

QUASI-ISOMETRY INVARIANCE OF RELATIVE FILLING FUNCTIONS

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ABSTRACT. For a finitely generated group G and collection of subgroups \mathcal{P} we prove that the relative Dehn function of a pair (G, \mathcal{P}) is invariant under quasi-isometry of pairs. Along the way we show quasi-isometries of pairs preserve almost malnormality of the collection and fineness of the associated coned off Cayley graphs. We also prove that for a cocompact simply connected combinatorial G -2-complex X with finite edge stabilisers, the combinatorial Dehn function is well-defined if and only if the 1-skeleton of X is fine.

1. INTRODUCTION

The main objects of study in this article are pairs (G, \mathcal{P}) where G is a **finitely generated group** with a chosen word metric dist_G , and \mathcal{P} is a **finite collection of subgroups**, *note that these assumptions will stand throughout the introduction.*

Let hdist_G denote the Hausdorff distance between subsets of G , and let G/\mathcal{P} denote the collection of left cosets gP for $g \in G$ and $P \in \mathcal{P}$.

For constants $L \geq 1$, $C \geq 0$ and $M \geq 0$, an (L, C, M) -*quasi-isometry of pairs* $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is an (L, C) -quasi-isometry $q: G \rightarrow H$ such that the relation

$$\{(A, B) \in G/\mathcal{P} \times H/\mathcal{Q}: \text{hdist}_H(q(A), B) < M\}$$

satisfies that the projections to G/\mathcal{P} and H/\mathcal{Q} are surjective.

This article is part of the program of investigating which properties of pairs (G, \mathcal{P}) are invariant under quasi-isometry of pairs. There are recent results in this direction. For example, it is a consequence of the quasi-isometric rigidity of relative hyperbolicity [BDM09], that if (G, \mathcal{P}) is a relatively hyperbolic pair, \mathcal{P} is a collection of non-relatively hyperbolic groups, and (G, \mathcal{P}) and (H, \mathcal{Q}) are quasi-isometric pairs, then H is hyperbolic relative to \mathcal{Q} . Under natural assumptions, quasi-isometries of pairs between relatively hyperbolic pairs induce canonical homomorphisms between their Bowditch boundaries [HH20] and canonical isomorphisms of JSJ trees [HH19]. Outside the framework of relatively hyperbolic groups, it is known that quasi-isometries of pairs preserve the number of Bowditch's filtered ends [MS21].

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For a pair (G, \mathcal{P}) , Osin introduced the notions of *finite relative presentation* and *relative Dehn function* as natural generalizations of their standard counterparts for finitely generated groups, see [Osi06]. These notions characterise relatively hyperbolic pairs (G, \mathcal{P}) as the ones which are relatively finitely presented and have relative Dehn function bounded from above by a linear function. By quasi-isometric rigidity of relative hyperbolicity, among relatively finitely presented pairs, quasi-isometries of pairs preserve having linear relative Dehn function.

The main result of this article confirms the natural expectation that among relatively finitely presented pairs, quasi-isometric pairs have equivalent relative Dehn functions. This is not an elementary statement, as we describe below.

Convention 1.1 ($\Delta_{G, \mathcal{P}}$ is well defined). By $\Delta_{G, \mathcal{P}}$ is well defined we mean that G is finitely presented relative to \mathcal{P} and $\Delta_{G, \mathcal{P}}$ takes only finite values with respect to a finite relative presentation of G and \mathcal{P} . From here on, when we refer to a relative Dehn function, we always assume that it has been defined using a finite relative presentation.

Let \mathcal{P} be a collection of subgroups of group G . A *refinement* \mathcal{P}^* of \mathcal{P} is a set of representatives of conjugacy classes of the collection of subgroups $\{\text{Comm}_G(gPg^{-1}) : P \in \mathcal{P} \text{ and } g \in G\}$ where $\text{Comm}_G(P)$ denotes the commensurator of the subgroup P in G .

Theorem A. *Let $(G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a quasi-isometry of pairs and let \mathcal{P}^* be a refinement of \mathcal{P} . If the relative Dehn function $\Delta_{H, \mathcal{Q}}$ is well defined, then $\Delta_{G, \mathcal{P}^*}$ is well defined and $\Delta_{G, \mathcal{P}^*} \asymp \Delta_{H, \mathcal{Q}}$.*

A phenomenon that occurs for pairs (G, \mathcal{P}) is that being relatively finitely presented does not imply that the relative Dehn function is well-defined. This is in sharp contrast with the standard framework where a finitely presented group always has a well-defined Dehn function. The proof of Theorem A provides an insight into this phenomenon via the following results on which our argument relies on.

In the framework of relatively hyperbolic groups, Bowditch introduced the notion of fine graph [Bow12]. A simplicial graph Γ is *fine* if for every $n \geq 0$ and every edge e in Γ there are finitely many circuits of length less than or equal to n which contain e . This is weaker than the graph being locally finite. The relationship between this notion and isoperimetric functions was made explicit by Groves and Manning [GM08, Proposition 2.50, Question 2.51]. The following result can be interpreted as a homotopical version of [Mar16, Theorem 1.3] where an analogous statement is proved for homological Dehn functions.

Theorem B (Theorem 2.1). *Let X be a cocompact simply connected combinatorial G -2-complex with finite edge stabilisers. The combinatorial Dehn function Δ_X of X takes only finite values if and only if the 1-skeleton of X is a fine graph.*

It is obvious that being fine is not a property preserved by quasi-isometries in the class of graphs. For a pair (G, \mathcal{P}) , together with a finite generating set S of G , one can assign a connected and cocompact G -graph known as the coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{P}, S)$; a notion introduced by Farb [Far98], see Definition 4.2. It is an observation that the quasi-isometry type of $\hat{\Gamma}(G, \mathcal{P}, S)$ is independent of the finite generating set S ; *throughout the introduction* $\hat{\Gamma}(G, \mathcal{P})$ denotes the coned-off Cayley graph with respect to some finite generating set of G . In this framework, under some assumptions, we are able to prove that fineness is preserved under quasi-isometry of pairs in the class of coned-off Cayley graphs. A collection of subgroups \mathcal{P} of a group G is *reduced* if for any $P, Q \in \mathcal{P}$ and $g \in G$, then P and gQg^{-1} being commensurable subgroups implies $P = Q$ and $g \in P$.

Theorem C (Theorem 5.15). *Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a quasi-isometry of pairs. Suppose \mathcal{P} and \mathcal{Q} are reduced. Then there is an induced quasi-isometry of graphs $\hat{q}: \hat{\Gamma}(G, \mathcal{P}) \rightarrow \hat{\Gamma}(H, \mathcal{Q})$, and if $\hat{\Gamma}(H, \mathcal{Q})$ is a fine graph then $\hat{\Gamma}(G, \mathcal{P})$ is a fine graph.*

The condition that the coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{P})$ is fine forces the collection \mathcal{P} to be almost malnormal (see Definition 6.4). It is an observation that any almost malnormal collection of infinite subgroups is reduced. We prove that the property of being almost malnormal is preserved under quasi-isometry of pairs up to taking a refinement.

Theorem D (Theorem 6.11). *Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a quasi-isometry of pairs. If \mathcal{Q} is an almost malnormal collection of infinite subgroups, then any refinement \mathcal{P}^* of \mathcal{P} is almost malnormal and $q: (G, \mathcal{P}^*) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs.*

The previous results can be linked to Osin's definition of relative Dehn function $\Delta_{G, \mathcal{P}}$ of a relatively finitely presented pair (G, \mathcal{P}) via the following result. A connected graph Γ is called *fillable* if, when considering Γ with the length metric obtained by regarding each edge as an segment of length one, there is an integer k such that the coarse isoperimetric function f_k^Γ takes only finite values, see Section 3 for definitions.

Theorem E (See Theorem 4.16). *If (G, \mathcal{P}) is a relatively finitely presented pair, then*

(1) *$\hat{\Gamma}(G, \mathcal{P})$ is fillable.*

(2) *The relative Dehn function $\Delta_{G, \mathcal{P}}$ is well defined if and only if $\hat{\Gamma}(G, \mathcal{P})$ is fine graph.*

Conversely, if $\hat{\Gamma}(G, \mathcal{P})$ is fine and fillable, then (G, \mathcal{P}) is a relatively finitely presented pair and hence $\Delta_{G, \mathcal{P}}$ is well defined.

The following result is a re-statement of a result of Osin [Osi06, Theorem 2.53], see Proposition 4.8. This statement allow us translate his definition of relative Dehn function to the realm of coarse isoperimetric functions of coned-off Cayley graphs.

Theorem F (Osin). *Let G be a group and let \mathcal{P} be a collection of subgroups. Suppose that $\Delta_{G, \mathcal{P}}$ is well defined. Then $\Delta_{G, \mathcal{P}}$ is equivalent to the coarse isoperimetric function $f_N^{\hat{\Gamma}(G, \mathcal{P})}$ of $\hat{\Gamma}(G, \mathcal{P})$ for all sufficiently large integers N .*

Let us describe the argument proving Theorem A using the results that have been stated.

Proof of Theorem A. Let us first observe that we can assume that the collections \mathcal{P} and \mathcal{Q} contain only infinite subgroups. First note that if \mathcal{P}_∞ and \mathcal{Q}_∞ are the collections obtained by removing finite subgroups from \mathcal{P} and \mathcal{Q} respectively, then $q: (G, \mathcal{P}_\infty) \rightarrow (H, \mathcal{Q}_\infty)$ is a quasi-isometry of pairs as well. Moreover, for an arbitrary pair (K, \mathcal{L}) , adding or removing a finite subgroup of K to \mathcal{L} preserves having well-defined relative Dehn function, and if the functions are well-defined they are equivalent, see for example [Osi06, Theorem 2.40].

Assume that \mathcal{P} and \mathcal{Q} consist only of infinite subgroups. Since (H, \mathcal{Q}) is relatively finitely presented and $\Delta_{H, \mathcal{Q}}$ is well-defined, Theorem E implies that $\hat{\Gamma}(H, \mathcal{Q})$ is fillable and fine. Since $\hat{\Gamma}(H, \mathcal{Q})$ is a fine graph, it follows that \mathcal{Q} is an almost malnormal collection. Then Theorem D implies that \mathcal{P}^* is an almost malnormal collection. Hence, both \mathcal{Q} and \mathcal{P}^* are reduced collections and $q: (G, \mathcal{P}^*) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs. Now we can invoke Theorem C to obtain a quasi-isometry $\hat{q}: \hat{\Gamma}(G, \mathcal{P}^*) \rightarrow \hat{\Gamma}(H, \mathcal{Q})$ and also obtain that $\hat{\Gamma}(G, \mathcal{P}^*)$ is fine. It is a standard result in the literature that being fillable is a property preserved by quasi-isometry in the class of connected graphs, and any two quasi-isometric graphs have equivalent coarse isoperimetric functions (see for instance [BH99, Proposition III.H.2.2]). The quasi-isometry \hat{q} implies that $\hat{\Gamma}(G, \mathcal{P}^*)$ is fillable and both $\hat{\Gamma}(G, \mathcal{P}^*)$ and $\hat{\Gamma}(H, \mathcal{Q})$ have equivalent coarse isoperimetric inequalities. Then Theorem E implies that (G, \mathcal{P}) is relatively finitely presented and $\Delta_{G, \mathcal{P}}$ is well-defined. The proof concludes by invoking Theorem F. \square

In the class of finitely generated groups, being finitely presented is a quasi-isometry invariant. We do not know the answer to the following general question:

Question 1.2. Suppose that $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs and (H, \mathcal{Q}) is relatively finitely presented. Is (G, \mathcal{P}) relatively finitely presented?

There is a rich class of pairs (G, \mathcal{P}) with well defined relative Dehn function. Hyperbolically embedded subgroups were introduced in [DGO17] by Dahmani, Guirardel and Osin. Given a group G , $X \subset G$ and $H \leq G$, let $H \hookrightarrow_h (G, X)$ denote that H is a hyperbolically embedded subgroup of G with respect to X .

Theorem G (Theorem 7.2). *Let G be a finitely presented group and $H \leq G$ be a subgroup. If $H \hookrightarrow_h G$ then the relative Dehn function $\Delta_{G, H}$ is well-defined.*

In the context of Theorem G, the relative Dehn function $\Delta_{G, \mathcal{P}}$ is bounded from above by a linear function if and only if G is hyperbolic relative to H , see [Osi06]. It is well known that the class of pairs (G, H) such that $H \hookrightarrow_h G$ properly extends relative hyperbolicity, for examples see [DGO17].

Question 1.3. Is there a pair (G, \mathcal{P}) such that $\Delta_{G, \mathcal{P}}$ is well-defined, but \mathcal{P} is not hyperbolically embedded in G ?

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2. COMBINATORIAL DEHN FUNCTIONS AND FINE GRAPHS

The goal of this section is to prove Theorem B. We use the notion of disk diagram in a combinatorial complex, for definitions see for example [MW02]. We begin by recalling the definition of a combinatorial Dehn function, then we prove each direction of Theorem B individually as Lemma 2.3 and Lemma 2.4. Note that Lemma 2.4 does not require the hypothesis of finite edge stabilisers.

Suppose X is a combinatorial 2-complex and let $c : S^1 \rightarrow X$ be a circuit in $X^{(1)}$ that is null-homotopic in X . Then there is D a disk diagram $i : D^2 \rightarrow X$ *spanning* c , that is, i is a combinatorial map and $i(\partial D^2) = c$. Let $\text{Area}(D)$ denote the number of faces of D and define

$$\delta_X(c) := \min\{\text{Area}(D) : D \text{ is a disk spanning } c\},$$

the *combinatorial Dehn function* Δ_X of X is defined to be

$$\Delta_X(n) := \max\{\delta_X(c) : c \text{ is a circuit in } X^{(1)}, \text{ null-homotopic in } X, \text{ with } |c| \leq n\}.$$

Unless otherwise stated all graphs in this article are assumed to be simplicial. We recall the following definition due to Bowditch [Bow12, Proposition 2.1]. A graph Γ is *fine* if for every $n \geq 0$ and every edge e in Γ there are finitely many circuits of length less than or equal to n which contains e .

Theorem 2.1 (Theorem B). *Let X be a cocompact simply connected combinatorial G -2-complex with finite edge stabilisers. The combinatorial Dehn function Δ_X of X takes only finite values if and only if the 1-skeleton of X is a fine graph.*

The next three lemmas prove the theorem. The method is essentially a van Kampen diagram approach to the proof of [Mar16, Theorem 1.3]. The first lemma is a triviality.

Lemma 2.2. *Let X be a cocompact simply connected combinatorial G -2-complex with finite edge stabilisers, then each edge is contained in finitely many 2-cells.*

The next lemma proves the “only if” direction of Theorem 2.1.

Lemma 2.3. *Let X be a cocompact simply connected combinatorial G -2-complex with finite edge stabilisers. If the combinatorial Dehn function Δ_X of X is well defined then $X^{(1)}$ is a fine graph.*

Proof. Let D be a cellular 2-disc. We say D is *golden* if D has an enumeration of its 2-cells f_1, \dots, f_k with the property that ∂f_{i+1} contains a 1-cell of the subcomplex induced by $f_1 \cup \dots \cup f_i$, and there is a cellular map $D \rightarrow X$.

Let R be a 2-cell and $f_1 : R \rightarrow X$, then a simple counting argument yields there are only finitely many golden disks with at most $n \geq 0$ faces making the following diagram commute:

$$\begin{array}{ccc} R & \xrightarrow{f_1} & X \\ \downarrow & \nearrow & \\ D & & \end{array}$$

Observe that by taking the minimal area filling for a circuit c of length n in X gives rise to a golden disk D with at most $\Delta_X(n)$ many 2-cells. Now, there are only finitely many 2-cells containing a given edge e , so by the previous paragraph there are only finitely many golden disks D containing e with at most $\Delta_X(n)$ many 2-cells. In particular, for each $n \geq 0$ there are only finitely many circuits in X of length less than or equal to n containing e . It follows that $X^{(1)}$ is a fine graph. \square

The next lemma proves the “if” direction of Theorem 2.1. Note that we can drop the hypothesis of finite edge stabilisers.

Lemma 2.4. *Let X be a cocompact simply connected combinatorial G -2-complex. If $X^{(1)}$ is a fine graph then the combinatorial Dehn function Δ_X of X is well defined.*

Proof. Let Y_n denote the set of circuits of length less than or equal to n in X .

Claim: Y_n is a G -set with finitely many orbits.

Let $\{e_1, \dots, e_r\}$ be edges representing the orbits of the G -action on $X^{(1)}$. Every circuit in X of length less than or equal to n can be translated to contain some e_i , the claim now follows from fineness of $X^{(1)}$. \blacklozenge

Let A_n be an upper bound for the area of a circuit of length less than or equal to n in X , this is well defined by the previous claim. Let γ be a closed path without backtracks in X , then γ can be expressed as a concatenation of closed paths $\gamma_1 \dots \gamma_k$, such that $1 \leq k \leq \text{Len}(\gamma)$, for $i = 1, \dots, k$ we have $\text{Len}(\gamma_i) \leq \text{Len}(\gamma)$ and each γ_i is a circuit. Now, filling each γ_i we have

$$\text{Area}(\gamma) \leq \sum_{i=1}^k \text{Area}(\gamma_i) \leq \sum_{i=1}^k A_{\text{Len}(\gamma_i)} \leq k A_{\text{Len}(\gamma)} \leq \text{Len}(\gamma) A_{\text{Len}(\gamma)}.$$

This yields a finite upper bound for $\Delta_X(\ell)$ and so we conclude that Δ_X is well defined. \square

3. COARSE ISOPERIMETRIC FUNCTIONS

To prove quasi-isometry invariance we will use the less general version of ϵ -fillings for graphs and 2-complexes defined in [Osi06]. The original definition, set up for essentially arbitrary metric spaces, can be found in [BH99, Chapter III.H.2]. The main result of

this section is Proposition 3.2 - a generalisation of a result of Osin [Osi06, Theorem 2.53] alluded to in the introduction.

Let X be a 2-complex. A *singular combinatorial loop* $c : S^1 \rightarrow X$ is a combinatorial structure on S^1 and a continuous map such that for every open n -cell of S^1 , either $f|_e$ is a homeomorphism onto an open cell of X , or else $f(e)$ is contained in the $(n-1)$ -skeleton of X .

Let c be a combinatorial cycle in X . An ϵ -*filling* of c is a pair (P, Φ) consisting of a triangulation P of a 2-disc D^2 and a singular combinatorial map $\Phi : P^{(1)} \rightarrow X^{(1)}$, such that $\Phi|_{S^1} = c$ and the image under Φ of each face of P is a set of diameter at most ϵ . Define $|\Phi|$ to be the number of faces of Φ and

$$\text{Area}_\epsilon(c) := \min\{|\Phi| : \phi \text{ an } \epsilon\text{-filling of } c\}.$$

The *coarse isoperimetric function* of X is then defined to be

$$f_\epsilon^X(\ell) := \sup\{\text{Area}_\epsilon(c) : \text{Len}(c) \leq \ell\}.$$

Definition 3.1. For two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$, we say that f is *asymptotically less than* g , and we write $f \leq g$ if there exist constants $C, K, L \in \mathbb{N}$ such that

$$f(n) \leq Cg(Kn) + Ln.$$

Further we say f is *asymptotically equivalent* to g , and write $f \asymp g$ if $f \leq g$ and $g \leq f$.

Proposition 3.2. *Let X be a cocompact simply connected combinatorial G -2-complex. If Δ_X takes only finite values, then for $N \in \mathbb{N}$ large enough f_N^X takes only finite values and $f_N^X \asymp \Delta_X$.*

Proof. Since X is a cocompact G -2-complex there are only finitely many G -orbits of 2-cells in X . Let $\{D_1, \dots, D_n\}$ denote a representative set of orbits and let N be an integer greater than the maximum diameter of each disc D_i for $i = 1, \dots, n$.

First, we will show $f_N^X \leq \Delta_X$. Let $c : S^1 \rightarrow X$ be a singular combinatorial loop of length ℓ . Let $\Phi : D \rightarrow X$ be a disk diagram of minimal area that fills $c : S^1 \rightarrow X$. Barycentric subdividing D twice to obtain D'' yields a simplicial disk such that the image of each face in X has diameter less than N , i.e. (D'', Φ) is an N -filling of c . It follows that $\text{Area}_\epsilon(c) \leq 12N\Delta_X(\text{Len}(c))$. In particular, $f_N^X \leq \Delta_X$.

It remains to show that $\Delta_X \leq f_N^X$. Consider an N -filling (P, Φ) of a combinatorial loop c in $X^{(1)}$. Considering (P, Φ) as a $3N$ -filling we may assume that each 0-cell of P maps to a 0-cell of X and each 1-cell of P maps to an edge path in X of length at most N . Thus, after subdividing P at most N times we may assume that Φ is cellular on $P^{(1)}$. For each 2-cell of the subdivided P , its boundary map determines a cellular loop in X with length bounded by $3N$. Now, we fill each such loop with some disc diagram $D \rightarrow X$ to obtain a diagram for c in X which has area at most $\Delta_X(3N)f_{3N}^X(\text{Len}(c))$. In particular we conclude that $\Delta_X(n) \leq \Delta_X(3N)f_{3N}^X(c)$. \square

A connected graph Γ is *fillable* if, when considering Γ with the length metric obtained by regarding each edge as a segment of length one, there is an integer k such that the coarse isoperimetric function f_k^Γ takes only finite values.

Proposition 3.3. [BH99, Proposition III.H.2.2] *If Γ and Γ' are quasi-isometric connected graphs such that Γ is fillable, then Γ' is fillable and $f_k^\Gamma \asymp f_k^{\Gamma'}$ for large enough k .*

Remark 3.4. If a connected graph Γ is fillable, then there is a positive integer m such that the complex obtained by attaching 2-cells to all circuits of length less than or equal to m is simply connected.

4. RELATIVE DEHN FUNCTIONS OF GROUPS

Definition 4.1 (Finite relative presentation). Let G be a group, \mathcal{P} an arbitrary collection of subgroups of G , and let S be a subset of G . We say that G is *generated by S relative to \mathcal{P}* if G is generated by the set $\mathcal{S} = S \cup \bigcup_{P \in \mathcal{P}} P$, equivalently, the natural homomorphism

$$(1) \quad F = F(S) * \bigast_{P \in \mathcal{P}} P \rightarrow G$$

is surjective. In the case that S is finite, G is relatively finitely generated with respect to \mathcal{P} .

Let $R \subseteq F$ be a set that normally generates the kernel of the above homomorphism, then we say

$$(2) \quad G = \langle S, \mathcal{P} \mid R \rangle$$

is a *presentation of G relative to \mathcal{P}* . If both S and R are finite we say G is *relatively finitely presented with respect to \mathcal{P}* , or just, *relatively finitely presented* if the collection \mathcal{P} is clear from the context, and (1) is a *relative finite presentation*.

Definition 4.2 (Coned-off Cayley graph). Let G be a group, let \mathcal{P} be an arbitrary collection of subgroups of G , and let S be a generating set of G . Denote by G/\mathcal{P} the set of all cosets gP with $g \in G$ and $P \in \mathcal{P}$. The *coned-off Cayley graph of G with respect to \mathcal{P}* is the graph $\hat{\Gamma}(G, \mathcal{P}, S)$ with vertex set $G \cup G/\mathcal{P}$ and edges are of the following type

- $\{g, gs\}$ for $s \in S$,
- $\{x, gP\}$ for $g \in G$, $P \in \mathcal{P}$ and $x \in gP$.

We call vertices of the form gP *cone points*.

Note that $\hat{\Gamma}(G, \mathcal{P}, S)$ contains the Cayley graph of G with respect to the generating set S , and the quasi-isometry type of $\hat{\Gamma}(G, \mathcal{P}, S)$ is independent of the finite generating set S of G . This justifies the notation $\hat{\Gamma}(G, \mathcal{P})$ that we use through the article.

Definition 4.3 (Relative Dehn function of a relative presentation). Let $G = \langle X, \mathcal{P} \mid R \rangle$ be a relative presentation. For a word $W \in F$ that represents 1 in G , there is an expression

$$(3) \quad W = \prod_{i=1}^k f_i^{-1} R_i f_i$$

where $R_i \in R$ and $f_i \in F$.

We say a function $f: \mathbb{N} \rightarrow \mathbb{N}$ is a *relative isoperimetric function* of the presentation $G = \langle S, \mathcal{P} \mid R \rangle$ if, for any $n \in \mathbb{N}$, and any word W as above of length $\leq n$, one can write W as in (3) with $k \leq f(n)$. The smallest relative isoperimetric function of $G = \langle S, \mathcal{P} \mid R \rangle$ is called the *relative Dehn function of G with respect to \mathcal{P}* , and it is denoted $\Delta_{G, \mathcal{P}}$.

Definition 3.1 and Theorem 4.4 below justify the notation $\Delta_{G, \mathcal{P}}$ for the relative Dehn function of G with respect to \mathcal{P} .

Theorem 4.4. [Osi06, Theorem 2.34] *Let G be a finitely presented group relative to \mathcal{P} . Let Δ_1 and Δ_2 be the relative Dehn functions associated to two finite relative presentations. If Δ_1 takes only finite values, then Δ_2 takes only finite values, and $\Delta_1 \asymp \Delta_2$.*

Definition 4.5 (Osin-Cayley graph and Osin-Cayley complex). Assume G has a relative presentation as in (2). We call the Cayley graph $\Gamma(G, \mathcal{S})$ with $\mathcal{S} = S \sqcup \bigsqcup_{P \in \mathcal{P}} (P - \{1\})$ the *Osin-Cayley graph* and we denote it by $\bar{\Gamma}(G, \mathcal{P}, S)$. Note that in general this graph is not simplicial.

For each $P \in \mathcal{P}$, denote by R_P the set of all words in the alphabet $P - \{1\}$ that represent the identity in P , that is, we have the presentation $P = \langle P - \{1\} \mid R_P \rangle$. Also we have the following presentation

$$F = \langle S, \bigsqcup_{P \in \mathcal{P}} (P - \{1\}) \mid \bigsqcup_{P \in \mathcal{P}} R_P \rangle.$$

The Osin-Cayley complex $\bar{X}(G, \mathcal{P}, S)$ is the 2-complex with 1-skeleton $\bar{\Gamma}(G, \mathcal{P}, S)$ and we attach:

- One 2-cell for each loop labelled with a word in R , which we call from now on *R-cells*.
- One 2-cell for each loop labelled by a word in $\bigsqcup_{P \in \mathcal{P}} R_P$, which we call from now on *P-cells*.

Remark 4.6. By [Osi06, Definition 2.31] the relative Dehn function $\Delta_{G, \mathcal{P}}$ can be described as follows. For any combinatorial loop $\gamma: S^1 \rightarrow \bar{X}(G, \mathcal{P}, S)$, the relative area $\text{Area}^{\text{rel}}(\gamma)$ of γ is the number of R -cells in a minimal disk diagram for γ , where minimality is with respect to the number of R -cells. Then

$$\Delta_{G, \mathcal{P}}(n) = \max\{\text{Area}^{\text{rel}}(\gamma) : \gamma \text{ is a loop in } \bar{X}(G, \mathcal{P}, S) \text{ of length at most } n\}.$$

Definition 4.7 (A natural quasi-isometry between $\bar{\Gamma}(G, \mathcal{P}, S)$ and $\hat{\Gamma}(G, \mathcal{P}, S)$). Assume G is generated by S relatively to \mathcal{P} . Let

$$\varphi: \bar{\Gamma}(G, \mathcal{P}, S) \rightarrow \hat{\Gamma}(G, \mathcal{P}, S)$$

be the map defined as follows. Add a vertex at the midpoint of each edge $e = \{g, gh\}$ of $\bar{\Gamma}(G, \mathcal{P}, S)$ with $h \in P$, $p \in \mathcal{P}$, and label in P . Consider the inclusion of the vertex set of $\bar{\Gamma}(G, \mathcal{P}, S)$ into the vertex set of $\hat{\Gamma} = \hat{\Gamma}(G, \mathcal{P}, S)$. Observe that this map extends to a G -equivariant cellular map between $\bar{\Gamma}(G, \mathcal{P}, S)$ and $\hat{\Gamma}(G, \mathcal{P}, S)$. Specifically, for an edge $e = \{g, gh\}$ with $h \in P$ and label in P of $\bar{\Gamma}(G, \mathcal{P}, S)$, the midpoint of e maps to the vertex gP ; an edge $\{g, gs\}$ with label in S is an edge that is common to both $\bar{\Gamma}(G, \mathcal{P}, S)$ and $\hat{\Gamma}(G, \mathcal{P}, S)$. Observe that the map $\varphi: \bar{\Gamma}(G, \mathcal{P}, S) \rightarrow \hat{\Gamma}(G, \mathcal{P}, S)$ is indeed a $(1, 1)$ -quasi-isometry.

Proposition 4.8. *Let G be a group and let \mathcal{P} be a collection of subgroups. If $\Delta_{G, \mathcal{P}}$ is well defined, then $\Delta_{G, \mathcal{P}} \asymp f_N^{\hat{\Gamma}(G, \mathcal{P})}$ for all sufficiently large integers N .*

Proof. This is a re-statement of Osin's result [Osi06, Theorem 2.53] modulo the fact that $\hat{\Gamma}(G, \mathcal{P}, S)$ and the Cayley graph $\bar{\Gamma}(G, \mathcal{P}, S)$ are quasi-isometric graphs, see Definition 4.7. \square

Definition 4.9 (Coned-off Cayley complex $\hat{X}(G, \mathcal{P}, S)$). Consider the finite relative presentation $G = \langle S, \mathcal{P} \mid R \rangle$. For $F = F(S) * \ast_{P \in \mathcal{P}} P$ we consider the splitting as the fundamental group of the graph of groups \mathbf{Y} that consists of a vertex v labelled with the trivial group, one vertex v_P labelled with each $P \in \mathcal{P}$ respectively, one edge e_P that joins v with v_P for each $P \in \mathcal{P}$ labelled with the trivial group, and one edge e_s for each $s \in S$ labelled with the trivial group. Let \mathcal{T} be the Bass-Serre tree of \mathbf{Y} , see [Ser03].

Since each subgroup P of F survives in the quotient G , we have that $N = \langle\langle R \rangle\rangle$ acts freely on \mathcal{T} . Also note that G acts on the quotient $X_1 = \mathcal{T}/N$. We leave the reader to verify that X_1 is G -homeomorphic to the coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{P}, S)$.

The *coned-off Cayley complex* $\hat{X}(G, \mathcal{P}, S)$ of G is a 2-dimensional G -complex with 1-skeleton X_1 defined as follows. Since the natural morphisms $P \rightarrow G$ with $P \in \mathcal{P}$ are injective, the subgroup $\langle\langle R \rangle\rangle$ intersects trivially with every $P \in \mathcal{P}$. Thus the action of $\langle\langle R \rangle\rangle$ on the Bass-Serre tree \mathcal{T} is free, and the quotient map $\rho: \mathcal{T} \rightarrow \mathcal{T}/N$ is a covering map. Fix a vertex x_0 of \mathcal{T} and consider it as a base point. Then any element g of $\langle\langle R \rangle\rangle$ induces a unique path α_g from x_0 to gx_0 . Let $\gamma_g = \rho \circ \alpha_g$ be the closed path in \hat{X}_1 induced by α_g based at $\rho(x_0)$. This induces an isomorphism $N \rightarrow \pi_1(X_1, \rho(x_0))$ defined by $g \mapsto \gamma_g$. For $g \in G$ and $h \in N$, let $g \cdot \gamma_h$ be the translated closed path in X_1 without an initial point, i.e., these are cellular maps from $S^1 \rightarrow X_1$. Consider the G -set $\Omega = \{g \cdot \gamma_r \mid r \in R, g \in G\}$ of closed paths in X_1 . The complex \hat{X} is then obtained by attaching a 2-cell to X_1 for every closed path in Ω . In particular, the pointwise G -stabilizer of a 2-cell of \hat{X} coincides with the pointwise G -stabilizer of its boundary path. The natural isomorphism from N to

$\pi_1(X^{(1)}, \rho(x_0))$ implies that \hat{X} is simply connected. Moreover, the G -action is cocompact since R is finite.

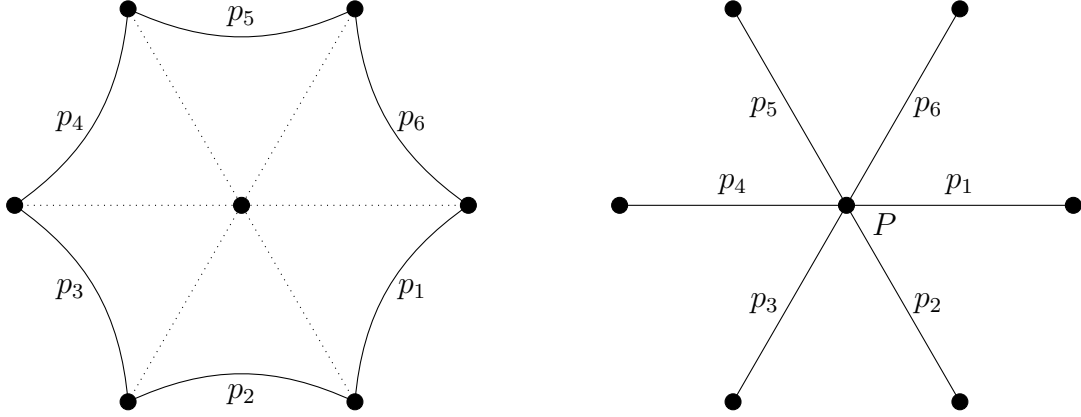


FIGURE 1. The image of the boundary of a \mathcal{P} -cell on $\bar{\Gamma}(G, \mathcal{P}, S)$ under the quasi-isometry ϕ .

Definition 4.10 (A natural map between $\bar{X}(G, \mathcal{P}, S)$ and $\hat{X}(G, \mathcal{P}, S)$). There exists a G -map $\varphi: \bar{X}(G, \mathcal{P}, S) \rightarrow \hat{X}(G, \mathcal{P}, S)$ that extends the natural quasi-isometry $\varphi: \bar{\Gamma}(G, \mathcal{P}, S) \rightarrow \hat{\Gamma}(G, \mathcal{P}, S)$. In particular, we have a commutative diagram

$$\begin{array}{ccc} \bar{\Gamma}(G, \mathcal{P}, S) & \xrightarrow{\varphi} & \hat{\Gamma}(G, \mathcal{P}, S) \\ \downarrow & & \downarrow \\ \bar{X}(G, \mathcal{P}, S) & \xrightarrow{\varphi} & \hat{X}(G, \mathcal{P}, S). \end{array}$$

Specifically every R -cell in $\bar{X}(G, \mathcal{P}, S)$ is sent homeomorphically to the corresponding 2-cell in $\hat{X}(G, \mathcal{P}, S)$, while every \mathcal{P} -cell in $\bar{X}(G, \mathcal{P}, S)$ is collapsed to a star-like 1-complex as we see in Figure 1.

Remark 4.11. The following statements are straightforward to verify from the definition of $\varphi: \bar{X}(G, \mathcal{P}, S) \rightarrow \hat{X}(G, \mathcal{P}, S)$ and Figure 1. Denote by $\Delta_{\hat{X}}$ the combinatorial Dehn function of $\hat{X}(G, \mathcal{P}, S)$.

- (1) Let $\hat{\gamma}: S^1 \rightarrow \hat{X}(G, \mathcal{P}, S)$ be a loop with no backtracks in the coned-off Cayley complex. Then we can pull-back $\hat{\gamma}$ to a loop $\gamma: S^1 \rightarrow \bar{X}(G, \mathcal{P}, S)$ in such a way that the following diagram commutes

$$\begin{array}{ccc} & S^1 & \\ \swarrow \exists! \gamma & & \searrow \hat{\gamma} \\ \bar{\Gamma}(G, \mathcal{P}, S) & \xrightarrow{\varphi} & \hat{\Gamma}(G, \mathcal{P}, S). \end{array}$$

Let $D \rightarrow \bar{X}(G, \mathcal{P}, S)$ be a disk diagram filling a combinatorial loop $\gamma : S^1 \rightarrow \bar{X}(G, \mathcal{P}, S)$. Then there exists a disk diagram $\hat{D} \rightarrow \hat{X}(G, \mathcal{P}, S)$ so that the following diagram commutes

$$\begin{array}{ccccc} S^1 & \xrightarrow{\gamma} & \bar{X}(G, \mathcal{P}, S) & \xrightarrow{\varphi} & \hat{X}(G, \mathcal{P}, S) \\ \parallel & & \uparrow & & \uparrow \\ \partial D & \longrightarrow & D & \dashrightarrow & \hat{D}. \end{array}$$

- (2) Let $\gamma : S^1 \rightarrow \bar{X}(G, \mathcal{P}, S)$ be a combinatorial loop of length n , then we can push it to a loop $\hat{\gamma} = \varphi \circ \gamma : S^1 \rightarrow \hat{X}(G, \mathcal{P}, S)$ of length at most $2n$, that is, we have the following commutative diagram

$$\begin{array}{ccc} & S^1 & \\ \gamma \swarrow & & \searrow \varphi \circ \gamma \\ \bar{\Gamma}(G, \mathcal{P}, S) & \xrightarrow{\varphi} & \hat{\Gamma}(G, \mathcal{P}, S) \end{array}.$$

Let $\hat{D} \rightarrow \hat{X}(G, \mathcal{P}, S)$ be a disk diagram filling the cycle $\hat{\gamma} = \varphi \circ \gamma$. Then there exists a disk diagram $D \rightarrow \bar{X}$ such that the following diagram commutes

$$\begin{array}{ccccc} \bar{X}(G, \mathcal{P}, S) & \xrightarrow{\varphi} & \hat{X}(G, \mathcal{P}, S) & \xleftarrow{\hat{\gamma}} & S^1 \\ \uparrow & & \uparrow & & \parallel \\ D & \dashrightarrow & \hat{D} & \longleftarrow & \partial \hat{D}. \end{array}$$

- (3) In both items above, $\text{Area}^{\text{rel}}(D) = \text{Area}(\hat{D})$.

Proposition 4.12. *Let $G = \langle S, \mathcal{P} \mid R \rangle$ be a finite relative presentation, and let $\Delta_{G, \mathcal{P}}$ and X be the corresponding relative Dehn function and coned-off Cayley complex respectively. Then $\Delta_{G, \mathcal{P}}(n) \asymp \Delta_X(n)$ for every $n \in \mathbb{N}$.*

Proof. Let $\hat{\gamma} : S^1 \rightarrow \hat{X}(G, \mathcal{P}, S)$ be a loop of length n with no backtracks in the coned-off Cayley complex. By the first item of Remark 4.11 and considering a minimal relative area disk diagram $D \rightarrow \bar{X}(G, \mathcal{P}, S)$ filling a pull-back cycle $\hat{\gamma} : S^1 \rightarrow \hat{\Gamma}(G, \mathcal{P}, S)$ of γ , it follows that

$$\Delta_{G, \mathcal{P}}(|\hat{\gamma}|) \geq \Delta_{G, \mathcal{P}}(|\gamma|) \geq \text{Area}^{\text{rel}}(D) = \text{Area}(\hat{D}) \geq \text{Area}(\hat{\gamma})$$

where the equality comes from the third item of Remark 4.11. Therefore $\Delta_{G, \mathcal{P}}(n) \geq \Delta_{\hat{X}}(n)$ for all $n \in \mathbb{N}$. Analogously, let $\gamma : S^1 \rightarrow \bar{X}(G, \mathcal{P}, S)$ be a combinatorial loop. By the second item of Remark 4.11 and considering a minimal area disk diagram $\hat{D} \rightarrow \hat{X}(G, \mathcal{P}, S)$ filling $\hat{\gamma} = \varphi \circ \gamma$, it follows that

$$\text{Area}^{\text{rel}}(\gamma) \leq \text{Area}^{\text{rel}}(D) = \text{Area}(\hat{D}) \leq \Delta_{\hat{X}}(|\hat{\gamma}|) \leq \Delta_{\hat{X}}(2|\gamma|)$$

and hence $\Delta_{G, \mathcal{P}}(n) \leq \Delta_{\hat{X}}(2n)$ for all $n \in \mathbb{N}$.

□

The following corollary is a direct consequence of Theorem 2.1 and Proposition 4.12.

Corollary 4.13. *Let G be finitely presented relative to a collection of subgroups \mathcal{P} . The following statements are equivalent:*

- (1) *The relative Dehn function $\Delta_{G,\mathcal{P}}$ takes only finite values.*
- (2) *The graph $\hat{\Gamma}(G, \mathcal{P})$ is fine.*

In the proof of [GM08, Proposition 2.50] is implicit that (1) implies (2) of the previous theorem.

The following corollary is a straightforward consequence of Proposition 4.12 and Proposition 3.2.

Corollary 4.14. *Let G be a group finitely presented relative to a finite collection of subgroups \mathcal{P} . If $\Delta_{G,\mathcal{P}}$ takes only finite values, then $\hat{\Gamma}(G, \mathcal{P})$ is fillable for some integer m .*

Proposition 4.15. *Let G be a group finitely generated by S with respect to \mathcal{P} . If $\hat{\Gamma}(G, \mathcal{P}, S)$ is connected, fine, cocompact, and k -fillable, then G is finitely presented relative to \mathcal{P} .*

Proof. Fix a vertex x_0 of $\hat{\Gamma}$ with trivial stabiliser. Since $\hat{\Gamma}$ is fine and there are finitely many G -orbits of edges, there are finitely many G -orbits of circuits of length at most k . Let $\{\gamma_1, \dots, \gamma_\ell\}$ be a collection of representatives, and after translations assume that each γ_i contains the vertex x_0 corresponding to the identity element of G . Then each γ_i defines an element of the fundamental group $\pi_1(\hat{\Gamma}, x_0)$.

Since $\hat{\Gamma} = \hat{\Gamma}(G, \mathcal{P})$ is cocompact, the collection \mathcal{P} is finite. Let \tilde{G} be the free product $F(S) * \ast_{P \in \mathcal{P}} P$, and let \mathcal{T} be the corresponding Bass-Serre tree of \tilde{G} . Consider the short exact sequence,

$$1 \rightarrow N \hookrightarrow \tilde{G} \xrightarrow{\varphi} G \rightarrow 1$$

where φ is the homomorphism induced by the inclusion $S \cup \bigcup_{i=1}^n P_i$ into G . Since $\hat{\Gamma}$ is connected, it is an observation that $\hat{\Gamma}$ can be identified with the quotient \mathcal{T}/N . The quotient map $\rho: \mathcal{T} \rightarrow \hat{\Gamma}$ is equivariant with respect to $\varphi: \tilde{G} \rightarrow G$. Let \tilde{x}_0 be a vertex of \mathcal{T} such that $\rho(\tilde{x}_0) = x_0$. Since the restriction of φ to each P_i is injective, the action of N on \mathcal{T} is free. In particular, ρ is a universal covering map. Note that \tilde{x}_0 has trivial \tilde{G} -stabiliser. Hence, via liftings, ρ yields a bijection ψ between the elements of \tilde{G} (or equivalently the \tilde{G} -orbit of \tilde{x}_0) and the set of homotopy classes of paths in $\hat{\Gamma}$ with initial vertex x_0 and terminal vertex in the G -orbit of x_0 . In particular, a well known fact of covering space theory, shows that the restriction $\psi: N \rightarrow \pi_1(\hat{\Gamma}, x_0)$ is a group isomorphism. Let $r_i \in N$ be defined by $\psi(r_i) = \gamma_i$ (a similar argument to this was used in [CHK20]).

Since $\hat{\Gamma}$ is k -fillable, the complex \hat{X} obtained by attaching 2-cells with boundary paths the circuits of length at most k is simply connected. That implies that $\pi_1(\hat{\Gamma}, x_0)$ is

generated by the closed paths arising as concatenations of the form $\alpha_g \cdot \gamma_i \cdot \bar{\alpha}_g$ for $g \in \tilde{G}$, where α_g is the projection via ρ of the unique path from \tilde{x}_0 to $g.\tilde{x}_0$. Equivalently, N is generated by the elements $gr_i g^{-1}$ for $g \in \tilde{G}$. We have shown that N is normally generated by $\mathcal{R} = \{r_1, \dots, r_\ell\}$ and therefore $\langle S, \mathcal{P} \mid \mathcal{R} \rangle$ is a finite relative presentation of G . \square

Summarising the results of this section we obtain Theorem 4.16 below.

Theorem 4.16 (Theorem E). *Let G be a group finitely generated relative to a finite collection of subgroups \mathcal{P} . If G is finitely presented relative to \mathcal{P} , then*

(1) $\hat{\Gamma}(G, \mathcal{P})$ is fillable.

(2) The relative Dehn function $\Delta_{G, \mathcal{P}}$ is well defined if and only if $\hat{\Gamma}(G, \mathcal{P})$ is fine graph.

Conversely, if $\hat{\Gamma}(G, \mathcal{P})$ is fine and fillable, then G is finitely presented relative to \mathcal{P} and hence $\Delta_{G, \mathcal{P}}$ is well defined.

Proof. This follows from Corollary 4.13, Corollary 4.14 and Proposition 4.15. \square

Note that Theorem E from the introduction is a particular case Theorem 4.16.

5. FINENESS AND QUASI-ISOMETRIES OF PAIRS

In this section we will prove Theorem C from the introduction. The heart of the argument is establishing Proposition 5.6 which gives conditions on a quasi-isometry of pairs $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ to induce a quasi-isometry of coned off Cayley graphs. The remainder of the section then works towards replacing the geometric-set-theoretic conditions on q with algebraic conditions on \mathcal{P} and \mathcal{Q} . This yields Proposition 5.12. Finally, we give a proof of Theorem C.

Another equivalent definition of Bowditch's fine graphs is used in this section [Bow12, Proposition 2.1].

Definition 5.1 (Fine). Let Γ be a graph and let v be a vertex of Γ . Let

$$T_v \Gamma = \{w \in V(\Gamma) \mid \{v, w\} \in E(\Gamma)\}.$$

denote the set of the vertices adjacent to v . For $x, y \in T_v \Gamma$, the *angle metric* $\angle_v(x, y)$ is the length of the shortest path in the graph $\Gamma \setminus \{v\}$ between x and y , with $\angle_v(x, y) = \infty$ if there is no such path. The graph Γ is *fine at v* if $(T_v \Gamma, \angle_v)$ is a locally finite metric space. The graph Γ is *fine at $C \subseteq V(\Gamma)$* if Γ is fine at v for all $v \in C$. The graph Γ is a *fine graph* if it is fine at every vertex.

Definition 5.2 (Quasi-isometry of Pairs). Consider two pairs (G, \mathcal{P}) and (H, \mathcal{Q}) where G and H are finitely generated groups with chosen word metrics dist_G and dist_H with respect to some finite generating sets. Denote the Hausdorff distance between subsets of H by hdist_H . An (L, C) -quasi-isometry $q: G \rightarrow H$ is an (L, C, M) -quasi-isometry of pairs

$q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ if the relation

$$\dot{q} = \{(A, B) \in G/\mathcal{P} \times H/\mathcal{Q}: \text{hdist}_H(q(A), B) < M\}$$

satisfies that the projections into G/\mathcal{P} and H/\mathcal{Q} are surjective.

Remark 5.3. Note that in Definition 5.2 the notion of a quasi-isometry of pairs is independent of the chosen finite generating sets for G and H . In the case where we want to keep track of specific generating sets we use the following notation. If G and H are groups generated by finite generating sets S_0 and T_0 respectively, by a quasi-isometry of pairs $(G, \mathcal{P}, S_0) \rightarrow (H, \mathcal{Q}, T_0)$ we mean a quasi-isometry of pairs $(G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ with respect to the word metrics induced by S_0 and T_0 .

Remark 5.4. If \mathcal{P} is a finite collection, then the metric space $(G/\mathcal{P}, \text{hdist})$ is locally finite. Indeed, fixing $P \in \mathcal{P}$ and $r > 0$, there are finitely many left cosets in G/\mathcal{P} such that $\text{hdist}(P, gP) < r$. Moreover, the left G -action on G/\mathcal{P} by multiplication on the left preserves the Hausdorff distance hdist between subsets of G and hence it is an action by isometries.

Remark 5.5. If $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is an (L, C, M) -quasi-isometry of pairs, and \dot{q} is a function $G/\mathcal{P} \rightarrow H/\mathcal{Q}$, then

$$\frac{1}{L} \text{hdist}(A, B) - C - 2M \leq \text{hdist}(\dot{q}(A), \dot{q}(B)) \leq L \text{hdist}(A, B) + C + 2M.$$

In particular, $\dot{q}: (G/\mathcal{P}, \text{hdist}) \rightarrow (H/\mathcal{Q}, \text{hdist})$ is a quasi-isometry.

The main technical result of this section is the following proposition. Note that given a connected graph Γ we consider the vertex set as a metric space with metric induced by the path metric. In particular, a quasi-isometry between graphs is a function of the vertex sets satisfying the usual axioms.

Proposition 5.6. *Let G and H be groups, let $S \subset G$ and $T \subset H$, and let $S_0 \subset S$ and $T_0 \subset T$ be finite generating sets of G and H respectively. Consider collections \mathcal{P} and \mathcal{Q} of subgroups of G and H respectively. Let $q: G \rightarrow H$ be a function.*

Suppose q is a quasi-isometry $\Gamma(G, S) \rightarrow \Gamma(H, T)$, is a quasi-isometry of pairs $(G, \mathcal{P}, S_0) \rightarrow (H, \mathcal{Q}, T_0)$, and \dot{q} is a bijection $G/\mathcal{P} \rightarrow H/\mathcal{Q}$.

- (1) *If $\hat{q} = q \cup \dot{q}$, then \hat{q} is a quasi-isometry $\hat{\Gamma}(G, \mathcal{P}, S) \rightarrow \hat{\Gamma}(H, \mathcal{Q}, T)$.*
- (2) *If $\hat{\Gamma}(H, \mathcal{Q}, T)$ is fine at cone vertices, then $\hat{\Gamma}(G, \mathcal{P}, S)$ is fine at cone vertices.*

Remark 5.7. There are algebraic conditions on \mathcal{P} and \mathcal{Q} that imply that \dot{q} is a bijection, see Proposition 5.12.

Corollary 5.8. *Suppose that $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is an quasi-isometry of pairs and \dot{q} is a bijection. Then $\hat{q}: \hat{\Gamma}(G, \mathcal{P}) \rightarrow \hat{\Gamma}(H, \mathcal{Q})$ is a quasi-isometry, and if $\hat{\Gamma}(H, \mathcal{Q})$ is a fine graph, then $\hat{\Gamma}(G, \mathcal{P})$ is a fine graph.*

Proof of Proposition 5.6. Suppose $q: \Gamma(G, S) \rightarrow \Gamma(Q, T)$ is a (\bar{L}, \bar{C}) -quasi-isometry and $q: (G, \mathcal{P}, S_0) \rightarrow (H, \mathcal{Q}, T_0)$ is a (L, C, M) -quasi-isometry of pairs.

For any path $\alpha = [v_0, v_1, \dots, v_\ell]$ in $\hat{\Gamma}(G, \mathcal{P}, S)$, let $\hat{q}(\alpha)$ denote a path in $\hat{\Gamma}(H, \mathcal{Q}, T)$ from $\hat{q}(v_0)$ to $\hat{q}(v_\ell)$ obtained as the concatenation of paths $\beta_0, \dots, \beta_{\ell-1}$ where β_i is a path from $\hat{q}(v_i)$ to $\hat{q}(v_{i+1})$ defined as follows:

- (1) If v_i and v_{i+1} are elements of G , then β_i is a geodesic in $\Gamma(H, T)$ from $q(v_i)$ to $q(v_{i+1})$. Since $q: \Gamma(G, S) \rightarrow \Gamma(Q, T)$ is a (\bar{L}, \bar{C}) -quasi-isometry, β_i has length bounded by $\bar{L} + \bar{C}$.
- (2) Suppose $v_i \in G$ and $v_{i+1} \in G/\mathcal{P}$. Observe that v_i is an element of the left coset v_{i+1} . Since $q: (G, \mathcal{P}, S_0) \rightarrow (H, \mathcal{Q}, T_0)$ is an (L, C, M) -quasi-isometry of pairs, there is a geodesic of length at most M in $\Gamma(H, T_0)$ from $q(v_i)$ to an element w of the left coset $\dot{q}(v_{i+1})$. Let β_i be the concatenation of this geodesic in $\Gamma(H, T_0)$ followed by the edge between w and the cone vertex $\dot{q}(v_{i+1})$. Observe that β_i is a path of length at most $M + 1$ in $\hat{\Gamma}(H, \mathcal{Q}, T)$.
- (3) If $v_i \in G/\mathcal{P}$ and $v_{i+1} \in G$ then β_i is defined in an analogous way as in the previous case, and also has length at most $M + 1$.

Observe that

$$|\hat{q}(\alpha)| \leq (\bar{L} + \bar{C} + M + 1)|\alpha|.$$

The above inequality applied in the case that α is a geodesic between vertices x and y of $\hat{\Gamma}(G, \mathcal{P}, S)$ implies that

$$\text{dist}_{\hat{\Gamma}(H, \mathcal{Q}, T)}(\hat{q}(x), \hat{q}(y)) \leq (\bar{L} + \bar{C} + M + 1) \text{dist}_{\hat{\Gamma}(G, \mathcal{P}, S)}(x, y)$$

for any pair of vertices x, y of $\hat{\Gamma}(G, \mathcal{P}, S)$. By symmetry an analogous inequality holds for vertices of $\hat{\Gamma}(H, \mathcal{Q})$. Since \dot{q} is a bijection, the definition of $\hat{q}(\alpha)$ shows that α passes through a cone vertex A if and only if $\hat{q}(\alpha)$ passes through the cone vertex $\dot{q}(A)$. We summarise this discussion in the following lemma.

Lemma 5.9. *There are constants $\hat{L} \geq 1$ and $\hat{C} \geq 0$ such that:*

- (1) *The function \hat{q} is a (\hat{L}, \hat{C}) -quasi-isometry from $\hat{\Gamma}(G, \mathcal{P}, S)$ to $\hat{\Gamma}(H, \mathcal{Q}, T)$.*
- (2) *Let α be a path in $\hat{\Gamma}(G, \mathcal{P}, S)$.*
 - (a) *For any $A \in G/\mathcal{P}$, α passes through the cone vertex A if and only if $\hat{q}(\alpha)$ passes through the cone vertex $\dot{q}(A)$.*
 - (b) *$|\hat{q}(\alpha)| \leq \hat{L} |\alpha|$.*

We prove the contrapositive of the second statement of the proposition. Suppose that $\hat{\Gamma}(G, \mathcal{P}, S)$ is not fine at cone vertices. Then there is $P \in \mathcal{P}$ such that $(T_P \Gamma, \angle_P)$ is not locally-finite. Let $r > 0$ and let $\{g_i\} \subseteq P$ be an infinite subset such that $\angle_P(g_i, g_j) \leq r$ for every i, j . Let $\alpha_{i,j}$ be a path in $\hat{\Gamma}(G, \mathcal{P}, S)$ from g_i to g_j of length at most r that does not contain the cone vertex P . Let Q denote the left coset $\dot{q}(P)$. Let γ_i be a geodesic

in $\Gamma(H, T_0)$ from an element h_i of Q to $q(g_i)$ such that $\text{dist}_H(h_i, q(g_i)) = \text{dist}_H(Q, q(g_i))$. Since q is a (L, C, M) quasi-isometry of pairs, each γ_i has length at most M .

Let us prove that the set $\{h_i\}$ is infinite. Suppose, for contradiction, that $\{h_i\}$ is a finite set. Since T_0 is a finite generating set, $\Gamma(H, T_0)$ is a locally finite graph and hence it admits only finitely many paths of length at most M with initial vertex in $\{h_i\}$. Since each γ_i has length at most M and initial vertex in $\{h_i\}$, it follows that the set $\{q(g_i)\}$ is finite and in particular, bounded. Since q is a quasi-isometry $\Gamma(G, S_0) \rightarrow \Gamma(G, T_0)$, it follows that the set $\{g_i\}$ is a bounded subset of vertices in the locally finite graph $\Gamma(G, S_0)$, hence the set $\{g_i\}$ is finite, a contradiction.

To conclude the proof, we show that $\hat{\Gamma}(H, \mathcal{Q}, T)$ is not fine at the cone vertex Q . Since $\{h_i\}$ is an infinite subset of Q , it is enough to show that $\angle_Q(h_i, h_j) \leq r\hat{L} + M$ for any i, j . Consider the path $\beta_{i,j}$ from h_i to h_j obtained as the concatenation of the path γ_i from h_i to $\hat{q}(g_i)$, followed by the path $\hat{q}(\alpha_{i,j})$ from $\hat{q}(g_i)$ to $\hat{q}(g_j)$, and then the path $\bar{\gamma}_j$ from $\hat{q}(g_j)$ to h_j . The paths γ_i and γ_j have length bounded by M , and they do not contain the cone vertex Q as they are paths in $\Gamma(H, T_0)$; the path $\hat{q}(\alpha_{i,j})$ has length at most $r\hat{L}$ and does not contain the cone vertex Q by Lemma 5.9. Therefore $\angle_Q(h_i, h_j) \leq |\gamma_i| + |\hat{q}(\alpha_{i,j})| + |\gamma_j| \leq 2M + r\hat{L}$ as desired. \square

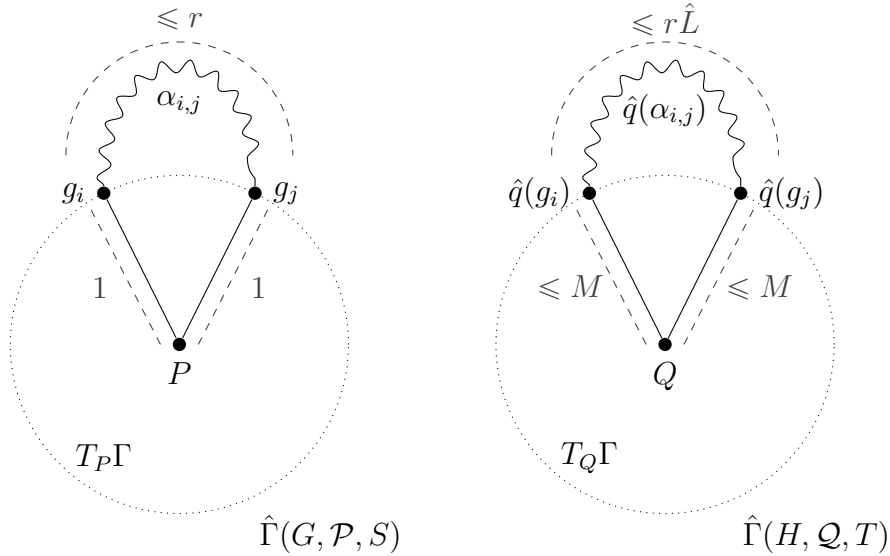


FIGURE 2. Illustration of the proof of Proposition 5.6

The goal for the remainder of this section is to give algebraic conditions on \mathcal{P} and \mathcal{Q} to ensure \dot{q} is a bijection. The following key definition will provide such a criteria.

Definition 5.10 (Reduced Collection). A collection of subgroups \mathcal{P} of a group G is *reduced* if for any $P, Q \in \mathcal{P}$ and $g \in G$, then P and gQg^{-1} being commensurable subgroups implies $P = Q$ and $g \in P$.

Remark 5.11. If \mathcal{P} is a reduced collection of subgroups of a group G , then $P = \text{Comm}_G(P)$ for any $P \in \mathcal{P}$.

Proposition 5.12. Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a (L, C, M) -quasi-isometry of pairs. Then

- (1) \dot{q} is a surjective function $G/\mathcal{P} \rightarrow H/\mathcal{Q}$ if \mathcal{Q} is reduced.
- (2) \dot{q} is a bijection $G/\mathcal{P} \rightarrow H/\mathcal{Q}$ if \mathcal{P} and \mathcal{Q} are reduced.

There are different versions of the following lemma in the literature: [MSW11, Lemma 2.2], [Mar09, Lemma 4.7] and [Hru10, Proposition 9.4], the statement below is taken from the later reference. For $A \subset G$, $\mathcal{N}_k(A)$ denotes the closed neighborhood of A in (G, dist_G) .

Lemma 5.13. Let G be a finitely generated group with word metric dist_G . Let gP and fQ are arbitrary left cosets of subgroups of G . Then for any $k > 0$ there is $M > 0$ such that

$$\mathcal{N}_k(gP) \cap \mathcal{N}_k(fQ) \subseteq \mathcal{N}_M(gPg^{-1} \cap fQf^{-1}).$$

Lemma 5.14. Let G be a finitely generated group with a word metric dist_G , let P and Q be subgroups, and let $g \in G$. Then P and gQg^{-1} are commensurable subgroups if and only if $\text{hdist}_G(P, gQ) < \infty$.

Proof. Suppose K is a finite index subgroup of P and gQg^{-1} . Then $\text{hdist}(K, P) < \infty$ and $\text{hdist}(K, gQg^{-1})$ are finite. Since $\text{hdist}(gQg^{-1}, gQ) \leq \text{dist}(1, g) < \infty$, it follows that

$$\text{hdist}(P, gQ) \leq \text{hdist}(P, K) + \text{hdist}(K, gQg^{-1}) + \text{hdist}(gQg^{-1}, gQ) < \infty.$$

Conversely, suppose $\text{hdist}(P, gQ)$ is finite. Then $P \subset P \cap \mathcal{N}_k(gQ)$ for some k , and therefore Lemma 5.13 implies that $P \subseteq \mathcal{N}_M(P \cap gQg^{-1})$ for some M . It follows that $P \cap gQg^{-1}$ is a finite index subgroup of P . In an analogous way one shows that $P \cap gQg^{-1}$ is a finite index subgroup of gQg^{-1} . Whence, P and gQg^{-1} are commensurable subgroups. \square

Proof of Proposition 5.12. To prove the first statement, we only need to show that the relation \dot{q} is a function. Suppose that \mathcal{Q} is reduced and the pairs (A, h_1Q_1) and (A, h_2Q_2) belong to \dot{q} . Then $h_1Q_1, h_2Q_2 \in H/\mathcal{Q}$ and $\text{hdist}_H(h_1Q_1, h_2Q_2) < \infty$. Lemma 5.14 implies that $h_1Q_1h_1^{-1}$ and $h_2Q_2h_2^{-1}$ are commensurable subgroups. Since \mathcal{Q} is reduced, it follows that $Q_1 = Q_2$ and $h_2 \in h_1Q_1$. In particular, $h_1Q_1 = h_2Q_2$ and hence \dot{q} is a function. The second statement of the lemma follows from the first one. \square

We are now ready to prove Theorem C from the introduction.

Theorem 5.15 (Theorem C). Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a quasi-isometry of pairs. Suppose \mathcal{P} and \mathcal{Q} are reduced finite collections. Then there is an induced quasi-isometry of graphs $\hat{q}: \hat{\Gamma}(G, \mathcal{P}) \rightarrow \hat{\Gamma}(H, \mathcal{Q})$, and if $\hat{\Gamma}(H, \mathcal{Q})$ is a fine graph then $\hat{\Gamma}(G, \mathcal{P})$ is a fine graph.

Proof. The result follows from applying Proposition 5.12 to Corollary 5.8. \square

6. ALMOST MALNORMAL COLLECTIONS AND QUASI-ISOMETRIES OF PAIRS

In this section we will prove Theorem D from the introduction. First, we introduce a refinement \mathcal{P}^* of a collection \mathcal{P} . In Proposition 6.2 we show under mild hypothesis (G, \mathcal{P}) and (G, \mathcal{P}^*) are quasi-isometric pairs under the identity map.

Definition 6.1. Let \mathcal{P} be a collection of subgroups of group G . A *refinement* \mathcal{P}^* of \mathcal{P} is a set of representatives of conjugacy classes of the collection of subgroups $\{\text{Comm}_G(gPg^{-1}) : P \in \mathcal{P} \text{ and } g \in G\}$.

Proposition 6.2. Let \mathcal{P}^* be a refinement of a finite collection of subgroups \mathcal{P} of a finitely generated group G . If P is a finite index subgroup of $\text{Comm}_G(P)$ for every $P \in \mathcal{P}$, then (G, \mathcal{P}) and (G, \mathcal{P}^*) are quasi-isometric pairs via the identity map on G .

Proof. Let $\mathcal{P} = \{P_1, \dots, P_k\}$. Since $\text{Comm}_G(gPg^{-1}) = g \text{Comm}_G(P)g^{-1}$, we can assume that every subgroup in \mathcal{P}^* is of the form $\text{Comm}_G(P)$ for some $P \in \mathcal{P}$. Let $q: G \rightarrow G$ be the identity map. Since q is a $(1, 0)$ -quasi-isometry, it is enough to show that there is $M > 0$ such that the relation

$$\dot{q} = \{(A, B) \in G/\mathcal{P} \times G/\mathcal{P}^* : \text{hdist}(A, B) < M\}$$

satisfies that it projects surjectively on G/\mathcal{P} and on G/\mathcal{P}^* .

For any $P_i \in \mathcal{P}$, note that $\text{hdist}(P_i, \text{Comm}_G(P_i)) < \infty$ since P_i has finite index in $\text{Comm}_G(P_i)$. Let

$$M_1 = \max\{\text{hdist}(P_i, \text{Comm}_G(P_i)) : 1 \leq i \leq k\}.$$

By definition of \mathcal{P}^* , for any P_i , there is $Q_i \in \mathcal{P}^*$ and $g_i \in G$ such that $\text{Comm}(P_i) = g_i Q_i g_i^{-1}$. In particular $\text{hdist}(\text{Comm}_G(P_i), g_i Q_i)$ is finite. Let

$$M_2 = \max\{\text{hdist}(\text{Comm}_G(P_i), g_i Q_i) : 1 \leq i \leq k\}.$$

Let $M > M_1 + M_2$. Then for any $gP_i \in G/\mathcal{P}$, $(gP_i, gg_i Q_i) \in \dot{q}$. On the other hand, if $gQ \in G/\mathcal{P}^*$ then $Q = \text{Comm}_G(P)$ for some $P \in \mathcal{P}$ and hence $(gP, gQ) \in \dot{q}$. \square

Remark 6.3. Note that in the previous proposition if \mathcal{P} is infinite the map $\dot{q} : G/\mathcal{P} \rightarrow G/\mathcal{P}^*$ must be finite-to-one. Otherwise after conjugating, there will be a sequence of subgroups $P_i \leq \text{Comm}_G(P_0)$ such that $|\text{Comm}_G(P_0) : P_i| \rightarrow \infty$, in particular, the sequence of Hausdorff distances $\text{hdist}(\text{Comm}_G(P_0), P_i)$ is not bounded.

Definition 6.4. A collection of subgroups \mathcal{P} of a group G is *almost malnormal* if for any $P, P' \in \mathcal{P}$ and $g \in G$, either $gPg^{-1} \cap P'$ is finite, or $P = P'$ and $g \in P$.

Remark 6.5. If \mathcal{P} is an almost malnormal collection of infinite subgroups of a group G , then \mathcal{P} is reduced.

Remark 6.6. If a group G acts by automorphisms on a fine graph Γ such that edge stabilizers are finite and \mathcal{P} is a collection of representatives of conjugacy classes of vertex stabilizers, then \mathcal{P} is an almost malnormal collection.

Proposition 6.7. *Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a quasi-isometry of pairs. If \mathcal{Q} is an almost malnormal finite collection of infinite subgroups and \mathcal{P} is a finite collection, then any refinement \mathcal{P}^* of \mathcal{P} is almost malnormal.*

The proof of Proposition 6.7 relies on the following lemmas.

Lemma 6.8. *Let \mathcal{P} be a collection of subgroups of a group G . Suppose P is a finite index subgroup of $\text{Comm}_G(P)$ for every $P \in \mathcal{P}$. Then any refinement \mathcal{P}^* of \mathcal{P} is a reduced collection.*

Proof. Since commensurable subgroups have equal commensurator, $\text{Comm}_G(\text{Comm}_G(P))$ equals $\text{Comm}_G(P)$ for every $P \in \mathcal{P}$. Let $P_1, P_2 \in \mathcal{P}$ and $g \in G$, and suppose $\text{Comm}_G(P_1)$ and $g \text{Comm}_G(P_2) g^{-1}$ are commensurable subgroups.

$$\begin{aligned} \text{Comm}_G(P_1) &= \text{Comm}_G(\text{Comm}_G(P_1)) \\ &= \text{Comm}_G(g \text{Comm}_G(P_2) g^{-1}) \\ &= \text{Comm}_G(\text{Comm}_G(g P_2 g^{-1})) \\ &= \text{Comm}_G(g^{-1} P_2 g) \\ &= g \text{Comm}_G(P_2) g^{-1}. \end{aligned}$$

Hence \mathcal{P}^* is reduced. □

Lemma 6.9. *Let \mathcal{P} be a finite collection of infinite subgroups of a finitely generated group G . Then \mathcal{P} is almost malnormal if and only if for any $A, B \in G/\mathcal{P}$, either $A = B$ or $\mathcal{N}_n(A) \cap \mathcal{N}_n(B)$ is a finite subset of G for every n .*

Proof. Suppose that \mathcal{P} is an almost malnormal collection of infinite subgroups. Let $g_1 P_1, g_2 P_2 \in G/\mathcal{P}$ and suppose that $\mathcal{N}_n(g_1 P_1) \cap \mathcal{N}_n(g_2 P_2)$ is an infinite (and hence unbounded) subset of G for some integer n . By Lemma 5.13, there is an integer m such that $\mathcal{N}_n(g_1 P_1) \cap \mathcal{N}_n(g_2 P_2) \subset \mathcal{N}_m(g_1 P_1 g_1^{-1} \cap g_2 P_2 g_2^{-1})$. It follows that $g_1 P_1 g_1^{-1} \cap g_2 P_2 g_2^{-1}$ is an infinite subgroup and hence $P_1 = P_2$ and $g_1^{-1} g_2 \in P_1$ by almost malnormality. Therefore $g_1 P_1 = g_2 P_2$.

Conversely, suppose that for any $A, B \in G/\mathcal{P}$, either $A = B$ or $\mathcal{N}_n(A) \cap \mathcal{N}_n(B)$ is a finite set for every n . Let $P, P' \in \mathcal{P}$ and $g \in G$ and suppose that $g P g^{-1} \cap P'$ is an infinite subgroup. It follows that there is $n > 0$ such that $\mathcal{N}_n(g P) \cap \mathcal{N}_n(P')$ is an infinite subset of G . Hence $g P = P'$ and in particular $P = P'$ and $g \in P$. □

Lemma 6.10. *Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a quasi-isometry of pairs. Suppose that \mathcal{P} and \mathcal{Q} are finite collections, and \mathcal{Q} is reduced. If Q is finite index in $\text{Comm}_H(Q)$ for every $Q \in \mathcal{Q}$, then P is finite index in $\text{Comm}_G(P)$ for every $P \in \mathcal{P}$.*

Proof. Since \mathcal{Q} is reduced, \dot{q} is a function from $G/\mathcal{P} \rightarrow G/\mathcal{Q}$. Since both \mathcal{P} and \mathcal{Q} are finite collections, it follows that $\dot{q}: (G/\mathcal{P}, \text{hdist}) \rightarrow (H/\mathcal{Q}, \text{hdist})$ is a quasi-isometry between locally finite metric spaces. Suppose that $P \in \mathcal{P}$ has infinite index in $\text{Comm}_G(P)$. Lemma 5.14 implies that there is an infinite collection of left cosets $\mathcal{A} = \{g_i P : i \in I\}$ such that $\text{hdist}(g_i P, g_j P) < \infty$ for any $i, j \in I$. By local finiteness of $(G/\mathcal{P}, \text{hdist})$, the collection \mathcal{A} is an unbounded subset of G/\mathcal{P} . It follows that $\mathcal{B} = \{\dot{q}(g_i P) : i \in I\}$ is an unbounded subset of H/\mathcal{Q} . Since \mathcal{Q} is a finite collection, and $\dot{q}(g_i P) = h_i Q_i$ for some $h_i \in H$ and $Q_i \in \mathcal{Q}$, the pigeon hole principle implies that we can assume that all Q_i 's are a fixed $Q \in \mathcal{Q}$. By Lemma 5.14, the subgroup Q has infinite index in $\text{Comm}_H(Q)$. \square

Proof of Proposition 6.7. Suppose that $q: (G, \mathcal{P}) \rightarrow (G, \mathcal{Q})$ is a quasi-isometry of pairs. Since \mathcal{Q} is an almost malnormal collection of infinite subgroups, it is a reduced collection and every element of \mathcal{Q} has finite index in its commensurator. Since \mathcal{P} and \mathcal{Q} are finite collections, Lemma 6.10 implies that every element of \mathcal{P} has finite index in its commensurator. Let \mathcal{P}^* be a refinement of \mathcal{P} in G . By Proposition 6.2 there is a quasi-isometry of pairs $p: (G, \mathcal{P}^*) \rightarrow (G, \mathcal{P})$. Then the composition $r = p \circ q$ is an (L, C, M) -quasi-isometry of pairs $(G, \mathcal{P}^*) \rightarrow (H, \mathcal{Q})$. Lemma 6.8 implies that \mathcal{P}^* is a reduced collection. Therefore \dot{r} is a bijection $G/\mathcal{P}^* \rightarrow H/\mathcal{Q}$ by Proposition 5.12. To conclude that \mathcal{P}^* is an almost malnormal we verify the hypothesis of Lemma 6.9.

Claim: \mathcal{P}^* is a finite collection of infinite subgroups.

Since \mathcal{P} is finite, then \mathcal{P}^* is finite. Every element of \mathcal{P}^* is a conjugate of a subgroup of the form $\text{Comm}_G(P)$ for some $P \in \mathcal{P}$, hence it is enough to show that \mathcal{P} contains only infinite subgroups. Observe that any $P \in \mathcal{P}$ is an infinite subgroup since $\text{hdist}(\dot{q}(P), Q) < \infty$ for some $Q \in H/\mathcal{Q}$ and every subgroup in \mathcal{Q} is infinite. \blacklozenge

Claim: For any $A, B \in G/\mathcal{P}^*$, either $A = B$ or $\mathcal{N}_n(A) \cap \mathcal{N}_n(B)$ is a finite subset of G for every n .

Let $A, B \in G/\mathcal{P}^*$ and suppose that $A \neq B$. Since $\dot{r}: G/\mathcal{P}^* \rightarrow H/\mathcal{Q}$ is a bijection, it follows that $\dot{r}(A)$ and $\dot{r}(B)$ are distinct elements of H/\mathcal{Q} . Since \mathcal{Q} is an almost malnormal collection, Lemma 6.9 implies that for any integer m the intersection $\mathcal{N}_m(\dot{r}(A)) \cap \mathcal{N}_m(\dot{r}(B))$ is a finite (and hence bounded) subset of H . Since $r: G \rightarrow H$ is a quasi-isometry, it follows that for every n , the intersection $\mathcal{N}_n(A) \cap \mathcal{N}_n(B)$ is a bounded (and hence finite) subset of G . \blacklozenge \square

Theorem 6.11 (Theorem D). *Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a quasi-isometry of pairs. If \mathcal{Q} is an almost malnormal finite collection of infinite subgroups and \mathcal{P} is a finite collection, then any refinement \mathcal{P}^* of \mathcal{P} is almost malnormal and $q: (G, \mathcal{P}^*) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs.*

Proof. The result follows from Proposition 6.2 and Proposition 6.7. \square

7. HYPERBOLICALLY EMBEDDED SUBGROUPS

In this section we show that there are examples of pairs (G, H) with well defined relative Dehn function outside of the context of relatively hyperbolic groups. Hyperbolically embedded subgroups were introduced in [DGO17] by Dahmani, Guirardel and Osin. Given a group G , $X \subset G$ and $H \leq G$, let $H \hookrightarrow_h (G, X)$ denote that H is a hyperbolically embedded subgroup of G with respect to X . There is a characterisation in [MR21] of H being hyperbolically embedded into G that fits into the context of our Theorem E, namely, in terms of fine vertices in coned-off Cayley graphs (see Definition 5.1).

Proposition 7.1. [MR21, Proposition 1.4] *Let G be a group, $X \subset G$ and $H \leq G$. Then $H \hookrightarrow_h (G, X)$ if and only if $\hat{\Gamma}(G, H, X)$ is connected, hyperbolic, and fine at cone vertices.*

The following theorem provides our examples.

Theorem 7.2. *Let G be a finitely presented group and $H \leq G$ be a subgroup. If $H \hookrightarrow_h G$ then the relative Dehn function $\Delta_{G,H}$ is well-defined.*

The proof of the theorem is discussed after the following lemma.

Lemma 7.3. *Let G be a finitely generated group and H a finitely generated subgroup. Then G is finitely presented if and only if G is finitely presented relative to H .*

Proof (sketch). Suppose that S and $\{h_1, \dots, h_k\}$ are finite generating sets of G and H respectively. Let w_i be a word in S that represents $h_i \in G$, let R be a collection of relations in G over S , and let R_H be the collection of all relations in H over the generating set H , that is, $H = \langle H \mid R_H \rangle$. Observe that the following statements are equivalent.

- (1) $\langle S \sqcup H \mid h_i = w_i, R, R_H \rangle$ is a finite relative presentation of G with respect to H .
- (2) $\langle S \sqcup \{h_1, \dots, h_k\} \mid h_i = w_i, R \rangle$ is a finite presentation of G .
- (3) $\langle S \mid R \rangle$ is finite presentation of G .

Then, the lemma follows by observing that G is finitely presented relative to H if and only if G admits a finite relative presentation of the form $\langle S \sqcup H \mid h_i = w_i, R, R_H \rangle$. \square

Proof of Theorem 7.2. First, note that the theorem is trivial in the case that H is a finite subgroup of G . Indeed, any finite subgroup is hyperbolically embedded by definition and a finite relative presentation of a group with respect to a finite subgroup is in fact a finite presentation. In particular, the relative Dehn function coincides with the Dehn function and the Dehn function of a finitely presented group is always well defined.

Since G is finitely generated and $H \hookrightarrow G$, it follows from [DGO17, Corollary 4.32] that H is finitely generated. Hence, by Lemma 7.3, G is finitely presented relative to H .

Let S be a finite generating set of G . In view of Theorem E(2), to conclude that $\Delta_{G,H}$ is well defined, it is enough to prove that $\hat{\Gamma}(G, \mathcal{P}, S)$ is a fine graph.

Suppose that $H \hookrightarrow (G, X)$ for some $X \subset G$. Without loss of generality, assume that X contains the finite generating set S , see [DGO17, Corollary 4.27]. It follows

that $\hat{\Gamma}(G, H, S)$ is a subgraph of $\hat{\Gamma}(G, H, X)$. Since S is finite, observe that every vertex of $\hat{\Gamma}(G, H, S)$ has either finite degree or is cone-vertex. By Proposition 7.1, the graph $\hat{\Gamma}(G, H, X)$ is fine at every cone vertex, and hence so is $\hat{\Gamma}(G, H, S)$. Therefore $\hat{\Gamma}(G, H, S)$ is a fine graph. \square

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