Lattices in non-positive curvature

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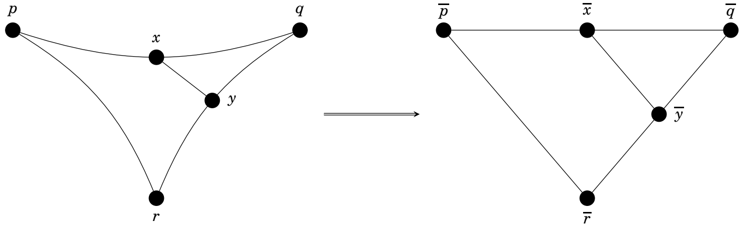
$\mathrm{CAT}(0)$ spaces

A geodesic metric space X is $\mathrm{CAT}(0)$ if for every geodesic triangle $P = \triangle(p, q, r) \subseteq X$ there exists a comparison triangle in $\mathbb{E}^2$ with the same side lengths as P such that for each pair of points $x, y \in \partial P$ we have

$$d_X(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$



Properties and examples

$\mathrm{CAT}(0)$ spaces are contractible and have unique geodesics.

Examples

- $\mathbb{E}^n$
- Trees
- Non-compact symmetric spaces (e.g. $\mathbb{R}\mathbf{H}^2$)
- Euclidean and hyperbolic buildings

Lattices in locally compact groups

Let $H = \mathrm{Isom}(X)$ be a locally compact group with Haar measure μ. A discrete subgroup $\Gamma \leq H$ is:

- a *lattice* if X/Γ has finite covolume;
- a *uniform lattice* if X/Γ is compact.

For a lattice Γ in a product $\prod_{i=1}^{n} H_i$ we say Γ is:

- *irreducible* if the projection to each subproduct of the H_i is non-discrete;
- *reducible* otherwise.

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Key point

Uniform lattices in the isometry group of a $\mathrm{CAT}(0)$ space are $\mathrm{CAT}(0)$ groups.

Idea: Study lattices in $\mathrm{Isom}(X)$ where X is a $\mathrm{CAT}(0)$ space.

Questions about lattices

Let X be a metric space and $H = \mathrm{Isom}(X)$ with Haar measure μ.

1. Does H have lattices?
2. What are properties of a generic lattice in H?
3. Do the properties of lattices in H reflect properties of H?
4. Can we classify lattices in H up to isomorphism, commensurability, or quasi-isometry?

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Crystallographic groups

- Let $\mathbb{E}^n$ denote Euclidean n-space, then $\mathrm{Isom}(\mathbb{E}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$.
- A lattice $\Gamma < \mathrm{Isom}(\mathbb{E}^n)$ admits a short exact sequence $\{1\} \to \mathbb{Z}^n \to \Gamma \to \pi_{\mathrm{O}(n)}(\Gamma) \to \{1\}$.
- A virtually abelian group Γ admits a short exact sequence $\{1\} \to \mathbb{Z}^n \to \Gamma \to P \to \{1\}$.
- **Zassenhaus:** If G is a finitely generated virtually abelian group, then G acts properly discontinuously cocompactly on $\mathbb{E}^n$ and so is $\mathrm{CAT}(0)$. Moreover, Γ is an $\mathrm{Isom}(\mathbb{E}^n)$-lattice if and only if $P \rightarrowtail \mathrm{GL}_n(\mathbb{R})$.

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Lattices in Lie groups

Let $G(\mathbb{R})$ be a non-compact semisimple real Lie group with finite centre.

e.g. $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})^2$.

Let K be a finite extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$ and let $\Gamma = G(\mathcal{O}_K)$

e.g. $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ or $\Lambda = \mathrm{SL}_2(\mathbb{Z})^2$.

For each place s_i of K we have an embedding $\sigma_i : K \rightarrowtail \mathbb{R}$ or $\mathbb{C}$

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Lattices in Lie groups (cont.)

Both $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ and $\Lambda = \mathrm{SL}_2(\mathbb{Z})^2$ are lattices in $\mathrm{SL}_2(\mathbb{R})^2$.

Γ is irreducible and Λ is reducible.

Both of them are *arithmetic groups*.

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Answers for Lie Groups

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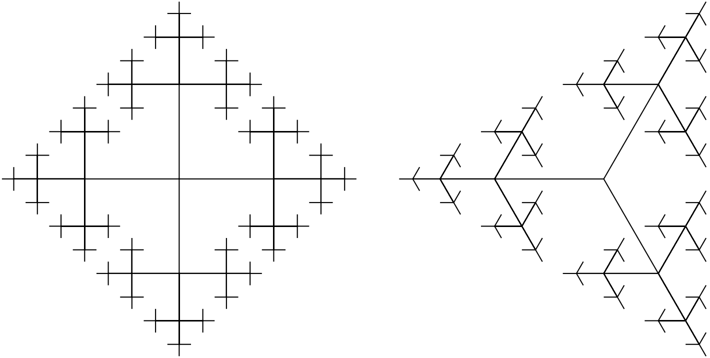
3. They have property (T) if H does. Homomorphisms of higher rank lattices induce homomorphisms of H. [Margulis Super Rigidity]
4. In theory in higher rank yes. All lattices are arithmetic [Margulis 1984] so possible using number theoretic techniques.

Tree lattices

Let $\mathcal{T}$ be a regular or biregular infinite tree e.g.



Let $T = \mathrm{Aut}(\mathcal{T})$. We want to study uniform lattices in T, i.e. groups which act cocompactly with finite stabilisers on $\mathcal{T}$.

Graphs of groups

Let $\mathcal{A} = (V\mathcal{A}, E\mathcal{A}, \iota, \tau)$ be a graph equipped with the following data:

- For each vertex v and edge e a group A_v or A_e;
- For each edge e, monomorphisms $\psi_e : A_e \to A_{\iota(e)}$ and $\psi_{\overline{e}} : A_e \to A_{\tau(e)}$

We denote this by A and call it a *graph of groups over $\mathcal{A}$*.

Graphs of groups (cont.)

From a graph of groups A we obtain the *fundamental group*. Let $\mathcal{X}$ be a spanning tree of $\mathcal{A}$ then $\pi_1(A)$ has generators

$$A_v,\ t_e \text{ for } v \in V\mathcal{A},\ e \in E\mathcal{A},$$

and relations

- $t_e = 1$ if $e \in E\mathcal{X}$;
- the relations in the groups A_v;
- $t_e \psi_e(g) t_e^{-1} = \psi_{\overline{e}}(g)$ for all $g \in A_e$.

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We also obtain a tree by taking the graph with vertices $\coprod_{v \in V\mathcal{A}} \pi_1(A)/A_v$ and edges $\coprod_{e \in E\mathcal{A}} \pi_1(A)/A_e$.

Bass-Serre Theory

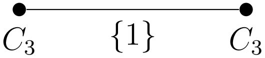
From a group action on a tree we can also construct a graph of groups from the vertex and edge stabilisers in a fundamental domain.

Theorem (Bass-Serre)

The processes of constructing of the Bass-Serre tree from a graph of groups and constructing a graph of groups from a group action on a tree are mutually inverse.

A quick example

- Let $\mathcal{A}$ be a single edge e with two vertices v and w.
- Let $A_v = A_w = C_3$ and $A_e = \{1\}$;



- $\pi_1(A) = \langle a, b \mid a^3, b^3 \rangle = C_3 * C_3$;
- The Bass-Serre tree is the 3-regular tree.
- Every vertex is stabilised by a conjugate of $\langle a \rangle$ or $\langle b \rangle$.

Tree lattices (revisited)

Theorem (Bass-Kulkarni 1990)

Every tree lattice splits as a faithful graph of finite groups, and conversely every faithful graph of finite groups is a tree lattice.

Corollary

Uniform tree lattices are virtually free.

Caprace–Monod's Decomposition Theorem (2009)

Let Y be proper $\mathrm{CAT}(0)$ space and assume $\mathrm{Isom}(Y)$ acts cocompactly and minimally. There exists a $\mathrm{Isom}(Y)$-stable subset X and $H \trianglelefteq_{\mathrm{f.i}} \mathrm{Isom}(X)$ such that:

$$X = \mathbb{R}^n \times M_1 \times \cdots \times M_p \times X_1 \times \cdots \times X_q,$$

and

$$H \cong \mathrm{Isom}(\mathbb{E}^n) \times L_1 \times \cdots \times L_p \times D_1 \times \cdots \times D_q,$$

where each M_i is a symmetric space of non-compact type with associated Lie group L_i and X_j is an irreducible $\mathrm{CAT}(0)$ space with totally disconnected isometry group D_j. Moreover, $\mathrm{AR(H)} = \mathrm{Isom(\mathbb{E}^n)}$.

Irreducibility

Theorem (Caprace-Monod 2009)

Let X and H be as on the previous slide. Suppose Γ is a uniform H-lattice, then Γ is a irreducible if and only if Γ does not virtually split as a direct product of two infinite groups.

On $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices

- It was thought every $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice was reducible i.e. virtually $\mathbb{Z}^n \times F_m$.

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- **Leary-Minasyan 2019:** Constructed groups $\mathrm{LM}(A)$ using a graph with a single vertex and edge which are irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices.

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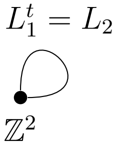
- **Leary-Minasyan 2019:** Constructed groups $\mathrm{LM}(A)$ using a graph with a single vertex and edge which are irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices.
- These groups are not linear, residually finite, or biautomatic answering a 25 year old question about whether every $\mathrm{CAT}(0)$ group is biautomatic.

Leary-Minasyan Groups



Let $A \in \mathrm{O}(2)$ and L_1 be a finite index subgroup of $\mathbb{Z}^2$ and $L_2 = A(L_1)$.

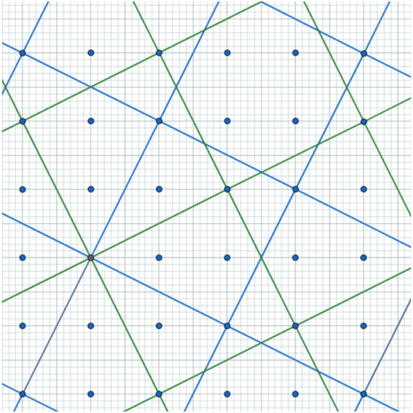
$$\mathrm{LM}(A) = \langle a, b, t \mid [a, b],\ tL_1t^{-1} = L_2 \rangle$$

$$\phi : \mathrm{LM}(A) \to \mathrm{Isom}(\mathbb{E}^n) \quad \text{by} \quad \begin{cases} a \mapsto & [1, 0]^T \\ b \mapsto & [0, 1]^T \\ t \mapsto & A. \end{cases}$$

Leary-Minasyan Groups (cont.)

$$A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix},\ L_1 = \left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle \text{ and } L_2 = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\rangle$$

$$G := \mathrm{LM}(A) = \langle a, b, t \mid [a,b],\ ta^2b^{-1}t^{-1} = a^2b,\ tab^2t^{-1} = a^{-1}b^2 \rangle$$



Irreducibility

Theorem (Leary-Minasyan 2019)

G is an irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T_{10})$-lattice.

Here $T_{10} := \mathrm{Aut}(\mathcal{T}_{10})$ the Bass-Serre tree of G.

The Flat Torus Theorem

Theorem

Let L be a free abelian group acting properly by semisimple isometries on $\mathrm{CAT}(0)$ space X. If a group Γ of isometries of X normalizes L, then a finite-index subgroup of Γ centralizes L. If Γ is finitely generated, then ΓL has a finite-index subgroup containing L as a direct factor.

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- Now $C \trianglelefteq \mathrm{LM}(A)$ so by *the flat torus theorem* there exists some $k > 0$ such that t^k centralises L;
- Thus, t^k acts trivially on $C \otimes \mathbb{R}$. Which is nonsensical unless $C = 0$.
- It follows L acts faithfully on $\mathcal{T}_{10}$ as a vertex stabiliser. Hence the projection is non-discrete. $\square$

Graphs of lattices

Definition (Graph of lattices)

Let H be a locally compact group with Haar measure μ. A graph of H-lattices $(A, \mathcal{A}, \psi)$ is a graph of groups $(A, \mathcal{A})$ equipped with a morphism $\psi : \mathcal{A} \to H$ such that:

1. *Each local group $A_\sigma \in \mathcal{A}$ is covirtually an H-lattice and the image $\psi(A_\sigma)$ is an H-lattice;*
2. *The local groups are commensurable in $\Gamma = \pi_1(\mathcal{A})$ and their images are commensurable in H;*
3. *For each $e \in EA$ the element t_e of the path group $\pi(\mathcal{A})$ is mapped under ψ to an element of $\mathrm{Comm}_H(\psi_e(A_e))$.*

A rough classification

Let $(A, \mathcal{A}, \psi)$ be a graph of $\mathrm{Isom}(\mathbb{E}^n)$-lattices with Bass-Serre tree $\mathcal{T}$ and fundamental group Γ. Let $T = \mathrm{Aut}(\mathcal{T})$.

Theorem (H. 2021)

Assume A is finite. If for each local group A_σ, the kernel $\mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$, then Γ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice and hence a $\mathrm{CAT}(0)$ group. Conversely, if Λ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice, then Λ splits as a finite graph of uniform H-lattices with Bass-Serre tree $\mathcal{T}$.

Generic properties?

Theorem (H. 2021)

Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{E}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. The following are equivalent:

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6. *and if* $n = 2$, Γ *is non-residually finite and not virtually biautomatic.*

Thanks for listening!