PROFINITE RIGIDITY OF KÄHLER GROUPS: RIEMANN SURFACES AND SUBDIRECT PRODUCTS

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ABSTRACT. This paper establishes strong profinite rigidity results for Kähler groups, showing that certain groups are determined within the class of residually finite Kähler groups by their profinite completion. Examples include products of surface groups and certain groups with exotic finiteness properties studied earlier by Dimca–Papadima–Suciu and Llosa Isenrich. Consequently, there are aspherical smooth projective varieties that are determined up to homeomorphism by their algebraic fundamental group. The main tool is the following: for a closed Kähler manifold X, the holomorphic fibrations of X over hyperbolic 2-orbifolds can be recovered from the profinite completion of its fundamental group. We also prove profinite invariance of the BNS invariant.

CONTENTS

1. INTRODUCTION

For a smooth complex projective variety X, its topological fundamental group will be denoted by $\pi_1(X)$ and $\pi_1^{\mathrm{alg}}(X)$ will denote its algebraic fundamental group. Motivated by the seminal work of Grothendieck [Gro70], this paper considers the following questions:

Question 1.1. *Given an aspherical smooth projective variety with $\pi_1(X)$ residually finite:*

- *To what extent does $\pi_1^{alg}(X)$ determine X up to homeomorphism?*
- *What about $\pi_1(X)$ up to isomorphism?*

Varieties for which the above questions have positive answers must be very special. Indeed, building on work of Serre [Ser64] that uses the action of

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$\mathrm{Aut}(\mathbb{C})$, it is known that there exist arbitrarily large collections of aspherical smooth projective varieties that are pairwise not homotopy equivalent but have isomorphic algebraic fundamental groups. See [BCG15, MS10, Sto19, Sto] for a plethora of examples. On the other hand, the main results of this paper show that a strong form of rigidity holds for direct products of hyperbolic Riemann surfaces.

The techniques in this paper apply not only to smooth projective varieties but also to compact Kähler manifolds, so our results are stated in this more general context. Moreover, since the algebraic fundamental group of a smooth complex projective variety is isomorphic to the profinite completion of its topological fundamental group, from here forward this paper will use the language of profinite completions instead of algebraic fundamental groups. The notation $\widehat{G}$ will denote the profinite completion of a group G.

Recall that a *Kähler group* is a group that can be realised as the fundamental group of a compact Kähler manifold; see [ABC+96] and [Py25] for the essential properties of this class of groups. The first main result of this paper is profinite rigidity of products of hyperbolic surface groups amongst residually finite Kähler groups.

Theorem A. *Let X be a compact Kähler manifold with residually finite fundamental group. Assume that*

$$\widehat{\pi_1(X)} \cong \prod_{i=1}^{r} \widehat{\pi_1(S_i)}$$

for some $r \geq 1$, where each S_i is a closed oriented surface of genus at least 2. Then $\pi_1(X)$ is isomorphic to $\prod_{i=1}^{r} \pi_1(S_i)$. Moreover, if X is aspherical then it is biholomorphic to $\prod_{i=1}^{r} S_i$ for a suitable choice of complex structure on each of the S_i.

Note that Toledo constructed examples of projective varieties with non-residually finite fundamental group [Tol93], so the hypothesis that $\pi_1(X)$ is residually finite is natural. It remains a major open problem whether every infinite Kähler group admits a nontrivial finite quotient, but all known examples of infinite Kähler groups admit linear representations over $\mathbb{C}$ with infinite image and hence have nontrivial profinite completion. Partial results still hold without the residual finiteness assumption; e.g., see Theorem D below.

Interestingly, Theorem A is virtually false. Amongst the aforementioned examples building on those of Serre, Bauer–Catanese–Grunewald constructed varieties X_1, X_2 that are both quotients of a fixed product of curves by distinct free actions of a certain finite group such that X_1 and X_2 are not homotopy equivalent but $\widehat{\pi_1(X_1)} \cong \widehat{\pi_1(X_2)}$ [BCG15, Thm. 7.9]. Quotients of this kind are said to be *isogenous to a product*. Still, the natural bijection between the sets of finite index subgroups of residually finite groups with the same profinite completion [Rei15, Prop. 4.4] and Theorem A immediately implies that their construction is the only way to produce examples isogenous to a product that are not homotopy equivalent but have the same algebraic fundamental group. To be precise:

Corollary 1.2. *Let Y be a product of finitely many hyperbolic Riemann surfaces, F be a finite group acting freely and holomorphically on Y, and $X_1 =$*

Y/F. *Suppose that X_2 is an aspherical compact Kähler manifold with $\pi_1(X_2)$ residually finite such that $\widehat{\pi_1(X_2)} \cong \widehat{\pi_1(X_1)}$. Then there is a deformation Y' of the complex structure on Y so that F still acts holomorphically and freely on Y' and X_2 is biholomorphic to Y'/F. In particular, X_2 is also isogenous to a product and has the same associated finite group.*

In the case where X_1 is a rigid variety it would be interesting to know whether or not X_1 and X_2 must be equivalent under the action of $\mathrm{Aut}(\mathbb{C})$.

Before stating further results, some classical definitions regarding the *profinite genus* of a group are required.

Genus of a Kähler group. Recall that the *profinite genus* $\mathcal{G}(G)$ of a finitely generated residually finite group G is:

$$\mathcal{G}(G) = \left\{ H \text{ finitely generated residually finite } \mid \widehat{H} \cong \widehat{G} \right\} / \cong$$

In other words, $\mathcal{G}(G)$ is the set of isomorphism classes of finitely generated residually finite groups with profinite completion isomorphic to $\widehat{G}$. If $\mathcal{B}$ is a class of residually finite groups, then $\mathcal{G}_{\mathcal{B}}(G)$ denotes $\mathcal{G}(G) \cap \mathcal{B}$. A finitely generated residually finite group G is then *profinitely rigid* (resp. $\mathcal{B}$-*profinitely rigid*) if $\mathcal{G}(G) = \{G\}$ (resp. $\mathcal{G}_{\mathcal{B}}(G) = \{G\}$), and G is *almost profinitely rigid* (resp. almost $\mathcal{B}$-profinitely rigid) if $|\mathcal{G}(G)| < \infty$ (resp. $|\mathcal{G}_{\mathcal{B}}(G)| < \infty$).

There are very few examples of groups that are profinitely rigid. A well-known open problem, attributed to Remeslennikov, is whether free groups and surface groups are profinitely rigid. A recent breakthrough is work of Bridson, McReynolds, Reid, and Spitler [BMRS20] that provides examples of profinitely rigid hyperbolic 3-manifold and 3-orbifold groups. See [BMRS20, Rei18, Rém24] for history and further references on profinite rigidity.

This paper considers the question of $\mathcal{K}$-profinite rigidity, where $\mathcal{K}$ is the class of residually finite Kähler groups, proving $\mathcal{K}$-profinite rigidity for many important examples. This includes the direct products appearing in Theorem A and certain Kähler subgroups of direct products of surface groups. The examples of Serre [Ser64], Bauer–Catanese–Grunewald [BCG15], and Stover [Sto19, Sto] mentioned above show that the genus of a Kähler group can consist of more than one isomorphism class, so the strongest possible result for general Kähler groups is that the $\mathcal{K}$-profinite genus is finite.

This paper also studies the following notion, which was introduced by Grothendieck [Gro70].

Definition 1.3. A group H is *Grothendieck rigid* if any homomorphism $u\colon K \to H$ from a finitely generated residually finite group K to H that induces an isomorphism on profinite completions is necessarily an isomorphism.

Lastly, in this paper a hyperbolic orbisurface group will always mean a discrete *cocompact* subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Note that a lattice in $\mathrm{PSL}_2(\mathbb{R})$ is a Kähler group if and only if it is cocompact, so the fact that an orbisurface group is always cocompact in this paper is no loss of generality. The rank of a finitely generated abelian group H means the minimal number of generators of the quotient of H by its torsion subgroup. Another main result of this paper is the following.

Theorem B. *Let G be a Kähler group that is either:*

- *a hyperbolic orbisurface group,*
- *a direct product of a finitely generated abelian group of even rank with finitely many hyperbolic orbisurface groups, or*
- *the kernel of a factorwise surjective homomorphism from a product of hyperbolic orbisurface groups onto $\mathbb{Z}^2$.*

Then G is profinitely rigid amongst residually finite Kähler groups, as well as Grothendieck rigid amongst residually finite Kähler groups.

Kähler groups of the form appearing in the third point of Theorem B do indeed exist. The first examples were constructed by Dimca, Papadima, and Suciu [DPS09], then studied further and generalised by Llosa Isenrich [LI19, LI20, LI24]. These groups are notable due to the fact that, despite being fundamental groups of closed Kähler manifolds and thus finitely presented, they lack higher finiteness properties: for each $n \geq 2$, there are examples that are of type F_n but not of type F_{n+1}. Recall that a group G is of type F_n if it admits a $K(G, 1)$ with finite n-skeleton. In light of this degeneracy, the fact that these coabelian subgroups are profinitely rigid amongst Kähler groups seems interesting.

Using earlier classification results due to Llosa Isenrich [LI24], one can use Theorem B to obtain the following:

Theorem C. *Let Γ_1, Γ_2, Γ_3 be three hyperbolic orbisurface groups and*

$$G \leq \Gamma_1 \times \Gamma_2 \times \Gamma_3$$

be a Kähler group. Then G contains a finite index subgroup H which is profinitely rigid amongst residually finite Kähler groups.

It is possible that our techniques could further lead to a proof of the finiteness of the Kähler profinite genus for many Kähler subgroups of direct products of hyperbolic orbisurface groups.

Fibrations. The main tool used to establish Theorems A and B is a profinite rigidity result for fibrings showing that holomorphic fibrations from a closed Kähler manifold X onto hyperbolic orbisurfaces are detected by $\widehat{\pi_1(X)}$. Before stating this more precisely, some classical facts about fibrations of Kähler manifolds over Riemann surfaces are now recalled.

A closed orbi-Riemann surface (from here forward, *orbisurface*) is a closed Riemann surface S together with a finite set of points $P \subset S$ and a collection of natural numbers $(m_p)_{p \in P}$ such that $m_p \geq 2$ for all $p \in P$. This orbifold is sometimes denoted (S, P), but more often the symbol S is used both for the orbifold and for the underlying (unmarked) Riemann surface. Then S is *hyperbolic* if its orbifold Euler characteristic

(1.3a) $$\chi^{orb}(S, P) = \chi(S) - \sum_{p \in P} \left(1 - \frac{1}{m_p}\right)$$

is negative.

Definition 1.4. A *fibration* from a closed complex manifold X onto a Riemann surface S is a surjective holomorphic map with connected fibres.

Remark 1.5. When S has positive genus, fibrations also appear under the name *irrational pencils* in the literature.

If X is a closed complex manifold and $f : X \to S$ a fibration onto a closed Riemann surface, one can define an orbifold structure $S_f^* = (S, P)$ on S using the multiplicites of singular fibres of f, in such a way that the following properties hold:

(1) The map $f : X \to S_f^*$ is holomorphic in the orbifold sense.
(2) The natural morphism $\pi_1(X) \to \pi_1(S)$ can be lifted to a morphism

$$f_* : \pi_1(X) \longrightarrow \pi_1^{orb}(S_f^*)$$

with finitely generated kernel, which is called the morphism *induced by f*.
(3) Up to isomorphism of the base, there are only finitely many fibrations $f : X \to S$ for which the induced orbifold S_f^* is hyperbolic.
(4) When X is Kähler, the morphisms induced by fibrations to hyperbolic orbisurfaces depend only on $\pi_1(X)$ and not on the specific Kähler manifold X.

See Section 2 for the definition of the group $\pi_1^{orb}(S_f^*)$. The reader should consult [Del08], [CS08] or [Py25, Ch. 2] for the definitions of a holomorphic map to an orbisurface and of the orbifold S_f^*, along with proofs of the above results.

Consequently, if G is a Kähler group then there is a canonical finite collection $\Gamma_1, \ldots, \Gamma_r$ of hyperbolic orbisurface groups equipped with surjective homomorphisms $\varrho_i : G \to \Gamma_i$ ($1 \le i \le r$) each induced by a holomorphic fibration f with S_f^* hyperbolic. In particular, each ϱ_i has finitely generated kernel.

Definition 1.6. The homomorphism $(\varrho_1, \ldots, \varrho_r) : G \to \Gamma_1 \times \cdots \times \Gamma_r$ defined in the previous paragraph is called the *universal homomorphism* for G.

The universal homomorphism was introduced in [LI20, Th. 9.1] to rephrase the results of Corlette–Simpson [CS08] and Delzant [Del16] in terms of a homomorphism to a direct product. Section 2 will precisely describe the sense in which this homomorphism is universal. The universal homomorphism played a fundamental role in many recent advances in the study of Kähler groups [LI20, LI24], and the next result shows that it is a profinite invariant amongst Kähler groups; note that this result holds amongst *all* Kähler groups, not just those which are residually finite.

Theorem D. *Let G and H be Kähler groups,*

$$\phi = (\phi_1, \ldots, \phi_r) : G \longrightarrow \Gamma_1 \times \cdots \times \Gamma_r$$
$$\psi = (\psi_1, \ldots, \psi_s) : H \longrightarrow \Sigma_1 \times \cdots \times \Sigma_s$$

be their universal homomorphisms, and $\widehat{\phi} = (\widehat{\phi_1}, \ldots, \widehat{\phi_r}), \widehat{\psi} = (\widehat{\psi_1}, \ldots, \widehat{\psi_s})$ be the induced homomorphisms on profinite completions. If there is an isomorphism $\Theta : \widehat{G} \to \widehat{H}$, then $r = s$ and, up to reordering factors, Γ_i is isomorphic to Σ_i for each $i \in \{1, \ldots, r\}$. Moreover, $\widehat{\psi} \circ \Theta = \widehat{\phi}$ after identifying the groups $\widehat{\Gamma}_i$ and $\widehat{\Sigma}_i$ by applying suitable isomorphisms of profinite orbisurface groups. In particular $\mathrm{im}(\widehat{\psi}) \cong \mathrm{im}(\widehat{\phi})$.

The expression *profinite orbisurface group* is used as shorthand for *profinite completion of an orbisurface group*. The proof of Theorem D will use the fact that hyperbolic orbisurface groups with the same profinite completion are isomorphic, a result due to Bridson, Conder and Reid [BCR16]. Theorem D should be contrasted with the long-standing open question of Delzant–Gromov [DG05] asking for a characterisation of the possible images of universal homomorphisms of Kähler groups; this question was resolved by the second author in the case of a product of three orbisurface groups [LI24].

The BNS invariant. By work of Delzant [Del10], the universal homomorphism of a Kähler group G is closely related to its Bieri–Neumann–Strebel invariant $\Sigma^1(G)$ (or BNS-invariant for short). See [BNS87] for the original article on $\Sigma^1(G)$, as well as [Py25, Ch. 11] for an introduction. If Γ is a finitely generated group, its BNS-invariant is a subset

$$\Sigma^1(\Gamma) \subset H^1(\Gamma, \mathbb{R}) - \{0\}$$

which is invariant by multiplication by positive scalars. It is often viewed as a subset of the sphere

$$\left(H^1(\Gamma, \mathbb{R}) - \{0\}\right)/\mathbb{R}_+^*,$$

but this paper considers it as a subset of $H^1(\Gamma, \mathbb{R}) - \{0\}$.

The BNS-invariant is famously related to finite generation of kernels of homomorphisms to infinite abelian groups. Now, consider a Kähler group G with universal homomorphism

$$(\varrho_1, \ldots, \varrho_r) : G \longrightarrow \Gamma_1 \times \cdots \times \Gamma_r,$$

where the Γ_i's are hyperbolic orbisurface groups. The morphism ϱ_i induces an injection $H^1(\Gamma_i, \mathbb{R}) \hookrightarrow H^1(G, \mathbb{R})$ whose image is denoted by V_i. Delzant proved that the BNS invariant $\Sigma^1(G)$ coincides with the complement of the union

$$(1.7) \qquad\qquad \bigcup_{i=1}^{r} V_i.$$

Also note that if G and H are groups with isomorphic profinite completions, then there is a linear isomorphism $L : H^1(G, \mathbb{Q}) \to H^1(H, \mathbb{Q})$. Let $L_\mathbb{R}$ denote the $\mathbb{R}$-linear extension $H^1(G, \mathbb{R}) \to H^1(H, \mathbb{R})$ of L. One can then ask:

Question 1.8. *Can L be chosen so that $L_\mathbb{R}(\Sigma^1(G)) = \Sigma^1(H)$?*

According to Theorem D, the respective complements of the BNS invariants of G and H in $H^1(G, \mathbb{R})$ and $H^1(H, \mathbb{R})$ consist of the same number of linear subspaces, and the set of dimensions of these subspaces are the same for G and H. Note that it is possible to have a triple V_1, V_2, V_3 such that $V_3 \cap (V_1 \oplus V_2) \neq \{0\}$, as shown by the examples built in [DPS09], so it is not obvious that an isomorphism on first cohomology must respect BNS invariants. However, the answer is indeed affirmative.

Theorem E. *Let G and H be Kähler groups such that $\widehat{G} \cong \widehat{H}$. Then there is a linear isomorphism $L : H^1(G, \mathbb{Q}) \to H^1(H, \mathbb{Q})$ such that*

$$L_\mathbb{R}(\Sigma^1(G)) = \Sigma^1(H).$$

As we shall see, the proof of Theorem E relies on the work of Hughes and Kielak [HK22], which itself was inspired by earlier work on 3-manifold groups by Liu [Liu23] and Jaikin–Zapirain [JZ20]. We shall not give a detailed survey of these works here, but simply mention two facts:

- the existence of a fibring over the circle of a closed 3-manifold M can be detected on the profinite completion of $\pi_1(M)$, this goes back to [JZ20];
- Theorem E is, in a sense, very similar to Theorem 1.3 from [Liu23].

Outline. Section 2 recalls necessary background about hyperbolic orbisurface groups, subgroups of direct products, and the universal homomorphism. Section 3 establishes profinite invariance of the universal homomorphism (Theorem D). Using Theorem D, Sections 4 and 5 prove $\mathcal{K}$-profinite rigidity of products of hyperbolic orbisurface groups (possibly with a free abelian factor) and of the Kähler coabelian subgroups appearing in Theorem B. These sections also prove that certain Kähler groups commensurable with groups in these classes have finite $\mathcal{K}$-genus and are Grothendieck rigid amongst residually finite Kähler groups. This establishes part of Theorem A by obtaining an isomorphism on fundamental groups and proves Theorem B. Theorem C is also established at the end of Section 5. Section 6 concludes the proof of Theorem A by producing the desired biholomorphism in the aspherical case. Section 7 proves profinite invariance of the BNS invariant for Kähler groups (Theorem E). Finally, Section 8 contains a few additional remarks and questions regarding Kodaira fibrations.

2. Preliminaries

This section collects results required later in the paper. Specifically, it describes results concerning ramified covers of surfaces (Section 2.A), subgroups of direct products (Section 2.B), and the universal homomorphism for Kähler groups (Section 2.C).

2.A. **Orbisurfaces and ramified covers.** Beyond classical material on orbifolds and ramified covers, this subsection also describes a finiteness result for certain ramified covers that is needed later in the paper.

Suppose that S is a closed oriented surface and $P \subset S$ is a finite set of points together with integer multiplicities $(m_p)_{p \in P}$, where $m_p \geq 2$ for each p. Let S^* be the orbifold associated with the data consisting of the set P and

the multiplicities $(m_p)_{p \in P}$ and let γ_p denote a small loop in $S - P$ enclosing p once. The loop γ_p defines a conjugacy class in the group $\pi_1(S - P)$, and the orbifold fundamental group of S^* is the quotient

$$\pi_1^{orb}(S^*) = \pi_1(S - P)/\langle\!\langle \gamma_p^{m_p} = 1, p \in P \rangle\!\rangle.$$

When S^* is hyperbolic, i.e., when its orbifold Euler characteristic is negative (see (1.3a)), the group $\pi_1^{orb}(S^*)$ can be realised as a cocompact discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Conversely, any cocompact discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ can be realised as the orbifold fundamental group of a hyperbolic orbifold. When the names of the orbifold points will only clutter notation, $S_{g,\underline{n}}$ will denote the orbifold whose underlying surface has genus $g \geq 0$ and $k \geq 0$ marked points of order $n_i \geq 2$ for $\underline{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$.

The following finiteness result is required in Section 4.

Proposition 2.1. *Let S_1 be a closed oriented surface of genus one. For every $g, d \geq 2$, there are only finitely many isomorphism classes of ramified covers $f : S_g \to S_1$ of degree at most d by a closed oriented surface S_g of genus g.*

Proof. Let $f : S_g \to S_1$ be as in the statement of the proposition, $\{p_1, \ldots, p_k\}$ be the ramification points of f, and $\{m_1, \ldots, m_k\}$ be the corresponding multiplicities. The Riemann-Hurwitz formula yields the equation

$$2g - 2 = \sum_{i=1}^k (m_i - 1),$$

hence $0 \leq k \leq 2g - 2$. If $\{q_1, \ldots, q_\ell\} = f(\{p_1, \ldots, p_k\})$ is the collection of branch points, then $\ell \leq 2g - 2$. The surface S_g and the map f are completely determined by the covering space

$$S_g - f^{-1}(\{q_1, \ldots, q_\ell\}) \longrightarrow S_1 - \{q_1, \ldots, q_\ell\}$$

induced by f. Each such covering space is associated with a conjugacy class of index d subgroups of the free group $\pi_1(S_1 - \{q_1, \ldots, q_\ell\})$ of rank $\ell + 1$. There are finitely many such subgroups for each ℓ and finitely many ℓ, hence finitely many coverings. This proves the result. $\qquad\square$

2.B. **Direct products of groups and their subgroups.** This section presents a characterisation of finitely presented subgroups of direct products due to Bridson, Howie, Miller and Short [BHMS13]. Before stating it, some notation is needed.

Consider a direct product $G = G_1 \times \cdots \times G_r$ of finitely many groups. For any choice of indices $1 \leq i_1 < \cdots < i_k \leq r$,

$$p_{i_1, \ldots, i_k} : G \longrightarrow G_{i_1} \times \cdots \times G_{i_k}$$

denotes the appropriate projection and $G_i \leq G$ denotes the canonical embedding of G_i in G as the ith direct factor. A subgroup $H \leq G$

- is *subdirect* if $p_i(H) = G_i$ for $1 \leq i \leq r$;
- is *full* if $(G_i \cap H) \neq \{1\}$ for $1 \leq i \leq r$; and
- has the virtual surjection to pairs (*VSP*) property if $p_{i,j}(H) \leq G_i \times G_j$ has finite index for each $1 \leq i < j \leq r$.

Given an arbitrary (nontrivial) subgroup $H \leq G$, after possibly reordering factors, there is always an embedding of H as a full subdirect product $H \leq G'_1 \times \cdots \times G'_s$ in a direct product of subgroups $G'_i \leq G_i$, $1 \leq i \leq s \leq r$. To see this, if there is a single factor that intersects H trivially, first project away from that factor, replace H with its isomorphic image under the projection, and replace the remaining factors by the projections $G'_i := p_i(H)$. Repeating this process eventually yields an embedding of H as a full subdirect product.

In [BHMS13], Bridson, Howie, Miller and Short carried out a detailed analysis of the properties of finitely presented residually free groups. In the course of this study, they proved the following general result about subgroups of direct products. Note that the only hypothesis about the factors is that they are finitely presented.

Theorem 2.2. *Let* $\Gamma_1, \ldots, \Gamma_r$ *be finitely presented groups and*

$$H \leq \Gamma_1 \times \cdots \times \Gamma_r$$

be a full subdirect product. If H has the VSP property, then it is finitely presented.

There is a converse to Theorem 2.2 when the groups Γ_i are residually free. Since it is not used in this paper, the reader is referred to [BHMS13] for a precise statement. Finally, the following result about subgroups of direct products of orbisurface groups, whose key argument follows almost verbatim from [BHMS13, Prop. 6.12], is required later. The proof is included for completeness.

Theorem 2.3. *If* $H \leq \Gamma_1 \times \cdots \times \Gamma_r$ *and* $H' \leq \Gamma'_1 \times \cdots \times \Gamma'_s$ *are isomorphic full subdirect products of hyperbolic orbisurface groups, then $r = s$ and, up to permuting factors, there are isomorphisms $\Gamma'_i \cong \Gamma_i$ that induce an isomorphism $H \cong H'$ of subgroups.*

Proof. To see that $r = s$, first note that $H_i = H \cap \Gamma_i$ is a nontrivial normal subgroup of H, since H is full. Since H is subdirect,

$$p_i(H_i) \cong H_i$$

is a nontrivial normal subgroup of the hyperbolic orbisurface group Γ_i. Standard facts about Fuchsian groups then imply that $p_i(H_i)$ contains an element of infinite order, hence H_i also does. The H_i commute with one another, thus H contains a free abelian subgroup isomorphic to $\mathbb{Z}^r$. Maximal abelian subgroups of hyperbolic orbisurface groups are cyclic. This implies that r is the maximal possible rank of a free abelian subgroup of either $\Gamma_1 \times \cdots \times \Gamma_r$ or H. The same argument holds for H', and thus $r = s$.

To complete the proof of the theorem, it suffices to prove that any surjective homomorphism from H (resp. H') to a hyperbolic orbisurface group factors through one of the projections $H \to \Gamma_i$ (resp. $H' \to \Gamma'_i$). To see that this claim suffices to prove the theorem, we fix an isomorphism $\psi: H \to H'$, and consider the epimorphism $\varrho'_j \circ \psi: H \to \Gamma'_j$. We obtain a factorisation through some $\varrho_i: H \to \Gamma_i$. Applying the same logic to the epimorphism

$\varrho_i \circ \psi^{-1} : H' \to \Gamma_i$ yields an epimorphism $\Gamma'_k \to \Gamma_i$ and leads to a commutative diagram

(2.3a)
$$
\begin{array}{ccc}
H' \longrightarrow H \longrightarrow H' \\
{\scriptstyle \varrho'_k}\downarrow \qquad \downarrow \qquad \downarrow{\scriptstyle \varrho'_j} \\
\Gamma'_k \longrightarrow \Gamma_i \longrightarrow \Gamma'_j
\end{array}
$$

in which the isomorphisms in the top row compose to give the identity map on H'. Since H' is full, $H'_j = H' \cap \Gamma'_j \le H'$ is nontrivial, and it is mapped to the nontrivial subgroup $p'_j(H'_j) \le \Gamma'_j$ in the above diagram, which implies that $\varrho'_k(H'_j)$ must be nontrivial. This requires that $k = j$. The composition of the morphisms in the lower row equals the identity, since the isomorphisms in the top row are inverses of each other. This implies that the morphism $\Gamma_i \to \Gamma'_j$ in (2.3a) is an isomorphism. After relabeling, this gives an isomorphism

$$\Gamma_1 \times \cdots \times \Gamma_r \to \Gamma'_1 \times \cdots \times \Gamma'_r$$

inducing ψ as desired.

We now turn to the proof of the factorisation claim. We content ourselves with the case $r = 2$, the general case being similar. If $H \le \Gamma_1 \times \Gamma_2$ and $\phi : H \to \Lambda$ is a surjective homomorphism onto a hyperbolic orbisurface group, then the images of $H_i = H \cap \Gamma_i$ in Λ are commuting normal subgroups. Again, standard facts about Fuchsian groups imply that at least one $\phi(H_i)$ is trivial, say $\phi(H_1)$. Since H_1 is the kernel of the projection of H onto Γ_2, this shows that ϕ factors through the projection of H onto Γ_2. $\qquad\square$

2.C. **The universal homomorphism.** Let G be a Kähler group. Definition 1.6 introduced the universal homomorphism of G, denoted

$$\varrho := (\varrho_1, \ldots, \varrho_r) : G \longrightarrow \Gamma_1 \times \cdots \times \Gamma_r.$$

This section collects several of its key properties.

Theorem 2.4. *The morphism $\varrho = (\varrho_1, \ldots, \varrho_r)$ has the following properties.*

(1) Each homomorphism ϱ_i is surjective with finitely generated kernel.

(2) The image $\varrho(G) \le \Gamma_1 \times \cdots \times \Gamma_r$ is full subdirect and virtually surjects onto pairs. In particular, it is finitely presented.

(3) Every epimorphism $\psi : G \to \Gamma$ onto a hyperbolic orbisurface group Γ factors through one of the ϱ_i and thus also through ϱ. If, moreover, ψ has finitely generated kernel, then there is an index $i \in \{1, \ldots, r\}$ and an isomorphism $\theta : \Gamma \to \Gamma_i$ such that $\theta \circ \psi = \varrho_i$.

Proof. Part (1) was already mentioned in the introduction. For example, see [Cat03, Lem. 4.2] or [Del08, Thm. 3] for a proof. Part (2) is [LI20, Lemma 9.2].

For Part (3), realise Γ as a cocompact discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ acting on the hyperbolic plane $\mathbb{H}^2$ and G as the fundamental group of a closed Kähler manifold X. Note that if ψ factors through a morphism induced by a holomorphic fibration of X onto an orbisurface, the base must be hyperbolic and hence ψ factors through one of the ϱ_i. Now suppose for a contradiction that ψ does not factor through such a fibration. According to [Py25, Thm. 9.31], there exists a ψ-equivariant map $f : \widetilde{X} \to \mathbb{H}^2$ that is holomorphic for

one of the two $\mathrm{PSL}_2(\mathbb{R})$-invariant complex structures on $\mathbb{H}^2$. Moreover, the proof of [Py25, Thm. 9.31] shows in this case that the image of ψ cannot be discrete, which is a contradiction since $\Gamma = \psi(G)$ is discrete by assumption. Therefore ψ must factor through through a morphism induced by a holomorphic fibration onto a hyperbolic orbisurface, i.e., must factor through some ϱ_i. For the last claim, dealing with the case where ψ has finitely generated kernel, see the original proof in [Cat08] or [Py25, Thm. 2.23]. $\square$

Notice that the universal homomorphism for a Kähler full subdirect product of orbisurface groups is simply the inclusion in the direct product, as shown by the next lemma. This observation will be used in Section 4.

Lemma 2.5. *Let $G < \Gamma_1 \times \cdots \times \Gamma_r$ be a Kähler full subdirect product of hyperbolic orbisurface groups Γ_i, $1 \le i \le r$. Then the inclusion*

$$\phi: G \longrightarrow \Gamma_1 \times \cdots \times \Gamma_r$$

is the universal homomorphism for G.

Proof. Since $G < \Gamma_1 \times \cdots \times \Gamma_r$ is finitely presented and full subdirect, it follows from [LI20, Thm. 6.4] (see also [BM09, Thm. 4.6] for the case of hyperbolic surface groups) that the projections $\phi_i : G \to \Gamma_i$ have finitely generated kernel. Thus the ϕ_i are factors of the universal homomorphism, and it remains to show that there are no other factors. Applying the exact same argument as in the proof of [BHMS13, Prop. 6.12] and Theorem 2.3, composing ϱ with projection onto a factor of the universal homomorphism factors through projection onto some Γ_i, which proves that there are indeed no other factors. $\square$

Remark 2.6. Studying the possible images of universal homomorphisms provides insights into the nature of Kähler groups. Constraints on such images were established in [LI20, LI24], while nontriviality of possible images was shown in [DPS09, LI20].

3. Profinite rigidity of the universal homomorphism

This section is dedicated to the proof of Theorem D, which asserts that the universal homomorphism of a Kähler group is a profinite invariant. Note that we do not assume residual finiteness of the Kähler groups involved here. We collect preliminary results about profinite groups in Section 3.A, before proving Theorem D over the course of Sections 3.B and 3.C.

3.A. **Preliminaries.** In what follows, $b_1^{(2)}(\Gamma)$ denotes the first ℓ^2-Betti number of a discrete group Γ. See [Lüc02, Pan96] for the definition of this invariant. The following theorem gathers information about the first (usual and ℓ^2) Betti number of a finitely generated subgroup of a profinite group.

Theorem 3.1. *Suppose that Γ is a finitely presented residually finite group and let $\Lambda < \widehat{\Gamma}$ be a dense finitely generated subgroup. Then $b_1(\Lambda) \ge b_1(\Gamma)$ and $b_1^{(2)}(\Lambda) \ge b_1^{(2)}(\Gamma)$.*

Proof. The inequality between usual Betti numbers is [Rei15, Lem. 4.3]. The inequality between the first ℓ^2-Betti numbers is [Rei15, Prop. 6.3] when Λ is finitely presented, but the proof is the same under the weaker assumption that Λ is finitely generated after applying [Rei15, Thm. 6.5] by Lück and

Osin instead of Lück's approximation theorem. (See also [LO11] for Lück and Osin's original work.) $\qquad\square$

The next result is proved in the last paragraph of the proof of [BCR16, Thm. 3.6] in the case where the inclusion of Λ induces an isomorphism $\widehat{\Lambda} \to G$. The proof is analogous in the present case and is included for completeness.

Lemma 3.2. *Let G be a profinite group and $\Lambda < G$ be a dense subgroup. If $N \unlhd \Lambda$ is a finite normal subgroup of Λ, then it is a normal subgroup of G.*

Proof. Assume that $N \unlhd \Lambda$ is a finite normal subgroup. If N fails to be normal in G, then there is $x \in G$ and $n \in N$ such that the finite set

$$S = \left\{ xnx^{-1}n' \mid n' \in N \right\}$$

does not contain the identity 1_G of G. Since G is profinite there is a surjective homomorphism $\psi : G \to Q$ to a finite group Q whose restriction to $S \cup \{1_G\}$ is injective. Since $\Lambda < G$ is dense in the profinite topology, the restriction $\psi|_\Lambda : \Lambda \to Q$ is surjective. In particular, $\psi(N) \unlhd Q$ is normal, which implies that there exists some $n' \in N$ such that

$$\psi(1_G) = 1_Q = \psi(x)\psi(n)\psi(x)^{-1}\psi(n') = \psi(xnx^{-1}n') \in \psi(S).$$

However, ψ is injective on $S \cup \{1_G\}$ and $1_G \notin S$, which is a contradiction. Thus N is normal in G. $\qquad\square$

The following will be critical in the proofs of several of the primary results of this paper.

Proposition 3.3. *Let Γ be a hyperbolic orbisurface group. Then $\widehat{\Gamma}$ has no nontrivial finite normal subgroups. In particular, if $\Lambda < \widehat{\Gamma}$ is a dense subgroup, then Λ has no nontrivial finite normal subgroups.*

Proof. The fact that $\widehat{\Gamma}$ has no nontrivial finite normal subgroups is [BGZ14, Lem. 2.1] (see also [BCR16, Cor. 5.2]). The second assertion then follows from Lemma 3.2. $\qquad\square$

3.B. **Homomorphisms to profinite orbisurface groups.** The following notation will remain fixed throughout this section. Suppose that G is a Kähler group and $\phi = (\phi_1, \dots, \phi_r) : G \to \Gamma_1 \times \cdots \times \Gamma_r$ is its universal homomorphism. Let

$$\widehat{\phi} = \left(\widehat{\phi}_1, \dots, \widehat{\phi}_r \right) : \widehat{G} \longrightarrow \widehat{\Gamma}_1 \times \cdots \times \widehat{\Gamma}_r$$

denote the induced morphism on profinite completions and $\iota_G : G \to \widehat{G}$ (resp. $\iota_{\Gamma_i} : \Gamma_i \to \widehat{\Gamma}_i$) be the canonical homomorphism from G (resp. Γ_i) to its profinite completion. The goal of this section is to show that $\widehat{\phi}$ has a universal property with respect to homomorphisms to profinite hyperbolic orbisurface groups analogous to the universal property of ϕ. From here onward, the expression *profinite hyperbolic orbisurface group* is used as shorthand for the profinite completion of a hyperbolic orbisurface group. The following result will be essential in all that follows.

Lemma 3.4. *Suppose that $\widehat{\Sigma}$ is a profinite hyperbolic orbisurface group with $b_1(\Sigma) > 0$ and let $\widehat{\psi} : \widehat{G} \to \widehat{\Sigma}$ be a surjective homomorphism. Then $\widehat{\psi} \circ \iota_G$ factors through a unique ϕ_i and thus $\widehat{\psi}$ factors through a unique $\widehat{\phi}_i$. Moreover, if $\ker(\widehat{\psi} \circ \iota_G)$ is finitely generated, then the kernels of $\widehat{\psi} \circ \iota_G$ and ϕ_i coincide.*

Proof. Define $\Lambda = \widehat{\psi} \circ \iota_G(G)$, which is a dense, finitely generated subgroup of $\widehat{\Sigma}$. Thus $b_1(\Lambda) \geq b_1(\Sigma) > 0$ and

$$b_1^{(2)}(\Lambda) \geq b_1^{(2)}(\Sigma) = -\chi(\Sigma) > 0$$

by Theorem 3.1, where $\chi(\Sigma)$ denotes the orbifold Euler characteristic. Fix a nontrivial homomorphism $\nu : \Lambda \to \mathbb{Z}$.

Since $b_1^{(2)}(\Lambda) \neq 0$, [Gab02, Thm. 6.8] implies that $\ker(\nu)$ either is a finite normal subgroup of Λ or is not finitely generated. Then $\ker(\nu)$ cannot be finite by Proposition 3.3, so it is not finitely generated. Since $\ker(\nu)$ is not finitely generated, neither is $\ker(\nu \circ \widehat{\psi} \circ \iota_G)$.

A theorem of Napier and Ramachandran [NR01, Thm. 4.3] then implies that $\nu \circ \widehat{\psi} \circ \iota_G$ factors through one of the Γ_i for some index i (see [Del10] for a more general statement), yielding a commutative diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\widehat{\psi} \circ \iota_G} & \Lambda \\
\phi_i \downarrow & & \downarrow \nu \\
\Gamma_i & \longrightarrow & \mathbb{Z}
\end{array}
$$

Since $\ker(\phi_i) \trianglelefteq G$ is a finitely generated normal subgroup,

$$N := (\widehat{\psi} \circ \iota_G)(\ker(\phi_i)) \trianglelefteq \Lambda$$

is a finitely generated normal subgroup of Λ. It follows again from [Gab02, Thm. 6.8] that N is finite, and from Proposition 3.3 that N is trivial. Thus $\widehat{\psi} \circ \iota_G$ factors through ϕ_i. Functoriality of the profinite completions shows that $\widehat{\psi}$ factors through $\widehat{\phi}_i$, proving the existence part of the first assertion. For the uniqueness part, $\widehat{\psi} \circ \iota_G$ factoring through ϕ_i and ϕ_j produces a commutative diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\phi_j} & \Gamma_j \\
\phi_i \downarrow & & \downarrow u_j \\
\Gamma_i & \xrightarrow{u_i} & \Lambda
\end{array}
$$

The group $\phi_j(\ker(\phi_i))$ is a finitely generated normal subgroup of the hyperbolic orbisurface group Γ_j, hence is trivial or has finite index in Γ_j. The image of $\phi_j(\ker(\phi_i))$ under u_j is trivial, so if it had finite index in Γ_j then Λ would be finite, which is a contradiction. Therefore, $\phi_j(\ker(\phi_i))$ is trivial, and by symmetry $\phi_i(\ker(\phi_j))$ is trivial as well. This proves that ϕ_i and ϕ_j have the same kernel, hence they are equivalent and $i = j$. The uniqueness statement for $\widehat{\psi}$ follows from the uniqueness statement for $\widehat{\psi} \circ \iota_G$ and the observation that a factorisation of $\widehat{\psi}$ through $\widehat{\phi}_j$ induces a factorisation of $\widehat{\psi} \circ \iota_G$ through ϕ_j.

Finally, for the "moreover" part observe that if $\ker(\widehat{\psi} \circ \iota_G)$ is finitely generated, then $\phi_i(\ker(\widehat{\psi} \circ \iota_G))$ is a finitely generated normal subgroup of Γ_i. Since Λ is infinite, $\phi_i(\ker(\widehat{\psi} \circ \iota_G))$ must have infinite index in Γ_i. Therefore it is trivial, which implies that $\ker(\phi_i) = \ker(\widehat{\psi} \circ \iota_G)$ and so u_i is an isomorphism. $\square$

The next goal is to generalise Lemma 3.4 to arbitrary profinite hyperbolic orbisurface groups.

Corollary 3.5. *The conclusion of Lemma 3.4 holds with $\widehat{\Sigma}$ an arbitrary profinite hyperbolic orbisurface group.*

The proof of Corollary 3.5 requires the following lemma.

Lemma 3.6. *Suppose that G is a Kähler group, Λ is an infinite non-virtually abelian group with no nontrivial finite normal subgroups, and $\psi : G \to \Lambda$ is a surjective homomorphism. If G_0 is a finite index normal subgroup of G for which the restriction of ψ to G_0 factors through a component projection for the universal homomorphism of G_0, then ψ factors through a component projection for the universal homomorphism of G.*

Proof. Fix a closed Kähler manifold X with fundamental group G and consider the finite Galois covering $X_0 \to X$ corresponding to the subgroup $G_0 \le G$. By assumption, there exists a fibration $p : X_0 \to S_0$ onto a Riemann surface such that the restriction of ψ to G_0 factors through the morphism $p_* : G_0 \to \pi_1^{orb}(S_0)$ induced by p, where S_0 is given the orbifold structure defined by the multiple fibers of p and $\ker(p_*)$ is finitely generated. Write $\psi|_{G_0} = v \circ p_*$, for some morphism $v : \pi_1^{orb}(S_0) \to \Lambda$.

Fix $g \in G$ and consider the group

$$(3.6\text{a}) \qquad\qquad g \ker(p_*) g^{-1} \le G_0,$$

which is a subgroup of G_0 since G_0 is normal in G. It is normal in G_0, as the image of a normal subgroup by an (outer) automorphism, and finitely generated because $\ker(p_*)$ is. Hence

$$(3.6\text{b}) \qquad\qquad H := p_*(g \ker(p_*) g^{-1})$$

is a finitely generated normal subgroup of $\pi_1^{orb}(S_0)$, which implies that it is either trivial or has finite index.

Fix a base point $x_* \in X_0$ contained in a smooth fiber of p and a path α from x_* to $g(x_*)$, and let $F := p^{-1}(p(x_*))$. Each element from the subgroup defined in (3.6a) can be represented by a loop obtained by concatenating α, a loop in $g(F)$ based at $g(x_*)$, and then the inverse of α. Assume that $g(F)$ is not contained in a fibre of p. Then $p(g(F))$ equals S_0, and the map $g(F) \to S_0$ induces a morphism on fundamental groups whose image has finite index. Since the morphism $G_0 = \pi_1(X_0) \to \pi_1(S_0)$ can be written as a composition

$$G_0 \xrightarrow{\ p_*\ } \pi_1^{orb}(S_0) \xrightarrow{\ u\ } \pi_1(S_0)$$

and since the discussion above shows that $u \circ p_*(g \ker(p_*) g^{-1})$ has finite index in $\pi_1(S_0)$, then H cannot be trivial. Hence it has finite index in $\pi_1^{orb}(S_0)$. The equalities

$$\begin{aligned}
\{1\} &= v \circ p_*(\ker p_*) \\
&= \psi(\ker p_*) \\
&= \psi(g \ker(p_*) g^{-1}) \\
&= v(H)
\end{aligned}$$

then yields that $\{1\} = v(H) < \Lambda$ has finite index, which is a contradiction since Λ is infinite. Thus $g(F)$ must coincide with a fibre of p, and so g maps any smooth fibre of p to a fibre of p. This implies that G/G_0 acts on S_0 and there exists a holomorphic map $X \to S$ where S is the quotient of S_0 by the finite group G/G_0. Let $\pi : X \to \Sigma$ be the Stein factorisation of the map $X \to S$. To see that ψ factors through the morphism

$$(3.6c) \qquad\qquad \pi_* : \pi_1(X) \to \pi_1^{orb}(\Sigma)$$

induced by π, it is enough to prove that the fundamental group of a generic fibre of π, which generates $\ker(\pi_*)$, maps trivially to Λ. To see this, consider a generic fibre $F \subset X_0$ of p mapping to a generic fibre $F' \subset X$ of π under the covering map $X_0 \to X$. The map $F \to F'$ is a finite covering map and $\pi_1(F)$ maps trivially to Λ. Therefore the image of $\pi_1(F')$ in Λ is a finite normal subgroup, which must be trivial by hypothesis. This proves that ψ factors through the morphism (3.6c). The orbifold associated to the fibration $\pi : X \to \Sigma$ is hyperbolic because Λ is not virtually abelian, thus ψ factors through a component projection for the universal homomorphism for G. This completes the proof. $\qquad\square$

We are now prepared to prove Corollary 3.5.

Proof of Corollary 3.5. Consider a surjective homomorphism $\widehat{\psi} : \widehat{G} \to \widehat{\Sigma}$ to a profinite hyperbolic orbisurface group. Then, there is a finite index normal surface subgroup $\Sigma_0 \trianglelefteq \Sigma$ with $b_1(\Sigma_0) > 0$. Consider the subgroup $\widehat{\Sigma}_0 \triangleleft \widehat{\Sigma}$ and its inverse image

$$G_0 := (\widehat{\psi} \circ \iota_G)^{-1}(\widehat{\Sigma}_0).$$

The homomorphism $G_0 \to \widehat{\Sigma}_0$ factors through a component of the universal homomorphism for G_0 by Lemma 3.4. Lemma 3.6 then implies that $\widehat{\psi} \circ \iota_G : G \to \widehat{\Sigma}$ factors through a component ϕ_i of the universal homomorphism for G. The uniqueness of this component, as well as all other remaining assertions, are proved exactly as in Lemma 3.4. $\qquad\square$

3.C. **The universal homomorphism is a profinite invariant.** The purpose of this section is to prove Theorem D, whose statement is now recalled.

Theorem D. *Let G and H be Kähler groups,*

$$\phi = (\phi_1, \ldots, \phi_r) : G \longrightarrow \Gamma_1 \times \cdots \times \Gamma_r$$
$$\psi = (\psi_1, \ldots, \psi_s) : H \longrightarrow \Sigma_1 \times \cdots \times \Sigma_s$$

be their universal homomorphisms, and $\widehat{\phi} = (\widehat{\phi}_1, \ldots, \widehat{\phi}_r), \widehat{\psi} = (\widehat{\psi}_1, \ldots, \widehat{\psi}_s)$ be the induced homomorphisms on profinite completions. If there is an isomorphism $\Theta : \widehat{G} \to \widehat{H}$, then $r = s$ and, up to reordering factors, Γ_i is isomorphic to Σ_i for each $i \in \{1, \ldots, r\}$. Moreover, $\widehat{\psi} \circ \Theta = \widehat{\phi}$ after identifying the groups $\widehat{\Gamma}_i$ and $\widehat{\Sigma}_i$ by applying suitable isomorphisms of profinite orbisurface groups. In particular $\operatorname{im}(\widehat{\psi}) \cong \operatorname{im}(\widehat{\phi})$.

Proof. If Λ is a hyperbolic orbisurface group, then the homomorphisms $\widehat{\phi}$ and $\widehat{\psi}$ are universal with respect to surjective homomorphisms $\widehat{G} \cong \widehat{H} \to \widehat{\Lambda}$ by Corollary 3.5. For every $i \in \{1, \ldots, r\}$, the universal property of $\widehat{\psi}$ implies that there is a unique $k_i \in \{1, \ldots, s\}$ such that there exists a factorisation

$\widehat{\phi}_i \circ \Theta^{-1} = \widehat{\alpha}_{k_i} \circ \widehat{\psi_{k_i}}$ with $\widehat{\alpha}_{k_i} : \widehat{\Sigma}_{k_i} \to \widehat{\Gamma}_i$ surjective. Similarly, the universal property of $\widehat{\phi}$ implies that there is a unique $\ell_{k_i} \in \{1, \dots, r\}$ and an epimorphism $\widehat{\beta}_{\ell_{k_i}} : \widehat{\Gamma}_{\ell_{k_i}} \to \widehat{\Sigma}_{k_i}$ such that $\widehat{\psi}_{k_i} \circ \Theta = \widehat{\beta}_{\ell_{k_i}} \circ \widehat{\phi}_{\ell_{k_i}}$. This produces a commutative diagram

$$
\begin{array}{ccc}
\widehat{G} & \xrightarrow{\widehat{\phi}_{\ell_{k_i}}} & \widehat{\Gamma}_{\ell_{k_i}} \\
\widehat{\phi}_i \downarrow & & \downarrow \widehat{\alpha}_{k_i} \circ \widehat{\beta}_{\ell_{k_i}} \\
\widehat{\Gamma}_i & \xrightarrow{\mathrm{id}} & \widehat{\Gamma}_i
\end{array}
$$

of epimorphisms. The uniqueness in Corollary 3.5 implies that $\ell_{k_i} = i$. This proves that the map $i \mapsto k_i$ is injective and the map $k \mapsto l_k$ is surjective. Exchanging the roles of G and H proves that both maps are bijections, hence $r = s$. Thus, after reordering factors, we can assume that $k_i = i$ and $\ell_i = i (1 \le i \le r)$. The arguments above then yield $\widehat{\phi}_i = \widehat{\alpha}_i \circ \widehat{\beta}_i \circ \widehat{\phi}_i$. Thus $\widehat{\beta}_i$ is injective, hence an isomorphism. We then have

$$
\widehat{\psi} \circ \Theta = (\widehat{\beta}_1, \dots, \widehat{\beta}_r) \circ \widehat{\phi}
$$

as desired. Finally, since Γ_i and Σ_i are Fuchsian groups, it follows from [BCR16, Thm. 1.1] that $\Gamma_i \cong \Sigma_i$ for $1 \le i \le r$. $\qquad \square$

4. Profinite rigidity results for Kähler subgroups of direct products

This section studies the profinite genus of a Kähler subgroup of a direct product of hyperbolic orbisurface groups. The first result is that being a full subdirect product of orbisurface groups is a profinite invariant (Theorem 4.6). This is then used to establish profinite rigidity amongst residually finite Kähler groups for direct products of orbisurface groups and their co-$\mathbb{Z}^2$ Kähler subgroups.

4.A. **Preliminaries.** Here we recall general results on residually free groups and separability, which are needed in the next subsections.

Definition 4.1. A subgroup $H \le G$ is *separable* if it is the intersection of a family of finite index subgroups of G.

It follows from the definition of the profinite topology that a subgroup $H \le G$ is separable if and only if it is closed in the profinite topology. Moreover, a group is residually finite if and only if the trivial subgroup is separable. A crucial result in this paper is the following theorem of Bridson and Wilton [BW08, Thm. B], where the second statement follows from applying [Rei15, Lem. 4.6] to their result.

Theorem 4.2. *Let G be a finitely generated residually free group, then all finitely presented subgroups of G are separable. In particular, the inclusion $H \hookrightarrow G$ of a finitely presented subgroup induces an isomorphism between the profinite completion of H and the closure of its image in $\widehat{G}$.*

The next result follows from a straightforward adaptation of the proof of an analogous result for the finitely generated case [Sco78, Proof of Lem. 1.1]. A simple topological proof is given for completeness.

Lemma 4.3. *Let G be a group with a finite index subgroup $H < G$ such that every finitely presented subgroup of H is separable in H. Then every finitely presented subgroup of G is separable in G.*

Proof. Suppose $L < G$ is finitely presented. Then $L \cap H$ is a finite index subgroup of L, and thus is also finitely presented. In particular, $L \cap H$ is closed in the profinite topology on H and thus is by definition closed in the profinite topology on G. Then L is a finite union of cosets of $L \cap H$, and therefore is also closed. $\square$

The following result can be viewed as a version of Grothendieck rigidity for finitely presented groups.

Lemma 4.4. *Let G be a group with the property that every finitely presented subgroup is separable, and suppose that H is finitely presented and residually finite. If there is a homomorphism $\phi : H \to G$ that induces an isomorphism on profinite completions, then ϕ is an isomorphism.*

Proof. First observe that ϕ is injective. Indeed, first G is residually finite by hypothesis since the trivial group is finitely presented. Thus there is a commutative diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\ \phi\ } & G \\
{\scriptstyle \iota_H}\downarrow & & \downarrow{\scriptstyle \iota_G} \\
\widehat{H} & \xrightarrow{\ \widehat{\phi}\ } & \widehat{G}
\end{array}
$$

with the vertical homomorphisms injective and the bottom homomorphism an isomorphism, hence ϕ is injective as claimed. Therefore $H \cong \phi(H) < G$ is a finitely presented subgroup, and thus $\phi(H)$ is separable in G by hypothesis. It then follows from [LR11, Lem. 2.5] that $\phi(H) = G$. $\square$

Since direct products of hyperbolic orbisurface groups are virtually residually free, the following is a direct consequence of Theorem 4.2 and Lemma 4.3.

Corollary 4.5. *Let G be a direct product of finitely many hyperbolic orbisurfaces and $H < G$ be a finitely presented subgroup. Then H is separable in G. In particular, the inclusion $H \hookrightarrow G$ induces an embedding $\widehat{H} \hookrightarrow \widehat{G}$ of profinite completions.*

4.B. **Virtual direct products.** The previous results in this section are now used to prove profinite rigidity results for Kähler subgroups of direct products of orbisurface groups.

Theorem 4.6. *Let G and H be residually finite Kähler groups for which there is an isomorphism $\Theta : \widehat{G} \to \widehat{H}$. If*

$$G < \Gamma := \Gamma_1 \times \cdots \times \Gamma_r$$

is a full subdirect product of hyperbolic orbisurface groups Γ_i, $1 \leq i \leq r$, then there is an embedding $H < \Gamma$ as a full subdirect product. Furthermore, there is an automorphism Ξ of $\widehat{\Gamma}$, consisting of a reordering of the factors of $\widehat{\Gamma}$ followed by automorphisms of the $\widehat{\Gamma}_i$, such that $\Xi(\overline{G}) = \overline{H}$, where the closures of G and H are taken inside $\widehat{\Gamma}$. Finally, $\overline{G}$ (resp. $\overline{H}$) identifies with $\widehat{G}$ (resp. $\widehat{H}$).

Proof. The inclusion $\phi : G \hookrightarrow \Gamma_1 \times \cdots \times \Gamma_r$ is the universal homomorphism for G by Lemma 2.5. Theorem D then implies that, after possibly reordering factors, the universal homomorphism ψ for H has target Γ and there is a factor-preserving automorphism Ξ of $\widehat{\Gamma}$ inducing a commutative diagram

$$
\begin{array}{ccc}
\widehat{\Gamma} & \xrightarrow{\ \Xi\ } & \widehat{\Gamma} \\
\widehat{\phi} \uparrow & & \uparrow \widehat{\psi} \\
\widehat{G} & \xrightarrow{\ \Theta\ } & \widehat{H}.
\end{array}
$$

Corollary 4.5 implies that $\widehat{\phi}$ is injective with image $\overline{G}$. Thus $\widehat{\psi} = \Xi \circ \widehat{\phi} \circ \Theta^{-1}$ is injective and so is ψ. The closure of $H \cong \psi(H)$ is isomorphic to $\widehat{H}$ by Corollary 4.5 again and it coincides with $\Xi(\overline{G})$. Finally, Theorem 2.4 implies that the image of ψ is full subdirect, and this completes the proof. $\square$

A direct consequence of Theorem 4.6 is profinite rigidity of direct products of hyperbolic orbisurface groups among residually finite Kähler groups.

Corollary 4.7. *Let $G = \Gamma_1 \times \cdots \times \Gamma_r$ be a direct product of hyperbolic orbisurface groups and H be a residually finite Kähler group such that there is an isomorphism $\Theta : \widehat{G} \to \widehat{H}$. Then $H \cong \Gamma_1 \times \cdots \times \Gamma_r$.*

Proof. According to Theorem 4.6, there is an embedding $\psi : H \to \Gamma_1 \times \cdots \times \Gamma_r$ as a finitely presented full subdirect product inducing an isomorphism of profinite completions. Since Kähler groups are finitely presented, Corollary 4.5 and Lemma 4.4 imply that ψ is an isomorphism. $\square$

We observe that direct products of hyperbolic orbisurface groups are indeed Kähler groups; this follows for instance from a trick due to Kollár [ABC⁺96, p. 7].

As discussed in the introduction, general Kähler groups are not uniquely determined by their profinite completion. This is even false for subgroups of direct products of surface groups. Indeed, Bessa, Grunewald, and Zalesskii constructed nonisomorphic finite index subgroups $P_1, P_2 \leq \Gamma_1 \times \Gamma_2$ of a direct product of two hyperbolic orbisurface groups with $\widehat{P_1} \cong \widehat{P_2}$ [BGZ14, Ex. 3.14]. Since these groups are finite index in a Kähler group, they are Kähler. However, their examples do have finite $\mathcal{K}$-profinite genus.

Corollary 4.8. *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$ be a direct product of hyperbolic orbisurface groups. If G is a finite index subgroup of Γ, then $\mathcal{G}_\mathcal{K}(G)$ is finite.*

Proof. Suppose $H \in \mathcal{G}_\mathcal{K}(G)$. Since $G < \Gamma$ has finite index, by projecting onto a smaller direct product and replacing the Γ_i by suitable finite index subgroups if necessary, we can assume that G is a full subdirect product. Theorem 4.6 provides an embedding $H \hookrightarrow \Gamma$ and an automorphism Ξ of $\widehat{\Gamma}$ such that $\overline{H} = \Xi(\overline{G})$. Both $\widehat{G} \cong \overline{G}$ and $\widehat{H} \cong \overline{H}$ then have finite index in $\widehat{\Gamma}$, and the bijection between the finite index subgroups of a group and those of its profinite completion gives

$$|\Gamma : G| = |\widehat{\Gamma} : \widehat{G}| = |\widehat{\Gamma} : \widehat{H}| = |\Gamma : H|,$$

where the last equality uses the fact that H is separable in Γ by Corollary 4.5. Since Γ is finitely generated, it has only finitely many subgroups of index $|\Gamma : G|$, and therefore there are only finitely many possibilities for H. $\square$

4.C. **Co-$\mathbb{Z}^2$ subgroups.** We now address the $\mathcal{K}$-profinite genus of Kähler subdirect products of hyperbolic orbisurface groups which are virtually co-$\mathbb{Z}^2$.

Proposition 4.9. *Suppose G, H are finitely presented full subdirect products of $\Gamma_1 \times \cdots \times \Gamma_r$, a direct product of hyperbolic orbisurface groups, such that there are surjective homomorphisms $f_i : \Gamma_i \to \mathbb{Z}^n$ and $g_i : \Gamma_i \to \mathbb{Z}^n$ with $\ker\left(\sum_{i=1}^r f_i\right) = G$ and $\ker\left(\sum_{i=1}^r g_i\right) = H$. Then there is an isomorphism $\phi : G \to H$ if and only if, up to permuting factors of $\Gamma_1 \times \cdots \times \Gamma_r$, there are isomorphisms $\phi_i : \Gamma_i \to \Gamma_i$ and $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $\psi \circ f_i = g_i \circ \phi_i$.*

Proof. We deal with the "only if" direction, the other one being clear. If G and H are isomorphic, Theorem 2.3 says that (after possibly permuting factors) there are isomorphisms $\phi_i : \Gamma_i \to \Gamma_i$ so that the isomorphism

$$\phi := \prod_{i=1}^r \phi_i : \Gamma_1 \times \cdots \times \Gamma_r \to \Gamma_1 \times \cdots \times \Gamma_r$$

carries the normal subgroup G to the normal subgroup H. Hence ϕ induces an isomorphism $\psi : \mathbb{Z}^n \to \mathbb{Z}^n$ on the quotients, which satisfies

$$\psi \circ \sum_{i=1}^r f_i = \sum_{i=1}^r g_i \circ \phi.$$

This implies the relation $\psi \circ f_i = g_i \circ \phi_i$ for each $i \in \{1, \ldots, r\}$. $\square$

We now study homomorphisms from (orbi)surface groups to $\mathbb{Z}^2$. Notice that if Γ is a closed hyperbolic orbisurface group, then there is an epimorphism $p : \Gamma \to \Sigma = \pi_1(S)$ to the fundamental group of the (not necessarily hyperbolic) surface S obtained by forgetting the orbifold structure, and any surjective morphism $f : \Gamma \to \mathbb{Z}^2$ factors through p. Let $\mathrm{Epi}^d(\Gamma)$ denote the set of epimorphisms $f : \Gamma \to \mathbb{Z}^2$ inducing a degree d morphism $\Sigma \to \mathbb{Z}^2$.

Proposition 4.10. *For a closed hyperbolic orbisurface group Γ and a degree $d \in \mathbb{N}$ with $d \geq 2$, let $\mathrm{Aut}(\Gamma)$ act on $\mathrm{Epi}^d(\Gamma)$ by precomposition. Then the coset space $\mathrm{Epi}^d(\Gamma)/\mathrm{Aut}(\Gamma)$ consists of a single element.*

Proof. Let $f, g \in \mathrm{Epi}^d(\Gamma)$. We consider the induced injections

$$f^*, g^* : \mathrm{H}^1(\mathbb{Z}^2, \mathbb{Z}) \to \mathrm{H}^1(\Sigma, \mathbb{Z}).$$

By [BE84, Cor 8.2], there is a self-homeomorphism of the surface S, inducing $h \in \mathrm{Aut}(\Sigma)$ and $h^* : \mathrm{H}^1(\Sigma, \mathbb{Z}) \to \mathrm{H}^1(\Sigma, \mathbb{Z})$, so that

(4.10a) $$h^* \circ f^* = g^*.$$

This homeomorphism of S can be modified by an isotopy so that it preserves the marked points of the orbifold, and hence we can lift h to an automorphism $\overline{h}$ of Γ. Because $\mathbb{Z}^2$ is abelian, f and g are determined by f^* and g^* respectively, so (4.10a) yields the equality $f \circ \overline{h} = g$. This proves that $\mathrm{Epi}^d(\Gamma)/\mathrm{Aut}(\Gamma)$ is made of a single point. $\square$

The previous proposition immediately yields the following:

Corollary 4.11. *Let Γ_i $(1 \leq i \leq r)$ be closed hyperbolic orbisurface groups and $d_1, \ldots, d_r$ be integers greater than 1. Fix some surjections $f_i \in \mathrm{Epi}^{d_i}(\Gamma_i)$ $(1 \leq i \leq r)$, and consider the group $G = \ker(\sum_{i=1}^r f_i) < \Gamma_1 \times \cdots \times \Gamma_r$. Then the*

isomorphism type of G depends only on the degrees d_i and not on the specific choice of the f_i.

We are now ready to return to profinite rigidity results. We start with the following theorem, whose proof requires to restrict to surface groups, in contrast to most of our other results.

Theorem 4.12. *Let $G < \prod_{i=1}^r \Gamma_i$ be a Kähler full subdirect product of hyperbolic surface groups. Suppose that there are homomorphisms $f_i : \Gamma_i \to \mathbb{Z}^2$, which either have finite index image or are trivial, such that G is the kernel of the morphism $f = \sum_{i=1}^r f_i : \Gamma_1 \times \cdots \times \Gamma_r \to \mathbb{Z}^2$, assumed to be surjective. Then:*

(1) for each $H \in \mathcal{G}_\mathcal{K}(G)$, there are homomorphisms $g_i : \Gamma_i \to \mathbb{Z}^2$, which either have finite index image with $|\mathbb{Z}^2{:}\mathrm{im}(f_i)| = |\mathbb{Z}^2{:}\mathrm{im}(g_i)|$ or are trivial, such that H is isomorphic to the kernel of the morphism

$$g = \sum_{i=1}^r g_i{:}\Gamma_1 \times \cdots \times \Gamma_r \to \mathbb{Z}^2;$$

(2) $\mathcal{G}_\mathcal{K}(G)$ is finite;
(3) if, in addition, the nontrivial f_i are surjective, then $\mathcal{G}_\mathcal{K}(G) = \{G\}$.

Proof. We write $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$. To prove (1), we consider a group $H \in \mathcal{G}_\mathcal{K}(G)$ and fix an isomorphism $\Theta : \widehat{G} \to \widehat{H}$. By Theorem 4.6, we can find an embedding $H \leq \Gamma_1 \times \cdots \times \Gamma_r$ as a full subdirect product, and isomorphisms $\Xi_i : \widehat{\Gamma}_i \to \widehat{\Gamma}_i$ such that there is a commutative diagram

$$
\begin{array}{ccc}
\widehat{\Gamma} & \xrightarrow{\ \Xi\ } & \widehat{\Gamma} \\
\uparrow & & \uparrow \\
\widehat{G} & \xrightarrow{\ \Theta\ } & \widehat{H}.
\end{array}
$$

Here the vertical arrows are *embeddings*, induced by the inclusions of G and H in Γ, and $\Xi = \prod_{i=1}^r \Xi_i$. By right exactness of the profinite completion, we thus have a commutative diagram

(4.12a)
$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{G} & \longrightarrow & \widehat{\Gamma}_1 \times \cdots \times \widehat{\Gamma}_r & \xrightarrow{\ \widehat{f}\ } & \widehat{\mathbb{Z}^2} & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle\Theta} & & \downarrow{\scriptstyle\Xi} & & \downarrow{\scriptstyle\mathrm{id}} & & \\
1 & \longrightarrow & \widehat{H} & \longrightarrow & \widehat{\Gamma}_1 \times \cdots \times \widehat{\Gamma}_r & \xrightarrow{\ \phi\ } & \widehat{\mathbb{Z}^2} & \longrightarrow & 1,
\end{array}
$$

where ϕ is *defined* by $\phi = \widehat{f} \circ \Xi^{-1}$. Next, we claim that the kernel of the restriction of ϕ to Γ is equal to H. By definition, this kernel is given by

(4.12b) $$\widehat{H} \cap (\Gamma_1 \times \cdots \times \Gamma_r) = \overline{H} \cap (\Gamma_1 \times \cdots \times \Gamma_r).$$

Since H is separable in $\Gamma_1 \times \cdots \times \Gamma_r$ (by Corollary 4.5), the right-hand side of Equation (4.12b) is equal to H, thus proving our claim about the intersection of $\ker \phi$ with Γ. It follows that H is normal in $\Gamma_1 \times \cdots \times \Gamma_r$; we denote by A the quotient. We thus get an exact sequence

(4.12c) $$1 \to H \to \Gamma_1 \times \cdots \times \Gamma_r \to A \to 1.$$

The separability of H in Γ yields that A is residually finite. Comparing (4.12a) and (4.12c), one gets an isomorphism $\widehat{A} \cong \widehat{\mathbb{Z}^2}$. Since A is finitely

generated and residually finite, it is isomorphic to $\mathbb{Z}^2$. Now, there are morphisms $g_i : \Gamma_i \to \mathbb{Z}^2$ and an automorphism ψ of $\widehat{\mathbb{Z}^2}$ such that $H = \ker(\sum_{i=1}^r g_i)$ and $\phi = \psi \circ \widehat{g}$, where $g = \sum_{i=1}^r g_i$. Combining this with (4.12a), we get that

$$\widehat{f} = \psi \circ \widehat{g} \circ \Xi,$$

or equivalently that

(4.12d) $$\widehat{f_i} = \psi \circ \widehat{g_i} \circ \Xi_i$$

for each $i \in \{1, \dots, r\}$. In particular, $\widehat{f_i}$ and $\widehat{g_i}$ are simultaneously trivial or simultaneously have finite index images. Because $\widehat{f_i}$ has image of a given finite index or is trivial exactly when f_i does, we see that each g_i either has image of finite index or is trivial in the same way as f_i. We thus have

$$|\mathbb{Z}^2 : \operatorname{im}(f_i)| = |\widehat{\mathbb{Z}^2} : \operatorname{im}(\widehat{f_i})| = |\widehat{\mathbb{Z}^2} : \operatorname{im}(\widehat{g_i})| = |\mathbb{Z}^2 : \operatorname{im}(g_i)|$$

for those indices i such that f_i (or equivalently g_i) is nontrivial.

We now turn to the proof of (2). We write $\Gamma_i = \pi_1(S_i)$, where S_i is a closed oriented surface. By applying Theorem 2.4 and [LI24, Thm 6.2] to G and H, after potentially reordering the Γ_i, there is an $s \geq 3$ so that the maps f_i and g_i are induced by ramified covers of S_i over a torus for $i = 1, \dots, s$ and f_i and g_i are trivial for $i = s+1, \dots, r$. This yields that $G = G_0 \times \Gamma_{s+1} \times \cdots \times \Gamma_r$ and $H = H_0 \times \Gamma_{s+1} \times \cdots \times \Gamma_r$ where $G_0 = \ker(\sum_{i=1}^s f_i)$ and $H_0 = \ker(\sum_{i=1}^s g_i)$ are full subdirect products of $\Gamma_1 \times \cdots \times \Gamma_s$. We now want to argue that the degrees of the ramified coverings corresponding to f_i and g_i are the same for each $i = 1, \dots, s$.

Denote by H^{1+1} the image of the cup product $H^1 \times H^1 \to H^2$. By (4.12a) and naturality of the cup product (see also [KW16]), for every natural number $n \geq 1$ we have a commutative diagram

$$
\begin{array}{ccc}
H^{1+1}(\mathbb{Z}^2; \mathbb{Z}/n) & \xrightarrow{f_i^{1+1}} & H^{1+1}(\Gamma_i; \mathbb{Z}/n) \\
\uparrow & & \uparrow \\
H^{1+1}(\widehat{\mathbb{Z}^2}; \mathbb{Z}/n) & \xrightarrow{\widehat{f_i}^{1+1}} & H^{1+1}(\widehat{\Gamma_i}; \mathbb{Z}/n) \\
\downarrow & & \downarrow \\
H^{1+1}(\widehat{\mathbb{Z}^2}; \mathbb{Z}/n) & \xrightarrow{\widehat{g_i}^{1+1}} & H^{1+1}(\widehat{\Gamma_i}; \mathbb{Z}/n) \\
\downarrow & & \downarrow \\
H^{1+1}(\mathbb{Z}^2; \mathbb{Z}/n) & \xrightarrow{g_i^{1+1}} & H^{1+1}(\Gamma_i; \mathbb{Z}/n).
\end{array}
$$

Here the vertical maps are isomorphisms: the middle ones are induced by ψ and Ξ_i (see (4.12d)) and the top and bottom ones come from the goodness of the groups involved. (The definition of the notion of goodness will be recalled in Section 5.A.) Now, since $H^{1+1}(\Gamma_i; \mathbb{Z}/n) = H^2(\Gamma_i; \mathbb{Z}/n)$ and $H^{1+1}(\mathbb{Z}^2; \mathbb{Z}/n) = H^2(\mathbb{Z}^2; \mathbb{Z}/n)$, the degree of the ramified cover inducing f_i (resp. g_i) can be recovered from the values of n for which the map f_i^{1+1} (resp. g_i^{1+1}) is equal to 0. Indeed, given a prime n, its highest power dividing these degrees can be recovered by considering the powers of n for which f_i^{1+1} (or g_i^{1+1}) vanishes. Since the vertical maps above are isomorphisms, the maps

f_i^{1+1}, $\widehat{f}_i^{1+1}$, $\widehat{g}_i^{1+1}$, and g_i^{1+1} vanish for the same values of n. Hence, the degree of the ramified covers inducing f_i and g_i are equal and Proposition 2.1 implies that $\mathcal{G}_{\mathcal{K}}(G)$ is finite.

Finally, to establish (3) we observe that if the f_i are surjective or trivial, then combining the equality of the degrees of f_i and g_i with Corollary 4.11, we obtain that H_0 and G_0 are isomorphic, hence so are H and G.   □

5. Once more with tori

The primary goal of this section is to prove $\mathcal{K}$-profinite rigidity of direct products of a free abelian group with finitely many hyperbolic orbisurface groups (see Theorem 5.3 in Section 5.A). We then combine this with the results from Section 4 to establish Theorem C (Section 5.B).

5.A. **Central extensions and profinite groups.** We need to understand profinite properties of central extensions of groups by $\mathbb{Z}^k$. Before stating the first relevant result, some notions related to *goodness* are required. Let G be a discrete group. As noted by Serre [Ser94, p. 15], for $j = 0, 1$ and all finite G-modules M, the restriction map $H^j(\widehat{G}, M) \to H^j(G, M)$ on group cohomology induced by $G \to \widehat{G}$ is an isomorphism. A group is then called *good* if this map is an isomorphism for all j and *n-good* if the restriction map continues to be an isomorphism up to and including $j = n$. See [Rei15, §7] for more on goodness.

The importance of 2-goodness lies in the connection between $H^2(G, M)$ and central extensions of G by M. Specifically, a group G is 2-good if and only if any group extension

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1$$

of G by a finitely generated group N induces an injection $\widehat{N} \hookrightarrow \widehat{E}$ [Rei15, Prop. 7.10]. Since the profinite completion is right exact, 2-goodness is precisely what is needed to deduce that there is always an induced exact sequence

$$1 \longrightarrow \widehat{N} \longrightarrow \widehat{E} \longrightarrow \widehat{G} \longrightarrow 1$$

of profinite completions.

The following lemma is essentially due to Wilton and Zalesskii [WZ17, Lem. 8.3] in the context of orientable Seifert fibre spaces. The proof given below extracts the essence of their argument.

Lemma 5.1. *Let G_0 be a residually finite group that is 2-good, and assume that for all nontrivial $\psi \in H^2(G_0, \mathbb{Z}^k)$ there exists a prime power p^m such that the image of ψ in $H^2(G_0, (\mathbb{Z}/p^m)^k)$ is nontrivial. If*

$$(5.1a) \qquad\qquad 1 \longrightarrow \mathbb{Z}^k \longrightarrow G \longrightarrow G_0 \longrightarrow 1$$

is a central extension, then (5.1a) *splits if and only if the induced exact sequence*

$$(5.1b) \qquad\qquad 1 \longrightarrow \widehat{\mathbb{Z}}^k \longrightarrow \widehat{G} \longrightarrow \widehat{G}_0 \longrightarrow 1$$

of profinite completions splits.

Proof. As described before the statement of the lemma, exactness of the sequence of profinite completions follows from 2-goodness of G_0. Moreover, there is a commutative diagram

$$
\begin{array}{ccc}
H^2(G_0, \mathbb{Z}^k) & \longrightarrow & H^2(G_0, (\mathbb{Z}/p^m)^k) \\
\downarrow & & \uparrow{\scriptstyle\cong} \\
H^2(\widehat{G}_0, \widehat{\mathbb{Z}}^k) & \longrightarrow & H^2(\widehat{G}_0, (\mathbb{Z}/p^m)^k)
\end{array}
$$

where the horizontal maps are reduction of coefficients modulo p^m, the left vertical map is taking profinite completions, and the right vertical map is restriction. Furthermore, it is clear that (5.1a) splitting implies (5.1b) splits. For the remainder of the proof, assume that (5.1a) does not split, and hence its characteristic class $\psi \in H^2(G_0, \mathbb{Z}^k)$ is nontrivial.

Choose a prime power p^m so that the reduction ψ_{p^m} of ψ modulo p^m is nontrivial. Then 2-goodness implies that there is $\widehat{\psi}_{p^m} \in H^2(\widehat{G}_0, (\mathbb{Z}/p^m)^k)$ whose restriction to G_0 is ψ_{p^m}. The exact sequence

$$
1 \longrightarrow (\mathbb{Z}/p^m)^k \longrightarrow \widehat{G}_p \longrightarrow \widehat{G}_0 \longrightarrow 1
$$

associated with $\widehat{\psi}_{p^m}$ then does not split, but it is also the reduction modulo p^m of (5.1b), and so that sequence cannot split. This proves the lemma. □

The content of [WZ17, Lem. 8.3] is the case where G_0 is a surface group and $k = 1$. The proof of the following corollary to Lemma 5.1 corrects a mistake in the direction of a cohomology exact sequence in [WZ17] and generalises their result to $k \geq 1$ and G_0 a product of hyperbolic orbisurface groups.

Corollary 5.2. *Let $k \geq 1$ be an integer and G_0 be a finite product of hyperbolic orbisurface groups. Then, a central extension given by a short exact sequence*

(5.2a) $$ 1 \longrightarrow \mathbb{Z}^k \longrightarrow G \longrightarrow G_0 \longrightarrow 1 $$

splits if and only if the induced short exact sequence

(5.2b) $$ 1 \longrightarrow \widehat{\mathbb{Z}}^k \longrightarrow \widehat{G} \longrightarrow \widehat{G}_0 \longrightarrow 1 $$

of profinite completions splits.

Proof. It is well-known that hyperbolic orbisurface groups are good, and the direct product of finitely many good groups is good [GJZZ08, Prop. 3.4], so G_0 is good. In particular, G_0 is residually finite and 2-good. It remains to check that every nontrivial class in $H^2(G_0, \mathbb{Z}^k)$ has nontrivial reduction modulo some prime power p^m.

For any prime power p^m, the exact sequence

$$
1 \longrightarrow \mathbb{Z}^k \longrightarrow \mathbb{Z}^k \longrightarrow (\mathbb{Z}/p^m)^k \longrightarrow 1
$$

coming from reduction modulo p^m induces a long exact sequence

$$\cdots \longrightarrow H^1(G_0, (\mathbb{Z}/p^m)^k) \longrightarrow H^2(G_0, \mathbb{Z}^k) \longrightarrow H^2(G_0, \mathbb{Z}^k)$$
$$\longrightarrow H^2(G_0, (\mathbb{Z}/p^m)^k) \longrightarrow H^3(G_0, \mathbb{Z}^k) \longrightarrow \cdots$$

in cohomology. We shall only be interested in the following piece of this long exact sequence:

$$(5.2c) \qquad H^2(G_0, \mathbb{Z}^k) \longrightarrow H^2(G_0, \mathbb{Z}^k) \longrightarrow H^2(G_0, (\mathbb{Z}/p^m)^k).$$

The group $H^2(G_0, \mathbb{Z}^k)$ is a finitely generated abelian group and the first arrow in (5.2c) is simply the multiplication by p^m. Choose a nonzero class $\psi \in H^2(G_0, \mathbb{Z}^k)$. If ψ is not a torsion element then it is not divisible by p for p large enough, hence has a nontrivial image in $H^2(G_0, (\mathbb{Z}/p)^k)$. Now suppose that ψ is torsion. Identify the torsion subgroup of $H^2(G_0, \mathbb{Z}^k)$ with

$$(5.2d) \qquad \prod_{t=1}^{r} \mathbb{Z}/q_t^{m_t}\mathbb{Z}$$

for some prime numbers q_t and integers m_t. Since ψ is nonzero, it has a nonzero component under the identification of the torsion subgroup of $H^2(G_0, \mathbb{Z}^k)$ with the group in (5.2d). Let us assume for instance that it has a nonzero component in the first factor. Then ψ is not divisible by $q_1^{m_1}$, hence has nontrivial image in $H^2(G_0, (\mathbb{Z}/q_1^{m_1})^k)$. It follows that every nontrivial element of $H^2(G_0, \mathbb{Z}^k)$ has nontrivial reduction modulo some prime power, and thus Lemma 5.1 applies to give the desired conclusion. $\square$

Corollary 5.2 is now used to prove the main result of this section.

Theorem 5.3. *Let* $G = \mathbb{Z}^k \times \Gamma_1 \times \cdots \times \Gamma_r$, *where each* Γ_i *is a hyperbolic orbisurface group. If* H *is a residually finite Kähler group and* $\widehat{\phi} : \widehat{H} \to \widehat{G}$ *is an isomorphism, then* $H \cong G$.

Proof. Since H is residually finite, the isomorphism $\widehat{H} \cong \widehat{G}$ induces an embedding $\iota_H : H \hookrightarrow \widehat{\mathbb{Z}}^k \times \widehat{\Gamma}_1 \times \cdots \times \widehat{\Gamma}_r$. Theorem D then implies that the composition of ι_H with projection away from $\widehat{\mathbb{Z}}^k$ coincides with the universal homomorphism

$$\psi : H \to \prod_{i=1}^{r} \Gamma_i$$

for H, up to post-composition by some automorphisms of the $\widehat{\Gamma}_i$. In particular $\widehat{\psi}$ is onto. Note that $\psi(H)$ is a finitely presented subgroup of $\prod_{i=1}^{r} \Gamma_i$ by Theorem 2.4, hence is closed for the profinite topology by Corollary 4.5. If $\psi(H)$ were a proper subgroup of $\prod_{i=1}^{r} \Gamma_i$, then the image of $\widehat{\psi}$ would be a proper subgroup of $\prod_{i=1}^{r} \widehat{\Gamma}_i$. This is a contradiction. It follows that ψ is surjective. We now consider the composition of ι_H with the projection

$$p_1 : \widehat{\mathbb{Z}}^k \times \widehat{\Gamma}_1 \times \cdots \times \widehat{\Gamma}_r \longrightarrow \widehat{\mathbb{Z}}^k.$$

It has dense image. Since $\widehat{\mathbb{Z}}^k$ is torsion-free, $\mathrm{im}(p_1 \circ \iota_H)$ is isomorphic to $\mathbb{Z}^m$ for some $m \geq k$. In particular, we obtain an embedding

$$(5.3a) \qquad (p_1 \circ \iota_H, \psi) : H \hookrightarrow \mathbb{Z}^m \times \Gamma_1 \times \cdots \times \Gamma_r$$

as a subdirect product.

The proof is now completed using a simplified version of the argument for [FM22, Thm. 5.14]. Since the Γ_i have trivial centre, the centre of H is given by $Z(H) = \ker(\psi) = H \cap \mathbb{Z}^m$ where H is identified with its image under the embedding (5.3a). We examine the short exact sequence

$$(5.3b) \qquad 1 \longrightarrow Z(H) \longrightarrow H \longrightarrow \Gamma_1 \times \cdots \times \Gamma_r \longrightarrow 1,$$

where the surjection from H to $\Gamma_1 \times \cdots \times \Gamma_r$ is given by ψ. By right-exactness of taking profinite completions and 2-goodness of the group $\Gamma_1 \times \cdots \times \Gamma_r$, this yields the short exact sequence

$$(5.3c) \qquad 1 \longrightarrow \widehat{Z(H)} \longrightarrow \widehat{H} \longrightarrow \widehat{\Gamma}_1 \times \cdots \times \widehat{\Gamma}_r \longrightarrow 1.$$

Thus the image of $\widehat{Z(H)}$ in $\widehat{H}$ coincides with $\ker(\widehat{\psi}) \cong \widehat{\mathbb{Z}}^k$. It follows that $Z(H)$ is isomorphic to $\mathbb{Z}^k$.

Since $\widehat{H}$ is isomorphic to $\widehat{\mathbb{Z}}^k \times \widehat{\Gamma}_1 \times \cdots \times \widehat{\Gamma}_r$ in such a way that the projection $\widehat{H} \to \widehat{\Gamma}_1 \times \cdots \times \widehat{\Gamma}_r$ coincides with $\widehat{\psi}$ (up to an automorphism of the target), the sequence (5.3c) splits. Corollary 5.2 then implies that (5.3b) splits. In other words, we have proved that H is isomorphic to $\mathbb{Z}^k \times \Gamma_1 \times \cdots \times \Gamma_r$. $\qquad\square$

Finally, observe that Lemma 4.4 and finite presentability of Kähler groups imply the following Grothendieck rigidity result.

Corollary 5.4. *Suppose that G is a Kähler subgroup contained in the product of finitely many hyperbolic orbisurface groups with possibly a free abelian group of finite rank. If $H \in \mathcal{G}_{\mathcal{K}}(G)$ and if there is a morphism $\phi : H \to G$ that induces an isomorphism on profinite completions, then ϕ is an isomorphism.*

Using the vocabulary from Definition 1.3, Corollary 5.4 says that G is Grothendieck rigid amongst residually finite Kähler groups.

Proof. We consider the inclusion

$$G < P := \mathbb{Z}^k \times \Gamma_1 \times \cdots \times \Gamma_r,$$

where the Γ_i are hyperbolic orbisurface groups. According to Theorem 4.2 and Lemma 4.3, every finitely presented subgroup of P is separable in P. In particular every finitely presented subgroup of G is separable in P, hence separable in G. Lemma 4.4 now yields the desired conclusion. $\qquad\square$

At this point we have completely proved Theorem B. We summarize here where the various statements from that theorem have been proved.

Proof of Theorem B. The first two points are consequences of Theorem 5.3 (or of Corollary 4.7 for the first one), while the third follows from Theorem 4.12. The last claim in Theorem B follows from Corollary 5.4. $\qquad\square$

5.B. **The three factors case.** We use here the results from the previous sections to prove the following theorem, already stated in the introduction.

Theorem C. *Let Γ_1, Γ_2, Γ_3 be three hyperbolic orbisurface groups and*

$$G \leq \Gamma_1 \times \Gamma_2 \times \Gamma_3$$

be a Kähler group. Then G contains a finite index subgroup H which is profinitely rigid amongst residually finite Kähler groups.

Proof. Let G be as in the statement of the theorem. According to Corollary 1.3 from [LI24], we have the following dichotomy:

(1) either G has a finite index subgroup G_0 which is isomorphic to a direct product

$$\mathbb{Z}^{2k} \times \Sigma_{g_1} \times \cdots \times \Sigma_{g_s}$$

where $2k + s \leq 3$ and Σ_{g_i} is a surface group of genus $g_i \geq 2$.

(2) or G has a finite index subgroup G_0 which embeds as a normal full subdirect product of three hyperbolic surface groups $\Sigma_{g_1} \times \Sigma_{g_2} \times \Sigma_{g_3}$ with quotient isomorphic to $\mathbb{Z}^2$ and such that the quotient morphism $\phi : \Sigma_{g_1} \times \Sigma_{g_2} \times \Sigma_{g_3} \to \mathbb{Z}^2$ is non-trivial on factors.

In Case (1) the assertion is an immediate consequence of Theorem 5.3. In Case (2), the restriction of ϕ to factors is non-trivial and thus has finite index image $A_i := \phi(\Sigma_{g_i}) \leq \mathbb{Z}^2$. Let $A := \bigcap_{i=1}^3 A_i \cong \mathbb{Z}^2$. Then the subgroups $\Sigma_i' := \phi^{-1}(A) \cap \Sigma_i \leq \Sigma_i$ have finite index, the restriction $\phi' : \Sigma_1' \times \Sigma_2' \times \Sigma_3' \to A \cong \mathbb{Z}^2$ of ϕ to $\Sigma_1' \times \Sigma_2' \times \Sigma_3'$ is surjective on factors, and $G_0' := \ker(\phi') \leq G_0$ has finite index. The assertion then follows from Theorem 4.12. $\square$

6. Strong rigidity for direct products of Riemann surfaces

Theorem 5.3 says in particular that the profinite completion completely characterises direct products of hyperbolic surface groups (possibly with a free Abelian factor) among all residually finite Kähler groups. We prove here that for closed *aspherical* Kähler manifolds, more can be said. Namely, we establish the following theorem, which is more general than Theorem A.

Theorem 6.1. *Let X be a closed aspherical Kähler manifold with residually finite fundamental group. Let $S_1, \ldots, S_r$ be closed oriented surfaces of genus greater than 1 and let k be a natural integer.*

(1) *If the profinite completion of $\pi_1(X)$ is isomorphic to that of*

$$\pi_1(S_1) \times \cdots \times \pi_1(S_r) \times \mathbb{Z}^{2k},$$

then for each $j \in 1, \ldots, r$, there exists a closed Riemann surface F_j homeomorphic to S_j such that X is biholomorphic to a principal torus bundle over $F_1 \times \cdots \times F_r$ (with fiber of complex dimension k).

(2) *If moreover X is projective, it has a finite covering space X_0 which is biholomorphic to a direct product of Riemann surfaces with an Abelian variety.*

As we shall see, the proof relies on a classical argument going back to Siu's work [Siu80], as well as on some arguments taken from the work of Delzant and Py [DP19]. Note that when $k = 0$, the first item above says that X is biholomorphic to the direct product of the F_j, and we recover the statement of Theorem A.

Proof. Let X be as in the statement of Theorem 6.1. According to Theorem 5.3, there is an isomorphism

$$\psi : \pi_1(X) \to \pi_1(S_1) \times \cdots \times \pi_1(S_r) \times \mathbb{Z}^{2k}.$$

According to a result by Catanese (see Theorem 5.14 in [Cat08]), there exist closed Riemann surfaces $F_1, \ldots, F_r$ respectively homeomorphic to $S_1, \ldots, S_r$,

such that the surjections $\pi_1(X) \to \pi_1(S_i)$ defined by ψ are induced by some holomorphic fibrations $p_i : X \to F_i$ ($1 \leq i \leq r$). We now argue as in [DP19, §4.2]. Let $a : X \to A$ be the Albanese map of X and let $\alpha_i : F_i \to A_i$ be the Albanese map of F_i. Note that α_i is an embedding. By the universal property of α, there exists a holomorphic map $\varphi_i : A \to A_i$ making the following diagram commutative:

$$
\begin{array}{ccc}
X & \xrightarrow{\ a\ } & A \\
\downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \varphi_i} \\
F_i & \xrightarrow{\ \alpha_i\ } & A_i.
\end{array}
$$

We denote by $\varphi : A \to A_1 \times \cdots \times A_r$ the product of the maps φ_i and by $\alpha : F_1 \times \cdots \times F_r \to A_1 \times \cdots \times A_r$ the product of the maps α_i. We can and do assume that φ is a homomorphism. Consider the following submanifold of A:

$$Y := \{y \in A \mid \varphi(y) \in \mathrm{Im}(\alpha)\}.$$

The map a can be written as $a = i \circ \Phi$ where $\Phi : X \to Y$ is holomorphic and $i : Y \to A$ is the inclusion. One proves as in Lemma 35 from [DP19] that Φ is a homotopy equivalence. Finally, exactly as in [DP19], we invoke Lemma 6.2, stated below, to conclude that Φ is a biholomorphism. This proves the first item of the theorem, noting that the projection $Y \to \mathrm{Im}(\alpha)$ is a principal torus bundle. To prove the second item, we invoke (again as in [DP19]) Poincaré's reducibility theorem [Deb05, VI.8]. If X is projective, so is A. Poincaré's theorem implies that there is subtorus $B \subset A$ intersecting $\ker \varphi$ transversely. Hence the natural product map $\ker \varphi \times B \to A$ is a finite covering space and B is isogenous to the product of the A_i. Thus, after replacing B by a finite covering space we can assume that

$$B = B_1 \times \cdots \times B_r,$$

where B_i is a finite covering space of A_i. We thus have a natural covering map

$$\pi : \ker \varphi \times B_1 \times \cdots \times B_r \to A.$$

The preimage of Y under π is a finite covering space of Y (hence of X) which has the desired product structure. $\qquad\square$

A proof of the following classical lemma can be found in [Siu80] (see the proof of Theorem 8 there).

Lemma 6.2. *Let $f : X_1 \to X_2$ be a holomorphic map between two closed Kähler manifolds of the same dimension n. Assume that f has degree 1 and that the induced map $f_* : H_{2n-2}(X_1, \mathbb{R}) \to H_{2n-2}(X_2, \mathbb{R})$ is injective. Then f is a holomorphic diffeomorphism. This holds in particular if f is a homotopy equivalence.*

Remark 6.3. In (real) dimension greater or equal to 6, one could appeal to the results of Farrell–Jones [FJ93] to prove that a manifold X as in Theorem 6.1 must be homeomorphic to the direct product of the S_i's with a torus. However, our proof based on Kähler geometry yields a stronger result.

7. Profinite invariance of the BNS invariant

The goal of this section is to prove that the BNS invariant $\Sigma^1(G)$ of a Kähler group G is a profinite invariant, in a suitable sense, thereby proving Theorem E.

We first recall a few notions related to *Alexander polynomials* and present the $\mathsf{TAP}_1(R)$ property, for a ring R (Section 7.A). This property was introduced by Hughes and Kielak [HK22]. The definition will be recalled here (Definition 7.4), but a more detailed introduction to it can be found in [HK22]. We then establish in Section 7.B that Kähler groups have the $\mathsf{TAP}_1(R)$ property for every Noetherian unique factorization domain (UFD) R. The corresponding result for 3-manifold groups was proven in a series of papers by Friedl and Vidussi [FV08a, FV11b, FV13]. Finally, Theorem E is established in Section 7.C.

7.A. **Preliminaries and the TAP property.** The following definitions are taken from Friedl and Vidussi's survey [FV11a] but coaxed into the language of group homology. Throughout Section 7, R is a unique factorisation domain (UFD), which is moreover assumed to be Noetherian. For a finitely generated R-module M we can find a presentation

$$R^r \xrightarrow{\ P\ } R^s \longrightarrow M \longrightarrow 0,$$

where P is an $r \times s$ matrix with coefficients in R. This simply uses the fact that R is Noetherian.

Definition 7.1. The *first elementary ideal* $E(M)$ of M is the ideal of R generated by all $(s-1) \times (s-1)$ minors of the matrix P if $0 < s-1 \le r$, and is defined to be $\{0\}$ if $s-1 > r$ and R if $s-1 \le 0$.

Note that $E(M)$ is independent of the presentation. See [CF63] or Lemma 4.4 in [Tur01] for a proof of this fact. Now, since R is a UFD, there exists a unique smallest principal ideal of R which contains $E(M)$. A generator of this principal ideal is defined to be an *order* of M. It is well defined up to muliplication by a unit in R [Tur01, p. 18].

In what follows, if A is an arbitrary ring, we denote by $A[t^{\pm 1}]$ the ring of Laurent polynomials in the indeterminate t, with coefficients in A. It is a Noetherian UFD if A is so.

Let G be a finitely generated group and let $\alpha \colon G \twoheadrightarrow Q$ be an epimorphism onto a finite group. We consider the ring of Laurent polynomials $RQ[t^{\pm 1}]$ with coefficients in the group ring RQ. For each element $\varphi \in H^1(G; \mathbb{Z})$, we can define a G-action on $RQ[t^{\pm 1}]$ by the formula:

$$g \cdot x = t^{\varphi(g)} \alpha(g) x$$

for $g \in G, x \in RQ[t^{\pm 1}]$. The first twisted (homological) Alexander module of φ and α is defined to be $H_1(G; RQ[t^{\pm 1}])$. Observe that $H_1(G; RQ[t^{\pm 1}])$ has the structure of a left $R[t^{\pm 1}]$-module. It is a finitely generated $R[t^{\pm 1}]$-module since G is finitely generated.

Definition 7.2. The first twisted Alexander polynomial $\Delta_{1,R}^{\varphi;\alpha}(t)$ over R with respect to φ and α is defined to be the order of the twisted (homological) Alexander module of φ and α, treated as a left $R[t^{\pm 1}]$-module.

Remark 7.3. The twisted Alexander polynomial, or rather merely its vanishing or non-vanishing, can be defined for more general rings, see [HK22, §2.B]. However, we content ourselves here with the case where R is a Noetherian UFD.

In the following, we shall say that a nonzero character $\varphi : G \to \mathbb{Z}$ is fibred if its kernel is finitely generated (although the terminology algebraically fibred is more commonly used in the literature, we stick to the term fibred here). This is equivalent to the fact that φ is contained in the intersection

$$\Sigma^1(G) \cap -\Sigma^1(G).$$

See [BNS87] or [Py25, Ch. 11] for this result. Here, and as in the introduction, the BNS invariant $\Sigma^1(G)$ is viewed as a subset of $H^1(G, \mathbb{R}) - \{0\}$. Following [HK22], we say that φ is semi-fibred if it lies in $\Sigma^1(G) \cup -\Sigma^1(G)$. When $\Sigma^1(G) = -\Sigma^1(G)$, the two notions are equivalent. When φ is semi-fibred, the Alexander polynomial $\Delta_{1,R}^{\phi,\alpha}(t)$ is nonzero for any surjective morphism to a finite group $\alpha : G \to Q$; see [FV16, Lemma 4.1], or [HK22, Proposition 2.13]. Property $\mathsf{TAP}_1(R)$ deals with the converse of this implication:

Definition 7.4. We say that a finitely generated group G is in the class $\mathsf{TAP}_1(R)$ if for every nontrivial character $\varphi \in H^1(G; \mathbb{Z})$ the following property holds: φ is semi-fibred if and only if for every surjective homomorphism $\alpha : G \to Q$ to a finite group Q its twisted Alexander polynomial $\Delta_{1,R}^{\varphi,\alpha}(t)$ does not vanish.

Applications of this notion to profinite rigidity problems are given in [HK22]. Here we use it as a key step to establish Theorem E.

7.B. **Large characters and the TAP property.** We continue to consider a finitely generated group G. We shall say that a character $\varphi : G \twoheadrightarrow \mathbb{Z}$ is *large* if there exists a finite-index subgroup H of G, and a free noncyclic quotient $\psi : H \twoheadrightarrow F$, such that $\varphi|_H$ factors through ψ. Note that if φ is large, the group H can be taken to be characteristic.

As in Section 7.A, in what follows R is a Noetherian UFD.

Proposition 7.5. *Let G be a finitely generated group and let $\varphi \in H^1(G; \mathbb{Z})$ be nontrivial and primitive. If φ is large, then there exists a finite quotient $\alpha : G \twoheadrightarrow Q$ such that $\Delta_{1,R}^{\varphi,\alpha}(t) = 0$.*

Proof. Since φ is large there exists a finite index characteristic subgroup H of G such that $\varphi|_H$ factors through a surjection $H \twoheadrightarrow F_k$. Let n be the divisibility of the character $\varphi|_H$. Applying a suitable automorphism of F_k and then projecting onto a smaller free quotient, we can assume that $k = 2$ and that $\varphi|_H$ equals n times the composition of the surjection $H \twoheadrightarrow F_2 = F(a,b)$ with the morphism $F(a,b) \to \mathbb{Z}$ sending a to 1 and b to 0. Let $Q = G/H$ and let α denote the quotient map $G \to Q$. By [HK22, Lemma 2.11] or [FV08b, Lemma 3.3] we have

(7.5a) $$\Delta_{1,R}^{\varphi|_H,\mathbf{id}}(t) = \Delta_{1,R}^{\varphi,\alpha}(t).$$

Thus, it suffices to prove that $H_1(H; R[t^{\pm 1}])$ is not a torsion $R[t^{\pm 1}]$-module. By [HK22, Lemma 2.9], this implies that the element in (7.5a) is zero and concludes the proof of the proposition.

By Shapiro's Lemma we have an isomorphism

$$H_1(H; R[t^{\pm 1}]) \cong H_1(\ker \varphi|_H; R)$$

of $R[t^{\pm 1}]$-modules. There is a commutative diagram

$$
\begin{array}{ccc}
H & \twoheadrightarrow & F_2 \\
\uparrow & & \uparrow \\
\ker \varphi|_H & \twoheadrightarrow & F_\infty,
\end{array}
$$

and $H_1(\ker \varphi|_H; R)$ maps onto the module $H_1(F_\infty, R)$. Note that the conjugates

$$a^{b^j} = b^j a b^{-j} \quad (j \in \mathbb{Z})$$

form a basis of F_∞. This yields an isomorphism

$$H_1(F_\infty, R) = \oplus_{\mathbb{Z}} R,$$

the $R[t^{\pm 1}]$-module structure is obtained by letting t act by permutation of the factors. Hence $H_1(F_\infty, R)$ is not a torsion module so $H_1(\ker \varphi|_H; R)$ is not a torsion module. This completes the proof. $\square$

Corollary 7.6. *Let G be a finitely generated group. If every nontrivial primitive non-semifibred character of G is large, then G is in $\mathsf{TAP}_1(R)$.*

Proof. This is an immediate consequence of Proposition 7.5 and [HK22, Proposition 2.13], which says that the first twisted Alexander polynomials of semi-fibred characters do not vanish. $\square$

To prove that Kähler groups are $\mathsf{TAP}_1(R)$, we will use a characterisation of large characters for Kähler groups in terms of factorisations through hyperbolic orbisurface groups. The following result, which is well-known, follows from the work of Napier–Ramachandran [NR01] or from Delzant's description of the BNS-invariant for Kähler groups [Del10], that was recalled in the introduction.

Theorem 7.7. *Let G be a Kähler group and let $\chi \in H^1(G; \mathbb{Z})$ be an integral character. The following are equivalent*

   (1) χ is large;
   (2) χ is not fibred;
   (3) χ is not semi-fibred;
   (4) χ factors through one of the components of the universal homomorphism of G.

Proof. We have seen in the introduction that $\Sigma^1(G) = -\Sigma^1(G)$. This implies that the second and third points are equivalent. The first point implies the second since infinite index normal subgroups of nonabelian finitely generated free groups are not finitely generated. The third point implies the fourth by [NR01] or [Del10, Theorem 1.1]. Finally, it is a classical fact that any homomorphism from a hyperbolic surface group to $\mathbb{Z}$ factors through a surjection to a nonabelian free group. Since hyperbolic orbisurface groups are virtually hyperbolic surface groups, this proves that the fourth point implies the first. $\square$

Combining Corollary 7.6 and Theorem 7.7, we obtain:

Proposition 7.8. *If G is a Kähler group, then G is in $\mathsf{TAP}_1(R)$.*

7.C. **Proving Theorem E.** We start by recalling the statement we aim at:

Theorem E. *Let G and H be Kähler groups such that $\widehat{G} \cong \widehat{H}$. Then there is a linear isomorphism $L : H^1(G, \mathbb{Q}) \to H^1(H, \mathbb{Q})$ such that*

$$L_{\mathbb{R}}(\Sigma^1(G)) = \Sigma^1(H).$$

In the statement above, $L_{\mathbb{R}}$ stands for the extension of L to a map $H^1(G, \mathbb{R}) \to H^1(H, \mathbb{R})$.

Before proving Theorem E, we need to recall several notions introduced by Liu [Liu23]. Let A and B be a pair of finitely generated free abelian groups. Let $\Phi \colon \widehat{A} \to \widehat{B}$ be a continuous homomorphism between their profinite completions. Any basis of A (as a $\mathbb{Z}$-module) can be considered as a basis of $\widehat{A}$ as a $\widehat{\mathbb{Z}}$-module (and similarly for B). Hence, if we fix two bases $v = (v_j)_{1 \leq j \leq a}$ and $w = (w_t)_{1 \leq t \leq b}$ of A and B respectively, the morphism Φ can be represented by a matrix $M(\Phi, v, w)$ of size $b \times a$, with coefficients in $\widehat{\mathbb{Z}}$. The following definition is due to Liu [Liu23, §3].

Definition 7.9. The *matrix coefficient module* $\mathrm{MC}(\Phi)$ of Φ is the $\mathbb{Z}$-submodule of $\widehat{\mathbb{Z}}$ spanned by the coefficients of $M(\Phi, v, w)$.

One readily checks that this module does not depend on the choices of the bases v and w. Alternatively, one can say that $\mathrm{MC}(\Phi)$ is the smallest $\mathbb{Z}$-submodule $L \subset \widehat{\mathbb{Z}}$ such that $\Phi(A) \subset B \otimes_{\mathbb{Z}} L$. By definition, $\mathrm{MC}(\Phi)$ is finitely generated. It is moreover free, since $\widehat{\mathbb{Z}}$ is torsion-free.

We now consider the homomorphism

$$\mathrm{MC}(\Phi) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

obtained by composing the inclusion of $\mathrm{MC}(\Phi)$ in $\widehat{\mathbb{Z}}$ with the canonical morphism $\widehat{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$. Since $\mathrm{MC}(\Phi)$ is free, we can choose a lift $\varepsilon : \mathrm{MC}(\Phi) \to \mathbb{Z}$ of this morphism. Once ε is fixed, we define the *ε-specialization of Φ* as the homomorphism $\Phi_{\varepsilon} : A \to B$ obtained by composing the two morphisms in the following diagram:

$$A \xrightarrow{\ \Phi\ } B \otimes_{\mathbb{Z}} \mathrm{MC}(\Phi) \xrightarrow{\ 1 \otimes \varepsilon\ } B \otimes_{\mathbb{Z}} \mathbb{Z} = B.$$

We also consider the dual $\Phi^{\varepsilon} : H^1(B, \mathbb{Z}) \to H^1(A, \mathbb{Z})$ of Φ_{ε}, called the *dual ε-specialization of Φ*. Both Φ_{ε} and Φ^{ε} depend on ε. If Φ is an isomorphism, then Φ_{ε} and Φ^{ε} have finite cokernel (see Lemma 4.10 in [HK22]).

Proof of Theorem E. Let G and H be Kähler groups as in the statement of the theorem and let $A := H_1(G, \mathbb{Z})/\mathrm{Torsion}$ and $B := H_1(H, \mathbb{Z})/\mathrm{Torsion}$. Fix an isomorphism $\Psi \colon \widehat{G} \to \widehat{H}$ between the profinite completions of G and H. The map Ψ induces an isomorphism between the groups

$$\widehat{H_1(G, \mathbb{Z})} \text{ and } \widehat{H_1(H, \mathbb{Z})}$$

which in turns induces an isomorphism

$$\Phi : \widehat{A} \to \widehat{B}.$$

We can apply the previous discussion to Φ and its coefficient module $\mathrm{MC}(\Phi) \subset \widehat{\mathbb{Z}}$. We fix a morphism $\varepsilon : \mathrm{MC}(\Phi) \to \mathbb{Z}$ as above and consider the dual ε-specialization $\Phi^\varepsilon : H^1(H, \mathbb{Z}) \to H^1(G, \mathbb{Z})$. Since Φ^ε has finite cokernel, we obtain an isomorphism $\Phi^\varepsilon_{\mathbb{R}} : H^1(H; \mathbb{R}) \to H^1(G; \mathbb{R})$, induced by Φ^ε. Before continuing the proof we recall that if $(\varrho_1, \ldots, \varrho_r) : G \to \Gamma_1 \times \cdots \times \Gamma_r$ is the universal homomorphism of G, and if $V_i = \mathrm{Im}(\varrho_i^* : H^1(\Gamma_i, \mathbb{R}) \to H^1(G, \mathbb{R}))$ $(1 \le i \le r)$, then the BNS invariant of G is the complement of the union

$$(7.9a) \qquad \bigcup_{i=1}^r V_i \subset H^1(G, \mathbb{R}).$$

Similarly the BNS invariant of H is the complement of a finite union

$$(7.9b) \qquad \bigcup_{j=1}^r W_j \subset H^1(H, \mathbb{R}),$$

where the W_j are obtained using the universal homomorphism of H. The number of subspaces in (7.9a) and (7.9b) are equal thanks to Theorem D. We note that the V_i and the W_j are defined over $\mathbb{Q}$. We shall use the following lemma:

Lemma 7.10. *Let $\varphi \in H^1(H, \mathbb{Z})$. If $\Phi^\varepsilon(\varphi)$ is fibred, then φ is fibred.*

Proof. This is a direct consequence of Theorem 4.12 in [HK22], applied with $n = 1$. $\qquad \square$

We now conclude our proof of Theorem E using the lemma. We want to prove that

$$(7.10a) \qquad \Phi^\varepsilon_{\mathbb{R}}(\bigcup_{i=1}^r W_i) = \bigcup_{j=1}^r V_j.$$

We will first prove that this equality holds for rational points. So let $a \in \bigcup_{i=1}^r W_i$ be a rational point. According to Theorem 7.7 and Lemma 7.10, $\Phi^\varepsilon(a)$ belongs to the union of the V_j. Since the W_j are rationally defined, this implies that

$$(7.10b) \qquad \Phi^\varepsilon_{\mathbb{R}}(\bigcup_{j=1}^r W_j) \subset \bigcup_{i=1}^r V_i.$$

We note that each subspace V_i (or W_j) has (positive) even real dimension, hence $V_i - \{0\}$ is connected. Since the V_i have pairwise trivial intersection, the connected components of

$$\left(\bigcup_{i=1}^r V_i \right) - \{0\}$$

are the $V_i - \{0\}$. Since $\Phi^\varepsilon_{\mathbb{R}}(W_j - \{0\})$ is connected it must be contained in one of the $V_i - \{0\}$. Note that Theorem D implies that the set of dimensions of the V_i coincides with the set of dimension of the W_j. Up to permutation, we assume that

$$\dim W_1 = \max_{1 \le j \le r} \dim W_j = \max_{1 \le j \le r} \dim V_j.$$

If i is such that $\Phi_{\mathbb{R}}^{\varepsilon}(W_1 - \{0\}) \subset V_i$, we then necessarily have $\Phi_{\mathbb{R}}^{\varepsilon}(W_1) = V_i$. Up to reordering the V_t we assume $i = 1$. We thus have $\Phi_{\mathbb{R}}^{\varepsilon}(W_1) = V_1$ and

$$\Phi_{\mathbb{R}}^{\varepsilon}(\bigcup_{j=2}^{r} W_j) \subset \bigcup_{i=2}^{r} V_i.$$

By now choosing a subspace W_{j_0} of maximal dimension among $W_2, \dots, W_r$, we prove as above that $\Phi_{\mathbb{R}}^{\varepsilon}(W_{j_0}) = V_i$ for some index $i \in \{2, \dots, r\}$. By repeating this argument, we see that the inclusion (7.10b) is promoted to equality (7.10a), as desired. $\qquad\square$

8. Kodaira fibrations

One class of Kähler groups not covered by the previous results in this paper are fundamental groups of Kodaira fibrations. Recall that a *Kodaira fibration* X is a complex surface endowed with a holomorphic submersion onto a Riemann surface, which has connected fibers and is not isotrivial. It is known that the genus of both the base and the fiber must then be greater than 1. See e.g. [Cat17] for an introduction to these complex surfaces. The fundamental group of a Kodaira fibration X is called a *Kodaira fibration group*. If b and f are respectively the genus of the base and fiber of the fibration, we say that X and $\pi_1(X)$ have type (f, b). Note that X may fibre over a surface in more than one way, and so it may have multiple types.

The goal of this section is to raise the following structural question about normal subgroups of Kähler groups and explain its relation to the profinite rigidity problem for fundamental groups of Kodaira fibrations.

Question 8.1. *Let G be a Kähler group and $N \trianglelefteq G$ be a finitely generated normal subgroup. If $b_1^{(2)}(N) > 0$, is N virtually isomorphic to a surface group?*

One could also consider the more restricted problem where N is the kernel of the homomorphism to a hyperbolic orbisurface group induced by a fibration. A related question appears in [Nic22, p. 3000]: it is unknown if a finitely generated nonabelian free group can appear as a normal subgroup of a Kähler group. The final result of this paper proves that a positive answer to Question 8.1 yields a weak form of profinite rigidity for Kodaira fibration groups.

Proposition 8.2. *Suppose that Question 8.1 has a positive answer. Let G be a Kodaira fibration group of type (f, b). The following properties hold.*

(1) If $H \in \mathcal{G}_{\mathcal{K}}(G)$, then H fits into an extension

$$1 \longrightarrow \pi_1(\Sigma_f) \longrightarrow H \longrightarrow \pi_1(\Sigma_b) \longrightarrow 1,$$

where Σ_f and Σ_b are closed oriented surfaces of genus f and b respectively.

(2) If X is a closed aspherical Kähler manifold such that $\pi_1(X) \in \mathcal{G}_{\mathcal{K}}(G)$, then X is a Kodaira fibration.

Proof. First, we observe that the group G is good in the sense of Serre; this follows from Lemma 7.2 and Theorem 7.3 in [Rei15]. Then, by [Rei15, Prop.

7.10], there is a short exact sequence:

$$1 \longrightarrow \widehat{\pi_1(\Sigma_f)} \longrightarrow \widehat{G} \longrightarrow \widehat{\pi_1 \Sigma_b)} \longrightarrow 1,$$

where Σ_f and Σ_b are closed oriented surfaces of genus f and b respectively. Now suppose $H \in \mathcal{G}_{\mathcal{K}}(G)$. Since $\pi_1(\Sigma_b)$ appears as a factor for the universal homomorphism of G, Theorem D implies that there is a homomorphism $H \twoheadrightarrow \pi_1(\Sigma_b)$ with finitely generated kernel N. The closure of N in $\widehat{H}$ is moreover isomorphic to $\widehat{\pi_1(\Sigma_f)}$. Theorem 3.1 implies that $b_1^{(2)}(N) > 0$. If Question 8.1 has a positive answer, then N is commensurable with a surface group, hence itself isomorphic to a surface group by the Nielsen realization problem. Note that N is torsion-free, as a subgroup of the torsion-free group $\widehat{\pi_1(\Sigma_f)}$ (see [Rei15, Cor. 7.6] for the latter point). It follows that H is an extension

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow H \longrightarrow \pi_1(\Sigma_b) \longrightarrow 1$$

where Σ_g is a closed oriented surface of genus g. Lemma 7.2 from [Rei15] implies that H is good, which in turn yields that the profinite completion of

$$N \cong \pi_1(\Sigma_g)$$

is isomorphic to its closure in $\widehat{H}$, i.e. to $\widehat{\pi_1(\Sigma_f)}$. This implies $g = f$ and proves the first point of the proposition. To prove the second point of the proposition, we observe that if X is a closed aspherical Kähler manifold such that $\pi_1(X) \in \mathcal{G}_{\mathcal{K}}(G)$, then $\pi_1(X)$ is a surface-by-surface group by the first point, hence X has complex dimension 2. Applying [Kot99, Prop. 1] or [Kap98] yields that X is a Kodaira fibration. $\qquad \square$

Remark 8.3. Proposition 8.2 suggests the following problem: is it true that a Kähler group that is also a surface-by-surface group can be realized as a Kodaira fibration group? Although this might be true, we have not tried to establish this fact. See also [Vid19] for a related discussion.

Despite the conditional result above, it is not clear whether or not one should expect fundamental groups of Kodaira fibrations to be profinitely rigid amongst residually finite Kähler groups. A result of González-Diez [GD20] shows that for each nontrivial element $\sigma \in \text{Aut}(\mathbb{C})$ not conjugate to complex conjugation, there exists a Kodaira fibration X with a Galois conjugate X^{σ} such that their universal covers are not isomorphic. It would be interesting to understand the fundamental groups of these examples in terms of whether or not Kodaira fibration groups can have arbitrarily large $\mathcal{K}$-profinite genus.

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