THE FIRST ℓ^2-BETTI NUMBER AND GROUPS ACTING ON TREES

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ABSTRACT. We generalise results of Thomas, Allcock, Thom–Petersen, and Kar–Niblo to the first ℓ^2-Betti number of quotients of certain groups acting on trees by subgroups with free actions on the edge sets of the graphs.

1. INTRODUCTION

The ℓ^2-Betti numbers $b_i^{(2)}(G)$ of a group G are defined in [11]. The ℓ^2-Euler characteristic $\chi^{(2)}$ of G is the alternating sum of these Betti numbers and is denoted $\chi^{(2)}(G)$. Let $\mathfrak{C}$ denote the class of groups F such that

- $b_1^{(2)}(F) = b_2^{(2)}(F) = 0$, and
- either $\chi^{(2)}(F) = 0$ or F is finite.

Note that that $\mathfrak{C}$ contains all ℓ^2-acyclic groups (i.e. the groups for which $b_i^{(2)} = 0$ for all $i > 0$) and in particular it contains all amenable groups. Relevant background on ℓ^2-cohomology is included in Section 2. In this note we prove the following theorem.

Theorem 1.1. *Let F be a group acting cocompactly on a tree, with vertex and edge stabilisers in $\mathfrak{C}$, let N be a subgroup normally generated by m elements, intersecting the vertex stabilisers trivially. Let G denote F/N and set $k := \chi^{(2)}(F) + m$. Then the following conclusions hold:*

(i) *If $k \leq 0$, then G is infinite.*
(ii) *If $k < 0$, then $b_1^{(2)}(G) \geq -k > 0$.*
(iii) *If G is finite, then $k > 0$ and $|G| \geq \frac{1}{k}$.*

Note that the hypotheses of this theorem guarantee that N acts freely on the specified tree and in particular N is necessarily a free group. Note also that, according to [2, Corollary 1.4], if $b_1^{(2)}(G) > 0$ then G has no commensurated infinite amenable subgroup and according to [4, Corollary 6] does not have property (T). If we also have $b_2^{(2)}(G) = 0$, then G is in the class $\mathscr{D}_{\mathrm{reg}}$ by [14, Lemma 2.8]. We refer the reader to [3] for background on property (T) and to [14, Definition 2.6] for the definition of the class $\mathscr{D}_{\mathrm{reg}}$. By the main result of Osin's paper [12] we have the following corollary.

Corollary 1.2. *Let G, F and N be as in Theorem 1.1. Assume that G is finitely presented, (virtually) indicable and that $\chi^{(2)}(F) + m < 0$. Then G is (virtually) acylindrically hyperbolic.*

The simplest way in which the indicability hypothesis may arise is through *stable letters*: Let T denote the F-tree of Theorem 1.1. Let K denote the (necessarily

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normal) subgroup generated by the vertex stabilisers. Then there is a subgroup $E \leq F$ that complements K and all such subgroups are free of uniquely determined rank. Such a subgroup may be referred to as *a subgroup of stable letters of the action*. The group G has an infinite cyclic quotient when $N \cap E$ has infinite index in E, in other words when there is a stable letter that is faithfully represented in G, and in this case G is indicable.

Recall that a group G is C^**-simple* if the reduced group C^*-algebra, denoted $C^*_r(G)$, has exactly two norm closed 2-sided ideals, 0, and the algebra $C^*_r(G)$ itself. By [5, Corollary 6.7] we obtain the following.

Corollary 1.3. *With G, F and N as before, G is* C^**-simple if and only if it has trivial amenable radical.*

Theorem 1.1 has some historical pedigree. It originally began life as a result about quotients of free groups due to Thomas (see Theorem 1.4(i)) and was proved using combinatorial methods [15]. The result was generalised by Allcock to incorporate a bound on the rank of the abelianisation of the quotient group [1]. The introduction of ℓ^2-cohomology came when Peterson–Thom [13, Theorem 3.6] and Kar–Niblo [10] independently linked the inequality of Thomas to the first ℓ^2-Betti number. These discoveries are summarized in the following result.

Theorem 1.4 (Thomas [15], Allcock [1], Peterson–Thom [13], Kar–Niblo [10])**.** *Let G be a group with a presentation*

$$\langle x_1,\dots,x_n;\ r_1^{k_1},\dots,r_m^{k_m}\rangle$$

in which the elements r_i have order k_i when interpreted in G.

(i) *If $n-\sum_{i=1}^m \frac{1}{k_i} \geq 1$ then G is infinite.*
(ii) *If G is finite then $|G| \geq \frac{1}{1-n+\sum_{i=1}^m k_i}$.*
(iii) *If $n-\sum_{i=1}^m \frac{1}{k_i} > 1$ then G is non-amenable.*

Deduction of Theorem 1.4 from Theorem 1.1. Let G be a group with a presentation

$$G=\langle x_1,\dots,x_n|\ r_1^{k_1},\dots,r_m^{k_m}\rangle.$$

Adding m fresh generators $y_1,\dots,y_m$, we can give the following alternative presentation of the same group:

$$G=\langle x_1,\dots,x_n,y_1,\dots,y_m|\ y_1^{k_1},\dots,y_m^{k_m},r_1y_1^{-1},\dots,r_my_m^{-1}\rangle.$$

Let F be the group with presentation

$$F=\langle x_1,\dots,x_n,y_1,\dots,y_m|\ y_1^{k_1},\dots,y_m^{k_m}\rangle$$

and let N be the subgroup of F normally generated by $r_1y_1^{-1},\dots,r_my_m^{-1}$. Then F is a free product of cyclic groups: in particular it is virtually free and has Euler characteristic $\chi(F)=\sum_{i=1}^m \frac{1}{k_i}-n-m+1$. The condition that the r_i have order k_i in the original presentation ensures that N does not meet any of the finite subgroups of F and so is torsion-free. Applying Theorem 1.1 with these choices of F, N, G yields Theorem 1.4. □

Finally, we also provide a computation of the first ℓ^2-Betti number for certain groups acting on trees. This generalises a result of Lück [7], which covers the case of an amalgamated free product, and a result of Tsouvalas [16, Corollary 3.7]. Tsouvalas assumes the vertex stabilisers are either residually finite or virtually torsion-free and the edge stabilisers are finite. Here we replace both of these assumptions with Lück's less restrictive assumption that the first ℓ^2-Betti numbers of the edge stabilisers vanish. So, for example, the theorem applies to fundamental groups of graphs of $\mathfrak{C}$-groups.

Theorem 1.5. *Let F be a group acting on a tree and let V and E denote sets of representatives of F-orbits of vertices and edges. Assume for each $e \in E$ that $b_1^{(2)}(F_e) = 0$, then*

$$b_1^{(2)}(F) = \sum_{v \in V} \left(b_1^{(2)}(F_v) - \frac{1}{|F_v|} \right) + \sum_{e \in E} \frac{1}{|F_e|},$$

where for a group G, $\frac{1}{|G|}$ is interpreted as 0 if G is infinite.

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2. Background on ℓ^2-homology

Let G be a group. Then both G and the complex group algebra $\mathbb{C}G$ act by left multiplication on the Hilbert space $\ell^2 G$ of square-summable sequences. The group von Neumann algebra $\mathcal{N}G$ is the ring of G-equivariant bounded operators on $\ell^2 G$. The regular elements of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}G$ can be identified with the *ring of affiliated operators*, and is denoted by $\mathcal{U}G$. One has the inclusions $\mathbb{C}G \subseteq \mathcal{N}G \subseteq \ell^2 G \subseteq \mathcal{U}G$ and it is also known that $\mathcal{U}G$ is a self-injective ring which is flat over $\mathcal{N}G$. For more details concerning these constructions we refer the reader to [11] and especially to Theorem 8.22 of §8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [11, §8.3]. Now let Y be a G-CW complex as defined in [11, Definition 1.25 of §1.2]. The ℓ^2-homology groups of Y are then defined to be the equivariant homology groups $H_i^G(Y; \mathcal{U}G)$, and we have

$$b_i^{(2)}(Y) = \dim_{\mathcal{U}G} H_i^G(Y; \mathcal{U}G).$$

The ℓ^2-Betti numbers of a group G are then defined to be the ℓ^2-Betti numbers of EG. By [11, Theorem 6.54(8)], the zeroth ℓ^2-Betti number of G is equal to $1/|G|$ where $1/|G|$ is defined to be zero if G is infinite. Moreover, if G is finite then $b_n^{(2)}(G) = 0$ for $n \geq 1$.

Let $C_*(Y; \mathcal{U}G)$ denote the standard cellular chain complex of Y with coefficients in $\mathcal{U}G$. We have the formula

$$\dim_{\mathcal{U}G} C_i(Y; \mathcal{U}G) = \sum_\sigma \frac{1}{|G_\sigma|}$$

where σ runs through a set of orbit representatives of i-dimensional cells in Y and if G_σ is infinite then $1/|G_\sigma|$ is taken to be equal to 0. Standard arguments of homological algebra give the connection between two Euler characteristic computations:

$$\sum_i (-1)^i b_i^{(2)}(Y) = \sum_i (-1)^i \dim_{\mathcal{U}G} C_i(Y; \mathcal{U}G). \tag{1}$$

We record two other consequences, both of which can be found in [11, Section 6.6].

Lemma 2.1. *Let G be a group and let Y be a cocompact G-complex with finite stabilizers. Then we have*

(i) $\chi^{(2)}(G) = \sum_{i=0}^{\infty}\left((-1)^i \sum_{v\in S_i} \frac{1}{|G_{v_i}|}\right) = \sum_{i=0}^{\infty}(-1)^i b_i^{(2)}(G)$;

(ii) *if G has a finite index subgroup H with a finite $K(H,1)$ then*

$$\chi^{(2)}(G) = \chi(G).$$

We will need the following lemma for the proofs in the next section. One should think of it as a mild generalisation of Theorem 6.54(2) in [11]

Lemma 2.2 (Comparison with the Borel construction up to rank)**.** *Let X be a G-CW complex. Suppose for all $x \in X$ the isotropy group G_x is finite or $b_p^{(2)}(G_x) = 0$ for all $0 \le p \le n$, then*

$$b_p^{(2)}(X) = b_p^{(2)}(EG \times X) \quad \textit{for } 0 \le p \le n.$$

Proof. It suffices to prove that the dimension of the kernel and cokernel of the map

$$\mathrm{pr}_p : H_p^G(EG \times X; \mathcal{N}G) \to H_p^G(X; \mathcal{N}G)$$

induced by the projection $EG \times X \to X$ are trivial for $0 \le p \le n$. By an identical argument to [11, Theorem 6.54(2)] it suffices to prove for each isotropy subgroup $H \le G$ and $0 \le p \le n$ the kernel and cokernel of the map $\mathrm{pr}_p : H_p^H(EH; \mathcal{N}H) \to H_p^H(*; \mathcal{N}H)$ have trivial dimension. If H is finite this follows from [11, Theorem 6.54(8a)], and is immediate if $b_p^{(2)}(H) = 0$ for all $0 \le p \le n$. □

3. The Main Theorem

To prove Theorem 1.1, one needs the following method of computing the ℓ^2-Euler characteristic of a group acting on a tree analogous to Chiswell's result [9] for rational Euler characteristic.

Proposition 3.1 (Chatterji–Mislin [8])**.** *Let F be a group acting on a tree and let V and E denote sets of representatives of F-orbits of vertices and edges. If the ℓ^2-Euler characteristic of each vertex and edge group is finite, then*

$$\chi^{(2)}(F) = \sum_{v\in V} \chi^{(2)}(F_v) - \sum_{e\in E} \chi^{(2)}(F_e).$$

Proof of Theorem 1.1. There is a cocompact action of F on a tree T with vertex and edge stabilizers in the class $\mathfrak{C}$. Let V and E denote the vertex and edge sets. Let $\overline{T}$ denote the quotient graph T/N and write $\overline{V}$ and $\overline{E}$ for its vertex and edge sets. Now $G = F/N$ acts cocompactly on $\overline{T}$ with vertex and edge stabilizers in $\mathfrak{C}$. The augmented chain complex of T is the short exact sequence

$$0 \to \mathbb{Z}E \to \mathbb{Z}V \to \mathbb{Z} \to 0.$$

Upon factoring out the action of N we have the exact sequence

$$H_1(N,\mathbb{Z}) \to \mathbb{Z}\overline{E} \to \mathbb{Z}\overline{V} \to \mathbb{Z} \to 0, \tag{2}$$

this being the tail end of the long exact sequence of homology of N. Let $\{r_i : i = 1,..m\}$ denote a normal generating set for N. Choose a vertex v_0 in T to be a fixed basepoint. For $1 \le i \le m$ consider the geodesic from v_0 to $v_0 r_i$. In the quotient graph $\overline{T}$ this geodesic descends to a loop because v_0 and $v_0 r_i$ become identified in $\overline{T}$. Now 2-discs can be glued to each loop. By adjoining free G-orbits of 2-discs equivariantly we can build a 2-complex Y with an action of G, whose 1-skeleton is $\overline{T}$, and which has augmented cellular chain complex

$$\mathbb{Z}G^m \to \mathbb{Z}\overline{E} \to \mathbb{Z}\overline{V} \to \mathbb{Z} \to 0. \tag{3}$$

The exactness of (2) ensures the exactness of 3.

Let V_0 and E_0 be sets of orbit representatives of vertices and edges in Y. Now, applying [11, Theorem 6.80(1)], then(1) and finally Proposition 3.1, we have that

$$\begin{aligned} \chi^{(2)}(Y) &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y) \\ \sum_{v\in V_0} \frac{1}{|G_v|} - \sum_{e\in E_0} \frac{1}{|G_e|} + m &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y) \\ \chi^{(2)}(F) + m &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y). \end{aligned}$$

Applying Lemma 2.2, we have

$$\begin{aligned} \chi^{(2)}(F) + m &= b_0^{(2)}(EG\times Y) - b_1^{(2)}(EG\times Y) + b_2^{(2)}(EG\times Y). \\ &\geq b_0^{(2)}(EG\times Y) - b_1^{(2)}(EG\times Y). \end{aligned}$$

Applying [11, Theorem 6.54(1a)] to the diagonal map $EG \to EG\times Y$ we see that

$$\chi^{(2)}(F) + m \geq b_0^{(2)}(G) - b_1^{(2)}(G).$$

Let $k = \chi^{(2)}(F) + m$. If $k \leq 0$, then $b_0^{(2)}(G) - b_1^{(2)}(G) \leq 0$ and so G is infinite, this proves (i). Now, assume $k < 0$. In this case G is infinite and therefore $b_0^{(2)}(G) = 0$. It follows that $b_1^{(2)}(G) \geq -k > 0$, this proves (ii).

If G is finite, then $b_0^{(2)}(G) = \frac{1}{|G|}$, $b_1^{(2)}(G) = 0$, and $k > 0$. In particular, $k \geq \frac{1}{|G|} > 0$ and (iii) follows. $\square$

4. On the ℓ^2-invariants for certain groups acting on trees

Proof of Theorem 1.5. Let V and E denote sets of representatives of F-orbits of vertices and edges for the action of F on the tree. We consider the relevant part of the E^1-page for the F-equivariant spectral-sequence (see Chapter VII.9 of [6]) applied to the tree:

$$\begin{array}{c|ccc} 1 & \bigoplus_{v\in V} H_1^F(F\times_{F_v} EF_v; \mathcal{U}F) & & 0 \\ 0 & \bigoplus_{v\in V} H_0^F(F\times_{F_v} EF_v; \mathcal{U}F)) & \xleftarrow[d^1]{} & \bigoplus_{e\in E} H_0^F(F\times_{F_e} F_e; \mathcal{U}F)) \\ \hline & 0 & & 1 \end{array}$$

If F is finite then $b_1^{(2)}(F) = 0$, so d^1 is injective and $E^2_{1,0} = 0$. The result follows from the fact $E^1_{0,1} = 0$.

Now, assume F is infinite, then d^1 is surjective since $b_0^{(2)}(F) = 0$. Thus,

$$\dim_{\mathscr{U}F}(\mathrm{Ker}(d^1)) = \sum_{e\in E} b_0^{(2)}(F_e) - \sum_{v\in V} b_0^{(2)}(F_v).$$

Now, the spectral sequence obviously collapses on the E^2-page and $E^1_{0,1} = E^2_{0,1}$. Since von Neumann dimension is additive over short exact sequences, we have

$$\begin{aligned} b_1^{(2)}(F) &= \dim_{\mathscr{U}F}(\mathrm{Ker}(d^1)) + \dim_{\mathscr{U}F}(E^2_{0,1}) \\ &= \left(\sum_{e\in E} b_0^{(2)}(F_e) - \sum_{v\in V} b_0^{(2)}(F_v)\right) + \sum_{v\in V} b_1^{(2)}(F_v), \end{aligned}$$

and the result follows. □

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