PROFINITE RIGIDITY OF AFFINE COXETER GROUPS

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Abstract. We prove that affine Coxeter groups are profinitely rigid.

1. Introduction

For a group G we denote by $\mathcal{F}(G)$ the set of isomorphism classes of finite quotients of G. A group G is called *profinitely rigid relative to a class of groups* $\mathcal{C}$ if $G \in \mathcal{C}$ and for any group H in the class $\mathcal{C}$ whenever $\mathcal{F}(G) = \mathcal{F}(H)$, then $G \cong H$. A finitely generated residually finite group G is called *profinitely rigid* if G is profinitely rigid among all finitely generated residually finite groups.

Theorem 1.1. *Affine Coxeter groups are profinitely rigid.*

Given a finite graph Γ with the vertex set $V(\Gamma)$, the edge set $E(\Gamma)$ and an edge-labeling $m\colon E(\Gamma) \to \mathbb{N}_{\geqslant 3} \cup \{\infty\}$, the associated *Coxeter group* W_Γ is given by the presentation

$$W_\Gamma = \left\langle V(\Gamma) \;\middle|\; \begin{array}{l} v^2 \text{ for all } v \in V(\Gamma), (vw)^2 \text{ if } \{v,w\} \notin E(\Gamma), \\ (vw)^{m(\{v,w\})} \text{ if } \{v,w\} \in E(\Gamma) \text{ and } m(\{v,w\}) < \infty \end{array} \right\rangle.$$

The Coxeter groups associated to the graphs in Figure 1 are precisely the *irreducible affine Coxeter groups*. More generally, a Coxeter group W_Γ is *affine* if Γ is a disjoint union of those graphs. It was shown in [17] that irreducible affine Coxeter groups are profinitely rigid relative to the class consisting of all Coxeter groups, our main result generalises this. Other work on profinite rigidity of Coxeter groups can be found in [2, 3, 6, 24].

An *n-dimensional crystallographic group* G is a discrete, cocompact subgroup of the group of isometries of the Euclidean space $\mathbb{E}^n$. An n-dimensional crystallographic group G always gives rise to the short exact sequence $1 \hookrightarrow \mathbb{Z}^n \hookrightarrow G \twoheadrightarrow P \twoheadrightarrow 1$ where P is finite and is called the *point group* of G. By definition, G is *symmorphic* if the above short exact sequence splits. Note that affine Coxeter groups are examples of symmorphic crystallographic groups.

The next proposition collects old and new profinite invariants of crystallographic groups. A group G is said to be *just infinite* if G itself is infinite but all proper quotients of G are finite. Let G and H be crystallographic groups with point groups $P_1, P_2 \leqslant \mathrm{GL}_n(\mathbb{Z})$. By definition, P_1 and P_2 are in the same $\mathbb{Q}$-class if they are conjugate in $\mathrm{GL}_n(\mathbb{Q})$. For a group G, the poset $\mathcal{CF}(G)$ is the lattice of finite subgroups of G modulo the conjugacy relation.

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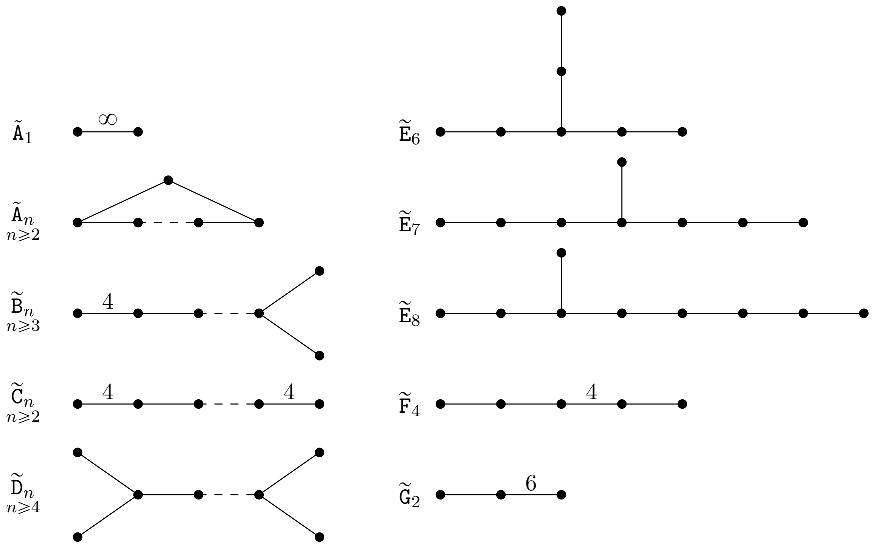

FIGURE 1. Coxeter graphs of affine type.

Proposition 1.2. *Let G be an n-dimensional crystallographic group and H be a finitely generated residually finite group. If $\mathcal{F}(G) = \mathcal{F}(H)$, then H is an n-dimensional crystallographic group whose point group is isomorphic to the point group of G. In particular, if G is profinitely rigid relative to the class of n-dimensional crystallographic groups, then G is profinitely rigid in the absolute sense. Moreover, the following statements hold*

(1) $\mathcal{CF}(G) = \mathcal{CF}(H)$;
(2) G is torsion free if and only if H is torsion free;
(3) G is centreless if and only if H is centreless;
(4) G is just infinite if and only if H is just infinite;
(5) $G^{\mathrm{ab}} \cong H^{\mathrm{ab}}$;
(6) G is symmorphic if and only if H is symmorphic;
(7) the point group of G is in the same $\mathbb{Q}$-class as the point group of H.

Proof. The first paragraph of the proposition is given by Proposition 2.10. We now, prove the “moreover”. (1) is Proposition 2.8, (2) is given by Proposition 2.10, (3) is Proposition 2.12, (4) is Proposition 2.13, (5) is classical (see for example [23]), (6) essentially follows from Grunewald–Zalesskii [13] but we include a proof for completeness (see Proposition 2.10), and finally (7) is due to Piwek–Popovic–Wilkes and follows from the fact that the point groups of G and H are conjugate in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ by [21, page 554] and conjugacy in $\mathrm{GL}_n(\mathbb{Q})$ is a necessary condition for conjugacy over $\widehat{\mathbb{Z}}$ [21, page 558]. □

Remark 1.3. If follows from Proposition 1.2 and [21] that every crystallographic group in dimension at most 4 is profinitely rigid. Note that [11] provides an example of an 11-dimensional crystallographic group with point group of order 55 which is not profinitely rigid. Further, for each prime number $p \geqslant 23$ there exist non profinitely rigid crystallographic groups of shape $\mathbb{Z}^{p-1} \rtimes \mathbb{Z}_p$, see [5, Theorem 1].

Remark 1.4. Since we wrote this paper it has come to light that Theorem 1.1 was proven independently in [19]. The authors there in fact prove *first order rigidity* of affine Coxeter groups and then appeal to a result of Oger [18] which states the two notions are equivalent for abelian-by-finite groups.

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2. The ingredients in Proposition 1.2

The following lemma characterises crystallographic groups among virtually free abelian groups of positive rank algebraically.

Lemma 2.1. *Let G be a virtually free abelian group of rank $n \geqslant 1$. The group G is an n-dimensional crystallographic group if and only if G does not have non-trivial finite normal subgroups.*

Proof. If G is an n-dimensional crystallographic group, then by definition G acts faithfully, cocompactly and by isometries on $\mathbb{E}^n$. Let $\Phi\colon G \to \mathrm{Isom}(\mathbb{E}^n)$ denote a faithful cocompact action. Note that the set of fixed points of an isometry of $\mathbb{E}^n$ is always either empty or an affine subspace, see [1, Part II, Proposition 6.5].

Suppose for a contradiction that $N \trianglelefteqslant G$ is a non-trivial finite normal subgroup.

Claim 2.2. $\mathrm{Fix}(\Phi(N)) = \{x \in \mathbb{E}^n \mid \Phi(n)(x) = x \text{ for all } n \in N\}$ *is a non-empty affine subspace of* $\mathbb{E}^n$*.*

Proof. For $x \in \mathbb{E}^n$, the orbit $O = \Phi(N)(x)$ is bounded since N is finite. Hence the convex hull of O spans a convex polytope. Clearly, $\Phi(N)$ fixes the polytope (not necessarily pointwise but as a polytope). Hence, it fixes the geometric center (also known as the centroid) of the polytope. Since $\mathrm{Fix}(\Phi(N))$ is non-empty it must be an affine subspace. In particular, $\mathrm{Fix}(\Phi(N))$ is closed and convex. ∎

Claim 2.3. $\mathrm{Fix}(N)$ *is G invariant.*

Proof. Let $g \in G$. Then $\Phi(g)(\mathrm{Fix}(\Phi(N))) = \mathrm{Fix}(\Phi(gNg^{-1})) = \mathrm{Fix}(\Phi(N))$ by elementary calculations. ∎

Claim 2.4. *There exists a constant C such that for every $x \in \mathbb{E}^n$ we have $d(x, \mathrm{Fix}(\Phi(N))) \leqslant C$.*

Proof. Since the action is cocompact, there exists a compact set $K \subseteq \mathbb{E}^n$ such that $\bigcup_{g \in G} \Phi(g)(K) = \mathbb{E}^n$, hence the supremum of the distance of two points in K is the desired constant C. ∎

Claim 2.5. $\mathrm{Fix}(\Phi(N)) = \mathbb{E}^n$.

Proof. Since $\mathrm{Fix}(\Phi(N))$ is an affine subspace, there exists an orthogonal projection $\pi\colon \mathbb{E}^n \to \mathrm{Fix}(\Phi(N))$. Now suppose there exists a point $x_0 \in \mathbb{E}^n - \mathrm{Fix}(\Phi(N))$. Consider the unit speed straight line $c\colon [0,\infty) \to \mathbb{E}^n$ starting in $\pi(x_0)$ and going through x_0. By construction and convexity of $\mathrm{Fix}(\Phi(N))$ we obtain $d(c(t), \pi(x_0)) = t$ for all $t \in [0,\infty)$, contradicting Claim 2.4. ■

This is a contradiction, since the action is faithful and we just showed that N is contained in the kernel of Φ.

On the other hand, if G is not crystallographic, then by [26] there exists a non-trivial finite normal subgroup $N \trianglelefteq G$ such that G/N is crystallographic. Hence, if G does not have non-trivial finite normal subgroups, then G is crystallographic. □

Let G be a group and $\mathcal{N}$ be the set of all finite index normal subgroups of G. We equip each G/N, $N \in \mathcal{N}$ with the discrete topology and endow $\prod_{N\in\mathcal{N}} G/N$ with the product topology. We define a map

$$\iota\colon G \to \prod_{N\in\mathcal{N}} G/N \text{ by } g \mapsto (gN)_{N\in\mathcal{N}}.$$

The map ι is injective if and only if G is residually finite. The *profinite completion* of G, denoted by $\widehat{G}$, is defined as $\widehat{G} := \overline{\iota(G)}$. Let G and H be finitely generated residually finite groups. Then $\mathcal{F}(G) = \mathcal{F}(H)$ if and only if $\widehat{G} \cong \widehat{H}$, see [9].

Lemma 2.6. *Let G be a finitely generated residually finite group. Denote by $\iota\colon G \to \widehat{G}$ the canonical homomorphism. If $N \trianglelefteq G$ is a finite normal subgroup, then $\iota(N)$ is normal in $\widehat{G}$.*

Compare to the proof of Theorem 3.6 in [2].

Proof. Assume for contradiction that $\iota(N)$ is not normal in $\widehat{G}$. Then there exist $n \in \iota(N)$ and $g \in \widehat{G}$ such that $gng^{-1} \notin \iota(N)$. Hence the finite set $S := \{gng^{-1}m \mid m \in \iota(N)\}$ does not include the trivial element. We know that $\iota(N) = \{m_1, \dots, m_l\}$. Since $\widehat{G}$ is residually finite, there exists an epimorphism $\psi_k\colon \widehat{G} \twoheadrightarrow H_k$ with H_k finite and $\psi_k(gng^{-1}m_k) \neq 1$ for every $k \in \{1, \dots, l\}$.

Define $\psi = \psi_1 \times \ldots \times \psi_l\colon \widehat{G} \to H_1 \times \ldots \times H_l$ by $(\psi_1 \times \ldots \times \psi_l)(h) = (\psi_1(h), \dots, \psi_l(h))$. In particular, this map has finite image and $1 \notin \psi(S)$. But $\psi \circ \iota(N)$ is normal in the image $\psi \circ \iota(G)$, and $\psi \circ \iota(G) = \psi(\widehat{G})$ by [2, Lemma 2.1], so it is necessary that $1 \in \psi(S)$. This contradiction shows that $\iota(N)$ is normal in $\widehat{G}$. □

Given a group G we denote by $\mathcal{CF}(G)$ the set of conjugacy classes of all finite subgroups in G. We define a partial order on $\mathcal{CF}(G)$ as follows: $[A] \leqslant [B]$ if there exists a $g \in G$ such that $A \subseteq gBg^{-1}$.

Proposition 2.7. *Let G be a finitely generated virtually free abelian group. Then, $\mathcal{CF}(G) = \mathcal{CF}(\widehat{G})$.*

Proof. Let G be a finitely generated virtually free abelian group. We define a map $\psi\colon \mathcal{CF}(G) \to \mathcal{CF}(\widehat{G})$ via $\psi([A]) := [\iota(A)]$. Note ψ is clearly order preserving.

Virtually abelian groups are finite subgroup separable by [12, Theorem 1]. Thus by [6, Lemma 3.4] the map ψ is injective.

We follow the proof strategy used in [15, Theorem 2.7]. For the surjectivity we show that a finite subgroup of $\widehat{G}$ is conjugate to a finite subgroup of G. Let H denote a finite subgroup of $\widehat{G}$. Since G is virtually free abelian, there exists a normal subgroup $A \cong \mathbb{Z}^n$ such that $Q := G/A$ is finite. Thus, we have $H \subseteq G \cdot \widehat{A} = \widehat{A} \cdot G$. Define $\rho\colon H \times \widehat{A} \to \widehat{A}$ where $\rho(h,a) = hah^{-1}$. Since G and $\widehat{A}$ normalize $\widehat{A}$, so does H. Thus $\widehat{A}$ is an H-module, A is an H-submodule and $\widehat{A}/A$ is an H-module.

Let $h \in H$. There exist elements $g_h \in G$ and $x_h \in \widehat{A}$ such that $h = x_h g_h$. The element x_h is in general not uniquely determined by h, however, its image in $\widehat{A}/A$ is, since $G \cap \widehat{A} = A$.

Consider the map $D\colon H \to \widehat{A}/A$ by $h \mapsto x_h A$. A computation shows that the map D is a *derivation*, that is, $D(h_1h_2) = D(h_1) + h_1D(h_2)$, where $h_1D(h_2) = h_1x_{h_2}h_1^{-1}A$, for $h_1, h_2 \in H$. We claim that $H^1(H; \widehat{A}/A) = 0$. Indeed, let k denote the order of H, let $f \in H^1(H; \widehat{A}/A)$ denote a derivation and $g \in H$ an arbitrary element and set $x := \sum_{h\in H} f(h)$. Now, we can compute that $gx = \sum_{h\in H} f(h) - kf(g) = x - kf(g)$. So we obtain $kf(g) = gx' - x'$ for $x' := -x$ in $\widehat{A}/A$. Since $\widehat{A}/A$ is k-divisible, we can divide by k and obtain $f(g) = gy - y$ for $y = x'/k$. Thus, $f = 0$.

Since $H^1(H; \widehat{A}/A) = 0$, we see that D is an *inner derivation*, that is there exists a $b \in \widehat{A}$ such that $D(h) = hbh^{-1}b^{-1}A$ for every $h \in H$. It follows that $bhb^{-1} \in G$, since $D(h) = x_hA = hbh^{-1}b^{-1}A$, which implies $g_hbhb^{-1} \in A \subseteq G$. Hence, $bHb^{-1} \subseteq G$ as desired. This implies the surjectivity of ψ. □

Proposition 2.8. *Let G be a finitely generated virtually free abelian group and H be a finitely generated residually finite group such that $\widehat{G} \cong \widehat{H}$. Then, $\mathcal{CF}(G) = \mathcal{CF}(H)$.*

Proof. Since $\widehat{G} \cong \widehat{H}$ is virtually abelian and $H \hookrightarrow \widehat{H}$ it follows that H is a virtually free abelian group. Hence, by Proposition 2.7 we have order isomorphisms $\mathcal{CF}(G) \to \mathcal{CF}(\widehat{G}) \to \mathcal{CF}(\widehat{H}) \to \mathcal{CF}(H)$. Let α denote the composite isomorphism and note that for any $[A] \in \mathcal{CF}(G)$ and $B \in \alpha([A])$ we have $A \cong B$. □

Lemma 2.9. *Let G be a crystallographic group with point group P. Then G is symmorphic if and only if G has a subgroup isomorphic to P.*

Proof. Let $1 \hookrightarrow \mathbb{Z}^n \hookrightarrow G \overset{\pi}{\twoheadrightarrow} P \twoheadrightarrow 1$ be the short exact sequence associated to G. If there exists a group homomorphism $\varphi\colon P \to G$ such that $\pi \circ \varphi = id_P$, then φ is injective and therefore G has a subgroup $\varphi(P) \cong P$.

For the other direction let $H \leqslant G$ be a subgroup such that $H \cong P$. Since the kernel of π is torsion free, the map $\pi_{|H}\colon H \to P$ is injective and therefore an isomorphism since $|H| = |P|$. We define $\phi := \pi_{|H}^{-1}$. It is straightforward to verify that ϕ is a section. □

Proposition 2.10. *Let G be an n-dimensional crystallographic group with point group P and H be a finitely generated residually finite group. If $\widehat{G} \cong \widehat{H}$, then H is an n-dimensional crystallographic group with point group isomorphic to P. Moreover,*

(1) G is symmorphic if and only if H is symmorphic.
(2) G is torsion free if and only if H is torsion free.

Proof. Let H be a finitely generated residually finite group with $\widehat{G} \cong \widehat{H}$. By [13, Proposition 2.10] follows that H is a virtually free abelian group of rank n with quotient isomorphic to P.

By Lemma 2.1 the crystallographic group G does not have non-trivial finite normal subgroups, thus by Proposition 2.7 we know that $\widehat{G}$ does not have non-trivial finite normal subgroups. Hence, $\widehat{H}$ and therefore H does not have any non-trivial finite normal subgroups either. Thus, by Lemma 2.1 we see that H is an n-dimensional crystallographic group.

Now, Proposition 2.8 implies that G is torsion free if and only if H is torsion free. Further, G has a subgroup isomorphic to P if and only if H has a subgroup isomorphic to P. Thus, by Lemma 2.9 we obtain that G is symmorphic if and only if H is symmorphic. □

Theorem 2.11. [22, Theorem 6] *Let G be a crystallographic group. Then $Z(G) \cong \mathbb{Z}^n$, where n is the rank of the abelianization of G.*

Proposition 2.12. *Let G be an n-dimensional crystallographic group. Then $\widehat{G}$ is centreless if and only if G is centreless.*

Proof. We have $Z(G) \subseteq Z(\widehat{G})$ (see [4, Lemma 2.1]). Hence, if $\widehat{G}$ is centreless, then G is centreless as well.

Now, assume that $Z(G)$ is trivial. By Theorem 2.11 we know that G has finite abelianization, thus the commutator subgroup $[G, G]$ has finite index in G and therefore $\overline{[G,G]} = \widehat{[G,G]}$ has finite index in $\widehat{G}$. It follows that $\widehat{G}^{\mathrm{ab}}$ is finite.

The profinite completion $\widehat{G}$ has a normal subgroup N isomorphic to $\widehat{\mathbb{Z}^n}$ such that $\widehat{G}/N \cong P$ where P is the point group of G. Let $m = |P|$.

Assume for a contradiction that $\widehat{G}$ has a non-trivial centre. By Proposition 2.7 we know that $\widehat{G}$ does not have non-trivial finite normal subgroups, hence the torsion part of the centre of $\widehat{G}$ is trivial. Thus there exists a non-trivial $n_0 \in N \cap Z(\widehat{G})$.

Now we consider the transfer map $\mathrm{tr}\colon \widehat{G} \to N$ defined by Schur in [25] as follows: let $g_1, \ldots, g_m$ be a set of left coset representatives of N in $\widehat{G}$. For $g \in \widehat{G}$ and $i = 1, \ldots, m$, there exists $n_i \in N$ such that $gg_i = g_j n_i$ for some g_j. We define $\mathrm{tr}(g) := n_1 + \ldots + n_m$. In particular we have: $\mathrm{tr}(n_0) = m \cdot n_0$, thus the order of $\mathrm{tr}(n_0)$ is infinite. Hence, $\widehat{G}$ has an infinite abelian quotient which contradicts the fact that the abelianization of $\widehat{G}$ is finite. □

Proposition 2.13. *Let G be an n-dimensional crystallographic group with point group P and H be a finitely generated residually finite group. If $\widehat{G} \cong \widehat{H}$, then G is just infinite if and only if H is just infinite.*

Proof. By Proposition 2.10 we have that H is a crystallographic group. We denote by P' the point group of H. Since $\widehat{G} \cong \widehat{H}$, the point groups P and

P' are in the same $\mathbb{Q}$-class by [21, page 558]. A result of Ratcliffe–Tschantz [22, Theorem 11] shows that a crystallographic group is just infinite if and only if the corresponding representation of the point group $P \to \mathrm{GL}_n(\mathbb{Z})$ is $\mathbb{Z}$-irreducible. Moreover, by [7, page 497] we have that $\mathbb{Z}$-irreducibility is equivalent to $\mathbb{Q}$-irreducibility. Since $\mathbb{Q}$-irreducibility is preserved by conjugation in $\mathrm{GL}_n(\mathbb{Q})$, it follows that G is just infinite if and only if H is just infinite. □

3. Proof of Theorem 1.1

The following lemma follows from [8, Proposition 17.2.1], [20, Theorem 3.4], and Lemma 2.1.

Lemma 3.1. *A Coxeter group W_Γ is crystallographic if and only if every connected component of Γ is isomorphic to one of the graphs in Figure 1.*

Proof of Theorem 1.1. We first prove the result for the irreducible crystallographic Coxeter groups. Let $\widetilde{\mathtt{X}}_n$ be one of the graphs in Figure 1 and let $W = W_{\widetilde{\mathtt{X}}_n}$. Then $W \cong Q(\mathtt{X}_n^\vee) \rtimes W_{\mathtt{X}_n}$, where $Q(\mathtt{X}_n^\vee) \cong \mathbb{Z}^n$ is the corresponding coroot lattice and $W_{\mathtt{X}_n}$ is the corresponding finite Coxeter group. We denote by $Q(\mathtt{X}_n)$ the corresponding root lattice and by $P(\mathtt{X}_n)$ the weight lattice. See [14, pages 81 and 118] for the definitions of these lattices and for the Coxeter graphs of type $\mathtt{X}_n$. Note that by Lemma 3.1 the Coxeter group W is an n-dimensional crystallographic group.

Let G be a finitely generated residually finite group such that $\widehat{W} \cong \widehat{G}$. If $n \leqslant 4$, then $W \cong G$ by Remark 1.3. Now we assume that $n \geqslant 5$. By Proposition 1.2 it follows that G is an n-dimensional symmorphic crystallographic group whose point group is in the same $\mathbb{Q}$-class as $W_{\mathtt{X}_n}$ and $W^{\mathrm{ab}} \cong G^{\mathrm{ab}}$. We consider two cases:

Case 1: *Assume that $\widetilde{\mathtt{X}}_n$ is not of type $\widetilde{\mathtt{B}}_n$ or $\widetilde{\mathtt{C}}_n$.*

Since $n \geqslant 5$ the Coxeter graph $\widetilde{\mathtt{X}}_n$ is of type $\widetilde{\mathtt{A}}_n, \widetilde{\mathtt{D}}_n, \widetilde{\mathtt{E}}_6, \widetilde{\mathtt{E}}_7$ or $\widetilde{\mathtt{E}}_8$. Thus the corresponding root lattice $Q(\mathtt{X}_n)$ is equal to the coroot lattice $Q(\mathtt{X}_n^\vee)$, see [14, pages 102–105]. Hence $W \cong Q(\mathtt{X}_n) \rtimes W_{\mathtt{X}_n}$.

By [10, Theorem 1], there exists a $W_{\mathtt{X}_n}$ invariant lattice L such that $G \cong L \rtimes W_{\mathtt{X}_n}$ and $Q(\mathtt{X}_n) \subseteq L \subseteq P(\mathtt{X}_n)$. Thus W is a normal subgroup of G of index $|L/Q(\mathtt{X}_n)|$. Note that $W^{\mathrm{ab}} \cong W_{\mathtt{X}_n}^{\mathrm{ab}} \cong \mathbb{Z}_2$ by [17, Propositions 2.2 and 2.3]. Since $G^{\mathrm{ab}} \cong W^{\mathrm{ab}} \cong \mathbb{Z}_2$, it follows that $L/Q(\mathtt{X}_n)$ is trivial or is isomorphic to $\mathbb{Z}_2$.

The lattice $Q(\mathtt{X}_n)$ is a normal subgroup of G, thus $G/Q(\mathtt{X}_n) \cong L/Q(\mathtt{X}_n) \rtimes W_{\mathtt{X}_n}$. Since $|L/Q(\mathtt{X}_n)| \leqslant 2$, the semidirect product is indeed a direct product. Thus $G/Q(\mathtt{X}_n) \cong L/Q(\mathtt{X}_n) \times W_{\mathtt{X}_n} \twoheadrightarrow L/Q(\mathtt{X}_n) \times W_{\mathtt{X}_n}^{\mathrm{ab}} \cong L/Q(\mathtt{X}_n) \times \mathbb{Z}_2$. Since $G^{\mathrm{ab}} \cong W^{\mathrm{ab}} \cong \mathbb{Z}_2$ we conclude that $L/Q(\mathtt{X}_n)$ is trivial and therefore $G \cong W$.

Case 2: *Assume that $\widetilde{\mathtt{X}}_n$ is of type $\widetilde{\mathtt{B}}_n$ or $\widetilde{\mathtt{C}}_n$.*

First we note that $W_{\mathtt{B}_n} = W_{\mathtt{C}_n}$. Further, the irreducible affine Coxeter group $W_{\widetilde{\mathtt{X}}_n}$ does not have a quotient isomorphic to $(\mathbb{Z}_2^2) \rtimes W_{\mathtt{B}_n}$ or $\mathbb{Z}_4 \rtimes W_{\mathtt{B}_n}$ by [16, Proposition 7.2].

By [10, Theorem 1] and the discussion on page 217 in [10], there exist $W_{\mathtt{B}_n}$ invariant lattices $L_1 \subseteq L_2 \subseteq L_3$ such that $|L_i/L_{i-1}| = 2$ for $i = 2, 3$ and $W_{\widetilde{\mathtt{B}}_n} \cong L_l \rtimes W_{\mathtt{B}_n}$, $W_{\widetilde{\mathtt{C}}_n} \cong L_k \rtimes W_{\mathtt{B}_n}$ and $G \cong L_m \rtimes W_{\mathtt{B}_n}$ for $k, l, m \in \{1, 2, 3\}$. Moreover, $L_3/L_1 \cong \mathbb{Z}_4$ if n is odd and $L_3/L_1 \cong \mathbb{Z}_2^2$ if n is even. Thus the group $L_3 \rtimes W_{\mathtt{B}_n}$ has a quotient isomorphic to $\mathbb{Z}_2^2 \rtimes W_{\mathtt{B}_n}$ if n is odd and $\mathbb{Z}_4 \rtimes W_{\mathtt{B}_n}$ if n is even, namely $(L_3 \rtimes W_{\mathtt{B}_n})/L_1$.

Further, the group $L_2 \rtimes W_{\mathtt{B}_n}$ has a quotient isomorphic to $\mathbb{Z}_2^3$. More precisely: the abelianization of the point group $W_{\mathtt{B}_n}$ is $\mathbb{Z}_2^2$. Hence, $L_2 \rtimes W_{\mathtt{B}_n} \twoheadrightarrow (L_2 \rtimes W_{\mathtt{B}_n})/L_1 \cong L_2/L_1 \rtimes W_{\mathtt{B}_n} \twoheadrightarrow \mathbb{Z}_2 \times W_{\mathtt{B}_n}^{\mathrm{ab}} \cong \mathbb{Z}_2^3$.

Note that the abelianization of $W_{\widetilde{\mathtt{B}}_n}$ is isomorphic to $\mathbb{Z}_2^2$. Thus $W_{\widetilde{\mathtt{B}}_n} \cong L_1 \rtimes W_{\mathtt{B}_n}$.

Since $W_{\widetilde{\mathtt{C}}_n}$ does not have a quotient isomorphic to $\mathbb{Z}_2^2 \rtimes W_{\mathtt{B}_n}$ or $\mathbb{Z}_4 \rtimes W_{\mathtt{B}_n}$ we know that $W_{\widetilde{\mathtt{C}}_n} \cong L_2 \rtimes W_{\mathtt{B}_n}$. Thus the groups $W_{\widetilde{\mathtt{B}}_n} \cong L_1 \rtimes W_{\mathtt{B}_n}$ and $W_{\widetilde{\mathtt{C}}_n} \cong L_2 \rtimes W_{\mathtt{B}_n}$ can be distinguished from $L_3 \rtimes W_{\mathtt{B}_n}$ by their finite quotients.

Further, the abelianization of $W_{\widetilde{\mathtt{C}}_n}$ is $\mathbb{Z}_2^3$, thus the group $W_{\widetilde{\mathtt{B}}_n}$ can be distinguished from $W_{\widetilde{\mathtt{C}}_n}$ by the abelianisation. Finally, we obtain $G \cong W_{\widetilde{\mathtt{X}}_n}$.

It remains to deal with the case of a non-trivial direct product. Let $W_{\Gamma_1}, \ldots, W_{\Gamma_n}$ be irreducible affine Coxeter groups. Assume that $\widehat{W_{\Gamma_1}} \times \ldots \times \widehat{W_{\Gamma_n}} \cong \widehat{G}$. By Proposition 1.2, G is a symmorphic crystallographic group. We may decompose G as a direct product of directly indecomposable groups $G_1, \ldots, G_m$, thus $G \cong G_1 \times \ldots \times G_m$ and each G_i is a symmorphic crystallographic group for $i = 1, \ldots, m$. Now, applying [13, Proposition 2.17 (2)] we obtain $n = m$ and that there exists $\sigma \in \mathrm{Sym}(m)$ such that $\widehat{W_{\Gamma_i}} \cong \widehat{G_{\sigma(i)}}$. More precisely, the direct product structure gives rise to a blockmatrix structure in the representation of the point group in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$, this is preserved by conjugation and yields the desired conclusions. Since irreducible affine Coxeter groups are profinitely rigid we obtain $W_{\Gamma_i} \cong G_{\sigma(i)}$. Thus $W_{\Gamma_1} \times \ldots \times W_{\Gamma_m} \cong G$. □

References

[1] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Cited on Page 3.

[2] Martin R. Bridson, Marston D. E. Conder, and Alan W. Reid. Determining Fuchsian groups by their finite quotients. *Israel J. Math.*, 214(1):1–41, 2016. Cited on Pages 1 and 4.

[3] Martin R. Bridson, David B. McReynolds, Alan W. Reid, and Ryan Spitler. On the profinite rigidity of triangle groups. *Bull. Lond. Math. Soc.*, 53(6):1849–1862, 2021. Cited on Page 1.

[4] Martin R. Bridson, Alan W. Reid, and Ryan Spitler. Absolute profinite rigidity, direct products, and finite presentability, 2023, arXiv:2312.06058. Cited on Page 6.

[5] Robert C. Brigham. On the isomorphism problem for just-infinite groups. *Communications on pure and applied mathematics*, XXIV:789–796, 1971. Cited on Page 2.

[6] S. M. Corson, S. Hughes, P. Möller, and O. Varghese. Higman-Thompson groups and profinite properties of right-angled Coxeter groups, 2023, arXiv:2309.06213. Cited on Pages 1 and 5.

[7] C. W. Curtis and I. Reiner. *Representation Theory of finite groups and associative algebras*, volume 11 of *Pure and Applied Mathematics*. Interscience Publishers, New York, 1962. Cited on Page 7.

[8] Michael W. Davis. *The geometry and topology of Coxeter groups*, volume 32 of *London Mathematical Society Monographs*. Princeton University Press, 2008. Cited on Page 7.

[9] J. D. Dixon, E. W. Formanek, J. C. Poland, and L. Ribes. Profinite completions and isomorphic finite quotients. *Journal of Pure and Applied Algebra*, 23:227–231, 1982. Cited on Page 4.

[10] Walter Feit. Integral representations of Weyl groups rationally equivalent to the reflection representation. *J. Group Theory*, 1(3):213–218, 1998. Cited on Page 7.

[11] H. Finken, J. Neubüser, and W. Plesken. Space groups and groups of prime-power order. II. Classification of space groups by finite factor groups. *Arch. Math. (Basel)*, 35(3):203–209, 1980. Cited on Page 2.

[12] Fritz Grunewald and Daniel Segal. Conjugacy in polycyclic groups. *Comm. Algebra*, 6(8):775–798, 1978. Cited on Page 5.

[13] Fritz Grunewald and Pavel Zalesskii. Genus for groups. *J. Algebra*, 326:130–168, 2011. Cited on Pages 2, 6, and 8.

[14] Richard Kane. *Reflection groups and invariant theory*, volume 5 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer-Verlag, New York, 2001. Cited on Page 7.

[15] Peter H. Kropholler and John S. Wilson. Torsion in profinite completions. *J. Pure Appl. Algebra*, 88(1-3):143–154, 1993. Cited on Page 5.

[16] George Maxwell. The normal subgroups of finite and affine Coxeter groups. *Proc. London Math. Soc. (3)*, 76(2):359–382, 1998. Cited on Page 7.

[17] Philip Möller and Olga Varghese. On quotients of Coxeter groups. *J. Algebra*, 639:516–531, 2024. Cited on Pages 1 and 7.

[18] Francis Oger. Elementary equivalence and profinite completions: a characterization of finitely generated abelian-by-finite groups. *Proc. Amer. Math. Soc.*, 103(4):1041–1048, 1988. Cited on Page 3.

[19] Gianluca Paolini and Rizos Sklinos. Profinite rigidity of affine Coxeter groups, 2024, arXiv:2407.01141. Cited on Page 3.

[20] Luis Paris and Olga Varghese. Narrow normal subgroups of Coxeter groups and of automorphism groups of Coxeter groups. *Journal of Group theory*, 27:255–274, 2024. Cited on Page 7.

[21] Paweł Piwek, David Popović, and Gareth Wilkes. Distinguishing crystallographic groups by their finite quotients. *J. Algebra*, 565:548–563, 2021. Cited on Pages 2 and 7.

[22] John G. Ratcliffe and Steven T. Tschantz. Fibered orbifolds and crystallographic groups. *Algebr. Geom. Topol.*, 10(3):1627–1664, 2010. Cited on Pages 6 and 7.

[23] Alan W. Reid. Profinite rigidity. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pages 1193–1216. World Sci. Publ., Hackensack, NJ, 2018. Cited on Page 2.

[24] Yuri Santos Rego and Petra Schwer. The galaxy of Coxeter groups. *J. Algebra*, 656:406–445, 2024. Cited on Page 1.

[25] I. Schur. Neuer Beweis eines Satzes über endliche Gruppen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, pages 1013–1019, 1902. Cited on Page 6.

[26] H. Zassenhaus. Über einen Algorithmus zur Bestimmung der Raumgruppen. *Comment. Math. Helv.*, 21:117–141, 1948. Cited on Page 4.

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