqslant d_k \geqslant k - N$$

for all k, which is a contradiction. We conclude that for some k the $\mathcal{N}K$-dimension of $p(\ker \lambda_k)$ is greater than $1 - \operatorname{null} q' - \epsilon/2$, as claimed. ■

Let k be as above. Take $V \leqslant \ell^2(K)$ to be the closure of

$$\{p(x)(1) \mid x \in \ker \lambda_k\}.$$

Since $\ker \lambda_k$ is a K-module, the vector space V is K-invariant. It therefore has a von Neumann dimension, and the dimension is equal to the $\mathcal{N}K$-dimension of $p(\ker \lambda_k)$.

Take $x \in \ker \lambda_k$ such that

$$\|\pi_V(1) - p(x)(1)\|_2 \leqslant (\epsilon/2)^{1/2}.$$

Let $W = \ker p(x)^* \cap V$. We claim that $\dim_{\mathcal{N}K} W \leqslant \epsilon/2$; if the dimension is zero then we are done. Otherwise, let $w = \pi_W(1)$ and $w' = w/\|w\|_2$. We have

$$\begin{aligned}
\dim_{\mathcal{N}K} W &= \langle \pi_W(1), 1\rangle \\
&= \frac{\langle \pi_W(1), 1\rangle^2}{\langle \pi_W(1), 1\rangle} \\
&= \frac{\langle \pi_W(1), 1\rangle^2}{\langle \pi_W(1), \pi_W(1)\rangle} \\
&= \frac{\langle w, 1\rangle^2}{\langle w, w\rangle} \\
&= \langle w', 1\rangle^2 \\
&= \langle \pi_V(w'), 1\rangle^2 \\
&= \langle w', \pi_V(1)\rangle^2 \\
&= \big(\langle w', \pi_V(1) - p(x)(1)\rangle + \langle w', p(x)(1)\rangle\big)^2 \\
&= \big(\langle w', \pi_V(1) - p(x)(1)\rangle + \langle p(x)^* w', 1\rangle\big)^2 \\
&= \langle w', \pi_V(1) - p(x)(1)\rangle^2 \\
&\leqslant \big(\|w'\|_2 \cdot \|\pi_V(1) - p(x)(1)\|_2\big)^2 \\
&\leqslant \epsilon/2
\end{aligned}$$

with the penultimate line being the Cauchy–Schwarz inequality. Hence

$$\dim_{\mathcal{N}K} \ker p(x) = \dim_{\mathcal{N}K} \ker p(x)^* \leqslant \dim_{\mathcal{N}K} W + \dim_{\mathcal{N}K} V^\perp < \operatorname{null} q' + \epsilon$$

where the first inequality is obtained by projecting $\ker p(x)^*$ orthogonally onto $V^\perp$.

Write

$$x = (x_0, y_0, \dots, x_{k-1}, y_{k-1});$$

let $r = \sum_{i=0}^{k-1} t^i x_i$ and $r' = \sum_{i=0}^{k-1} t^i y_i$. We have shown that $\ker p(x) = \ker x_0$ has dimension less than $\operatorname{null} q' + \epsilon$, and hence less than 1. Therefore, $x_0 \neq 0$ and so the power associated to $\operatorname{init} r = x_0$ is zero. The power associated to $\operatorname{init} r'$ is at least zero. Also, $x \in \ker \lambda_k$ means precisely that

$$qr = q'r'.$$

Finally, if $q_0 x_0 \neq 0$ then the last equality implies that $q_0 \operatorname{init} r = q_0' \operatorname{init} r'$, and hence

$$\operatorname{null} r' \leqslant \operatorname{null} q + \operatorname{null} r < \operatorname{null} q + \operatorname{null} q' + \epsilon.$$

If however $q_0 x_0 = 0$, then $\operatorname{null} q + \operatorname{null} r \geqslant 1$, and so

$$\operatorname{null} r' \leqslant 1 \leqslant \operatorname{null} q + \operatorname{null} r < \operatorname{null} q + \operatorname{null} q' + \epsilon. \qquad \square$$

Corollary 3.8. *For every $q, q' \in \mathcal{U}K[t^{\pm 1}]$ and every $\epsilon > 0$ there exist $r, r' \in \mathcal{N}K[t^{\pm 1}]$ such that $\operatorname{null} r < \operatorname{null} q' + \epsilon$, $\operatorname{null} r' < \operatorname{null} q + \operatorname{null} q' + \epsilon$,*

$$qt^i r = q' t^{i'} r' \in \mathcal{N}K[t^{\pm 1}],$$

and the power associated to $\operatorname{init} r$ is zero and that associated to $\operatorname{init} r'$ is at least zero, where i is minus the power associated to $\operatorname{init} q$ and i' is minus the power associated to $\operatorname{init} q'$.

Proof. Since $\mathcal{U}K$ is the Ore localisation of $\mathcal{N}K$, and since q is a Laurent polynomial and hence has only finitely many terms, there exists an injective operator $p \in \mathcal{N}K$ such that $qp \in \mathcal{N}K[t^{\pm 1}]$; moreover, since p is injective we have $\operatorname{null} qp = \operatorname{null} q$. We similarly construct an injective operator $p' \in \mathcal{N}K$ such that $q'p' \in \mathcal{N}K[t^{\pm 1}]$ and $\operatorname{null} q'p' = \operatorname{null} q'$. We now apply Proposition 3.6 to the pair $(qp, q'p')$ to obtain $s, s' \in \mathcal{N}K[t^{\pm 1}]$ such that $\operatorname{null} s < \operatorname{null} q' + \epsilon$, $\operatorname{null} s' < \operatorname{null} q + \operatorname{null} q' + \epsilon$, and

$$qpt^i s = q'p't^{i'} s'.$$

We set $r = t^{-i}pt^i s$ and $r' = t^{-i'}p't^{i'}s'$, and note that $\operatorname{null} r = \operatorname{null} s$, $\operatorname{null} r' = \operatorname{null} s'$, the power associated to $\operatorname{init} r = t^{-i}pt^i \operatorname{init} s$ is zero, and that associated to $\operatorname{init} r' = t^{-i}pt^i \operatorname{init} s'$ is at least zero. ☐

Corollary 3.9. *Every two asymptotically injective sequences $(q_n)_n$ and $(q'_n)_n$ over $\mathcal{U}K[t^{\pm 1}]$ admit an* asymptotically injective common multiple*, that is, an asymptotically injective sequence $(x_n)_n$ over $\mathcal{N}K[t^{\pm 1}]$ such that there exist two asymptotically injective sequences $(y_n)_n$ and $(z_n)_n$ over $\mathcal{N}K[t^{\pm 1}]$ with $x_n = q_n y_n = q'_n z_n$ for every n.*

Proof. For every n we obtain x_n, y_n, and z_n with $x_n = q_n y_n = q'_n z_n$ from Corollary 3.8, setting $\epsilon = 2^{-n}$. This way

$$\sum_n \operatorname{null} x_n \leqslant \sum_n (\operatorname{null} q_n + \operatorname{null} y_n) \leqslant \sum_n \left(\operatorname{null} q_n + \operatorname{null} q'_n + 2^{-n}\right) < \infty$$

and similarly for $(y_n)_n$ and $(z_n)_n$.

Now let us investigate the powers associated to the initial parts of the operators. By Corollary 3.8, the power associated to $\operatorname{init} z_n$ is equal to minus that of $\operatorname{init} q'_n$, which is eventually zero. The power associated to $\operatorname{init} y_n$ is eventually at least zero, also by Corollary 3.8. However, since $(q'_n)_n$ and $(z_n)_n$, are asymptotically injective, the power associated to $\operatorname{init}(q'_n z_n)$ is also eventually zero, and as the power associated to $\operatorname{init} q_n$ is eventually zero as well, the power associated to $\operatorname{init} y_n$ must also be eventually zero. ☐

3.B. **Asymptotic agreement.** In this section we are dealing with sequences, but in reality we think of them as proxies, and we are really interested in their limits (which we will define later). Hence it is natural to introduce an equivalence relation on sequences.

Definition 3.10. We say that two sequences $(x_n)_n$ and $(y_n)_n$ in $\mathcal{U}G$ *asymptotically agree as operators*, written $(x_n)_n \approx (y_n)_n$, if

$$\sum_n \dim_{\mathcal{N}G}(\ker(x_n - y_n)^{\perp}) < \infty.$$

A sequence $(x_n)_n$ in $\mathcal{U}G$ *stabilises* if $(x_n)_n \approx (x_{n+1})_n$.

Note that if $(x_n)_n$ and $(y_n)_n$ are sequences in $\mathcal{U}K$, then

$$\dim_{\mathcal{N}G} \ker(x_n - y_n) = \dim_{\mathcal{N}K} \ker(x_n - y_n)$$

by Remark 3.5, and hence such sequences asymptotically agree as sequences in $\mathcal{U}K$ if and only if they asymptotically agree as sequences in $\mathcal{U}G$. Therefore, there is no need to specify over which group we are working.

Asymptotic agreement is an equivalence relation. Indeed, reflexivity and symmetry are trivial. We explain transitivity. Suppose that $(x_n)_n \approx (y_n)_n \approx (z_n)_n$. We have $\ker(x_n - z_n)^\perp \subseteq \ker(x_n - y_n)^\perp + \ker(y_n - z_n)^\perp$. Hence,

$$\sum_n \dim_{\mathcal{N}G} \ker(x_n - z_n)^\perp \leqslant \sum_n \dim_{\mathcal{N}G} \ker(x_n - y_n)^\perp + \sum_n \dim_{\mathcal{N}G} \ker(y_n - z_n)^\perp < \infty,$$

as required.

Lemma 3.11. *Let $(x_n)_n$, $(y_n)_n$, and $(z_n)_n$ be sequences in $\mathcal{U}G$. If $(x_n)_n \approx (y_n)_n$, then all of the following hold:*

(1) $(x_n + z_n)_n \approx (y_n + z_n)_n$,
(2) $(x_n z_n)_n \approx (y_n z_n)_n$,
(3) $(z_n x_n)_n \approx (z_n y_n)_n$.

Proof. This is immediate. For (2) and (3), it is enough to observe that the dimension of the kernel of a product of operators is bounded from below by the dimension of the kernel of either factor. □

We now extend the relation $\approx$ to power series.

Definition 3.12. Two sequences $(x_n)_n$ and $(y_n)_n$ over $\mathcal{U}K[\![t^{\pm 1}]\!]$ *asymptotically agree as power series*, written $(x_n)_n \approx_K (y_n)_n$, if for every fixed degree d, the sequence (over $\mathcal{U}K$) of coefficients of x_n by t^d and the sequence of coefficients of y_n by t^d asymptotically agree as operators.

A sequence $(x_n)_n$ over $\mathcal{U}K[\![t^{\pm 1}]\!]$ is *K-stabilising* if $(x_n)_n \approx_K (x_{n+1})_n$.

For power series, we add K as a subscript to $\approx$ due to the potential confusion for sequences of Laurent polynomials in $\mathcal{U}K[t^{\pm 1}]$. Such Laurent polynomials are at the same time elements of $\mathcal{U}G$, in which case the definition of $\approx$ applies, and Laurent power series, in which context we use $\approx_K$. It is clear that $\approx_K$ is again an equivalence relation.

We now collect basic arithmetic properties of the equivalence relation $\approx_K$.

Lemma 3.13. *Let $(x_n)_n$, $(y_n)_n$, and $(z_n)_n$ be admissible sequences in $\mathcal{U}K[\![t^{\pm 1}]\!]$. If $(x_n)_n \approx_K (y_n)_n$, then all of the following hold:*

(1) $(x_n + z_n)_n \approx_K (y_n + z_n)_n$,
(2) $(x_n z_n)_n \approx_K (y_n z_n)_n$,
(3) $(z_n x_n)_n \approx_K (z_n y_n)_n$.

Proof. We prove the items in turn.

(1) This is immediate.
(2) There exists a lower bound N for the powers associated to $\operatorname{init} x_n$, $\operatorname{init} y_n$, and $\operatorname{init} z_n$. Hence $2N$ is such a lower bound for $x_n z_n$ and $y_n z_n$.
The term of $x_n z_n$ next to t^d is a sum of appropriate products of terms of x_n and z_n next to t^i with $N \leqslant i \leqslant d - N$; an analogous statement holds for $y_n z_n$. The number of summands is thus bounded, and repeated application of Lemma 3.11 yields the result.
(3) This is analogous. □

Remark 3.14. The above result stops being true if the sequences are not admissible. If we take $x_n = t^n, y_n = 0$, and $z_n = t^{-n}$, then $(x_n)_n \approx_K (y_n)_n$ but

$$(x_n z_n)_n = (1)_n \not\approx_K (0)_n = (y_n z_n)_n.$$

3.C. **Partial inverse.** We are now approaching the main technical heart of the paper. We will introduce two constructions that play the role of inverses of elements in $\mathcal{U}K[t^{\pm 1}]$, one in $\mathcal{U}G$, and one in $\mathcal{U}K[\![t^{\pm 1}]\!]$. They will share many properties.

Recall from Section 2.D that in $\mathcal{U}G$ we have the notion of a partial inverse $x \mapsto x^\dagger$. The partial inverse of an element $x \in \mathcal{U}K$ lies in $\mathcal{U}K$, and for such an x we have $(t^i x)^\dagger = x^\dagger t^{-i}$.

Lemma 3.15. *Let $(x_n)_n$ and $(y_n)_n$ be sequences in $\mathcal{U}G$. If $(x_n)_n$ and $(y_n)_n$ asymptotically agree as operators, then so do the sequences of adjoints $({x_n}^*)_n$ and $({y_n}^*)_n$, and the sequences of partial inverses $({x_n}^\dagger)_n$ and $({y_n}^\dagger)_n$. In particular, if $(x_n)_n$ stabilises, then so do $({x_n}^*)_n$ and $({x_n}^\dagger)_n$.*

Proof. This is true for adjoints, since $\ker({x_n}^* - {y_n}^*) = (\operatorname{im}(x_n - y_n))^\perp$ has the same $\mathcal{N}G$-dimension as $\ker(x_n - y_n)$.

For partial inverses, we need to introduce some notation. Let

$$1 - d_n = \dim_{\mathcal{N}G} \ker(x_n - y_n),$$

and note that $\sum_n d_n < \infty$.

The subspace $\ker({x_n}^\dagger - {y_n}^\dagger)$ contains $(\operatorname{im} x_n)^\perp \cap (\operatorname{im} y_n)^\perp$, since this is the intersection of the kernels of ${x_n}^\dagger$ and ${y_n}^\dagger$. We claim it also contains

$$\overline{x_n\big(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp\big)}.$$

Indeed, since kernels of affiliated operators are closed, it is enough to show that $\ker({x_n}^\dagger - {y_n}^\dagger)$ contains

$$x_n\big(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp\big).$$

Take $w \in \ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp$ that also lies in the domain of x_n. We have $w \in \ker(x_n - y_n)$, and so $y_n w$ is defined and equals $x_n w$. Thus

$$({x_n}^\dagger - {y_n}^\dagger)(x_n w) = {x_n}^\dagger x_n w - {y_n}^\dagger y_n w = \pi_{(\ker x_n)^\perp} w - \pi_{(\ker y_n)^\perp} w = 0,$$

yielding the claim.

The spaces $(\operatorname{im} x_n)^\perp \cap (\operatorname{im} y_n)^\perp$ and

$$\overline{x_n\big(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp\big)}$$

are perpendicular. We will now bound the dimensions of these two spaces from below.

We have

$$\begin{aligned} \dim_{\mathcal{N}G} \operatorname{im} x_n &\geqslant \dim_{\mathcal{N}G}(x_n(\ker(x_n - y_n))), \\ \dim_{\mathcal{N}G}(x_n(\ker(x_n - y_n))) &= \dim_{\mathcal{N}G}(y_n(\ker(x_n - y_n))), \text{ and} \\ \dim_{\mathcal{N}G} \operatorname{im} y_n &\geqslant \dim_{\mathcal{N}G}(y_n(\ker(x_n - y_n))). \end{aligned}$$

Moreover, the definition of d_n tells us that the $\mathcal{N}G$-dimension of the orthogonal complement of $x_n(\ker(x_n - y_n))$ in $\overline{\operatorname{im} x_n}$ is bounded above by d_n, and so is the complement in $\overline{\operatorname{im} y_n}$. We conclude that

$$\begin{aligned}\dim_{\mathcal{N}G}\left((\operatorname{im} x_n)^\perp \cap (\operatorname{im} y_n)^\perp\right) &\geqslant \dim_{\mathcal{N}G}\left(x_n(\ker(x_n - y_n))\right)^\perp - 2d_n \\ &\geqslant \dim_{\mathcal{N}G}(\operatorname{im} x_n)^\perp - 2d_n.\end{aligned}$$

We will now focus on $x_n(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp)$. We have

$$(\ker x_n)^\perp \cap (\ker y_n)^\perp = \operatorname{im} {x_n}^* \cap \operatorname{im} {y_n}^* \geqslant {x_n}^*(\ker({x_n}^* - {y_n}^*)).$$

The $\mathcal{N}G$-codimension of this last subspace in $\operatorname{im} {x_n}^* = (\ker x_n)^\perp$ is bounded from above by d_n, and therefore

$$\begin{aligned}&\dim_{\mathcal{N}G} \overline{x_n(\ker(x_n - y_n) \cap (\ker x_n)^\perp \cap (\ker y_n)^\perp)} \\ &\geqslant \dim_{\mathcal{N}G} \overline{x_n(\ker(x_n - y_n) \cap (\ker x_n)^\perp)} - d_n \\ &\geqslant \dim_{\mathcal{N}G} \overline{x_n((\ker x_n)^\perp)} - 2d_n \\ &= \dim_{\mathcal{N}G} \overline{\operatorname{im} x_n} - 2d_n.\end{aligned}$$

Combining the last two inequalities gives $\dim_{\mathcal{N}G} \ker({x_n}^\dagger - {y_n}^\dagger) \geqslant 1 - 4d_n$, and hence $\dim_{\mathcal{N}G} \ker(x_n^\dagger - y_n^\dagger)^\perp \leqslant 4d_n$, and the result follows. $\square$

Lemma 3.16. *Let $(p_n)_n$ be a sequence in $\mathcal{U}G$, and let $(s_n)_n$ and $(q_n)_n$ be sequences in $\mathcal{U}G$ such that*

$$\sum_n \dim_{\mathcal{N}G} \ker s_n < \infty \text{ and } \sum_n \dim_{\mathcal{N}G} \ker q_n < \infty.$$

All of the following hold:

(1) $\left((q_n s_n)^\dagger\right)_n \approx \left({s_n}^\dagger {q_n}^\dagger\right)_n$.
(2) $(s_n {s_n}^\dagger)_n \approx (1)_n$.
(3) $({s_n}^\dagger s_n)_n \approx (1)_n$.
(4) $(p_n {q_n}^\dagger)_n \approx (p_n s_n (q_n s_n)^\dagger)_n$.
(5) $({{s_n}^\dagger}^\dagger)_n \approx (s_n)_n$.

Proof. We prove the items in turn.

(1) We have

$$(q_n s_n)(q_n s_n)^\dagger - q_n s_n {s_n}^\dagger {q_n}^\dagger = \pi_{\overline{\operatorname{im} q_n s_n}} - q_n \pi_{\overline{\operatorname{im} s_n}} {q_n}^\dagger.$$

The right-hand side is 0 on $(\operatorname{im} q_n)^\perp$, since both summands are 0 there, and on $q_n(\operatorname{im} s_n \cap (\ker q_n)^\perp)$, since the summands restrict to the identity there. Also, these two spaces are orthogonal, and so

$$\begin{aligned}&\dim_{\mathcal{N}G} \ker((q_n s_n)(q_n s_n)^\dagger - q_n s_n {s_n}^\dagger {q_n}^\dagger) \\ &\geqslant \dim_{\mathcal{N}G}(\operatorname{im} q_n)^\perp + \dim_{\mathcal{N}G} q_n(\operatorname{im} s_n \cap (\ker q_n)^\perp).\end{aligned}$$

Since q_n is injective on $(\ker q_n)^\perp$, we have

$$\dim_{\mathcal{N}G} q_n(\operatorname{im} s_n \cap (\ker q_n)^\perp) = \dim_{\mathcal{N}G}(\operatorname{im} s_n \cap (\ker q_n)^\perp).$$

Moreover,

$$\begin{aligned}\dim_{\mathcal{N}G}(\operatorname{im} s_n \cap (\ker q_n)^\perp) &\geqslant \dim_{\mathcal{N}G}(\ker q_n)^\perp - (1 - \dim_{\mathcal{N}G} \operatorname{im} s_n) \\ &\geqslant \dim_{\mathcal{N}G}(\ker q_n)^\perp - \dim_{\mathcal{N}G} \ker s_n.\end{aligned}$$

Putting these inequalities together yields

$$\begin{aligned}
&\dim_{\mathcal{N}G} \ker((q_n s_n)(q_n s_n)^\dagger - q_n s_n {s_n}^\dagger {q_n}^\dagger) \\
&\geqslant \dim_{\mathcal{N}G} (\operatorname{im} q_n)^\perp + \dim_{\mathcal{N}G} q_n(\operatorname{im} s_n \cap (\ker q_n)^\perp) \\
&\geqslant \dim_{\mathcal{N}G} (\operatorname{im} q_n)^\perp + \dim_{\mathcal{N}G} (\ker q_n)^\perp - \dim_{\mathcal{N}G} \ker s_n \\
&= 1 - \dim_{\mathcal{N}G} \ker s_n.
\end{aligned}$$

Since

$$\dim_{\mathcal{N}G} \ker q_n s_n \leqslant \dim_{\mathcal{N}G} \ker q_n + \dim_{\mathcal{N}G} \ker s_n,$$

we conclude that

$$\dim_{\mathcal{N}G} \ker((q_n s_n)^\dagger - {s_n}^\dagger {q_n}^\dagger) \geqslant 1 - 2\dim_{\mathcal{N}G} \ker s_n - \dim_{\mathcal{N}G} \ker q_n.$$

Thus,

$$\sum_n \dim_{\mathcal{N}G} \left(\ker((q_n s_n)^\dagger - {s_n}^\dagger {q_n}^\dagger)^\perp \right) \leqslant \sum_n (2\dim_{\mathcal{N}G} \ker s_n + \dim_{\mathcal{N}G} \ker q_n) < \infty.$$

(2) The kernel of the operator

$$1 - s_n {s_n}^\dagger = 1 - \pi_{\overline{\operatorname{im} s_n}}$$

is $\overline{\operatorname{im} s_n}$, and its dimension is $1 - \dim_{\mathcal{N}G} \ker s_n$; we finish the argument as above.

(3) For every element of $\mathcal{U}G$, its kernel and the kernel of its adjoint have the same dimension. We may therefore apply both of the above items to $({s_n}^*)_n$. Since partial inverse commutes with adjoint by [JZ19, Proposition 3.2(5)], using Lemma 3.15 we obtain

$$({s_n}^\dagger s_n)_n = ({s_n}^{*\dagger *} {s_n}^{**})_n = (({s_n}^* {s_n}^{*\dagger})^*)_n \approx (1^*)_n = (1)_n.$$

(4) This follows immediately from the items above and Lemma 3.11(2) and (3).

(5) By Lemma 3.11, associativity, and the items above,

$$({s_n}^{\dagger\dagger} - s_n)_n \approx (({s_n}^{\dagger\dagger} - s_n){s_n}^\dagger s_n)_n \approx (s_n - s_n)_n \approx (0)_n. \qquad \square$$

The above applies in particular to sequences $(q_n)_n$ and $(s_n)_n$ that are asymptotically injective, thanks to Lemma 3.4.

We will now introduce the first limit – it will later allow us to map our sequences to $\mathcal{U}G$.

Lemma 3.17. *For every stabilising sequence $(x_n)_n$ over $\mathcal{U}G$, there exists a unique element $x_\infty \in \mathcal{U}G$ such that $(x_n)_n$ and $(x_\infty)_n$ asymptotically agree.*

Proof. Let $V_n = \ker(x_{n+1} - x_n)$. By assumption, $\sum_n (1 - \dim_{\mathcal{N}G} V_n) < \infty$. Let $U_n = \overline{\sum_{m \geqslant n} {V_m}^\perp}$. The subspaces U_n are closed and G-invariant, and form a nested sequence with $\lim_n \dim_{\mathcal{N}G} U_n = 0$. Therefore $\bigcap_n U_n = \{0\}$.

Observe that the affiliated operators x_m with $m \geqslant n$ all agree on L_n, where L_n is defined to be ${U_n}^\perp$ intersected with all of their domains. Now, the subspaces L_n are G-invariant, form an ascending chain, and $\overline{L} = \ell^2 G$ with $L = \bigcup L_n$. We define $x'_\infty \colon L \to \ell^2 G$ by $x'_\infty|_{L_n} = x_n|_{L_n}$. It is clear that x'_∞ is densely defined and G-equivariant.

We now apply the same procedure to the stabilising sequence $({x_n}^*)_n$, and obtain a densely defined G-invariant operator x^*_∞. The domain of this operator is an ascending union $\bigcup R_m$ where every R_m lies in the domain of ${x_m}^*$. Take natural numbers n and m. For every $l \in L_n$ and r in R_m, we have

$$\langle l, x^*_\infty(r)\rangle = \langle l, {x_{\max(n,m)}}^*(r)\rangle = \langle x_{\max(n,m)}(l), r\rangle = \langle x'_\infty(l), r\rangle.$$

The equation holds for every m, and so l lies in the domain of the adjoint $x_\infty = (x^*_\infty)^*$. Therefore all of L lies in the domain of x_∞, and so x_∞ is densely defined. We also see that x_∞ agrees with x'_∞ on L. Moreover, since x^*_∞ is densely defined, x_∞ is closed. Hence x_∞ is the desired affiliated operator.

To prove uniqueness, suppose that we have another affiliated operator y_∞ such that $(x_n)_n$ and $(y_\infty)_n$ asymptotically agree. Then the sequences $(x_\infty)_n$ and $(y_\infty)_n$ asymptotically agree, forcing $\ker(x_\infty - y_\infty) = \ell^2 G$. This means that $x_\infty = y_\infty$ as affiliated operators. □

We will refer to the element x_∞ as the *limit* of the sequence $(x_n)_n$. The map $(x_n)_n \mapsto x_\infty$ will be denoted by $\lambda_{\mathcal{U}G}$. Note that if $x_n \in \mathcal{U}K$ for every n, then $x_\infty \in \mathcal{U}K$ as well.

3.D. **Expansion.** We are now ready to construct the second function that will serve as an inverse, this time in $\mathcal{U}K[\![t^{\pm 1}]\!]$.

Definition 3.18 (Expansion)**.** We define the *expansion map*

$$\mathcal{U}K[\![t^{\pm 1}]\!] \smallsetminus \{0\} \to \mathcal{U}K[\![t^{\pm 1}]\!]$$

by

$$q \mapsto \overline{q} = (\operatorname{init} q)^\dagger \sum_{k=0}^{\infty} \left((\operatorname{init} q - q)(\operatorname{init} q)^\dagger\right)^k.$$

We extend the definition by declaring $\overline{0} = 0$.

Recall that $\operatorname{init} q$ is of the form $t^i q_i$ for some $q_i \in \mathcal{U}K$. Also, the zeroth power of every operator is equal to 1.

The construction above may seem a little mysterious at first – it is instructive to consider the case in which $\operatorname{init} q$ lies in $\mathcal{U}K$ and is invertible in $\mathcal{U}K$.

Lemma 3.19. *For $q \in \mathcal{U}K[\![t^{\pm 1}]\!]$, suppose that $\operatorname{init} q$ is invertible in $\mathcal{U}K$. Then $\overline{q}$ is precisely the inverse of q in $\mathcal{U}K[\![t^{\pm 1}]\!]$.*

Proof. Observe that $(\operatorname{init} q)^\dagger$ is the inverse of $\operatorname{init} q$. Let $r = q(\operatorname{init} q)^\dagger$, and observe that we have $r = 1 + \sum_{i>0} t^i r_i$ with $r_i \in \mathcal{U}K$. Hence

$$\overline{r} = \sum_{k \geqslant 0} (1-r)^k$$

is the inverse of r. Thus, $(\operatorname{init} q)^\dagger \sum_{k\geqslant 0}(1-r)^k$ is the inverse of q. But

$$(\operatorname{init} q)^\dagger \sum_{k\geqslant 0} (1-r)^k = (\operatorname{init} q)^\dagger \sum_{k\geqslant 0} (\operatorname{init} q (\operatorname{init} q)^\dagger - q(\operatorname{init} q)^\dagger)^k = \overline{q}. \qquad \square$$

Lemma 3.20. *Let $(q_n)_n$ and $(r_n)_n$ be asymptotically injective sequences over $\mathcal{U}K[\![t^{\pm 1}]\!]$. All of the following hold:*

1. *If $(q_n)_n \approx_K (r_n)_n$ then $(\overline{q_n})_n \approx_K (\overline{r_n})_n$;*
2. *$(\overline{\overline{q_n}})_n \approx_K (q_n)_n$;*
3. *$(q_n\overline{q_n})_n \approx_K (\overline{q_n} q_n)_n \approx_K (1)_n$;*

(4) $(\overline{q_n r_n})_n \approx_K (\overline{r_n} \cdot \overline{q_n})_n$.

Proof. We prove the items in turn.

(1) The definition of asymptotic equivalence and the fact that $(q_n)_n$ and $(r_n)_n$ are asymptotically injective implies that for all n sufficiently large the powers associated to $\operatorname{init} r_n$ and $\operatorname{init} q_n$ are zero, and hence $(\operatorname{init} q_n)_n$ and $(\operatorname{init} r_n)_n$ asymptotically agree as operators. By Lemma 3.15, the sequences $((\operatorname{init} q_n)^\dagger)_n$ and $((\operatorname{init} r_n)^\dagger)_n$ asymptotically agree as well.

For every fixed d, the terms of $\overline{q_n}$ and $\overline{r_n}$ appearing next to t^d are obtained from finitely many corresponding terms in q_n and r_n, and from $(\operatorname{init} q_n)^\dagger$ and $(\operatorname{init} r_n)^\dagger$, via the same arithmetic operation. Since the powers associated to $\operatorname{init} q_n$ and $\operatorname{init} r_n$ are eventually zero, we may take these arithmetic operations to be independent of n. We now apply Lemma 3.11 for every degree d separately.

Since $\mathcal{U}K$ is the Ore localisation of $\mathcal{N}K$, there exists an injective operator $z_n \in \mathcal{N}K$ such that $p_n = (\operatorname{init} q_n) z_n \in \mathcal{N}K$. Polar decomposition gives us partial isometries u'_n in $\mathcal{N}K$ mapping $(\ker p_n)^\perp$ onto $\overline{\operatorname{im} p_n}$ and being trivial on $\ker p_n$. Arguing via Murray–von Neumann equivalence as in the proof of Lemma 3.4, we obtain partial isometries u_n in $\mathcal{N}K$ mapping $\ker p_n$ onto $(\operatorname{im} p_n)^\perp$ and being trivial on $(\ker p_n)^\perp$.

Analogously, we find injective operators $w_n \in \mathcal{N}K$ and partial isometries $v_n \in \mathcal{N}K$ mapping $\ker\big((\operatorname{init} r_n) w_n\big)$ onto $\big(\operatorname{im}((\operatorname{init} r_n) w_n)\big)^\perp$ and being trivial on $\big(\ker((\operatorname{init} r_n) w_n)\big)^\perp$.

The operators $(\operatorname{init} q_n) z_n + u_n$ and $(\operatorname{init} r_n) w_n + v_n$ are then invertible in $\mathcal{U}K$, and hence so are $\big((\operatorname{init} q_n) z_n + u_n\big) {z_n}^\dagger$ and $\big((\operatorname{init} r_n) w_n + v_n\big) {w_n}^\dagger$. Crucially, $(u_n)_n \approx (0)_n \approx (v_n)_n$, since $(q_n)_n$ and (r_n) are asymptotically injective and z_n and w_n are injective. Lemma 3.13 tells us that

$$\big((q_n z_n + u_n) {z_n}^\dagger\big)_n \approx_K (q_n z_n {z_n}^\dagger)_n \approx_K (q_n)_n$$

and

$$\big((r_n w_n + v_n) {w_n}^\dagger\big)_n \approx_K (r_n w_n {w_n}^\dagger)_n \approx_K (r_n)_n.$$

(2) Lemma 3.19 and the uniqueness of units yields $\overline{\overline{(q_n z_n + u_n) {z_n}^\dagger}} = (q_n z_n + u_n) {z_n}^\dagger$ for all n, and combining this with the first item and the equivalences above yields

$$(\overline{\overline{q_n}})_n \approx_K (q_n)_n.$$

(3) By Lemma 3.19, for all n we have

$$\big((q_n z_n + u_n) {z_n}^\dagger\big) \overline{(q_n z_n + u_n) {z_n}^\dagger} = \overline{(q_n z_n + u_n) {z_n}^\dagger} \big((q_n z_n + u_n) {z_n}^\dagger\big) = 1.$$

We are now done thanks to the first item, the equivalences above, and Lemma 3.13.

(4) Finally, by Lemma 3.19 and uniqueness of inverses we have

$$\overline{\big((q_n z_n + u_n) {z_n}^\dagger\big) \big((r_n w_n + v_n) {w_n}^\dagger\big)} = \overline{(r_n w_n + v_n) {w_n}^\dagger} \cdot \overline{(q_n z_n + u_n) {z_n}^\dagger}.$$

As before, we finish the proof using the first item, Lemma 3.13, and the two equivalences above. □

3.E. **Weakly rational elements.**

Definition 3.21. Let $\mathrm{WRat}_0(K,t)$ be the set of sequences $(p_n,q_n)_n$ such that all of the following hold:

- $(p_n)_n$ is an admissible sequence in $\mathcal{U}K[t^{\pm1}]$,
- $(q_n)_n$ is an asymptotically injective sequence in $\mathcal{U}K[t^{\pm1}]$,
- the sequence $(p_n(q_n)^\dagger)_n$ stabilises,
- the sequence $(p_n\overline{q_n})_n$ K-stabilises.

Recall that if we are given a stabilising sequence in $\mathcal{U}G$, we have the limit map $\lambda_{\mathcal{U}G}$ described in Lemma 3.17 returning a single element in $\mathcal{U}G$. The map $\Lambda_{\mathcal{U}G}\colon \mathrm{WRat}_0(K,t)\to\mathcal{U}G$ is defined by composing

$$(p_n,q_n)_n \mapsto (p_n{q_n}^\dagger)_n$$

with $\lambda_{\mathcal{U}G}$.

Similarly, given an admissible K-stabilising sequence in $\mathcal{U}K[t^{\pm1}]\!]$, we may apply the limit map over $\mathcal{U}K$ in every degree separately, and obtain a map $\lambda_{\mathcal{U}K[t^{\pm1}]\!]}$ returning an element of $\mathcal{U}K[t^{\pm1}]\!]$. The map

$$\Lambda_{\mathcal{U}K[t^{\pm1}]\!]}\colon \mathrm{WRat}_0(K,t)\to\mathcal{U}K[t^{\pm1}]\!]$$

is defined by composing

$$(p_n,q_n)_n \mapsto (p_n\overline{q_n})_n$$

with $\lambda_{\mathcal{U}K[t^{\pm1}]\!]}$. We will refer to $\Lambda_{\mathcal{U}G}$ and $\Lambda_{\mathcal{U}K[t^{\pm1}]\!]}$ as *limit functions.*

We let $\sim$ be a relation on $\mathrm{WRat}_0(K,t)$ given by $(p_n,q_n)_n\sim(p'_n,q'_n)_n$ if $\Lambda_{\mathcal{U}G}((p_n,q_n)_n)=\Lambda_{\mathcal{U}G}((p'_n,q'_n)_n)$ and $\Lambda_{\mathcal{U}K[t^{\pm1}]\!]}((p_n,q_n)_n)=\Lambda_{\mathcal{U}K[t^{\pm1}]\!]}((p'_n,q'_n)_n)$. Since equality is an equivalence relation it follows that $\sim$ is an equivalence relation too.

We define $\mathrm{WRat}(K,t)$ to be the set of $\sim$-equivalence classes. The elements of $\mathrm{WRat}(K,t)$ will be referred to as *weakly rational sequences.* We will abuse notation slightly and denote the elements of $\mathrm{WRat}(K,t)$ by a representing pair from $\mathrm{WRat}_0(K,t)$.

The maps $\Lambda_{\mathcal{U}G}$ and $\Lambda_{\mathcal{U}K[t^{\pm1}]\!]}$ descend to maps on $\mathrm{WRat}(K,t)$; we will denote these maps with the same symbols. Moreover,

$$\Lambda_{\mathcal{U}G}\times\Lambda_{\mathcal{U}K[t^{\pm1}]\!]}\colon \mathrm{WRat}(K,t)\to\mathcal{U}G\times\mathcal{U}K[t^{\pm1}]\!]$$

is injective.

Lemma 3.22. *For every $(p_n,q_n)_n\in\mathrm{WRat}_0(K,t)$ and every asymptotically injective sequence $(x_n)_n$ in $\mathcal{U}K[t^{\pm1}]$, the sequence $(p_nx_n,q_nx_n)_n$ lies in $\mathrm{WRat}_0(K,t)$ in the same equivalence class as $(p_n,q_n)_n$.*

Proof. As $(x_n)_n$ is asymptotically injective, the powers associated to $\operatorname{init} x_n$ are eventually zero, and therefore the powers associated to $\operatorname{init}(p_nx_n)$ are eventually bounded from below by those of $\operatorname{init} p_n$. This implies that $(p_nx_n)_n$ is admissible.

The sequence $(q_nx_n)_n$ is asymptotically injective by Lemma 3.3.

Lemma 3.16(4) implies that $(p_nx_n(q_nx_n)^\dagger)_n\approx(p_n(q_n)^\dagger)_n$, which implies both that $(p_nx_n(q_nx_n)^\dagger)_n$ is stable, and that

$$\Lambda_{\mathcal{U}G}((p_n,q_n)_n)=\Lambda_{\mathcal{U}G}((p_nx_n,q_nx_n)_n),$$

using the uniqueness part of Lemma 3.17.

Lemmata 3.13 and 3.20 tell us that $(p_n x_n \overline{q_n x_n})_n \approx_K (p_n \overline{q_n})_n$, which again implies K-stability and the equality

$$\Lambda_{\mathcal{U}K[\![t^{\pm 1}]\!]}((p_n, q_n)_n) = \Lambda_{\mathcal{U}K[\![t^{\pm 1}]\!]}((p_n x_n, q_n x_n)_n),$$

using the uniqueness part of Lemma 3.17 for the latter. □

Proposition 3.23. *There exists a ring structure on* $\mathrm{WRat}(K,t)$ *such that the map* $\iota\colon \mathcal{U}K[t^{\pm 1}]$, $p \mapsto (p,1)_n$ *is a ring monomorphism, and the maps* $\Lambda_{\mathcal{U}G}\colon \mathrm{WRat}(K,t) \to \mathcal{U}G$ *and* $\Lambda_{\mathcal{U}K[\![t^{\pm 1}]\!]}\colon \mathrm{WRat}(K,t) \to \mathcal{U}K[\![t^{\pm 1}]\!]$ *are ring homomorphisms.*

Proof. We will break the proof into three parts.

Additive structure. The proof will be a little involved. We start by defining a binary operation on $\mathrm{WRat}_0(K,t)$. We will denote it by $+$ and refer to it as addition. We will then show that it is respected by the limit functions, which will immediately tell us that it descends to a well-defined operation on $\mathrm{WRat}(K,t)$. Finally, we will check the necessary properties.

Let $(p_n, q_n)_n$ and $(p'_n, q'_n)_n$ in $\mathrm{WRat}_0(K,t)$ be given. We are first going to bring them to a common denominator: Corollary 3.9 gives us asymptotically injective sequences $(x_n)_n$, $(y_n)_n$, and $(z_n)_n$ over $\mathcal{N}K[t^{\pm 1}]$ with $x_n = q_n y_n = q'_n z_n$ for every n. Lemma 3.22 tells us that $(p_n y_n, x_n)_n$ and $(p'_n z_n, x_n)_n$ lie in $\mathrm{WRat}_0(K,t)$; we define the sum of $(p_n, q_n)_n$ and $(p'_n, q'_n)_n$ to be

$$(p_n y_n + p'_n z_n, x_n)_n.$$

We now need to check that this sequence lies in $\mathrm{WRat}_0(K,t)$.

To check that $(p_n y_n + p'_n z_n, x_n)_n$ lies in $\mathrm{WRat}_0(K,t)$, we need to verify the stability conditions of the definition (other parts of the definition being trivial to check). This is immediate, since

$$(p_n y_n + p'_n z_n) {x_n}^\dagger = p_n y_n {x_n}^\dagger + p'_n z_n {x_n}^\dagger,$$

and

$$(p_n y_n + p'_n z_n)\overline{x_n} = p_n y_n \overline{x_n} + p'_n z_n \overline{x_n}.$$

Bringing the sequences $(p_n, q_n)_n$ and $(p'_n, q'_n)_n$ to common denominators does not change their value under the limit maps by Lemma 3.22. Hence the computations above immediately show that the limit functions respect our addition. By definition of the equivalence class $\sim$ on $\mathrm{WRat}_0(K,t)$, we see that addition descends to a well-defined operation on $\mathrm{WRat}(K,t)$.

The addition has a neutral element $(0,1)_n$. Moreover,

$$(p_n, q_n)_n + (-p_n, q_n)_n = (0, q_n)_n \sim (0,1)_n.$$

Commutativity and associativity of our addition follow from the respective properties of addition in the ring $\mathcal{U}G \times \mathcal{U}K[\![t^{\pm 1}]\!]$.

Now we will investigate the behaviour of the function ι with respect to this newly defined addition – we have already checked that the limit functions respect it.

For $p, p' \in \mathcal{U}K[t^{\pm 1}]$, we have

$$\iota(p) + \iota(p') = (p,1)_n + (p',1)_n = (p+p',1)_n = \iota(p+p'). \quad \blacklozenge$$

Multiplicative structure. We now move to defining multiplication. We will do it in a similarly involved way as in the case of addition, starting with $\mathrm{WRat}_0(K,t)$.

Let $(p_n,q_n)_n$ and $(p'_n,q'_n)_n$ be given elements of $\mathrm{WRat}_0(K,t)$. Let i_n denote the power associated to $\operatorname{init} p'_n$, and note that the sequence $(i_n)_n$ is bounded from below, as $(p'_n)_n$ is admissible. Using Corollary 3.8, we get an asymptotically injective sequence $(s'_n)_n$ and an admissible sequence $(r'_n)_n$ such that

$$p'_n t^{-i_n} s'_n = q_n r'_n$$

for every n. Setting $s_n = t^{-i_n} s'_n t^{i_n}$ and $r_n = r'_n t^{i_n}$, we obtain an asymptotically injective sequence $(s_n)_n$ and an admissible sequence $(r_n)_n$ such that

$$p'_n s_n = q_n r_n$$

for every n. We define $(p_n,q_n)_n \cdot (p'_n,q'_n)_n = (p_n r_n, q'_n s_n)$. We first need to check that the output is an element of $\mathrm{WRat}_0(K,t)$. The sequence $(p_n r_n)_n$ is clearly admissible. The sequence $(q'_n s_n)_n$ is asymptotically injective by Lemma 3.3. By Lemma 3.16(1), $\big((q'_n s_n)^\dagger\big)_n \approx ({s_n}^\dagger {q'_n}^\dagger)_n$; by (2), $({q_n}^\dagger q_n)_n \approx ({s_n}^\dagger s_n)_n \approx (1)_n$. We therefore have

$$\begin{aligned}
\big(p_n r_n (q'_n s_n)^\dagger\big)_n &\approx \Big(p_n {q_n}^\dagger q_n r_n {s_n}^\dagger {q'_n}^\dagger\Big)_n \\
&= \Big(p_n {q_n}^\dagger p'_n s_n {s_n}^\dagger {q'_n}^\dagger\Big)_n \\
&\approx \Big(p_n {q_n}^\dagger p'_n {q'_n}^\dagger\Big)_n .
\end{aligned}$$

Since $\big(p_n {q_n}^\dagger\big)_n$ and $\Big(p'_n {q'_n}^\dagger\Big)_n$ stabilise, we have

$$\begin{aligned}
\Big(p_n {q_n}^\dagger p'_n {q'_n}^\dagger\Big)_n &\approx \Big(p_n {q_n}^\dagger p'_{n+1} {q'_{n+1}}^\dagger\Big)_n \\
&\approx \Big(p_{n+1} {q_{n+1}}^\dagger p'_{n+1} {q'_{n+1}}^\dagger\Big)_n \\
&\approx \big(p_{n+1} r_{n+1} (q'_{n+1} s_{n+1})^\dagger\big)_n .
\end{aligned}$$

The argument for

$$\big(p_n r_n \overline{q'_n s_n}\big)_n \approx_K \big(p_n \overline{q_n} p'_n \overline{q'_n}\big)_n \approx_K \Big(p_{n+1} r_{n+1} \overline{q'_{n+1} s_{n+1}}\Big)_n$$

is analogous.

These computations directly imply that multiplication is respected by the limit functions, and hence descends to a well-defined operation on $\mathrm{WRat}(K,t)$.

We have that $(1,1)_n$ is a neutral element of the multiplication, $\iota(1) = (1,1)_n$, and the limit functions take $(1,1)_n$ to the respective identities. Also, ι respects the multiplication.

Distributivity and associativity follow from the respective properties in $\mathcal{U}G \times \mathcal{U}K[\![t^{\pm 1}]\!]$. ◆

Embedding. For $p \in \mathcal{U}K[t^{\pm 1}]$ we have $(p,1)_n \in \mathrm{WRat}_0(K,t)$, and so ι is well defined.

Suppose that $(p,1)_n \sim (p',1)_n$. Then

$$p = \lambda_{\mathcal{U}G}((p)_n) = \Lambda_{\mathcal{U}G}((p,1)_n) = \Lambda_{\mathcal{U}G}((p',1)_n) = p'. \qquad ◆$$

As we have verified all of the claims, the proof is complete. □

Henceforth, we will always endow $\mathrm{WRat}(K,t)$ with the above ring structure.

It follows directly from the definition of the equivalence relation $\sim$ that $\Lambda_{\mathcal{U}G} \times \Lambda_{\mathcal{U}K[\![t^{\pm1}]\!]}\colon \mathrm{WRat}(K,t) \to \mathcal{U}G \times (\mathcal{U}K[\![t^{\pm1}]\!])$ is an embedding of rings. It would be interesting to establish that both $\Lambda_{\mathcal{U}G}$ and $\Lambda_{\mathcal{U}K[\![t^{\pm1}]\!]}$ are also injective – we leave it as an open question.

3.F. **Elements of positive support.**

Definition 3.24. An element $p \in \mathcal{U}K[\![t^{\pm1}]\!]$ is said to be of *positive support* if $p = 0$ or if the power associated to $\operatorname{init} p$ is positive. A sequence $(p_n, q_n)_n \in \mathrm{WRat}_0(K,t)$ is said to be of *positive support* if the elements p_n are eventually of positive support. An element of $\mathrm{WRat}(K,t)$ is said to be of *positive support* if it can be represented by a sequence of positive support.

Lemma 3.25. *The set of elements of positive support in* $\mathrm{WRat}(K,t)$ *is closed under addition, taking additive inverse, and multiplication.*

Proof. Let $(p_n, q_n)_n$ and $(p'_n, q'_n)_n$ be two elements of $\mathrm{WRat}_0(K,t)$ of positive support. The fact that $(-p_n, q_n)_n$ is also of positive support is immediate.

Let $(x_n)_n$ be an asymptotically injective sequence. By definition, the power associated to $\operatorname{init} x_n$ is eventually zero. Hence, for n large enough, the element $p_n x_n$ is zero or the power associated to $\operatorname{init}(p_n x_n)$ is bounded from below by the power associated to $\operatorname{init} p_n$. We conclude that $(p_n x_n, q_n x_n)_n \in \mathrm{WRat}_0(K,t)$ is also of positive support. This allows us to bring $(p_n, q_n)_n$ and $(p'_n, q'_n)_n$ to common denominators, and then the fact that their sum is of positive support is immediate, since the sum of two elements of $\mathcal{U}K[\![t^{\pm1}]\!]$ of positive support is itself of positive support.

Recall from the proof of Proposition 3.23 that to multiply the elements $(p_n, q_n)_n$ and $(p'_n, q'_n)_n$ we find an asymptotically injective sequence $(s_n)_n$ and an admissible sequence $(r_n)_n$ such that

$$q_n r_n = p'_n s_n$$

for every n. Moreover, the power associated to $\operatorname{init} r_n$ is at least that of $\operatorname{init} p'_n$. Hence, the powers associated to $\operatorname{init} r_n$ are eventually at least zero. This implies that $p_n r_n$ is eventually of positive support, and thus so is $(p_n r_n, q'_n s_n)_n$. □

Lemma 3.26. *If* $(p_n, q_n)_n \in \mathrm{WRat}_0(K,t)$ *is of positive support then the element* $1 + (p_n, q_n)_n$ *is invertible in* $\mathrm{WRat}(K,t)$*, and the inverse is of the form* $1 + (p'_n, q'_n)_n$ *with* $(p'_n, q'_n)_n$ *of positive support.*

Proof. Since p_n is eventually of positive support and $(q_n)_n$ is asymptotically injective, the sequence $(q_n + p_n)_n$ is asymptotically injective, and the sequence $(q_n)_n$ is admissible. Hence the sequence $(q_n, q_n + p_n)_n$ satisfies the first two bullet points of the definitions of lying in $\mathrm{WRat}_0(K,t)$.

As verified in the proof of Proposition 3.23, the sequence $(q_n + p_n, q_n)_n$ also lies in $\mathrm{WRat}_0(K,t)$, and it represents $1 + (p_n, q_n)_n$ in $\mathrm{WRat}(K,t)$. We thus know that $((q_n + p_n){q_n}^\dagger)_n$ stabilises and that $((q_n + p_n)\overline{q_n})_n$ K-stabilises.

By Lemma 3.15, the sequence $\big(((q_n + p_n){q_n}^\dagger)^\dagger\big)_n$ also stabilises, and since both $(q_n + p_n)_n$ and $(q_n)_n$ are asymptotically injective, Lemma 3.16 tells us that $\big(((q_n + p_n)q_n{}^\dagger)^\dagger\big)_n \approx \big(q_n(q_n + p_n)^\dagger\big)_n$, and so the latter sequence stabilises as well.

Since $((q_n+p_n)\overline{q_n})_n$ K-stabilises, by Lemma 3.20 the sequence $\big(\overline{(q_n + p_n)\overline{q_n}}\big)_n$ also K-stabilises. By Lemmata 3.13 and 3.20 we have

$$\big(\overline{(q_n + p_n)\overline{q_n}}\big)_n \approx_K \big(q_n\overline{(q_n + p_n)}\big)_n$$

and so the latter K-stabilises as well. We conclude that $(q_n, q_n + p_n)_n \in \mathrm{WRat}_0(K, t)$.

Finally, $(q_n+p_n, q_n)_n\cdot(q_n, q_n+p_n)_n = 1$ in $\mathrm{WRat}(K, t)$. Letting $p'_n = -p_n$ and $q'_n = q_n+p_n$ yields $1+(p'_n, q'_n)_n = (q_n, q_n+p_n)_n$ with $(p'_n, q'_n)_n$ of positive support. ☐

In $\mathcal{U}K[\![t^{\pm1}]\!]$ we also have a notion of being of positive support, which is either being zero or being supported on $\{t^i : i > 0\}$. When restricted to $\widehat{RG}^\varphi$, this coincides with the notion of positive support we defined earlier.

Lemma 3.27. *For $(p_n, q_n)_n \in \mathrm{WRat}(K, t)$, if $\Lambda_{\mathcal{U}K[\![t^{\pm1}]\!]}((p_n, q_n)_n)$ is of positive support, then so is $(p_n, q_n)_n$.*

Proof. Let k denote the lim inf of the powers associated to $\operatorname{init} p_n$; k is bounded from below since $(p_n)_n$ is admissible. If k is positive or equal to plus infinity then we are done. Otherwise, we will construct an asymptotically injective sequence $(r_n)_n$ such that the lim inf of the powers associated to $\operatorname{init}(p_nr_n)$ is strictly greater then k. Repeating this process at most $|k|$ times will conclude the proof, bearing in mind Lemma 3.22.

Let $a_n \in \mathcal{U}K$ be the term of p_n next to t^k. Since $\Lambda_{\mathcal{U}K[\![t^{\pm1}]\!]}((p_n, q_n)_n)$ is of positive support, we know that

$$(t^ka_n \operatorname{init} q_nt^{-k})_n \approx (0)_n.$$

Let $r_n = \operatorname{init} q_nt^{-k}\pi_{\ker(t^ka_n \operatorname{init} q_nt^{-k})}t^k$; these elements lie in $\mathcal{U}K$ eventually. Since the sequences $(\operatorname{init} q_n)_n$ and $(\pi_{\ker(t^ka_n \operatorname{init} q_nt^{-k})})_n$ are asymptotically injective, so is $(r_n)_n$. The initial term of p_nr_n has associated power greater than k for every n large enough. ☐

Proposition 3.28. *If P is a finite square matrix over* $\mathrm{WRat}(K, t)$ *such that all of its entries have positive support then $I + P$ is invertible over* $\mathrm{WRat}(K, t)$.

Proof. We will construct a left-inverse for $I + P$; the construction of a right-inverse is analogous. By Lemma 3.26, the first diagonal element of $I + P$ has an inverse $1 + x \in \mathrm{WRat}(K, t)$ with x of positive support. Let X be the diagonal matrix over $\mathrm{WRat}(K, t)$ all of whose diagonal elements are 1 with the exception of the first one, that is equal to $1 + x$. The matrix X is clearly invertible. The matrix $X(I + P)$ is again of the form $I + P'$ with entries of P' all having positive support (by Lemma 3.25), and the first diagonal entry of $I + P'$ is 1. Now elementary matrices with the off-diagonal entries of positive support (which are all invertible) transform $I + P'$ into a matrix $I + P''$ where the first column of P'' is zero, and all entries of P'' are of positive support, again by Lemma 3.25. We now argue in the same way for the second diagonal entry of $I + P''$, and in finitely many steps end with the identity matrix I, as desired. ☐

4. Passing vanishing of Novikov homology to other rings

In this section we will show that vanishing of Novikov homology up to a given degree can be determined by computing homology with coefficients being the division closure of RG in $\widehat{RG}^{\varphi}$. We will also show that for $R = \mathbb{Q}$, one can work with $\mathrm{WRat}(K, t)$ instead. We want to deal with both cases in one go.

As before, we have $\varphi \in S(G)$ with $K = \ker \varphi$, and $t \in G$ with $\varphi(t) = 1$.

Definition 4.1. A ring morphism $\Lambda\colon W \to \widehat{RG}^{\varphi}$ will be said to have *Neumann invertibility* if for every finite square matrix P over W, if all the entries of $\Lambda(P)$ have positive support over $\widehat{RG}^{\varphi}$ then $I + P$ is invertible over W.

Lemma 4.2. *The natural embedding of the division closure*

$$\mathcal{D} = \mathcal{D}(RG \subset \widehat{RG}^{\varphi})$$

into $\widehat{RG}^{\varphi}$ has Neumann invertibility.

Proof. Since $\Lambda\colon \mathcal{D} \to \widehat{RG}^{\varphi}$ is a natural embedding in this case, we will ignore it, and consider $\mathcal{D}$ to be a subring of $\widehat{RG}^{\varphi}$. Let P be a square matrix over $\mathcal{D}$ with entries of positive support over $\widehat{RG}^{\varphi}$. The upper-left corner entry of $I + P$ is of the form $1 + u \in \mathcal{D}$, where every element in the support of u has positive value under φ. In particular, $1 + u$ is invertible in $\widehat{RG}^{\varphi}$, and hence in $\mathcal{D}$, by Neumann series. It follows that we may multiply $I + P$ with a diagonal matrix X over $\mathcal{D}$, that is invertible over $\mathcal{D}$, to obtain $I + P'$ without changing the positivity of the support but simplifying the entry in the top-left corner, that is, such that the first diagonal entry of P' is 0 and the remaining entries have positive support.

The entries of $I + P'$ all lie in $\mathcal{D}$. We may now use elementary matrices over $\mathcal{D}$ with off-diagonal entries of positive support to clear the non-diagonal entries in the first column of $I + P'$. Repeating the process finitely many times for each diagonal entry of $I + P$ constructs a left-inverse of the matrix $I + P$ over $\mathcal{D}$; the argument for a right-inverse is analogous. □

We can use the above to show that the division and rational closures are equal.

Proposition 4.3. *Let R be a ring and let G be a finitely generated group. If $\varphi \in S(G)$, then $\mathcal{D}(RG \subset \widehat{RG}^{\varphi}) = \mathcal{R}(RG \subset \widehat{RG}^{\varphi})$.*

Proof. Let $\mathcal{D} = \mathcal{D}(RG \subset \widehat{RG}^{\varphi})$ and $\mathcal{R} = \mathcal{R}(RG \subset \widehat{RG}^{\varphi})$; clearly, $\mathcal{D} \subseteq \mathcal{R}$. Consider a square matrix $A \in \mathbf{M}_n(\mathcal{D})$ that is invertible over $\widehat{RG}^{\varphi}$. We need to show that A is invertible over $\mathcal{D}$, which will show that $\mathcal{D}$ is rationally closed, and hence that $\mathcal{D} = \mathcal{R}$. Let $B \in \mathbf{M}_n(\widehat{RG}^{\varphi})$ be such that $AB = I$ where I is the identity matrix.

Since A is in particular a finite matrix over $\widehat{RG}^{\varphi}$, there exists $r \in \mathbb{R}$ such that all entries of A are supported on $\varphi^{-1}\big((-r, \infty)\big)$. We truncate the entries of B at r, and obtain a matrix $\bar{B} \in \mathbf{M}_n(RG)$. Now, $A\bar{B} = I - P$ where the group elements in the supports of the entries of P are all positive with respect to φ. Indeed, let $Q = B - \bar{B}$. Then we have

$$I = AB = A(Q + \bar{B}) = AQ + A\bar{B}.$$

In particular, $A\bar{B} = I - AQ = I - P$. Moreover, since $A\bar{B} \in \mathbf{M}_n(\mathcal{D})$ we have $P \in \mathbf{M}_n(\mathcal{D})$.

By Lemma 4.2, the matrix $I - P$ has an inverse $(I - P)^{-1} \in \mathbf{M}_n(\mathcal{D})$. Hence, $A\bar{B}(I - P)^{-1} = I$ and $\bar{B}(I - P)^{-1} \in \mathbf{M}_n(\mathcal{D})$ since $\bar{B} \in \mathbf{M}_n(RG)$ and $(I - P)^{-1} \in \mathbf{M}_n(\mathcal{D})$. □

Proposition 4.4. *Let W be a ring containing RG as a subring, let $\Lambda\colon W \to \widehat{RG}^{\varphi}$ be a ring morphism such that $\Lambda|_{RG}$ is the usual embedding $RG \to \widehat{RG}^{\varphi}$, and suppose that Λ has Neumann invertibility.*

Let $C_\bullet = (C_n, \delta_n)_n$ be a chain complex of based free RG-modules with C_n and C_{n-1} finitely generated. Suppose that we are given a partial chain contraction $(H_i)_{i<n}$ of $C_\bullet \otimes_{RG} W$, and suppose that the resulting partial chain contraction $(\Lambda(H_i))_{i<n}$ of $C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}$ can be extended by one extra term $H'_n\colon C_n \otimes_{RG} \widehat{RG}^{\varphi} \to C_{n+1} \otimes_{RG} \widehat{RG}^{\varphi}$. Then the partial chain contraction $(H_i)_{i<n}$ can also be extended by one extra term

$$H_n\colon C_n \otimes_{RG} W \to C_{n+1} \otimes_{RG} W.$$

Proof. As all the modules are free and based, we will identify the homomorphisms with suitable matrices.

We will truncate the entries of the matrix H'_n to obtain a matrix $\bar{H}_n$ over RG, lift it to W, and construct a matrix A over W such that

$$\partial_{n+1}\bar{H}_n A + H_{n-1}\partial_n = \mathrm{id}_{C_n \otimes_{RG} W}.$$

Below we spell out the details.

Note that H'_n, when viewed as a matrix, has finitely many columns, and only finitely many non-zero entries (even though the number of rows might be infinite). Hence, when evaluating the product $\partial_{n+1}H_n$, only finitely many entries of the matrix ∂_{n+1} are multiplied by non-zero entries of H'_n and contribute to the matrix $\partial_{n+1}H'_n$. Let $v_1, \ldots, v_k$ be a list of all these entries of ∂_{n+1}; we additionally remove the elements that are zero from this list. Every such entry $v_i \in \widehat{RG}^{\varphi}$ has an element in its support with minimal value under φ; let $w_i \in \mathbb{R}$ denote this value. If this list is empty, we set $r = 0$; otherwise, take r to be a real number strictly greater than the maximum of the real numbers $-w_1, \ldots, -w_k$.

We truncate the entries of H'_n at r to obtain a matrix $\bar{H}_n$ over RG. We set

$$P = \partial_{n+1}(H'_n - \bar{H}_n).$$

By our choice of r, the entries of P are of positive support.

We have

$$I - P = \partial_{n+1}\bar{H}_n + \Lambda(H_{n-1})\partial_n = \Lambda(\partial_{n+1}\bar{H}_n + H_{n-1}\partial_n),$$

since W contains RG. Let $P' = I - \partial_{n+1}\bar{H}_n - H_{n-1}\partial_n$. By Neumann invertibility, $I - P'$ is invertible over W. Finally,

$$I = (\partial_{n+1}\bar{H}_n + H_{n-1}\partial_n)(I - P')^{-1} = \partial_{n+1}\bar{H}_n(I - P')^{-1} + H_{n-1}\partial_n(I - P')^{-1}.$$

We set $H_n = \bar{H}_n(I - P')^{-1}$ which is defined over W.

The only thing left to show is that $H_{n-1}\partial_n(I - P')^{-1} = H_{n-1}\partial_n$. Since $I - P'$ is invertible, this is equivalent to $H_{n-1}\partial_n = H_{n-1}\partial_n(I - P')$, which in turn is equivalent to $H_{n-1}\partial_n P' = 0$. Plugging in the definition of P', we see that this is equivalent to $H_{n-1}\partial_n = H_{n-1}\partial_n H_{n-1}\partial_n$. To prove this, we recall that H_{n-1} forms part of a chain contraction, and so we have H_{n-2} such that $\partial_n H_{n-1} + H_{n-2}\partial_{n-1} = I$. We obtain

$$H_{n-1}\partial_n H_{n-1}\partial_n = H_{n-1}(I - H_{n-2}\partial_{n-1})\partial_n = H_{n-1}\partial_n. \qquad \square$$

The inverse image $\Lambda^{-1}_{\mathcal{U}K[\![t^{\pm 1}]\!]}(\widehat{RG}^{\varphi})$ is a subring of $\mathrm{WRat}(K,t)$ that contains $\mathbb{Q}G$ (via the restriction of the map ι), and $\Lambda_{\mathcal{U}K[\![t^{\pm 1}]\!]} \circ \iota$ is the natural embedding.

Lemma 4.5. *The morphism $\Lambda_{\mathcal{U}K[\![t^{\pm 1}]\!]}$ restricted to $\Lambda^{-1}_{\mathcal{U}K[\![t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})$ has Neumann invertibility.*

Proof. Let P be a finite square matrix over the ring $\Lambda^{-1}_{\mathcal{U}K[\![t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})$ with $\Lambda_{\mathcal{U}K[\![t^{\pm 1}]\!]}(P)$ having positive supports. Then $I+P$ is invertible in $\mathrm{WRat}(K,t)$ by Lemma 3.27 and Proposition 3.28. The inverse is mapped by $\Lambda_{\mathcal{U}K[\![t^{\pm 1}]\!]}$ to the inverse of $I + \Lambda_{\mathcal{U}K[\![t^{\pm 1}]\!]}(P)$, and this inverse lies in $\widehat{\mathbb{Q}G}^{\varphi}$, since it is a Neumann series. Thus the inverse of $I + P$ lies in $\Lambda^{-1}_{\mathcal{U}K[\![t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})$. $\square$

We are now ready to prove the main result of this section.

Proposition 4.6. *Let G be a finitely generated group and let $\varphi \in S(G)$. Let $C_\bullet$ be a chain complex of free RG-modules finitely generated up to dimension n, with $C_k = 0$ for $k < 0$.*

(1) For a ring R,

$$H_j(C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}) = 0 \text{ for } j \leqslant n$$

if and only if

$$H_j\big(C_\bullet \otimes_{RG} \mathcal{D}(RG \subset \widehat{RG}^{\varphi})\big) = 0 \text{ for } j \leqslant n.$$

(2) For $R = \mathbb{Q}$,

$$H_j(C_\bullet \otimes_{\mathbb{Q}G} \widehat{\mathbb{Q}G}^{\varphi}) = 0 \text{ for } j \leqslant n$$

if and only if

$$H_j\big(C_\bullet \otimes_{\mathbb{Q}G} \Lambda^{-1}_{\mathcal{U}K[\![t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})\big) = 0 \text{ for } j \leqslant n.$$

Proof. The proofs are the same in both cases.

First, suppose that $H_i(C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}) = 0$ for $i \leqslant n$. Since the modules in the chain complex are free, the vanishing of homology is equivalent to the existence of a chain contraction, by Lemma 2.23. We will build a partial chain contraction over $\mathcal{D}$ or $\Lambda^{-1}_{\mathcal{U}K[\![t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})$ for $i \leqslant n$ (we need $R = \mathbb{Q}$ in the latter case) using Proposition 4.4 iteratively.

Suppose we already have such a chain contraction $(H_j)_{j<i}$ for a fixed i with $0 \leqslant i < n$. We can push it down to a partial chain contraction $(H'_j)_{j<i}$ over $\widehat{RG}^{\varphi}$, and since $H_i(C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}) = 0$, by Lemma 2.23 we can extend it by an extra term H'_i. Now Proposition 4.4 allows us to extend the initial partial chain contraction $(H_j)_{j<i}$ by an extra term H_i. This guarantees that the desired homology vanishes, again by Lemma 2.23.

For the other implication, suppose that $H_i\big(C_\bullet \otimes_{RG} \mathcal{D})\big) = 0$ for $i \leqslant n$. From Lemma 2.23 we obtain chain contractions over $\mathcal{D}$. Since $\mathcal{D} \subseteq \widehat{RG}^{\varphi}$, they are also chain contractions over $\widehat{RG}^{\varphi}$, and so $H_i(C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}) = 0$ for $i \leqslant n$.

When $R = \mathbb{Q}$, the argument is similar: we obtain a chain contraction over $\Lambda^{-1}_{\mathcal{U}K[t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})$, and push it to a chain contraction over $\widehat{\mathbb{Q}G}^{\varphi}$ using the map $\Lambda_{\mathcal{U}K[t^{\pm 1}]\!]}$. □

5. The main results

Proposition 5.1. *Let G be a countable discrete group and let*

$$\varphi\colon G \to \mathbb{Z}$$

be a non-trivial homomorphism. Let $C_\bullet$ be a complex of free $\mathbb{Q}G$-modules which is finitely generated up to degree n and such that $C_k = 0$ for $k < 0$. If

$$H_j(C_\bullet \otimes_{\mathbb{Q}G} \widehat{\mathbb{Q}G}^{\varphi}) = 0$$

for $j \leqslant n$, then $H_j(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{U}G) = 0$ for $j \leqslant n$.

Proof. By Proposition 4.6, we see that

$$H_j(C_\bullet \otimes_{\mathbb{Q}G} \Lambda^{-1}_{\mathcal{U}K[t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})) = 0$$

for $j \leqslant n$. Using Lemma 2.23, we realise this vanishing using chain contractions over $\Lambda^{-1}_{\mathcal{U}K[t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})$, and hence over $\mathrm{WRat}(K,t)$, since $\Lambda^{-1}_{\mathcal{U}K[t^{\pm 1}]\!]}(\widehat{\mathbb{Q}G}^{\varphi})$ is a subring of $\mathrm{WRat}(K,t)$. Now we apply $\Lambda_{\mathcal{U}G}$ to the chain contractions, and obtain chain contractions over $\mathcal{U}G$, yielding

$$H_j(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{U}G) = 0$$

for $j \leqslant n$, and the proof is finished. □

We are now ready to prove the results from the introduction.

Theorem A. *Let G be a group of type $\mathsf{FP}_n(\mathbb{Q})$. If $b_n^{(2)}(G) \neq 0$, then $\Sigma^n(G;\mathbb{Q}) = \varnothing$.*

Proof. As $\Sigma^k(G;\mathbb{Q}) \subseteq \Sigma^\ell(G;\mathbb{Q})$ for all $k \geqslant \ell$, we may assume n is the lowest dimension for which $b_n^{(2)}(G) \neq 0$. In particular, $H_n(G;\mathcal{U}G) \neq 0$. Since G is of type $\mathsf{FP}_n(\mathbb{Q})$, there is a resolution $C_\bullet$ of $\mathbb{Q}$ by free $\mathbb{Q}G$-modules that is finitely generated up to dimension n. We may compute $H_p(G;\mathcal{U}G)$ as $H_p(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{U}G)$. It follows that $H_n(C_\bullet \otimes_{\mathbb{Q}G} \mathcal{U}G) \neq 0$. Thus, by Proposition 5.1, for every character $\varphi\colon G \twoheadrightarrow \mathbb{Z}$, we have $H_n(G;\widehat{\mathbb{Q}G}^{\varphi}) = H_n(C_\bullet \otimes_{\mathbb{Q}G} \widehat{\mathbb{Q}G}^{\varphi}) \neq 0$. In particular, Sikorav's theorem implies that every integral character is not contained in $\Sigma^n(G;\mathbb{Q})$ and so necessarily every character on a ray from 0 to an integral character $\varphi \in H^1(G;\mathbb{R})$ is also not contained in $\Sigma^n(G;\mathbb{Q})$. Now, since the BNSR invariants are open subsets of $H^1(G;\mathbb{R})\smallsetminus\{0\}$ by Theorem 2.16(1), its complement in $H^1(G;\mathbb{R})$ must be a closed subset of $H^1(G;\mathbb{R})$, namely all of it. Hence, we have $\Sigma^n(G;\mathbb{Q}) = \varnothing$. □

The proof of Theorem B is entirely analogous once one replaces $C_\bullet$ with $C_\bullet(\widetilde{X};\mathbb{Q})$, the chain complex of the universal cover of X, viewed as a chain complex of free $\mathbb{Q}\pi_1 X$ modules.

Corollary C. *Let M be a closed connected $2n$-manifold or (more generally) a finite $\mathsf{PD}_{2n}(\mathbb{Q})$-complex. If $\chi(M) \neq 0$, then $\Sigma^n(M) = \Sigma^n(M;\mathbb{Q}) = \varnothing$. In particular, if M is additionally aspherical, then $\Sigma^n(\pi_1 M) = \Sigma^n(\pi_1 M;\mathbb{Q}) = \varnothing$.*

Proof. Let us start with M being a manifold. After passing to a finite cover we may assume that M is orientable and by [KS69, Theorem III] we may assume that M has the homotopy type of a finite CW complex. By [Lüc02, Remark 6.81] we have

$$\sum_{p \geqslant 0} (-1)^p b_p^{(2)}(\widetilde{M};\pi_1 M) = \chi(M) \neq 0.$$

In particular, there is some p where $b_p^{(2)}(\widetilde{M};\pi_1 M) \neq 0$. By Poincaré duality [Lüc02, Theorem 1.35(3)] we have $b_{2n-k}^{(2)}(M) = b_k^{(2)}(M)$. In particular, we may assume that $p \leqslant n$. Now, Theorem B implies that $\Sigma^p(M;\mathbb{Q}) = \varnothing$, of which $\Sigma^n(M)$ is a subset by Theorem 2.16(2). The argument for a $\mathsf{PD}_{2n}(\mathbb{Q})$-complex is identical.

Suppose in addition that M is aspherical. Then, M is a model for $K(\pi_1 M, 1)$ and so $b_p^{(2)}(\pi_1 M) = b_p^{(2)}(\widetilde{M};\pi_1 M)$. Now, by Theorem A we have $\Sigma^p(\pi_1 M;\mathbb{Q}) = \varnothing$. The result follows from Theorem 2.16(2). □

Corollary D. *Let G be a $\mathsf{PD}_n(\mathbb{Q})$-group and let $k = \lceil n/2 \rceil - 1$. If $\Sigma^k(G;\mathbb{Q})$ is non-empty, then the Singer Conjecture holds for G.*

Proof. Arguing by Poincaré duality as in the proof of Corollary C it suffices to show that $b_p^{(2)}(G) = 0$ for $p \leqslant k$. Now, by hypothesis there exists $\varphi \in \Sigma^k(G,\mathbb{Q})$. So, by Theorem A, $b_p^{(2)}(G) = 0$ for $p \leqslant k$. □

6. The Atiyah Conjecture and locally indicable groups

In this section we prove versions of Theorems A and B in positive characteristic. This relies on the existence of certain Hughes-free[1] skew fields.

Let R be a skew field and let G be a group. When it exists, we denote by $\mathcal{D}_{RG}$ the *Hughes-free skew field* of RG. We omit the technical definition of a Hughes-free skew field as we do not require it. However, we note that if it exists it is unique up to an RG-algebra isomorphism [Hug70]. We also mention that *Hughes-free* is equivalent to *strongly Hughes-free* by work of Gräter [Gra20].

We recall a property introduced by Agol [Ago08]. A group G is *residually finite rationally soluble* (or RFRS) if there exists a chain

$$G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \cdots$$

such that

1. G_i is a finite index normal subgroup of G;
2. $\bigcap_{i \geqslant 0} G_i = \{1\}$; and
3. $\ker(G_i \to G_i^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}) \leqslant G_{i+1}$,

[1] No relation to the first author.

where G_i^{ab} is the abelianisation of G_i. Note that by [OS25, Theorem 6.3], for a finitely generated group, being RFRS and being residually {locally indicable and virtually abelian} are equivalent. Without assuming finite generation, we still have all RFRS groups being residually {locally indicable and virtually abelian}.

Remark 6.1. It is conjectured that $\mathcal{D}_{RG}$ exists for any skew field R for all locally indicable groups [JZ21, Conjecture 1]; it is known to exist for residually {amenable and locally indicable} groups [JZ21, Corollary 1.3]. In particular, $\mathcal{D}_{RG}$ exists for RFRS groups.

Definition 6.2. A group G is *agrarian* over a ring R if there exists a skew field $\mathcal{D}$ and a monomorphism $\psi\colon RG \to \mathcal{D}$ of rings. If G is agrarian over R and X is a space with $\pi_1 X = G$, then we define the *agrarian $\mathcal{D}$-homology of X* to be

$$H_p^{\mathcal{D}}(X) = H_p(C_\bullet(\widetilde{X}; R) \otimes_{RG} \mathcal{D})$$

where $\mathcal{D}$ is viewed as an RG-$\mathcal{D}$-bimodule via the embedding $RG \to \mathcal{D}$. We also define the *agrarian $\mathcal{D}$-homology of G* to be

$$H_p^{\mathcal{D}}(G) = \mathrm{Tor}_p^{RG}(R, \mathcal{D}).$$

Since modules over a skew field have a canonical dimension function taking values in $\mathbb{N} \cup \{\infty\}$ we define

$$b_p^{\mathcal{D}}(X) = \dim_{\mathcal{D}} H_p^{\mathcal{D}}(X) \quad \text{and} \quad b_p^{\mathcal{D}}(G) = \dim_{\mathcal{D}} H_p^{\mathcal{D}}(G).$$

The Atiyah Conjecture. *Let G be a torsion-free countable group. Then, the ring $\mathcal{D}(\mathbb{C}G \subset \mathcal{U}G)$ is a skew-field.*

Remark 6.3. If a torsion-free countable group G satisfies the Atiyah Conjecture, then $\mathcal{D}_{\mathbb{C}G}$ exists and is isomorphic to $\mathcal{D}(\mathbb{C}G \subset \mathcal{U}G)$. This applies for instance to torsion-free subgroups of right-angled Artin and Coxeter groups [LOS12], torsion-free virtually special groups [Sch14], locally indicable groups [JZLÁ20], and more [Lin93, FL06, JZ19].

Theorem 6.4. *Let R be a skew field and let G be a group such that $\mathcal{D}_{RG}$ exists. Let $\varphi \in S(G)$.*

(1) Suppose that G is of type $\mathsf{FP}_n(R)$. If $b_n^{\mathcal{D}_{RG}}(G) \neq 0$, then $\Sigma^n(G; R) = \varnothing$.

(2) Let X be a connected CW complex with finite n-skeleton. If $b_n^{\mathcal{D}_{RG}}(X) \neq 0$, then $\Sigma^n(X; R) = \varnothing$.

In particular, if G satisfies the Atiyah Conjecture, then statements (1) and (2) hold with $R = \mathbb{C}$ and with ℓ^2-Betti numbers replacing $\mathcal{D}_{\mathbb{C}G}$-agrarian Betti numbers.

Proof. We proceed as in [HK25, Theorem 5.10]. Let $K = \ker \varphi$. Let $\mathbb{K}$ be the skew-field of twisted Laurent series with variable t and coefficients in the skew field $\mathcal{D}_{RK}$; here t is an element of G with $\varphi(t) = 1$ and the twisting extends the conjugation action of t on K – such an extension of the action exists since $\mathcal{D}_{RK}$ is Hughes-free (see [JZ21, Section 2.3] for an explanation). The skew field $\mathbb{K}$ naturally contains the Ore localisation $\mathrm{Ore}(\mathcal{D}_{RK}[t^{\pm 1}])$.

We have two embeddings, firstly $\widehat{RG}^{\varphi}$ embeds into $\mathbb{K}$, and secondly $\mathcal{D}_{RG}$ embeds into $\mathbb{K}$. To see the first embedding, $\widehat{RG}^{\varphi}$ may be viewed as a ring of twisted Laurent series in t with coefficients in RK. The second embedding exists because $\mathcal{D}_{RG}$ is isomorphic as an RG-algebra to $\mathrm{Ore}(\mathcal{D}_{RK}[t^{\pm 1}])$, by [JZ21, Proposition 2.2]. In particular, we may view $\mathbb{K}$ as a $\mathcal{D}_{RG}$-module.

Claim 6.5. *Let $C_\bullet$ be a chain complex of finitely generated free RG-modules such that $C_i = 0$ for $i < 0$. If $H_j(C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}) = 0$ for $j \leqslant n$, then $H_j(C_\bullet \otimes_{RG} \mathcal{D}_{RG}) = 0$.*

Proof of Claim 6.5. Since $H_j(C_\bullet \otimes_{RG} \widehat{RG}^{\varphi}) = 0$ for $j \leqslant n$ and $\widehat{RG}^{\varphi} \subset \mathbb{K}$ it follows from Lemma 2.23 that $H_j(C_\bullet \otimes_{RG} \mathbb{K}) = 0$ for $j \leqslant n$. Now, $\mathbb{K}$ and $\mathcal{D}_{RG}$ are skew-fields, and so

$$H_j(C_\bullet \otimes_{RG} \mathcal{D}_{RG}) \otimes_{\mathbb{K}} \mathbb{K} = H_j(C_\bullet \otimes_{RG} \mathbb{K}) = 0,$$

forcing $H_j(C_\bullet \otimes_{RG} \mathcal{D}_{RG}) = 0$ for $j \leqslant n$, as claimed. ■

The theorem is now proved either by taking $C_\bullet$ in the claim to be a free resolution of the trivial RG-module R, finitely generated up to degree n, in the case of (1); or taking $C_\bullet$ to be $C_\bullet(\widetilde{X}; R)$, viewed as a chain complex of free RG-modules, in the case of (2). Finally, the result follows from the appropriate version of Sikorav's Theorem and openness of the BNSR invariants (Theorem 2.16(1)). □

Versions of Corollary C and Corollary D in characteristic p can be formulated and proved for groups G where $\mathcal{D}_{\mathbb{F}_pG}$ exists by almost verbatim arguments – with the exception of substituting Theorems A and B with Theorem 6.4.

7. Some examples

In this section we detail a number of examples that both complement results already in the literature and might be of independent interest. The section is written with a reader well-versed in the respective topics in mind, and hence some standard terms will not be defined, or will be treated only very briefly.

An *elementarily free group* is a group with the same first order theory as a free group. Whilst we will not define what these are we mention the class includes free groups, surface groups, and many other hyperbolic limit groups; for more information see [BTW07]. Note that every finitely generated elementarily free group is isomorphic to a limit group and so is of type F (see [BTW07]). A *poly-elementarily-free group* of length n is a group G that admits a subnormal filtration $\{1\} = N_0 \lhd N_1 \lhd \cdots \lhd N_n = G$ with N_i/N_{i-1} isomorphic to a finitely generated elementarily free group. Note that poly-{finitely generated free or surface} groups are poly-elementarily-free.

Lemma 7.1. *If G is a poly-elementarily-free group of length n, then $b_p^{(2)}(G) = 0$ for $p \neq n$ and $b_n^{(2)}(G) = |\chi(G)|$.*

Proof. The case $n = 0$, where G is the trvial group, is easily dealt with, and so we may assume that $n > 0$.

Let $(N_i)_i$ be a subnormal chain with every $N_i/N_{i-1} = G_i$ non-trivial and finitely generated elementarily free, and with $N_0 = \{1\}$ and $N_n = G$. Each N_i is a group of type F since it is an extension of such groups. By [BK17, Corollary C] we have $b_p^{(2}(G_i) = 0$ unless $p = 1$, in which case the first ℓ^2-Betti number of G_i is $-\chi(G_i)$.

An inductive application of [Lüc02, Theorem 6.67] yields that for every i and every $p < i$,

$$b_p^{(2)}(N_i) = 0.$$

By [BTW07, Theorem B], an elementarily free group is measure equivalent to a free group. Now, inductively applying [ST10, Theorem 1.10] shows that for every i and every $p > i$ we have $b_p^{(2)}(N_i) = 0$. Thus, $b_i^{(2)}(N_i) = |\chi(N_i)|$, and hence taking $i = n$ we obtain $b_n^{(2)}(G) = |\chi(G)|$. □

The following result generalises [KV23, Proposition 1.5], dealing with free-by-free or surface-by-surface groups, and the first part of [KW19, Theorem 6.1], dealing with {free group of rank 2}-by-free groups.

Theorem 7.2. *Let G be a poly-elementarily-free group of length n. If $\chi(G) \neq 0$, then $\Sigma^n(G;\mathbb{Q}) = \Sigma^n(G) = \varnothing$.*

Proof. The result now follows from Lemma 7.1 and Theorem A. □

We remark that the conclusions of Lemma 7.1 and Theorem 7.2 remain valid if G is a poly-$\mathcal{X}$ group where $\mathcal{X}$ is the class of groups of type FP that are measure equivalent to a free group.

Example 7.3 (Pure mapping class group of a punctured sphere)**.** Let $m \geqslant 3$, and let S_m denote the m-punctured 2-sphere. Recall that the *pure mapping class group* $\Gamma_m := \mathrm{PMCG}(S_m)$ of S_m is the group of mapping classes of S_m which fix the m-punctures pointwise. It is well known (see e.g. [FM11, Section 9.3]) that Γ_m is poly-free of length $m-2$ and each subnormal quotient in the poly-free filtration is non-abelian. Hence, $\chi(\Gamma_m) \neq 0$. We have verified the hypotheses of Theorem 7.2 and conclude that $\Sigma^{m-2}(\Gamma_m;\mathbb{Q}) = \Sigma^{m-2}(\Gamma_m) = \varnothing$.

Example 7.4 (Real and complex hyperbolic lattices)**.** Let G be equal to $\mathrm{SO}(2n, 1)$ or $\mathrm{SU}(n, 1)$. If Γ is a uniform lattice in G, then by [Dod79] (see also [Bor85]), we have $b_p^{(2)}(\Gamma) = 0$ except when $p = n$, in which case $b_n^{(2)}(\Gamma) \neq 0$. Since a uniform lattice in G is measure equivalent to a non-uniform lattice in G, a theorem of Gaboriau [Gab02] implies that for a non-uniform lattice Γ we have $b_p^{(2)}(\Gamma) = 0$ except when $p = n$, in which case $b_n^{(2)}(\Gamma) \neq 0$. Thus for any lattice Γ in G, by Theorem A, we have $\Sigma^n(\Gamma) = \Sigma^n(\Gamma;\mathbb{Q}) = \varnothing$. This result was already known for 'simplest type lattices' in $\mathrm{SU}(n, 1)$, see [LIP24].

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