The Unstable Gromov-Lawson-Rosenberg Conjecture

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Scalar curvature

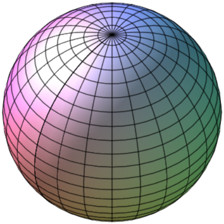
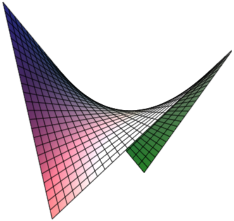
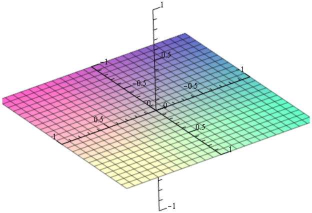
Definition (Scalar curvature)

Let (M, g) *be a connected Riemannian* n*-manifold. The scalar curvature* $\mathbf{S}$ *of* (M, g) *assigns to each point of* M *a real number defined by the local geometry. Precisely,* $\mathbf{S} = \mathrm{tr}_g(\mathrm{Ric})$*.*

Geometrically, we may compute S at a point p as the following derivative

$$S = -(3n+2)\, \frac{d^2}{d\varepsilon^2} \frac{\mathrm{Vol}(\mathrm{B}_\varepsilon(\mathrm{p}) \subset \mathrm{M})}{\mathrm{Vol}(\mathrm{B}_\varepsilon(\mathrm{p}) \subset \mathbb{R}^{\mathrm{n}})}\bigg|_{\varepsilon=0}$$

Scalar curvature

Examples

- S^n of radius r has constant scalar curvature equal to $\frac{n(n-1)}{r^2}$.
- Real hyperbolic space has negative scalar curvature.
- $\mathbb{E}^n$ has constant scalar curvature equal to 0.

Positive scalar curvature

Question

When does M admit a metric g of positive scalar curvature κ?

In dimension 2 this is completely solved.

Theorem (Gauss-Bonnet 1848)

Let M be a compact two-dimensional Riemannian manifold, then

$$\kappa = \int_M \mathbf{S} dA = 4\pi\chi(M).$$

The Euler characteristic (a topological invariant) is an obstruction to the geometric problem.

Dimension 3

A consequence of Perelman's proof of the Geometrization conjecture is the following:

Theorem (Perelman 2006, using work of Schoen, Yau, Gromov, and Lawson)

A closed orientable 3-manifold admits a metric of positive scalar curvature if and only if it is a connected sum of spherical 3-manifolds and copies of $S^1 \times S^2$.

Index theory

- Let M be a closed spin manifold and X a spinor bundle.
- Let $L^2(M, X)$ denote the space of square integrable sections $M \to X$.

$$L^2(M, X) = \left\{ f : M \to X \colon \int_M ||f(x)||^2 dx < \infty \right\}$$

- Let $D : L^2(M, X) \to L^2(M, X)$ be the Dirac operator.
- Define $\mathrm{Index(D)} = \dim \ker(\mathrm{D}) - \dim \operatorname{coker}(\mathrm{D})$.

Index theory

- $D^2 = \Delta + \kappa/4$ and $\Delta \geq 0$.
- Now, $\kappa > 0$ implies D^2 invertible.
- Hence, D invertible.
- So $\mathrm{Index(D)} = 0$

Theorem (Lichnerowicz 1963)

$\mathrm{Index(D)} \neq 0$ *implies* M *does not admit a metric with* $\kappa > 0$.

The Atiyah-Singer index theorem

Theorem (Atiyah-Singer 1963)

If M is a closed spin $4k$-manifold then $\mathrm{Index(D)} = \hat{\mathrm{A}}\mathrm{(M)}$.

Here $\hat{A}(M)$ is the "A-hat genus of M", a topological invariant.

A more general obstruction

Theorem (Rosenberg 1983)

Let M be a closed spin n-manifold and G a discrete group. Let $u : M \to BG$ be a continuous map. If M admits a metric of positive scalar curvature, then $\alpha[M, u] = 0 \in KO_n(C_r^* G)$.

Here $\alpha : \Omega_n^{\mathrm{Spin}}(BG) \to KO_n(C_r^* G)$ is the index of the Dirac operator.

The map α

We may factor α as

$$\Omega_n^{\mathrm{Spin}}(BG) \xrightarrow{D} ko_n(BG)$$

$$\xrightarrow{\quad p \quad} KO_n(BG) \xrightarrow{\mu_{\mathbb{R}}} KO_n(C_r^*(G;\mathbb{R})).$$

Here, D is the ko-orientation of spin bordism, p is the connective covering map of spectra and $\mu_{\mathbb{R}}$ is Rosenberg's assembly map.

Connective KO-theory

If $K^*(*) = \mathbb{Z}[x, x^{-1}]$ where x has degree 2. The *c*onnective K-theory of a point is $k^*(*) = \mathbb{Z}[x]$ where x has degree 2.

Similarly,

$$ko_*(*) = \begin{cases} KO_*(*) & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases}$$

G-CW complexes

Definition (G-CW complex)

A G-CW complex is a G-space X equipped with a filtration

$$\emptyset \subset X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq \bigcup_{n\in\mathbb{N}} X^{(n)} = X$$

Each $X^{(n)}$ is obtained from $X^{(n-1)}$ via a G-pushout of the form

$$\begin{tikzcd}
\coprod_{i\in I_n} G/H_i \times S^{n-1} \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\
\coprod_{i\in I_n} G/H_i \times D^n \arrow[r] & X^{(n)}
\end{tikzcd}$$

where $H_i < G$ are subgroups called the isotropy groups of X.

Families of subgroups

Definition (Family of subgroups)

A family $\mathcal{F}$ of subgroups of G is a collection of subgroups of G closed under taking subgroups and conjugation.

Examples:

- The trivial family $\mathcal{TRV} = \{\{1\}\}$.
- The family of finite subgroups $\mathcal{FIN}$.
- The family of virtually cyclic subgroups $\mathcal{VC}$.
- The family of all subgroups $\mathcal{ALL}$.

Classifying spaces for families

Definition

Let G be a group with family of subgroups $\mathcal{F}$. A G-CW complex X is a model for the classifying space $E_{\mathcal{F}}G$ if its isotropy groups are in $\mathcal{F}$ and for each $H \in \mathcal{F}$, the fixed point set X^H is contractible.

Note that for a discrete group G we have $E_{\mathcal{TRV}}G = EG$.

The universal property

Proposition

For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to E_{\mathcal{F}}G$.

It follows there is a unique composite map (up to G-homotopy):

$$EG \longrightarrow E_{\mathcal{FIN}}G \longrightarrow E_{\mathcal{VC}}G \longrightarrow E_{\mathcal{ALL}}G \simeq G/G \simeq \{*\}$$

Notation

We will denote $E_{\mathcal{FIN}}G$ by $\underline{\mathrm{E}}G$ and $\underline{\mathrm{E}}G/G$ by $\underline{\mathrm{B}}G$

A group G has property:
(M) if every finite subgroup is contained in a unique maximal finite subgroup.
(NM) if every maximal finite subgroup is self normalising.

The (Real) Baum-Connes Conjecture

Conjecture (Baum-Connes)

The assembly map $K_n^G(\underline{E}G) \to K_n(C_r^*(G))$*, induced by the projection* $\underline{E}G \to \{*\}$*, is an isomorphism.*

The Baum-Connes Conjecture implies that Rosenberg's assembly map is injective.

The map α

We may factor α as

$$\Omega_n^{\mathrm{Spin}}(BG) \xrightarrow{D} ko_n(BG)$$

$$\xrightarrow{\quad p \quad} KO_n(BG) \xrightarrow{\mu_{\mathbb{R}}} KO_n(C_r^*(G;\mathbb{R})).$$

Here, D is the ko-orientation of spin bordism, p is the connective covering map of spectra and $\mu_{\mathbb{R}}$ is induced by $EG \to \{*\}$.

The unstable Gromov-Lawson-Rosenberg conjecture

Conjecture (Gromov-Lawson-Rosenberg)

Let M be a closed spin n-manifold, $n \geq 5$ with $\pi_1 M = G$. Suppose that $u : M \to BG$ induces the identity on G, then M admits a metric of positive scalar curvature if and only if $\alpha[M, u] = 0 \in KO_n(C_r^* G)$.

Positive and negative results

The conjecture has been verified for:

- All simply connected M [Stolz 1992].
- When $\pi_1(M)$ is finite with periodic cohomology [Botvinnik-Gilkey-Stolz 1997].
- $G = \pi_1(M)$ is torsion free discrete and $\dim BG \leq 9$ [Joachim-Schick 1992].
- $\pi_1(M)$ is a Fuchsian group [Davis-Pearson 2003].

There is a counterexample due to [Schick 2004] with $\pi_1(M) = \mathbb{Z}^4 \oplus \mathbb{Z}_3$.

Towards a new result

Proposition

Let Γ be a group satisfying (M), (NM), the Baum-Connes conjecture, and be such that all maximal finite subgroups have periodic cohomology. If $\underline{B}G$ is finite and

$$p : \widetilde{ko}_n(\underline{B}\Gamma) \to \widetilde{KO}_n(\underline{B}\Gamma)$$

is an isomorphism for all $n \geq 6$ and injective for $n = 5$, then Γ satisfies the unstable GLR conjecture.

The proof

When Γ satisfies (M) and (NM), the p-chain spectral sequence of Davis and Lück is very well behaved. It collapses on the E^2-page.
Let $X = \underline{B}\Gamma$. Using this we get a commutative diagram:

$$\begin{tikzcd}
\widetilde{ko}_{n+1}(X) \arrow[r] \arrow[d, "p"] & \bigoplus_{(H)\in\Lambda} \widetilde{ko}_n(BH) \arrow[r] \arrow[d, "\mu_{\mathbb{R}}\circ p"] & \widetilde{ko}_n(B\Gamma) \arrow[r] \arrow[d, "\mu_{\mathbb{R}}\circ p"] & \widetilde{ko}_n(X) \arrow[d, "p"] \\
\widetilde{KO}_{n+1}(X) \arrow[r] & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n(C_r^*(H;\mathbb{R})) \arrow[r] & \widetilde{KO}_n(C_r^*(\Gamma;\mathbb{R})) \arrow[r] & \widetilde{KO}_n(X).
\end{tikzcd}$$

If $\beta \in \ker(c)$, then there exists $\gamma \in \ker(b)$.

The proof (cont.)

For a group L let $ko_n^+(BL)$ be the subgroup of $ko_n(BL)$ given by $D[M, f]$ where M is a positively curved spin manifold and f is a continuous map.

Now Botvinnik, Gilkey and Stolz (1997) prove for a finite group H of odd order with periodic cohomology that

$$ko_n^+(BH) = \ker(\mu_{\mathbb{R}} \circ p : ko_n(BH) \to KO_n^{\mathrm{top}}(C_r^*(H))$$

and so $\gamma \in ko_n^+(B\Gamma)$.

A result of Stolz (1995) states that if $D[M, f] \in ko_n^+(B\Gamma)$, then M admits a metric of positive scalar curvature. $\square$

A lemma

Lemma (Joachim-Schick 1998)

Let X be a finite CW complex of dimension at most 9, then $p : \widetilde{ko}_n(X) \to \widetilde{KO}_n(X)$ is an isomorphism for all $n \geq 6$ and an injection for $n = 5$.

- The map p induces a map of Atiyah-Hirzebruch spectral sequences $E^*_{*,*} \to F^*_{*,*}$ where,

$$E^2_{p,q} := H_p(X; ko_q) \quad \text{and} \quad F^2_{p,q} := H_p(X; KO_q)$$

- These converge to $ko_{p+q}(X)$ and $KO_{p+q}(X)$.

A lemma

The E^2 page for ko:

4	$H_0(X;\mathbb{Z})$	$H_1(X;\mathbb{Z})$	$H_2(X;\mathbb{Z})$	$H_3(X;\mathbb{Z})$	$H_4(X;\mathbb{Z})$
3	0	0	0	0	0
2	$H_0(X;\mathbb{Z}_2)$	$H_1(X;\mathbb{Z}_2)$	$H_2(X;\mathbb{Z}_2)$	$H_3(X;\mathbb{Z}_2)$	$H_4(X;\mathbb{Z}_2)$
1	$H_0(X;\mathbb{Z}_2)$	$H_1(X;\mathbb{Z}_2)$	$H_2(X;\mathbb{Z}_2)$	$H_3(X;\mathbb{Z}_2)$	$H_4(X;\mathbb{Z}_2)$
0	$H_0(X;\mathbb{Z})$	$H_1(X;\mathbb{Z})$	$H_2(X;\mathbb{Z})$	$H_3(X;\mathbb{Z})$	$H_4(X;\mathbb{Z})$
-1	0	0	0	0	0
-2	0	0	0	0	0
-3	0	0	0	0	0
-4	0	0	0	0	0

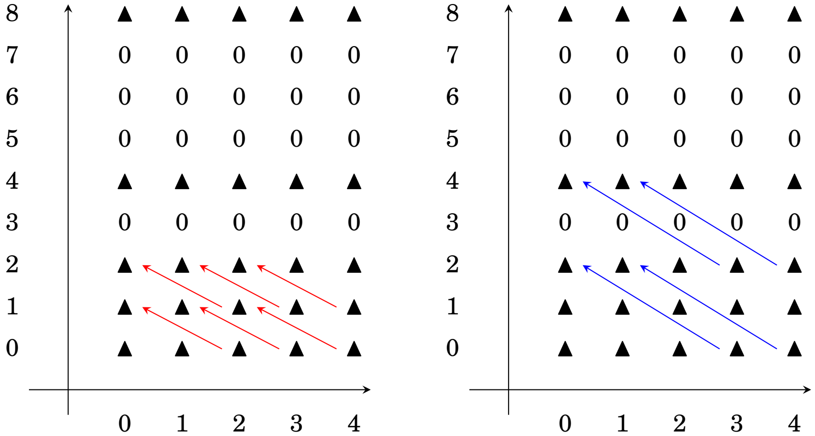
A lemma

The F^2 page for KO:

$$
\begin{array}{c|ccccc}
4 & H_0(X;\mathbb{Z}) & H_1(X;\mathbb{Z}) & H_2(X;\mathbb{Z}) & H_3(X;\mathbb{Z}) & H_4(X;\mathbb{Z}) \\
3 & 0 & 0 & 0 & 0 & 0 \\
2 & H_0(X;\mathbb{Z}_2) & H_1(X;\mathbb{Z}_2) & H_2(X;\mathbb{Z}_2) & H_3(X;\mathbb{Z}_2) & H_4(X;\mathbb{Z}_2) \\
1 & H_0(X;\mathbb{Z}_2) & H_1(X;\mathbb{Z}_2) & H_2(X;\mathbb{Z}_2) & H_3(X;\mathbb{Z}_2) & H_4(X;\mathbb{Z}_2) \\
0 & H_0(X;\mathbb{Z}) & H_1(X;\mathbb{Z}) & H_2(X;\mathbb{Z}) & H_3(X;\mathbb{Z}) & H_4(X;\mathbb{Z}) \\
\hline
-1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & 0 & 0 \\
-4 & H_0(X;\mathbb{Z}) & H_1(X;\mathbb{Z}) & H_2(X;\mathbb{Z}) & H_3(X;\mathbb{Z}) & H_4(X;\mathbb{Z})
\end{array}
$$

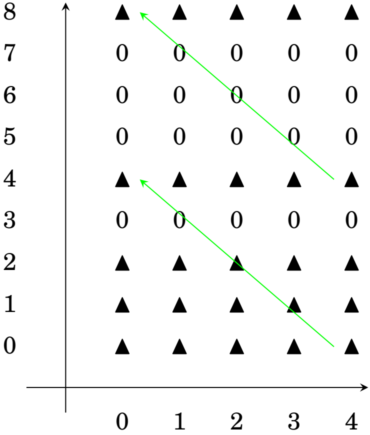
A lemma

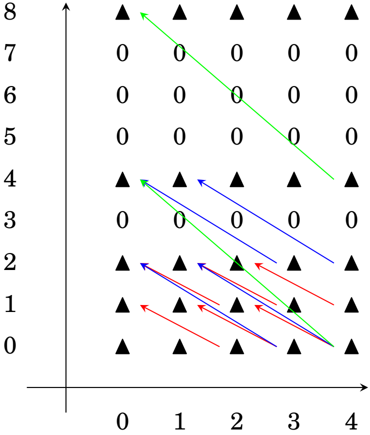
The d^2 and d^3 differentials:



A lemma

The differential d^5 and all of the differentials together:





A lemma

- Now, the differentials in both spectral sequences are given by the same duals of cohomology operations.
- It follows $E^n_{p,q} \cong F^n_{p,q}$ for $q \geq 6$ and $E^n_{p,q} \rightarrowtail F^n_{p,q}$ for $n = 5$.
- The spectral sequences converge to $E^\infty = \mathrm{Gr}\ \mathrm{ko(X)}$ and $F^\infty = \mathrm{Gr}\ \mathrm{KO(X)}$.
- The difference between $\mathrm{Gr}\ \mathrm{ko_n(X)}$ and $ko_n(X)$ is a sequence of extension problems. The Five Lemma yields that a solution to each extension problem in $ko_n(X)$ determines an isomorphic solution in $KO_n(X)$ for $n \geq 6$. $\square$

Putting it together

Theorem (H.)

Let G be a discrete group satisfying (M), (NM), the Baum-Connes conjecture, and be such that all maximal finite subgroups have periodic cohomology. If $\underline{B}G$ is finite and has dimension at most 9, then G satisfies the unstable GLR conjecture.

Examples

- 3-manifold groups with no elements of order 2;
- One relator groups;
- Many S-arithmetic subgroups of $\mathrm{PSL}_2(\mathbb{R})$;
- Graphs of groups of the above with torsion-free edge groups admitting a finite model for $\underline{\mathrm{B}}G$.

Thanks for listening!