CENTRALISERS AND THE VIRTUALLY CYCLIC DIMENSION OF $\mathrm{Out}(F_N)$

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ABSTRACT. We prove that the virtually cyclic (geometric) dimension of the finite index congruence subgroup $\mathrm{IA}_N(3)$ of $\mathrm{Out}(F_N)$ is $2N - 2$. From this we deduce the virtually cyclic dimension of $\mathrm{Out}(F_N)$ is finite. Along the way we prove Lück's property (C) holds for $\mathrm{Out}(F_N)$, we prove that the commensurator of a cyclic subgroup of $\mathrm{IA}_N(3)$ equals its centraliser, we give a weaker $\mathrm{IA}_N(3)$ analogue of various exact sequences arising from reduction systems for mapping class groups, and give a near complete description of centralisers of infinite order elements in $\mathrm{IA}_3(3)$.

1. INTRODUCTION

Let F_N denote the free group on N generators and let $\mathrm{Out}(F_N)$ denote its outer automorphism group. The study of $\mathrm{Out}(F_N)$ has been ubiquitous in geometric group theory and low dimensional topology; finding connections with arithmetic groups, mapping class groups, moduli spaces of graphs, and many others. Despite this, the topology of $\mathrm{Out}(F_N)$ has been notoriously hard to pin down.

It is known that the virtual cohomological dimension of $\mathrm{Out}(F_N)$ is $2N-3$ [CV86] and that $\mathrm{Out}(F_N)$ is a virtual duality group [BF00]. Much work has gone into computing various Euler characteristics of $\mathrm{Out}(F_N)$ [SV87b, SV87a, BV20] and in low dimensions the rational cohomology has been computed [HV98, Oha08, Bar16]. Moreover, some homological stability phenomena has been observed [Hat95, HV04, HVW06]. In this article we will be concerned with a topological property of a different flavour.

Virtually cyclic dimension. Given a group G, a collection of subgroups $\mathcal{G}$ is called a *family* if it is closed under conjugation and under taking subgroups. We say that a G-CW-complex X is a model for the classifying space $E_{\mathcal{G}}G$ if, given any G-CW-complex Y with isotropy in $\mathcal{G}$, there is up to homotopy a unique G-map $Y \to X$. Such a model always exists and it is unique up to

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G-homotopy equivalence. The *geometric dimension* of G with respect to the family $\mathcal{G}$, denoted $\mathrm{gd}_{\mathcal{G}}(G)$, is the minimum dimension n such that G admits an n-dimensional model for $E_{\mathcal{G}}G$.

The most studied families are: $\mathcal{TR}$ the family that contains only the trivial subgroup, $\mathcal{FIN}$ the family of finite subgroups, and $\mathcal{VC}$ the family of virtually cyclic subgroups. For the first, one recovers EG and all of the classical group cohomology to go with it. The latter two are relevant to the isomorphism conjectures in K-theory, $E_{\mathcal{FIN}}G$, denoted $\underline{E}G$, is relevant to the Baum–Connes Conjecture, and $E_{\mathcal{VC}}G$, the topic of this article and which we will henceforth denote by, $\underline{\underline{E}}G$, is relevant to the Farrell–Jones Conjecture.

The Farrell–Jones Conjecture, one of the most prominent conjectures in modern topology, predicts that a certain 'assembly map'

$$H_n^G(\mathrm{pr})\colon H_n^G(\underline{\underline{E}}G; \mathbf{K}_R) \to K_n(RG)$$

is an isomorphism. Whilst we will not explain all of the ramifications and developments of the Farrell–Jones Conjecture and instead refer the reader to the book project of Lück [Lüc]. We do point out that the Farrell–Jones Conjecture is still open for $\mathrm{Out}(F_N)$ but is known for mapping class groups of finite type surfaces [BB19]. Clearly, the left hand side is explicitly concerned with the classifying space for virtually cyclic actions and so understanding the minimal possible dimension for a model of $\underline{\underline{E}}G$ is of great importance.

In the present article we study the geometric $\mathcal{VC}$-dimension, which from now on we denote by $\underline{\underline{\mathrm{gd}}}(G)$ for a group G. Computations for $\underline{\underline{\mathrm{gd}}}$ are known for (relatively) hyperbolic groups [JPL06, LO07], elementary amenable groups [DP14], discrete linear groups [DKP15], CAT(0) groups [Lüc09, DP15], virtually polycyclic groups [LW12, CFH], mapping class groups of finite type surfaces [JPTN16, NP18], mapping class groups of punctured spheres [AJPTN18], systolic groups [OPa18], braid groups [FGM20], and orientable 3-manifold groups [JLSSn21].

Our goal is to establish that $\underline{\underline{\mathrm{gd}}}(\mathrm{Out}(F_N))$ is finite. For our purposes we are interested also in the finite index congruence subgroup

$$\mathrm{IA}_N(3) \coloneqq \ker(\mathrm{Out}(F_N) \to \mathrm{Aut}(H_1(F_N; \mathbb{F}_3))).$$

Our main result is the following:

Theorem A. *Let $N \geqslant 1$. Then, $\underline{\underline{\mathrm{gd}}}(\mathrm{IA}_N(3)) = 2N - 2$.*

Note that this theorem solves a problem of Lück [Lüc, Problem 10.51] for all torsion-free finite index subgroups of $\mathrm{Out}(F_N)$. From here we establish our desired result:

Corollary B. *Let $N \geqslant 1$. Then, $\underline{\underline{\mathrm{gd}}}(\mathrm{Out}(F_N))$ is finite.*

In a sense, our proof is similar to the analogous results for mapping class groups of finite type surfaces established by Juan-Pineda–Trujillo-Negrete [JPTN16] and Nucinkis–Petrosyan [NP18]. The proofs in the mapping class group case heavily rely on reduction systems and the Nielsen–Thurston classification of mapping classes. Two tools that are not available in the $\mathrm{Out}(F_N)$ setting.

One of the key steps in both proofs for mapping class groups is to the use Lück–Weiermann push-out construction [LW12]. This requires a description of the commensurators of infinite cyclic subgroups by means of short exact sequences coming from the reduction systems of various elements. This process allows for an inductive argument. To this end Juan-Pineda and Trujillo-Negrete prove that the commensurator of any infinite cyclic subgroup C of $\mathrm{MCG}(S)$ can be realized as the normaliser of a finite index subgroup of C. We obtain analogous results for $\mathrm{Out}(F_N)$ that we now describe in great detail.

Centralisers and commensurators. Towards proving the result of Juan-Pineda and Trujillo-Negrete for $\mathrm{Out}(F_N)$ we establish Lück's *Property (C)* (see Section 3 for a definition). We then use this and results in [Gue] to deduce the following theorem.

Theorem 3.5. *Let $\phi \in \mathrm{IA}_N(3)$. The commensurator of the cyclic group $\langle \phi \rangle$ in $\mathrm{IA}_N(3)$ is equal to its centraliser.*

Centralisers of elements of $\mathrm{Out}(F_N)$ have been widely studied in the literature (see for instance [BFH97, KL11, AKP17, RW15, AM, Mut22, Gue]). The centralisers of large families of elements of $\mathrm{Out}(F_N)$ are now completely understood, for example, fully irreducible outer automorphisms by Bestvina–Feighn–Handel [BFH97], atoroidal elements by the work of Feighn–Handel [FH09] or linearly growing elements by Rodenhausen–Wade [RW15] and Andrew–Martino [AM]. All these families are, in some sense, analogues of either pseudo-Anosov homeomorphisms or Dehn twists homeomorphisms and their centralisers have a similar structure. In all these cases, the proofs also imply that the centralisers have a finite index subgroup with a finite classifying space.

However, one major difficulty in understanding the centraliser of an arbitrary element of $\mathrm{Out}(F_N)$ is the lack of a complete analogue of the reduction system as in the case of the mapping class groups. This often prevents us from understanding the centraliser of an automorphism using its action on free groups of smaller ranks in analogy with understanding a mapping class by its action on subsurfaces.

In the present paper, we need to understand classifying spaces of centralisers of *arbitrary* elements of $\mathrm{Out}(F_N)$ in order to apply Lück–Weiermann push-out construction. Therefore, one of the main steps in the proof of Theorem A is to give a (weak) analogue of the reduction systems for mapping class groups. This is the content of the following theorem. We refer to Section 2 for definitions and notations.

Theorem 4.7. *Let $N \geqslant 2$ and let $\phi \in \mathrm{IA}_N(3)$. The centraliser $C(\phi)$ of ϕ in $\mathrm{IA}_N(3)$ satisfies one of the followings.*

   (1) The outer automorphism ϕ is a Dehn twist. There exist a JSJ tree T preserved by $C(\phi)$ and a short exact sequence

$$1 \to K \to C(\phi) \to \prod_{v \in V(F_N \backslash T)} \mathrm{IA}_v(3) \to 1,$$

where K is a free abelian group whose dimension is equal to $|E(F_N \backslash T)|$ and, for every $v \in V(F_N \backslash T)$, the group $\mathrm{IA}_v(3)$ is a finite index subgroup of the group $\mathrm{Out}(G_v, \mathrm{Inc}_v)$. Moreover, ϕ is contained in K.

(2) *There exist $A_1, A_2 \subseteq F_N$ with $F_N = A_1 * A_2$, $\mathrm{rk}(A_1), \mathrm{rk}(A_2) \leqslant N - 1$ and a homomorphism*

$$C(\phi) \to \mathrm{IA}(A_1, 3) \times \mathrm{IA}(A_2, 3)$$

whose kernel is a finite index subgroup of a direct product of two finitely generated free (maybe trivial or cyclic) groups.

(3) *There exist $A_1, \ldots, A_k, B \subseteq F_N$ nontrivial with $F_N = A_1 * \ldots A_k * B$ and a homomorphism*

$$C(\phi) \to \mathbb{Z} \times \prod_{i=1}^{k} \mathrm{IA}(A_i, 3)$$

whose kernel is abelian and ϕ projects onto the $\mathbb{Z}$ factor.

(4) *There exist a JSJ tree T preserved by $C(\phi)$, a partition $VT = V_1 \bigsqcup V_2$ and a homomorphism*

$$C(\phi) \to \mathbb{Z} \times \prod_{v \in V_2} \mathrm{Out}(G_v)$$

whose kernel is abelian and ϕ projects onto the $\mathbb{Z}$ factor.

We note that the case of a Dehn twist was proved by Rodenhausen–Wade [RW15] (see also the work of Cohen–Lustig [CL99]). Theorem 4.7 allows us to understand the centraliser of an element of $\mathrm{IA}_N(3)$ through its action on free groups of smaller ranks. Note the main limitation when compared to reduction theory is that the homomorphisms given in Theorem 4.7 are not necessarily surjective. In particular, we do not always have control on the image. We expect that Theorem 4.7 is a first step towards a complete understanding of centralisers *all* of elements of $\mathrm{Out}(F_N)$ and will be of independent interest. We remark that Guirardel–Horbez [GH21] also constructed dynamical decompositions for subgroups of $\mathrm{Out}(F_N)$ which are distinct from ours but, to our knowledge, could not be directly used to compute the geometric dimension of centralisers and Weyl groups.

Using Theorem 4.7 we are able to prove an inductive structural description of the Weyl group of an infinite order element in $\mathrm{IA}_N(3)$ (see Corollary 4.8). Our other major result on centralisers is a near comprehensive structure description of centralisers of infinite order elements in $\mathrm{IA}_3(3)$. This result is in a sense a specialisation of Theorem 4.7 but requires a careful and explicit analysis of stabilisers of free factor systems of F_3.

Theorem 5.1. *Let $\phi \in \mathrm{IA}_3(3)$. The centraliser $C(\phi)$ of ϕ in $\mathrm{IA}_3(3)$ is of type VF. Moreover, one of the following holds.*

(1) *The centraliser of ϕ is abelian.*

(2) *The centraliser of ϕ is isomorphic to $F \times \mathbb{Z}$ where F is a finitely generated free group.*

(3) *The centraliser of ϕ is isomorphic to a direct product $H \times \mathbb{Z}$ where H is a finite index subgroup of a direct product of two finitely generated free groups.*

(4) *The outer automorphism ϕ is a Dehn twist. There exist a JSJ tree T preserved by $C(\phi)$ and a short exact sequence*

$$1 \to K \to C(\phi) \to \prod_{v \in V(F_3 \backslash T)} \mathrm{IA}_v(3) \to 1,$$

where K is a free abelian group whose dimension is equal to $|E(F_3 \backslash T)|$ and, for every $v \in V(F_3 \backslash T)$, the group $\mathrm{IA}_v(3)$ is a finite index subgroup of the group $\mathrm{Out}(G_v, \mathrm{Inc}_v)$. Moreover, ϕ is contained in K.

We note that in Theorem 5.1, the conclusion that the centralisers are of type VF improves a result of Francaviglia–Martino–Syrigos [FMS21, Theorem 8.2.1]; where they prove that centralisers of infinite order elements of $\mathrm{Out}(F_3)$ are finitely generated (type F_1). This finiteness result crucially depends on work of Rodenhausen–Wade [RW15] where they prove centralisers of Dehn twists are of type VF.

Corollary 5.2. *Let $\phi \in \mathrm{Out}(F_3)$. The centraliser $C(\phi)$ is of type VF.*

Structure of the paper. In Section 2 we recall the necessary background on free factor systems, $\mathrm{IA}_N(3)$, and relative free factor graphs.

In Section 3 we establish Lück's property (C) for $\mathrm{Out}(F_N)$. We then use property (C) and results of [Gue] to prove Theorem 3.5.

In Section 4 we prove Theorem 4.7. The proof of this theorem is separated into two distinct subsections, according to the fixed subgroups of the considered outer automorphism. If the fixed subgroups fill the group F_N, then one can apply the theory of JSJ decompositions of groups. This is done in Section 4.A. Otherwise, the main input is work of Horbez–Wade [HW20] and Guirardel–Horbez (where they attribute some work to Guirardel and Levitt) [GH22] as well as a very careful study of the free factor systems that arise. This is done in Section 4.B. From here we deduce Corollary 4.8 which gives an inductive description of the Weyl groups of elements in $\mathrm{IA}_N(3)$. The work here is a key step towards being able to inductively apply the Lück–Weiermann construction. But, we suspect the results established here will be of independent interest.

In Section 5 we begin to specialise Theorem 4.7 to $\mathrm{IA}_3(3)$ which will form the base case of our induction later. The main result of this section is Theorem 5.1. A thorough analysis of elements of $\mathrm{IA}_3(3)$ allows us to refine Theorem 4.7.

In Section 6 we study the *proper geometric dimension* $\underline{\mathrm{gd}}(W(\phi))$, where $W(\phi)$ is the Weyl group of a Dehn twist ϕ in $\mathrm{IA}_N(3)$, proving it is at most $2N - 4$. Here $\underline{\mathrm{gd}}$ denotes the minimal dimension of a model for $\underline{E}G$. The need for this apparent diversion is that upper bounds on the proper geometric dimension feed into our inductive argument. The remaining cases of possible centralisers and their Weyl groups are dealt with later but the argument for Dehn twists turns out to be somewhat more technical.

Finally, in Section 7 we combine our analysis of the centralisers, commensurators, and Weyl groups of $\mathrm{IA}_N(3)$ to prove Theorem A and deduce Corollary B. The build up to the use of Lück–Weiermann turns out to be involved. The key steps being Proposition 7.3 and Proposition 7.4 where we study $\underline{\mathrm{gd}}(W(H))$ for the Weyl group of H an infinite cyclic subgroup of

$\mathrm{IA}_3(3)$, or a maximal infinite cyclic subgroup of $\mathrm{IA}_N(3)$ respectively. At this point the main theorem is at hand.

2. Preliminaries

2.A. **Free factor systems.** Let $N \geqslant 2$ and let F_N be a nonabelian free group of rank N. A *free factor system of F_N* is a finite set $\mathcal{F} = \{[A_1], \ldots, [A_k]\}$ of conjugacy classes of subgroups of F_N such that there exists a subgroup B of F_N with $F_N = A_1 * \ldots * A_k * B$.

A free factor system $\mathcal{F}$ of F_N is *sporadic* if either $\mathcal{F} = \{[A]\}$ and $F_N = A * \mathbb{Z}$ or $\mathcal{F} = \{[A], [B]\}$ and $F_N = A * B$. Otherwise, we say that $\mathcal{F}$ is *nonsporadic*.

The group $\mathrm{Out}(F_N)$ has a natural action on the set of free factor systems and we denote by $\mathrm{Out}(F_N, \mathcal{F})$ the stabiliser of a free factor system $\mathcal{F}$. Let $\phi \in \mathrm{Out}(F_N)$ and let $\mathcal{F}$ be a free factor system of F_N. Suppose that ϕ fixes every element of $\mathcal{F}$. Then, for every $[A] \in \mathcal{F}$, by malnormality of A, the element ϕ induces an element $\phi|_A \in \mathrm{Out}(A)$.

The collection of free factor systems is equiped with a natural partial order, where $\mathcal{F}_1 \leqslant \mathcal{F}_2$ if for every $[A] \in \mathcal{F}_1$, there exists $[B] \in \mathcal{F}_2$ such that $A \subseteq B$.

2.B. **Properties of the subgroup $\mathrm{IA}_N(3)$.** Let $N \geqslant 2$ and let

$$\mathrm{IA}_N(3) = \ker(\mathrm{Out}(F_N) \to \mathrm{Aut}(H_1(F_N; \mathbb{F}_3))).$$

In this section, we recall some properties of $\mathrm{IA}_N(3)$. Most of them show some aperiodic properties of the group $\mathrm{IA}_N(3)$ which will be of great interest in the rest of the paper.

Proposition 2.1. [BFH00, Corollary 5.7.6] *The group $\mathrm{IA}_N(3)$ is torsion free.*

Theorem 2.2. [HM20a, Theorem II.3.1] *Let H be a subgroup of $\mathrm{IA}_N(3)$.*

(1) *Suppose $\mathcal{F}$ is an H-periodic free factor system. Then $\mathcal{F}$ is fixed by H and every element $[A] \in \mathcal{F}$ is fixed by H.*

(2) *If $[g]$ is an H-periodic conjugacy class of some element $g \in F_N$, then $[g]$ is fixed by H.*

Theorem 2.3. [HM20b, Theorem 1.1] *Let H be a virtually abelian subgroup of $\mathrm{IA}_N(3)$. Then H is abelian and finitely generated.*

Lemma 2.4. [Gue, Lemma 2.9] *A subgroup H of $\mathrm{IA}_N(3)$ is abelian if and only if it does not contain a nonabelian free group.*

Theorem 2.5. [Gue, Theorem 1.1] *For all $\phi, \psi \in \mathrm{IA}_N(3)$, if there exists $m \in \mathbb{Z}^*$ such that $\phi^m = \psi^m$, then $\phi = \psi$.*

Proposition 2.6. [Gue, Corollary 4.2] *For all $\phi, \psi \in \mathrm{IA}_N(3)$, if there exist $m, n \in \mathbb{Z}^*$ such that ϕ^m and ψ^n commute, then ϕ and ψ commute.*

2.C. **The relative free factor graph.** In this section, we introduce a Gromov hyperbolic space on which $\mathrm{Out}(F_N)$ acts by isometries. This space will play a key role in the study of centralisers of elements of $\mathrm{Out}(F_N)$ as centralisers will fix points of its Gromov boundary.

Let $\mathcal{F}$ be a free factor system of F_N. An $(F_N, \mathcal{F})$-*free factor system* is a proper free factor system $\mathcal{F}'$ of F_N with $\mathcal{F} < \mathcal{F}'$. An $(F_N, \mathcal{F})$-*free factor* is a subgroup A of F_N such that there exists an $(F_N, \mathcal{F})$-free factor system $\mathcal{F}'$ of F_N with $[A] \in \mathcal{F}'$.

The *free factor graph of F_N relative to $\mathcal{F}$*, denoted by $\mathbb{F}(F_N, \mathcal{F})$, is the graph whose vertices are the conjugacy classes of $(F_N, \mathcal{F})$-free factors of F_N, two such conjugacy classes $[A], [B]$ being adjacent if either $A \subsetneq B$ or $B \subsetneq A$.

By a result of Handel–Mosher [HM], the graph $\mathbb{F}(F_N, \mathcal{F})$ is Gromov-hyperbolic (see also the work of Bestvina–Feighn [BF14] for the case $\mathcal{F} = \varnothing$ and the work of Guirardel–Horbez [GH22, Proposition 2.11] for general free products of groups).

The group $\mathrm{Out}(F_N, \mathcal{F})$ acts naturally on $\mathbb{F}(F_N, \mathcal{F})$ by isometries. An outer automorphism $\phi \in \mathrm{Out}(F_N, \mathcal{F})$ is *fully irreducible relative to $\mathcal{F}$* if there does not exist a proper free factor system $\mathcal{F} < \mathcal{F}'$ fixed by a power of ϕ. These elements are the loxodromic elements of $\mathbb{F}(F_N, \mathcal{F})$.

Theorem 2.7. [Gup18, Theorem A] *Let $\mathcal{F}$ be a nonsporadic free factor system of F_N. An element $\phi \in \mathrm{Out}(F_N, \mathcal{F})$ is a loxodromic element of $\mathbb{F}(F_N, \mathcal{F})$ if and only if ϕ is fully irreducible relative to $\mathcal{F}$.*

The following theorem was proved by Handel and Mosher [HM20a] when the subgroup is finitely generated case and by Guirardel and Horbez [GH22] in the general case.

Theorem 2.8. [GH22, Theorem 7.1][HM20a, Theorem A] *Let H be a subgroup of $\mathrm{IA}_N(3)$ and let $\mathcal{F}$ be a maximal proper H-invariant free factor system. Suppose that $\mathcal{F}$ is nonsporadic. Then H contains a fully irreducible outer automorphism relative to $\mathcal{F}$.*

We record the following fact, which is a consequence of the description of the Gromov boundary of $\mathbb{F}(F_N, \mathcal{F})$. It is due to Hamenstädt [Ham] for the case $\mathcal{F} = \varnothing$, and Guirardel and Horbez [GH22] for the general case. We refer to [GH22, Section 3] for the definition of an $(F_N, \mathcal{F})$-arational tree.

Proposition 2.9. [GH22, Theorem 3.4] *Let $\mathcal{F}$ be a nonsporadic free factor system of F_N and let H be a subgroup of $\mathrm{Out}(F_N, \mathcal{F})$. If H has a finite orbit in $\partial_\infty \mathbb{F}(F_N, \mathcal{F})$, then H has a finite index subgroup which fixes the homothety class of an $(F_N, \mathcal{F})$-arational tree.*

For the rest of the article, we only need to know some properties of the stabiliser in $\mathrm{Out}(F_N, \mathcal{F})$ of the homothety class $[T]$ of an $(F_N, \mathcal{F})$-arational tree T. We have a natural homomorphism

$$\mathrm{SF} \colon \mathrm{Stab}([T]) \to \mathbb{R}_+^\times$$

given by the stretching factor, whose kernel is denoted by $\mathrm{Stab}_{\mathrm{Isom}}(T)$. The homomorphism SF has the following properties.

Lemma 2.10. [GH22, Lemma 6.2, Proposition 6.3, Corollary 6.12] *The following hold:*

(1) *The image of* SF *is cyclic.*
(2) *For every* $\phi \in \mathrm{Stab}([T])$, *we have* $\mathrm{SF}(\phi) \neq 1$ *if and only if* ϕ *is fully irreducible relative to* $\mathcal{F}$.

2.D. **JSJ decompositions of free groups.** This section follows the work of Guirardel–Levitt [GL17].

An F_N-*tree* is a simplicial tree equipped with an action of F_N by isometries. Let $\mathcal{A}$ be a finite set of conjugacy classes of finitely generated subgroups of F_N. We say that F_N is *one-ended relative to* $\mathcal{A}$ if there does not exist an F_N-tree T with trivial edge stabilisers such that, for every $[A] \in \mathcal{A}$, the group A fixes a point in T. Otherwise, the group F_N is *infinitely-ended relative to* $\mathcal{A}$.

If F_N is one-ended relative to $\mathcal{A}$, by [GL17, Theorem 9.14], there exists an F_N-tree $T_{\mathcal{A}}$ with infinite cyclic edge stabilisers called the *JSJ tree relative to* $\mathcal{A}$. We record some of its properties in the following theorem. We denote by $\mathrm{Out}(F_N, \mathcal{A})$ the group of outer automorphisms of F_N preserving $\mathcal{A}$ and by $\mathrm{Out}(F_N, \mathcal{A}^{(t)})$ the subgroup of $\mathrm{Out}(F_N, \mathcal{A})$ such that, for every $[A] \in \mathcal{A}$, an element $\phi \in \mathrm{Out}(F_N, \mathcal{A}^{(t)})$ has a representative fixing A elementwise. We also denote by $\mathrm{IA}(\mathcal{A}, 3)$ (resp. $\mathrm{IA}(\mathcal{A}^{(t)}, 3)$) the group $\mathrm{Out}(F_N, \mathcal{A}) \cap \mathrm{IA}_N(3)$ (resp. $\mathrm{Out}(F_N, \mathcal{A}^{(t)}) \cap \mathrm{IA}_N(3)$).

Theorem 2.11. [GL17, Theorem 9.14] *Let* $\mathcal{A}$ *be a finite set of conjugacy classes of finitely generated subgroups of* F_N *such that* F_N *is one-ended relative to* $\mathcal{A}$. *The tree* $T_{\mathcal{A}}$ *satisfies the following properties.*

(1) *Edge stabilisers are infinite cyclic.*
(2) *For every* $[A] \in \mathcal{A}$, *the group* A *fixes a point in* $T_{\mathcal{A}}$.
(3) *The group* $\mathrm{Out}(F_N, \mathcal{A})$ *preserves the* F_N-*equivariant isometry class of* T.
(4) *We have a partition* $VT_{\mathcal{A}} = V_1 \bigsqcup V_2$ *of the vertices of* $T_{\mathcal{A}}$ *such that:*
    (a) *for every* $v \in V_1$, *the group* G_v *is isomorphic to the fundamental group of a compact hyperbolic surface* Σ_v *with infinite mapping class group such that for every* $e \in ET$ *adjacent to* v, *the group* G_e *is contained in a boundary subgroup;*
    (b) *for every* $[A] \in \mathcal{A}$ *and every* $v \in G_{v_1}$ *the intersection* $A \cap G_v$ *is contained in a boundary subgroup:*
    (c) *for every* $v \in V_1$, *the image of the homomorphism* $\mathrm{IA}(\mathcal{A}, 3) \to \mathrm{Out}(G_v)$ *is contained in* $\mathrm{MCG}(\Sigma_v)$;
    (d) *for every* $v \in V_2$, *the image* $\mathrm{IA}(\mathcal{A}^{(t)}, 3) \to \mathrm{Out}(G_v)$ *is trivial. In that case, we say that* v *is* rigid.

2.E. **Dehn twist outer automorphisms.** In this section, we describe some special types of outer automorphisms called *Dehn twist outer automorphisms*. Dehn twist outer automorphisms were intensively studied (see for instance [CL95, CL99, Lev05, RW15]).

Let $N \geqslant 2$. If $a \in F_N$, we denote by $\mathrm{Aut}(F_N, a)$ the subgroup of $\mathrm{Aut}(F_N)$ fixing a and by $\mathrm{Aut}(F_N, [a])$ the subgroup of $\mathrm{Aut}(F_N)$ preserving the conjugacy class $[a]$ of a. Let $\mathrm{Out}(F_N, [a])$ be the image of $\mathrm{Aut}(F_N, [a])$ in $\mathrm{Out}(F_N)$. These groups are generally called *McCool groups* in the literature [GL16, BFH20].

In order to define a Dehn twist outer automorphism, we use the JSJ decomposition described in the previous section.

Let $\phi \in \mathrm{IA}_N(3)$ and let $\mathcal{A} = \{[\mathrm{Fix}(\Phi)]\}_{\Phi \in \phi}$. The set $\mathcal{A}$ is a finite set of conjugacy classes of finitely generated subgroups of F_N by [BH92, GJLL98]. Note that $\mathcal{A}$ is stabilised by the centraliser $C(\phi)$ of ϕ in $\mathrm{IA}_N(3)$.

The outer automorphism ϕ is a *Dehn twist outer automorphism* if F_N is one ended relative to $\mathcal{A}$ and, for every vertex $v \in VT_{\mathcal{A}}$ of the JSJ tree associated with $\mathcal{A}$, the vertex v is rigid. Using Theorem 2.11 (d), we see in particular that, the homomorphism $\langle \phi \rangle \to \prod_{v \in V(F_N \backslash T_{\mathcal{A}})} \mathrm{Out}(G_v)$ is trivial. Thus, for every $v \in VT_{\mathcal{A}}$, there exists $[A] \in \mathcal{A}$ such that $G_v \subseteq A$. If $v \in V(F_N \backslash T_{\mathcal{A}})$, we denote by $\mathrm{Out}(G_v, \mathrm{Inc}_v)$ the group of outer automorphisms of G_v preserving the conjugacy classes of the incident edge stabilisers.

Our definition of Dehn twists outer automorphisms is not standard but is equivalent to the usual one for elements of $\mathrm{IA}_N(3)$ (this is a consequence of for instance [FH19, Lemma 5.33]).

Rodenhausen and Wade [RW15] described the centraliser of a Dehn twist $\phi \in \mathrm{IA}_N(3)$ in terms of its action on $T_{\mathcal{A}}$.

Theorem 2.12. *Let $\phi \in \mathrm{IA}_N(3)$ be a Dehn twist. Its centraliser $C(\phi)$ in $\mathrm{IA}_N(3)$ fits in a short exact sequence*

$$1 \to K \to C(\phi) \to \prod_{v \in V(F_N \backslash T_{\mathcal{A}})} \mathrm{IA}_v(3) \to 1,$$

where K is a free abelian group whose dimension is equal to $|E(F_N \backslash T_{\mathcal{A}})|$ and, for every $v \in V(F_N \backslash T_{\mathcal{A}})$, the group $\mathrm{IA}_v(3)$ is a finite index subgroup of the group $\mathrm{Out}(G_v, \mathrm{Inc}_v)$. Moreover, ϕ is contained in K.

We will also use a specific construction of Dehn twists which follows the work of Levitt [Lev05]. The group $\mathrm{Aut}(F_N)$ acts on the set of F_N-equivariant isometry classes of F_N-trees by precomposition of the action, and this action passes to the quotient to give an action of $\mathrm{Out}(F_N)$ on the set of F_N-equivariant isometry classes of F_N-trees.

Let $\mathcal{T}$ be an F_N-equivariant isometry class of an F_N-tree T. The stabiliser $\mathrm{Stab}(\mathcal{T})$ of $\mathcal{T}$ in $\mathrm{IA}_N(3)$ has a natural homomorphism $\mathrm{Stab}(\mathcal{T}) \to \prod_{v \in V(F_N \backslash T)} \mathrm{Out}(G_v)$. By [Lev05, Propositions 2.2, 2.3], the kernel of this homomorphism consists of Dehn twists. Note that, by Theorem 2.12, for every Dehn twist $\phi \in \mathrm{IA}_N(3)$, the kernel of the natural homomorphism $C(\phi) \to \prod_{v \in V(F_N \backslash T_{\mathcal{A}})} \mathrm{IA}_v(3)$ consists of Dehn twists. We have in fact the following result.

Lemma 2.13. [Lev05, Proposition 3.1] *Let $\mathcal{T}$ be the F_N-equivariant isometry class of an F_N-tree T with nontrivial edge stabilisers. The kernel of $\mathrm{Stab}(\mathcal{T}) \to \prod_{v \in V(F_N \backslash T)} \mathrm{Out}(G_v)$ is abelian and consists of Dehn twists.*

3. PROPERTY (C) FOR $\mathrm{Out}(F_N)$ AND CONSEQUENCES

Let $N \geqslant 2$. Following Lück [Lüc09, Condition 3.1], we say that a group G has *Property (C)* if, for every infinite order element $h \in G$ and all $g \in G$ and $k, \ell \in \mathbb{Z}$, we have

$$g h^k g^{-1} = h^\ell \Rightarrow |k| = |\ell|.$$

In this section, we prove the following.

Proposition 3.1. *Let $N \geqslant 2$. The group $\mathrm{Out}(F_N)$ satisfies Property (C).*

Proof. Let $\phi \in \mathrm{Out}(F_N)$ be an element of infinite order, and let $\psi \in \mathrm{Out}(F_N)$ and $k, \ell \in \mathbb{Z}$, be such that $\psi \phi^k \psi^{-1} = \phi^\ell$. We prove that $|k| = |\ell|$.

Suppose first that $\phi \in \mathrm{IA}_N(3)$. Since $\mathrm{IA}_N(3)$ is a normal subgroup of $\mathrm{Out}(F_N)$, we also have $\psi \phi \psi^{-1} \in \mathrm{IA}_N(3)$. Since $\psi \phi^k \psi^{-1} = \phi^\ell$, a power of $\psi \phi \psi^{-1}$ commutes with a power of ϕ. By Proposition 2.6, the group $\langle \psi \phi \psi^{-1}, \phi \rangle$ is abelian. Since $\mathrm{IA}_N(3)$ is torsion free, the group $\langle \psi \phi \psi^{-1}, \phi \rangle$ is cyclic. Thus, there exists $m \in \mathbb{Z}$ such that either $\psi \phi \psi^{-1} = \phi^m$ or $\phi = \psi \phi^m \psi^{-1}$.

We treat the case $\psi \phi \psi^{-1} = \phi^m$, the other one being similar. The group $\langle \psi, \phi \rangle$ is then a quotient of a metabelian Baumslag-Solitar group $\mathrm{BS}(1, m)$. In particular, it does not contain a nonabelian free group. By the Tits alternative for $\mathrm{Out}(F_N)$ [BFH00], the group $\langle \psi, \phi \rangle$ is virtually abelian.

Let $n \geqslant 1$ be such that $\psi^n \in \mathrm{IA}_N(3)$. By Theorem 2.3, the group $\langle \psi^n, \phi \rangle$ is abelian. Recall that $\psi \phi \psi^{-1} = \phi^m$. Thus, we have

$$\phi = \psi^n \phi \psi^{-n} = \phi^{m^n},$$

and so $m^n = 1$ and $|m| = 1$. As $\psi \phi \psi^{-1} = \phi^m$, this also implies that $|k| = |\ell|$.

Suppose now that $\phi \notin \mathrm{IA}_N(3)$ and let $m \geqslant 1$ be such that $\phi^m \in \mathrm{IA}_N(3)$. Then we also have

$$\psi \phi^{mk} \psi^{-1} = \phi^{m\ell}.$$

By the previous case, we have $|k| = |\ell|$ and this concludes the proof. $\qquad\square$

We now outline some consequences of Proposition 3.1. Similar statements in the case of the mapping class group were proved by Juan-Pineda and Trujillo-Negrete [JPTN16].

Lemma 3.2. *Let $\phi \in \mathrm{IA}_N(3)$. For every $n \geqslant 1$, we have $C_{\mathrm{Out}(F_N)}(\phi) = C_{\mathrm{Out}(F_N)}(\phi^n)$ and $N_{\mathrm{Out}(F_N)}(\langle \phi \rangle) = N_{\mathrm{Out}(F_N)}(\langle \phi^n \rangle)$.*

Proof. We prove the result for the centraliser, the proof for the normaliser being similar. Let $n \geqslant 1$. Since $C_{\mathrm{Out}(F_N)}(\phi) \subseteq C_{\mathrm{Out}(F_N)}(\phi^n)$, it suffices to prove the converse inequality. Let $\psi \in C_{\mathrm{Out}(F_N)}(\phi^n)$. Then $(\psi \phi \psi^{-1})^n = \phi^n$. Since $\phi, \psi \phi \psi^{-1} \in \mathrm{IA}_N(3)$, by Theorem 2.5, we have $\psi \phi \psi^{-1} = \phi$. $\qquad\square$

Lemma 3.3. [Lüc09, Lemma 4.2] *Let G be a group satisfying Property (C) and let C be an infinite virtually cyclic subgroup of G. For every $k \in \mathbb{N}$, let $k!C$ be the subgroup of C given by $\{h^{k!} \mid h \in C\}$. There exists a nested sequence*

$$N_G(C) \subseteq N_G(2!C) \subseteq \ldots \subseteq N_G(k!C) \subseteq \ldots$$

such that

$$N_G[C] = \bigcup_{k \geqslant 1} N_G(k!C).$$

Recall that, for a group G and a subgroup $C \subseteq G$, the group $N_G[C]$ is the commensurator of C in G.

Proposition 3.4. *Let $N \geqslant 2$, let $g \in \mathrm{Out}(F_N)$ be an infinite order element and let $n \geqslant 1$ be such that $g^n \in \mathrm{IA}_N(3)$. Then*

$$N_{\mathrm{Out}(F_N)}[\langle g \rangle] = N_{\mathrm{Out}(F_N)}(\langle g^n \rangle).$$

Proof. See [JPTN16, Proposition 4.8] for the mapping class group case. By Lemma 3.3, we have

$$N_{\mathrm{Out}(F_N)}(\langle g \rangle) \subseteq N_{\mathrm{Out}(F_N)}(\langle g^{2!} \rangle) \subseteq \ldots \subseteq N_{\mathrm{Out}(F_N)}(\langle g^{k!} \rangle) \subseteq \ldots$$

and

$$N_{\mathrm{Out}(F_N)}[\langle g \rangle] = \bigcup_{k \geqslant 1} N_{\mathrm{Out}(F_N)}(\langle g^{k!} \rangle).$$

Since $g^n \in \mathrm{IA}_N(3)$, by Lemma 3.2, for any $k \geqslant 1$, we have

$$N_{\mathrm{Out}(F_N)}(\langle g \rangle) \subseteq \ldots \subseteq N_{\mathrm{Out}(F_N)}(\langle g^{n!} \rangle) = N_{\mathrm{Out}(F_N)}(\langle g^{(n+k)!} \rangle).$$

By Lemma 3.2 again, we have $N_{\mathrm{Out}(F_N)}(\langle g^{n!} \rangle) = N_{\mathrm{Out}(F_N)}(\langle g^n \rangle)$. Therefore, we see that

$$N_{\mathrm{Out}(F_N)}[\langle g \rangle] = N_{\mathrm{Out}(F_N)}(\langle g^n \rangle),$$

which concludes the proof. $\square$

Combining Proposition 3.4 and [Gue, Corollary 4.2], we obtain the following.

Theorem 3.5. *Let $\phi \in \mathrm{IA}_N(3)$. The commensurator of the cyclic group $\langle \phi \rangle$ in $\mathrm{IA}_N(3)$ is equal to its centraliser.* $\square$

4. CENTRALISERS OF ELEMENTS IN $\mathrm{Out}(F_N)$

Let $N \geqslant 2$. In this section, we study the centraliser of elements in $\mathrm{IA}_N(3)$. Centralisers of elements of $\mathrm{IA}_N(3)$ play a key role in the construction of a model for $\underline{\underline{E}}\,\mathrm{IA}_N(3)$ by the Lück–Weiermann push-out construction (see Theorem 7.1 below). This is why we need a precise description of the centraliser of an arbitrary element of $\mathrm{IA}_N(3)$.

Let $\phi \in \mathrm{IA}_N(3)$ be of infinite order and let $\mathcal{A} = \{[\mathrm{Fix}(\Phi)]\}_{\Phi \in \phi}$. Recall that $\mathcal{A}$ is a finite set of conjugacy classes of finitely generated subgroups of F_N and that $\mathcal{A}$ is stabilised by the centraliser $C(\phi)$ of ϕ in $\mathrm{IA}_N(3)$.

The study of the centraliser of F_N will be divided into two parts, depending on whether F_N is one-ended relative to $\mathcal{A}$ or not.

4.A. **The one-ended case.** In this section, suppose that F_N is one-ended relative to $\mathcal{A} = \{[\mathrm{Fix}(\Phi)]\}_{\Phi \in \phi}$. We will study the action of $C(\phi)$, the centraliser of ϕ in $\mathrm{IA}_N(3)$ on the JSJ tree $T_{\mathcal{A}}$ associated with $\mathcal{A}$. The main result is the following.

Theorem 4.1. *Let $\phi \in \mathrm{IA}_N(3)$. Suppose that F_N is one-ended relative to $\mathcal{A}$. Suppose also that ϕ is not a Dehn twist. Let $C(\phi)$ be the centraliser of ϕ in $\mathrm{IA}_N(3)$. Recall the partition $VT_{\mathcal{A}} = V_1 \coprod V_2$.*

(1) The group $C(\phi)$ fits into an exact sequence

$$1 \to K' \to C(\phi) \to \mathbb{Z} \times \prod_{v \in V_2} \mathrm{Out}(G_v)$$

where K' is abelian.

(2) The image of the projection on the first coordinate

$$C(\phi) \to \mathbb{Z}$$

is generated by a root of ϕ. The kernel K satisfies one of the followings.

(a) *The group K is isomorphic to a subgroup of*
$$\mathrm{Out}(A) \times \mathrm{Out}(B),$$
where $A, B \subseteq F_N$ are such that $\mathrm{rk}(A) + \mathrm{rk}(B) = N + 1$ and $\mathrm{rk}(A), \mathrm{rk}(B) \leqslant N - 1$.

(b) *The group K is isomorphic to a subgroup of*
$$\mathrm{Out}(A * \langle tst^{-1} \rangle, [s], [tst^{-1}]),$$
where $s, t \in F_N$ $A \subseteq F_N$, $\mathrm{rk}(A) = N - 1$, $s \in A$ and t is a basis element of F_N.

Proof. Consider the action of $C(\phi)$ on $T_{\mathcal{A}}$. This action gives an exact sequence
$$1 \to K' \to C(\phi) \to \prod_{v \in VF_N \backslash T_{\mathcal{A}}} \mathrm{Out}(G_v).$$
By Lemma 2.13, since every edge of $T_{\mathcal{A}}$ has infinite cyclic edge stabiliser, the kernel K' is abelian.

Recall the partition $VT_{\mathcal{A}} = V_1 \coprod V_2$ given by Theorem 2.11. Note that $\phi \in \mathrm{Out}(F_N, \mathcal{A}^{(t)})$.

Suppose towards a contradiction that $VT_{\mathcal{A}} = V_2$. By Theorem 2.11 (3)(d), since $\phi \in \mathrm{Out}(F_N, \mathcal{A}^{(t)})$, for every $v \in V_2$, the image of ϕ in $\mathrm{Out}(G_v)$ is trivial. Hence ϕ is contained in the kernel of the homomorphism $C(\phi) \to \prod_{v \in VT} \mathrm{Out}(G_v)$. But the kernel of this homomorphism consists of Dehn twists by Lemma 2.13. This contradicts the assumption made on ϕ. Thus, the set V_1 is nontrivial.

Let $v \in V_1$ and let Σ_v be the associated compact hyperbolic surface given by Theorem 2.11 (3)(a). By Theorem 2.11 (3)(b), for every $[A] \in \mathcal{A}$, the intersection of A with G_v is contained in a boundary component of ϕ. Recall that every ϕ-periodic conjugacy class of F_N is in fact fixed by Theorem 2.2 (2). Thus, by definition of $\mathcal{A}$, the mapping class of Σ_v induced by ϕ does not virtually preserve the homotopy class of any curve nonhomotopic to a boundary component. Therefore, the image of ϕ in $\mathrm{MCG}(\Sigma_v)$ is a pseudo-Anosov homeomorphism. This implies that its centraliser in $\mathrm{MCG}(\Sigma_v)$ is virtually cyclic.

Note that, since $C(\phi) \subseteq \mathrm{IA}_N(3)$, by Theorem 2.2 (2), the image of $C(\phi)$ in $\mathrm{Out}(G_v)$ is torsion free. Combining this remark and the above paragraph, we see that the image of $C(\phi)$ in $\mathrm{Out}(G_v)$ is infinite cyclic, generated by a root of ϕ. Hence, for every $v \in V_1$, the image $C(\phi) \to \mathrm{Out}(G_v)$ is infinite cyclic, generated by a root of ϕ. Thus, we have the following exact sequence
$$1 \to K' \to C(\phi) \to \mathbb{Z} \times \prod_{w \in V_2} \mathrm{Out}(G_w)$$
where K' is abelian. This proves Assertion (1).

We now prove Assertion (2). Let $v \in V_1$ and let $K = \ker(C(\phi) \to \mathrm{Out}(G_v))$.

Note that Σ_v is not homeomorphic to a pair of pants as $\mathrm{MCG}(\Sigma_v)$ is infinite (see Theorem 2.11 (3)(a)). Thus, there exists a closed geodesic curve γ in Σ_v which is not homotopic to a boundary component. This curve induces a splitting S_v of Σ_v. Since edge groups are all contained in boundary components of Σ_v, one can blow up S_v at v to obtain a splitting T' of F_N

such that T is obtained from T' by collapsing the orbit of an edge e. Note that the stabiliser of e is infinite cyclic and its conjugacy class corresponds to the conjugacy class associated with γ. Moreover, T' is preserved by K since K acts as the identity on $\pi_1(\Sigma_v)$.

Let U be the splitting obtained from T' by collapsing every orbit of edges except the one of e. Then, U is preserved by K. Thus, we have a homomorphism $K \to \prod_{w \in F_N \backslash U} \mathrm{Out}(G_w)$ whose kernel is infinite cyclic and generated by a twist D about e. Since G_e is contained in the conjugacy class of γ, the twist D does not have a representative which acts as the identity on $\pi_1(\Sigma_v)$. As K acts as the identity on $\pi_1(\Sigma_v)$, the homomorphism $K \to \prod_{v \in F_N \backslash U} \mathrm{Out}(G_w)$ is injective.

Since U has one orbit of edges, U induces one of the following splittings of F_N: either $F_N = A *_{\mathbb{Z}} B$ where $\mathrm{rk}(A) + \mathrm{rk}(B) = N + 1$ and $\mathrm{rk}(A), \mathrm{rk}(B) \leqslant N-1$, which yields case (2)(a), or $F_N = A *_{\mathbb{Z}}$ where $\mathrm{rk}(A) = N-1$. Moreover, in the second case, there exist $s \in A$ and a basis element t of F_N such that the vertex stabiliser of U is conjugate to $A * \langle tst^{-1} \rangle$. Since K preserves U, the image of K in $\mathrm{Out}(A * \langle tst^{-1} \rangle)$ preserves s and tst^{-1} which yields case (2)(b). This concludes the proof. □

4.B. **The infinitely-ended-case.** Let $\phi \in \mathrm{IA}_N(3)$. Suppose now that F_N is not one-ended relative to $\mathcal{A} = \{[\mathrm{Fix}(\Phi)]\}_{\Phi \in \phi}$. Let $\mathcal{F}_{\mathcal{A}}$ be the minimal free factor system of F_N such that for every $[A] \in \mathcal{A}$, there exists $[B] \in \mathcal{F}_{\mathcal{A}}$ such that $A \subseteq B$. By minimality of $\mathcal{F}_{\mathcal{A}}$, we have $C(\phi) \subseteq \mathrm{Out}(F_N, \mathcal{F}_{\mathcal{A}})$. We will consider $C(\phi)$-invariant free factor systems $\mathcal{F}$ of F_N such that $\mathcal{F}_{\mathcal{A}} \leqslant \mathcal{F}$ in order to obtain a description of $C(\phi)$.

Theorem 4.2. *Let $\phi \in \mathrm{IA}_N(3)$. Let $C(\phi)$ be the centraliser of ϕ in $\mathrm{IA}_N(3)$. Let $\mathcal{F}_{\mathcal{A}} \leqslant \mathcal{F}$ be a maximal proper $C(\phi)$-invariant free factor system.*

(1) If $\mathcal{F}$ is nonsporadic, there exists a surjective homomorphism

$$C(\phi) \to \mathbb{Z}$$

whose image is generated by a root of ϕ and such that the kernel K of this homomorphism is isomorphic to a subgroup of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$, where $A, B \subseteq F_N$ are nontrivial subgroups such that $F_N = A * B$.

(2) If $\mathcal{F}$ is sporadic, then $\mathcal{F} = \{[A_1], [A_2]\}$ and there exists a homomorphism

$$C(\phi) \to \mathrm{IA}(A_1, 3) \times \mathrm{IA}(A_2, 3)$$

whose kernel is a finite index subgroup of a direct product of two finitely generated free (maybe trivial or cyclic) groups.

Remark 4.3. A key point in the proof of Theorem 4.2 is the fact that every free factor system which has a finite orbit under iteration of an element of $\mathrm{IA}_N(3)$ is in fact fixed (see Theorem 2.2). Therefore, it is not clear how to deduce an exact sequence similar to the one of Theorem 4.2 for centralisers in $\mathrm{Out}(F_N)$ instead of in $\mathrm{IA}_N(3)$.

Remark 4.4. In Theorem 4.2, the number of elements in the free factor system $\mathcal{F}$ depends on whether Case 1 or Case 2 holds. In Case 2 the group A_1 is never trivial but A_2 might be trivial. We also have a control on the

ranks of the free factors appearing in $\mathcal{F}$. Indeed, in Case 2, for every $[A] \in \mathcal{F}$, the rank of A is bounded by $N - 1$. Moreover,

$$\sum_{[A] \in \mathcal{F}} \mathrm{rank}(A) \leqslant N.$$

In order to prove Theorem 4.2, we need some results regarding the stabiliser of a relative arational tree in the Gromov boundary of a relative free factor graph. The first one is a combination of a result extracted from [HW20], where it is attributed to Guirardel and Levitt and a result of Guirardel and Horbez [GH22].

Proposition 4.5. [GH22, HW20] *Let H be a subgroup of $\mathrm{IA}_N(3)$. Let $\mathcal{F}$ be a maximal H-invariant free factor system. Suppose that $\mathcal{F}$ is nonsporadic and that H has a finite index subgroup which fixes the homothety class of an $(F_N, \mathcal{F})$-arational tree T.*

  (1) We have a homomorphism $H \to \mathbb{Z} \times \prod_{[A] \in \mathcal{F}} \mathrm{IA}(A, 3)$ whose kernel is abelian and consists of Dehn twists.
  (2) The image of the projection $p \colon H \to \mathbb{Z}$ is surjective and generated by a root of any fully irreducible outer automorphism relative to $\mathcal{F}$ contained in H.
  (3) The kernel K of p is isomorphic to a subgroup of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$, where $A, B \subseteq F_N$ are nontrivial subgroups such that $F_N = A * B$.
  (4) There exists a nonperipheral subgroup $C \subseteq F_N$ such that, for every $k \in K$, the outer automorphism k has a representative fixing C elementwise.

Proof. By maximality of $\mathcal{F}$ and Theorem 2.8, the group H contains a fully irreducible outer automorphism relative to $\mathcal{F}$. Let H_0 be a finite index subgroup of H which fixes the homothety class of T. By Lemma 2.10, the group H_0 splits as a semi-direct product $H' \rtimes \mathbb{Z}$, where H' is contained in the isometric stabiliser of T. Moreover, the $\mathbb{Z}$ factor is generated by a root of any fully irreducible outer automorphism relative to $\mathcal{F}$ contained in H_0.

By [HW20, Lemmas 5.3, 5.6, Theorem 5.4], the commensurator of H' in $\mathrm{Out}(F_N, \mathcal{F}) \cap \mathrm{IA}_N(3)$, denoted by $N_{\mathrm{IA}_N(\mathcal{F},3)}[H']$, preserves the F_N-equivariant isometry class of an F_N-tree S.

The tree S satisfies the following properties. Edge stabilisers in F_N are infinite. The quotient graph $F_N \backslash S$ is a tree with one central vertex, v_0, adjacent to every other vertex. The stabiliser of v_0 in F_N is finitely generated. Moreover, if $v \in V(F_N \backslash S) - \{v_0\}$, then the conjugacy class of the stabiliser G_v of v in F_N is contained in $\mathcal{F}$.

By Theorem 2.2, the group $N_{\mathrm{IA}_N(\mathcal{F},3)}[H']$ preserves the conjugacy class of every G_v with $v \in V(F_N \backslash S) - \{v_0\}$. Thus, the group $N_{\mathrm{IA}_N(\mathcal{F},3)}[H']$ acts trivially on the quotient graph $F_N \backslash S$. Therefore we have a homomorphism

$$N_{\mathrm{IA}_N(\mathcal{F},3)}[H'] \to \mathrm{Out}(G_v) \times \prod_{[A] \in \mathcal{F}} \mathrm{Out}(A)$$

induced by the action on the vertex stabilisers. Since $N_{\mathrm{IA}_N(\mathcal{F},3)}[H'] \subseteq \mathrm{IA}_N(3)$ and since we are considering the restriction homomorphism on vertex stabilisers, the restriction of the image of H in $\prod_{[A] \in \mathcal{F}} \mathrm{Out}(A)$ is contained in $\prod_{[A] \in \mathcal{F}} \mathrm{IA}(A, 3)$.

By [HW20, Theorem 5.4], the image of H' in $\mathrm{Out}(G_v)$ is trivial and the image of $H_0 = H' \rtimes \mathbb{Z}$ is infinite cyclic.

Thus, the group $N_{\mathrm{IA}_N(\mathcal{F},3)}[H']$ fits into an exact sequence

$$1 \to K' \to N_{\mathrm{IA}_N(\mathcal{F},3)}[H'] \to \mathrm{Out}(G_v) \times \prod_{[A]\in\mathcal{F}} \mathrm{IA}(A,3),$$

where G_v is a nonabelian free subgroup of F_N such that the image of H_0 in $\mathrm{Out}(G_v)$ is isomorphic to $\mathbb{Z}$.

Since H' is a normal subgroup of a finite index subgroup of H, the group H is contained in $N_{\mathrm{IA}_N(\mathcal{F},3)}[H']$. Therefore, we have an exact sequence

$$1 \to K_0 \to H \to \mathrm{Out}(G_v) \times \prod_{[A]\in\mathcal{F}} \mathrm{IA}(A,3).$$

The kernel K_0 is the subgroup of H acting trivially on the vertex groups of T. Since T has nontrivial edge stabilisers, by Lemma 2.13, the group K_0 consists of Dehn twists and is abelian.

Thus, in order to prove Assertion (1), it suffices to prove that the image of H in $\mathrm{Out}(G_v)$ is infinite cyclic. Since the image of H_0 in $\mathrm{Out}(G_v)$ is infinite cyclic and since H_0 is a finite index subgroup of H, we see that the image of H in $\mathrm{Out}(G_v)$ is virtually infinite cyclic.

Thus, it suffices to prove that the image of H in $\mathrm{Out}(G_v)$ is torsion free. Let $\psi \in H$ whose image in $\mathrm{Out}(G_v)$ is finite. Thus, ψ has a power which preserves the conjugacy class of every element of G_v. By Theorem 2.2, the outer automorphism ψ preserves the conjugacy class of every element of G_v. In particular, the image of ψ in $\mathrm{Out}(G_v)$ is trivial. This implies that the image of H in $\mathrm{Out}(G_v)$ is virtually infinite cyclic and torsion free, hence is infinite cyclic. This proves Assertion (1).

As explained above, the image of H in $\mathrm{Out}(G_v)$ is generated by a root of any fully irreducible outer automorphism relative to $\mathcal{F}$ whose power is contained in H_0. Thus, it is generated by a root of any fully irreducible outer automorphism relative to $\mathcal{F}$ contained in H. This proves Assertion (2).

We now prove that the kernel K of $p\colon H \to \mathrm{Out}(G_v)$ satisfies Assertions (3), (4). Note that every element of K has a representative which fixes elementwise G_v, which is a nonperipheral subgroup. Assertion (4) follows.

It remains to prove Assertion (3). It suffices to prove that K is isomorphic to a subgroup of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$, where $A, B \subseteq F_N$ are nontrivial subgroups such that $F_N = A * B$. By [HW20, Lemma 5.6], the edges groups of S induce a nonsporadic free factor system of G_v. Thus, there exist nontrivial subgroups $C, D \subseteq G_v$ such that $G_v = C * D$ and, for every $e \in ES$, a conjugate of the group G_e is contained in either C or D. Since K acts trivially on G_v, it also preserves this decomposition.

Let S' be the splitting obtained from S by blowing up at v the splitting S_v induced by $G_v = C * D$ and attaching the edges groups accordingly. Then S' is preserved by K since K preserves both S and S_v. Moreover, S' has a unique orbit of an edge e with trivial stabiliser. Since v meets every orbit of edges in S, the image of e in $\overline{F_N \backslash S'}$ is a separating edge.

Let U be the splitting obtained from S' by collapsing every orbit of edges except the one of e. Then U is preserved by K since $K \subseteq \mathrm{IA}_N(3)$. Moreover, the decomposition of F_N induced by U is $F_N = A * B$, where $A, B \subseteq F_N$ are

nontrivial subgroups such that $F_N = A * B$. The stabiliser of this splitting is isomorphic to $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ by a result of Levitt [Lev05]. This concludes the proof. $\square$

Lemma 4.6. *Let $N \geqslant 2$ and let $\phi \in \mathrm{IA}_N(3)$. Let $\mathcal{F}_{\mathcal{A}} \leqslant \mathcal{F}$ be a maximal $C(\phi)$-invariant free factor system. Suppose that $\mathcal{F}$ is nonsporadic. The element ϕ is fully irreducible relative to $\mathcal{F}$. Moreover, the group $C(\phi)$ virtually fixes a point in $\partial_\infty \mathbb{F}(F_N, \mathcal{F})$.*

Proof. By Theorem 2.8, the group $C(\phi)$ contains a fully irreducible element ψ relative to $\mathcal{F}$.

We claim that ϕ is also a fully irreducible element relative to $\mathcal{F}$. Indeed, by Theorem 2.7, the element ψ is a loxodromic element of $\mathbb{F}(F_N, \mathcal{F})$. Thus, ψ fixes exactly two points T_+, T_- in $\partial_\infty \mathbb{F}(F_N, \mathcal{F})$. Since ϕ commutes with ψ, the element ϕ preserves $\{T_+, T_-\}$.

By Proposition 2.9, the element ϕ virtually fixes the homothety class of an arational $(F_N, \mathcal{F})$-tree. Thus, we can apply Proposition 4.5 to see that $\langle \phi \rangle$ fits in an exact sequence

$$1 \to K \to \langle \phi \rangle \to \mathbb{Z}$$

whose kernel K fixes elementwise a nonperipheral group.

Since $\mathcal{F}_{\mathcal{A}} \leqslant \mathcal{F}$, we see that ϕ does not fix elementwise a nonperipheral subgroup. In particular, the group K is trivial. Since the image of $\langle \phi \rangle \to \mathbb{Z}$ is generated by any fully irreducible outer automorphism relative to $\mathcal{F}$ by Proposition 4.5 (2), we see that ϕ itself is fully irreducible relative to $\mathcal{F}$.

The moreover part follows from the fact that $C(\phi)$ must preserve the attracting and repelling fixed points of ϕ in $\partial_\infty \mathbb{F}(F_N, \mathcal{F})$. $\square$

Proof of Theorem 4.2. Let $\mathcal{F} \geqslant \mathcal{F}_{\mathcal{A}}$ be a maximal $C(\phi)$-invariant free factor system.

Case 1. *Suppose that $\mathcal{F}$ is sporadic.*

Thus, we have $\mathcal{F} = \{[A], [B]\}$ where A and B might be equal. By for instance [Lev05, Proposition 4.2], the stabiliser of $\mathcal{F}$ in $\mathrm{IA}_N(3)$ is isomorphic to a finite index subgroup of either $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ if $[A] \neq [B]$ or of $A \rtimes \mathrm{Aut}(A)$ otherwise. In both cases, we have a homomorphism $C(\phi) \to \prod_{[C] \in \mathcal{F}} \mathrm{Out}(C)$ whose kernel is a finite index subgroup of a direct product of two free (maybe cyclic or trivial) groups. Since $C(\phi) \subseteq \mathrm{IA}_N(3)$, the image of $C(\phi) \to \prod_{[C] \in \mathcal{F}} \mathrm{Out}(C)$ is contained in $\prod_{[C] \in \mathcal{F}} \mathrm{IA}(C, 3)$. Thus, it remains to show that both such free groups in the kernel are finitely generated. We treat both cases separately.

Suppose that $\mathrm{Stab}(\mathcal{F})$ is isomorphic to a finite index subgroup of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$. Let $\Phi \in \phi$ be the unique automorphism in the outer class ϕ such that $\Phi(A) = A$ and $\Phi(B) = B$. Then the kernel of the homomorphism $C(\phi) \to \mathrm{Out}(A) \times \mathrm{Out}(B)$ is isomorphic to the intersection of $\mathrm{IA}_N(3)$ with a subgroup isomorphic to $\mathrm{Fix}(\Phi|_A) \times \mathrm{Fix}(\Phi|_B)$. In particular, both direct factors are finitely generated by [BH92].

Suppose now that $\mathrm{Stab}(\mathcal{F})$ is isomorphic to $A \rtimes \mathrm{Aut}(A)$. In that case, we have $F_N = A * \langle g \rangle$ for some $g \in F_N$. Let $\Phi \in \phi$ be the unique representative of ϕ sending A to A and g to ga with $a \in A$. Then the kernel of the homomorphism $C(\phi) \to \mathrm{Out}(A)$ is isomorphic to the intersection of $\mathrm{IA}_N(3)$

with a subgroup isomorphic to $\mathrm{Fix}(\Phi|_A) \times \mathrm{Fix}(\mathrm{ad}_{a^{-1}} \circ \Phi|_A)$, so that both direct factors are finitely generated. This concludes the proof when $\mathcal{F}$ is sporadic.

Case 2. *Suppose that $\mathcal{F}$ is nonsporadic.*

By Lemma 4.6, the set of fixed points of $C(\phi)$ in $\partial_\infty \mathbb{F}(F_N, \mathcal{F})$ is nonempty. By Proposition 2.9, the group $C(\phi)$ virtually fixes the homothety class of an $(F_N, \mathcal{F})$-arational tree. Thus, we can apply Proposition 4.5 in order to obtain the desired homomorphism $C(\phi) \to \mathbb{Z}$. Since ϕ is fully irreducible relative to $\mathcal{F}$ by Lemma 4.6, a root of ϕ generates the image of the homomorphism. This concludes the proof. $\qquad\square$

Combining Proposition 4.5, Theorems 4.1 and 4.2, we obtain the following properties of centralisers of elements of $\mathrm{IA}_N(3)$.

Theorem 4.7. *Let $N \geqslant 2$ and let $\phi \in \mathrm{IA}_N(3)$. The centraliser $C(\phi)$ of ϕ in $\mathrm{IA}_N(3)$ satisfies one of the followings.*

(1) *The outer automorphism ϕ is a Dehn twist. There exist a JSJ tree T preserved by $C(\phi)$ and a short exact sequence*

$$1 \to K \to C(\phi) \to \prod_{v \in V(F_N \backslash T)} \mathrm{IA}_v(3) \to 1,$$

where K is a free abelian group whose dimension is equal to $|E(F_N \backslash T)|$ and, for every $v \in V(F_N \backslash T)$, the group $\mathrm{IA}_v(3)$ is a finite index subgroup of the group $\mathrm{Out}(G_v, \mathrm{Inc}_v)$. Moreover, ϕ is contained in K.

(2) *There exist $A_1, A_2 \subseteq F_N$ with $F_N = A_1 * A_2$, $\mathrm{rk}(A_1), \mathrm{rk}(A_2) \leqslant N - 1$, and a homomorphism*

$$C(\phi) \to \mathrm{IA}(A_1, 3) \times \mathrm{IA}(A_2, 3)$$

whose kernel is a finite index subgroup of a direct product of two finitely generated free (maybe trivial or cyclic) groups.

(3) *There exist $A_1, \ldots, A_k, B \subseteq F_N$ nontrivial with $F_N = A_1 * \ldots A_k * B$, and a homomorphism*

$$C(\phi) \to \mathbb{Z} \times \prod_{i=1}^{k} \mathrm{IA}(A_i, 3)$$

whose kernel is abelian and ϕ projects onto the $\mathbb{Z}$ factor.

(4) *There exist a JSJ tree T preserved by $C(\phi)$, a partition $VT = V_1 \coprod V_2$, and a homomorphism*

$$C(\phi) \to \mathbb{Z} \times \prod_{v \in V_2} \mathrm{Out}(G_v)$$

whose kernel is abelian and ϕ projects onto the $\mathbb{Z}$ factor.

Proof. To make this explicit suppose first that ϕ is a Dehn twist. Then we are in case (1) and the short exact sequence follows from Theorem 2.12.

Suppose now that ϕ is not a Dehn twist. Let $\mathcal{A} = \{[\mathrm{Fix}(\Phi)]\}_{\Phi \in \phi}$. Suppose that F_N is one-ended relative to $\mathcal{A}$. Then we can apply Theorem 4.1 to obtain case (4).

Suppose that F_N is not one-ended relative to $\mathcal{A}$ and let $\mathcal{F} \geqslant \mathcal{F}_\mathcal{A}$ be a maximal $C(\phi)$-invariant free factor system. If $\mathcal{F}$ is sporadic, we can apply Theorem 4.2 (2) to get case (2).

Suppose that $\mathcal{F}$ is nonsporadic. By Lemma 4.6, the group $C(\phi)$ virtually fixes a point in $\partial_\infty \mathbb{F}(F_N, \mathcal{F})$ and ϕ is fully irreducible relative to $\mathcal{F}$. By Proposition 2.9, the group $C(\phi)$ virtually fixes the homothety class of an $(F_N, \mathcal{F})$-arational tree. Thus, we can apply Proposition 4.5 (1) to get the homomorphism of case (3). Note that a root of ϕ projects onto the $\mathbb{Z}$ factor by Proposition 4.5 (2) since ϕ is fully irreducible relative to $\mathcal{F}$.   □

4.C. **Weyl groups.** We now adapt Theorems 4.1 and 4.2 to the Weyl group $W(\phi) = C(\phi)/\langle\phi\rangle$ in $\mathrm{IA}_N(3)$ of an infinite order element $\phi \in \mathrm{IA}_N(3)$. Let $\mathcal{F}$ be a free factor system as in Theorem 4.2. For every $[A] \in \mathcal{F}$, since $\mathrm{IA}_N(3)$ is torsion free, the image $\phi|_A$ of ϕ in $\mathrm{IA}(A, 3)$ is either trivial or infinite. Let $\mathcal{F}_\infty$ be the subset of $\mathcal{F}$ consisting of all $[A] \in \mathcal{F}$ such that $\phi|_A$ is infinite and let $\mathcal{F}_T = \mathcal{F} - \mathcal{F}_\infty$. Let $H_T = \prod_{[A]\in\mathcal{F}_T} \mathrm{IA}(A, 3)$ and let $H_\infty = \prod_{[A]\in\mathcal{F}_\infty} C(\phi|_A)$. We denote by $\rho_\infty \colon C(\phi) \to H_\infty$ and $\rho_T \colon C(\phi) \to H_T$ the homomorphisms given by Theorem 4.2 (2).

Corollary 4.8. *Let $N \geqslant 2$, let $\phi \in \mathrm{IA}_N(3)$ be a root-closed element of infinite order and let $\mathcal{A} = \{[\mathrm{Fix}(\Phi)]\}_{\Phi\in\phi}$. Let $\mathcal{F} \geqslant \mathcal{F}_\mathcal{A}$ be a (possibly trivial) maximal $C(\phi)$-invariant free factor system. The Weyl group $W(\phi) = C(\phi)/\langle\phi\rangle$ of ϕ in $\mathrm{IA}_N(3)$ satisfies one of the following.*

(1) The element ϕ is a Dehn twist.
(2) The group F_N is one-ended relative to $\mathcal{A}$. The group $W(\phi)$ is isomorphic to a subgroup of $\mathrm{Out}(A) \times \mathrm{Out}(B)$, where $A, B \subseteq F_N$ are such that $\mathrm{rk}(A) + \mathrm{rk}(B) = N + 1$ and $\mathrm{rk}(A), \mathrm{rk}(B) \leqslant N - 1$.
(3) The group F_N is one-ended relative to $\mathcal{A}$. The group $W(\phi)$ is isomorphic to a subgroup $\mathrm{Out}(A * \langle tst^{-1}\rangle, [s], [tst^{-1}])$, where $A \subseteq F_N$, $\mathrm{rk}(A) = N - 1$, $s \in A$ and t is a basis element of F_N.
(4) The free factor system $\mathcal{F}$ is nonsporadic. The group $W(\phi)$ is isomorphic to a subgroup of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$, where $A, B \subseteq F_N$ are nontrivial subgroups such that $F_N = A * B$.
(5) The free factor system $\mathcal{F}$ is sporadic and $\rho_\infty(\phi)$ is infinite. Then $W(\phi)$ fits into an exact sequence

$$1 \to K \to W(\phi) \to H_\infty/\rho_\infty(\langle\phi\rangle) \times H_T,$$

where K is a finite index subgroup of a direct product of two finitely generated free groups.
(6) The free factor system $\mathcal{F}$ is sporadic and $\rho_\infty(\phi)$ is trivial. Then $W(\phi)$ fits into an exact sequence

$$1 \to K \to W(\phi) \to \prod_{[A]\in\mathcal{F}} \mathrm{IA}(A, 3),$$

where K is a direct product of a finitely generated free (maybe cyclic or trivial) group and a finite group.

Proof. Suppose that ϕ is not a Dehn twist. We begin with the case that F_N is one-ended relative to $\mathcal{A}$. By Theorem 4.1 (2), we have a homomorphism $C(\phi) \to \mathbb{Z}$ whose image is generated by a root of ϕ. Since ϕ is root-closed,

the image is generated by ϕ. Thus, the Weyl group $W(\phi)$ is isomorphic to the kernel K of this homomorphism. By Theorem 4.1 (2)(a) or (b), the group $W(\phi)$ satisfies either Case (2) or (3) respectively.

We now deal with the case that F_N is not one-ended relative to $\mathcal{A}$. Consider the maximal $C(\phi)$-invariant free factor system $\mathcal{F} \geqslant \mathcal{F}_{\mathcal{A}}$.

Suppose first that $\mathcal{F}$ is nonsporadic. Theorem 4.2 (1) gives a homomorphism whose image is generated by a root of ϕ. As in the proof of Cases (2) and (3), the group $W(\phi)$ is isomorphic to the kernel of this homomorphism, which leads Case (4).

Suppose now that $\mathcal{F}$ is sporadic. Theorem 4.2 (2) gives a homomorphism $C(\phi) \xrightarrow{(\rho_\infty, \rho_T)} H_\infty \times H_T$. It induces a quotient map

$$C(\phi) \to (H_\infty \times H_T)/_{(\rho_\infty, \rho_T)(\langle\phi\rangle)}.$$

Since $\rho_T(\phi)$ is trivial, this gives a map

$$C(\phi) \to H_\infty/\rho_\infty(\langle\phi\rangle) \times H_T$$

which induces a quotient map

$$W(\phi) \xrightarrow{\rho_W} H_\infty/\rho_\infty(\langle\phi\rangle) \times H_T.$$

When $\rho_\infty(\phi)$ is infinite, the kernel of ρ_W is exactly the same as the homomorphism (ρ_∞, ρ_T), so Case (5) follows.

Suppose that $\rho_\infty(\phi)$ is trivial. Then ϕ is contained in the kernel of (ρ_∞, ρ_T) and the kernel of ρ_W is $\ker((\rho_\infty, \rho_T))/\langle\phi\rangle$. Moreover, H_∞ is the trivial group.

By Theorem 4.2 (2), the kernel $\ker((\rho_\infty, \rho_T))$ is isomorphic to a direct product of two finitely generated free groups. Note that that the centraliser of an infinite element in a direct product of two finitely generated free groups is contained in a group isomorphic to $F \times \mathbb{Z}$, where F is a finitely generated free group. Thus, the quotient $\ker((\rho_\infty, \rho_T))/\langle\phi\rangle$ is isomorphic to a direct product of a finitely generated free group and a finite group. This shows Case (6). $\qquad\square$

5. Centralisers of elements in $\mathrm{Out}(F_3)$

In this section, we focus on the study of outer automorphisms of F_3. We prove the following.

Theorem 5.1. *Let $\phi \in \mathrm{IA}_3(3)$. The centraliser $C(\phi)$ of ϕ in $\mathrm{IA}_3(3)$ is of type VF. Moreover, one of the following holds.*

(1) *The centraliser of ϕ is abelian.*

(2) *The centraliser of ϕ is isomorphic to $F \times \mathbb{Z}$ where F is a finitely generated free group.*

(3) *The centraliser of ϕ is isomorphic to a direct product $H \times \mathbb{Z}$ where H is a finite index subgroup of a direct product of two finitely generated free groups.*

(4) *The outer automorphism ϕ is a Dehn twist. There exist a JSJ tree T preserved by $C(\phi)$ and a short exact sequence*

$$1 \to K \to C(\phi) \to \prod_{v \in V(F_3 \backslash T)} \mathrm{IA}_v(3) \to 1,$$

where K is a free abelian group whose dimension is equal to $|E(F_3 \backslash T)|$ and, for every $v \in V(F_3 \backslash T)$, the group $\mathrm{IA}_v(3)$ is a finite index subgroup of the group $\mathrm{Out}(G_v, \mathrm{Inc}_v)$. Moreover, ϕ is contained in K.

We highlight the following immediate corollary.

Corollary 5.2. *Let $\phi \in \mathrm{Out}(F_3)$. The centraliser $C(\phi)$ is of type VF.*

The proof of Theorem 5.1 is decomposed into several propositions. The idea is to consider a maximal $C(\phi)$-invariant free factor system $\mathcal{F}$ and to treat separately the cases when $\mathcal{F}$ is sporadic or not. Observe that, since we are considering a nonabelian free group of rank 3, the free factor system $\mathcal{F}$ is sporadic if and only if it contains the conjugacy class of a nonabelian free factor.

Lemma 5.3. *Let $\phi \in \mathrm{IA}_3(3)$. Let $\mathcal{F}$ be a maximal $C(\phi)$-invariant free factor system. Suppose that $\mathcal{F}$ is nonsporadic. If ϕ is not a Dehn twist then $C(\phi)$ is abelian.*

Proof. We claim that ϕ is fully irreducible relative to $\mathcal{F}$. Indeed, by maximality of $\mathcal{F}$ and Theorem 2.7, the group $C(\phi)$ contains a loxodromic element ψ of $\mathbb{F}(F_N, \mathcal{F})$. As ψ commutes with ϕ, the element ϕ must fix the attracting point of ψ in $\partial_\infty \mathbb{F}(F_N, \mathcal{F})$. By Proposition 2.9, ϕ must virtually fix the homothety class of an $(F_N, \mathcal{F})$-arational tree. By Proposition 4.5 (1), the group ϕ fits in an exact sequence

$$1 \to K' \to \langle \phi \rangle \to \mathbb{Z} \times \prod_{[A] \in \mathcal{F}} \mathrm{Out}(A),$$

where K' consists of Dehn twists and the projection on the $\mathbb{Z}$ factor is non-trivial if and only if ϕ is fully irreducible relative to $\mathcal{F}$.

For every $[A] \in \mathcal{F}$, as A is cyclic, the group $\mathrm{Out}(A)$ is finite. Thus, since ϕ is not a Dehn twist and has infinite order, the image of ϕ in the $\mathbb{Z}$ factor is nontrivial and ϕ is fully irreducible relative to $\mathcal{F}$.

Thus, by Theorem 2.7, ϕ is a loxodromic element of $\mathbb{F}(F_N, \mathcal{F})$. Therefore, $C(\phi)$ fixes the attracting point of ϕ in $\partial_\infty \mathbb{F}(F_N, \mathcal{F})$.

As above, by Proposition 4.5, since $\mathcal{F}$ is nonsporadic, the group $C(\phi)$ fits into an exact sequence

$$1 \to K \to C(\phi) \to \mathbb{Z} \times \prod_{[A] \in \mathcal{F}} \mathrm{Out}(A)$$

whose kernel K is an abelian group. Since $\mathcal{F}$ is nonsporadic, for every $[A] \in \mathcal{F}$, the group A is cyclic. Thus, for every $[A] \in \mathcal{F}$, the group $\mathrm{Out}(A)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In particular, the image of the above homomorphism is virtually abelian. Thus, the group $C(\phi)$ does not contain a nonabelian free group. By Lemma 2.4, the group $C(\phi)$ is abelian. $\square$

Lemma 5.4. *Let H be a subgroup of $\mathrm{IA}_3(3)$ preserving a free factor A of F_3 of rank 2. The image of H in $\mathrm{Out}(A)$ is a free group.*

Proof. Since A is a free factor of F_3, the image of H in $\mathrm{Out}(A)$ is contained in $\mathrm{IA}_2(3)$. Since A has rank 2, the group $\mathrm{Out}(A)$ is isomorphic to $\mathrm{GL}_2(\mathbb{Z})$ and is virtually free. Since $\mathrm{IA}_2(3)$ is torsion free by Proposition 2.1, the group $\mathrm{IA}_2(3)$ is free and so is the image of H. $\square$

Lemma 5.5. *Let $\phi \in \mathrm{IA}_3(3)$. Let $\mathcal{F}$ be a maximal $C(\phi)$-invariant free factor system. Suppose that $\mathcal{F} = \{[A], [B]\}$ with $F_3 = A * B$ and A is a nonabelian free group of rank 2. Let $\Phi \in \phi$ be the unique representative of ϕ such that $\Phi(A) = A$ and $\Phi(B) = B$. One of the following holds.*

(1) The group $C(\phi)$ is isomorphic to $(F \cap \mathrm{IA}_3(3)) \times \mathbb{Z}$, where F is a free subgroup of rank at most 2.

(2) The element ϕ is a Dehn twist.

Proof. By for instance [Lev05], for every $\psi \in C(\phi)$, there exists a unique representative $\Psi \in \psi$ such that $\Psi(A) = A$ and $\Psi(B) = B$. In particular, if $\psi \in C(\phi)$, then Ψ commutes with Φ.

Note that, since the rank of A is equal to 2, the rank of B is equal to 1. By Theorem 2.2 (2), for every $\psi \in C(\phi)$, the automorphism Ψ fixes B elementwise. Thus, the homomorphism which sends $\psi \in C(\phi)$ to $\Psi|_A \in \mathrm{Aut}(A)$ is injective.

Suppose first that $\Phi|_A$ is inner: there exists $g \in A$ such that $\Phi|_A = \mathrm{ad}_g$. Then, ϕ is a Dehn twist.

Suppose now that $\Phi|_A$ is not inner. By Lemma 5.4, the image of $C(\phi)$ in $\mathrm{Out}(A)$ is free and contained in the centraliser of the image of ϕ. Thus, the image of $C(\phi)$ in $\mathrm{Out}(A)$ is infinite cyclic, generated by the outer class of a root $\sqrt{\Phi|_A}$ of $\Phi|_A$. By Proposition 2.6, every element of $C(\phi)$ commutes with $\sqrt{\Phi|_A}$.

Hence $C(\phi)$ is isomorphic to $K \times \langle \sqrt{\Phi|_A} \rangle$, where K is contained in the subgroup of inner automorphisms of A. An inner automorphism $\mathrm{ad}|_g$ commutes with $\Phi|_A$ if and only if $\Phi|_A(g) = g$. Thus, K is contained in $K_\Phi = \{\mathrm{ad}_g \in \mathrm{Aut}(A) \mid g \in \mathrm{Fix}(\Phi|_A)\}$. Conversely, any element of K_Φ extends to an automorphism of F_3 commuting with Φ. Thus, $C(\phi)$ is isomorphic to $(K_\phi \cap \mathrm{IA}_3(3)) \times \langle \sqrt{\Phi|_A} \rangle$. Moreover, the free group K_ϕ has rank at most 2 by the work of Bestvina and Handel [BH92]. $\square$

Lemma 5.6. *Let $\phi \in \mathrm{IA}_3(3)$. Let $\mathcal{F}$ be a maximal $C(\phi)$-invariant free factor system. Suppose that $\mathcal{F} = \{[A]\}$ with $F_3 = A * \mathbb{Z}$ and let t be a generator of the second factor. One of the following holds.*

(1) The group $C(\phi)$ is isomorphic to $((F_\ell \times F_r) \cap \mathrm{IA}_3(3)) \times \mathbb{Z}$, where F_ℓ and F_r are two finitely generated free groups of rank at most 2.

(2) The element ϕ is a Dehn twist.

Proof. Recall that, by [Lev05], the kernel K of $C(\phi) \to \mathrm{Out}(A)$ is isomorphic to the intersection of $\mathrm{IA}_3(3)$ with a direct product $F_\ell \times F_r$ of two free (maybe cyclic or trivial) normal subgroups of $C(\phi)$. Every element of $F_\ell \times F_r$ is a Dehn twist. Moreover, as explained in the last paragraph of the proof of Case 1 in the proof of Theorem 4.2, each factor of $F_\ell \times F_r$ corresponds to the fixed subgroup of an automorphism in the outer class ϕ. The bound on the rank of the free groups then follows from the work of Bestvina and Handel [BH92].

We may suppose that ϕ is not a Dehn twist, so that the image of ϕ in $\mathrm{Out}(A)$ is not trivial. By Lemma 5.4, as in the proof of Lemma 5.5, the image of $C(\phi)$ in $\mathrm{Out}(A)$ is infinite cyclic, generated by a root of ϕ which commutes with every element of $C(\phi)$.

Combining the above two paragraphs, we see that $C(\phi)$ is isomorphic to $((F_\ell \times F_r) \cap \mathrm{IA}_3(3)) \times \mathbb{Z}$, where F_ℓ and F_r are two finitely generated free groups of rank at most 2. $\qquad\square$

Proof of Theorem 5.1. Let $\phi \in \mathrm{IA}_3(3)$ and let $C(\phi)$ be the centraliser of ϕ in $\mathrm{IA}_3(3)$. If ϕ is a Dehn twist, its centraliser in $\mathrm{Out}(F_3)$ is of type VF by the work of Rodenhausen–Wade [RW15] (see also the work of Andrew–Martino [AM]). Moreover, the short exact sequence given in case (4) follows from Theorem 2.12.

So suppose that ϕ is not a Dehn twist. Let $\mathcal{F}$ be a maximal $C(\phi)$-invariant free factor system. If $\mathcal{F}$ is nonsporadic, by Lemma 5.3, the group $C(\phi)$ is abelian. By Theorem 2.3, it is finitely generated. If $\mathcal{F}$ is sporadic, by Lemmas 5.5 and 5.6, the centraliser of ϕ satisfies one of Assertions $(2),(3)$ of Theorem 5.1. This concludes the proof. $\qquad\square$

6. Proper geometric dimension of Weyl groups of Dehn twists

In this section, we specify our study to the case of Dehn twists outer automorphisms. We will prove the following proposition.

Proposition 6.1. *Let $N \geqslant 2$ and let $\phi \in \mathrm{IA}_N(3)$ be a Dehn twist. The geometric dimension $\underline{\mathrm{gd}}(W(\phi))$ of the Weyl group of ϕ is bounded by $2N-4$.*

Recall that for a group G, the *proper geometric dimension*, $\underline{\mathrm{gd}}(G)$, is defined to be the minimal $n \in \mathbb{N} \cup \{\infty\}$ such that G admits an n-dimensional model for $\underline{E}G = E_{\mathcal{FIN}}G$.

In order to prove Proposition 6.1, we take advantage of the short exact sequence given by Theorem 2.12. We need to understand more precisely the groups $\mathrm{Out}(G_v, \mathrm{Inc}_v)$ as defined in this theorem. We first recall a result, due to Meucci.

Theorem 6.2. [Meu11] *Let $\mathcal{F}$ be a free factor system of F_N, let $\mathcal{F}^{\mathbb{Z}}$ be the subset of $\mathcal{F}$ consisting of the conjugacy classes of cyclic subgroups and let $\mathcal{F}^{\geqslant 2} = \mathcal{F} - \mathcal{F}^{\mathbb{Z}}$. Then*

$$\underline{\mathrm{gd}}(\mathrm{Out}(F_N, \mathcal{F}^{(t)})) = 2N - 2 \sum_{[A] \in \mathcal{F}^{\geqslant 2}} (\mathrm{rk}(A) - 1) - 2 - |\mathcal{F}^{\mathbb{Z}}|.$$

The proof of the following proposition is implicit in [Lüc00, Proof of theorem 3.1], see also [MPSSn20, Theorem 2.3].

Proposition 6.3. *Let G be a group. Let $f : G \to Q$ be a group homomorphism. Let Y be a model for $\underline{E}Q$. Then*

$$\underline{\mathrm{gd}}(G) \leqslant \max\{\underline{\mathrm{gd}}(G_\sigma) + \dim(\sigma) \mid \sigma \text{ is a cell of } Y\},$$

where the stabilisers are taken with respect to the G-action on Y induced by the projection.

In particular if Q torsion free and K is the kernel of f, then $\underline{\mathrm{gd}}(G) \leqslant \underline{\mathrm{gd}}(Q) + \underline{\mathrm{gd}}(K)$.

For a subgroup $H \subseteq \mathrm{Out}(F_N)$, we say that H satisfies *Property (CF)* if, for every $\psi \in H$, every ψ-periodic conjugacy class of elements of F_N is fixed. By Theorem 2.2 (2), any subgroup of $\mathrm{IA}_N(3)$ satisfies Property (CF). Groups with Property (CF) satisfies the following result.

Lemma 6.4. *Let $H \subseteq \text{Out}(F_N)$ be a subgroup satisfying Property (CF) and let $A \subseteq F_N$ be a malnormal subgroup preserved by H. The image of $p \colon H \to \text{Out}(A)$ is torsion free and satisfies Property (CF).*

Proof. Let $\psi \in H$ and let $k \in \mathbb{N}^*$ be such that $p(\psi)^k = \text{id}$. Then every conjugacy class of elements of A is fixed by ψ by Property (CF). Thus, ψ has a representative which acts as the identity on A and $\psi = \text{id}$. The fact that $p(H)$ satisfies Property (CF) is immediate. $\square$

Let $\mathcal{C} = \{[x_1], \dots, [x_k]\}$ be a finite set of conjugacy classes of elements of F_N. We will denote by $\mathcal{F}_\mathcal{C}$ the minimal free factor system of F_N such that, for every $i \in \{1, \dots, k\}$, there exists $[A] \in \mathcal{F}_\mathcal{C}$ with $x_i \in A$.

Corollary 6.5. *Let $\mathcal{C}$ be a finite set of conjugacy classes of elements of F_N. Let H be a subgroup of $\text{Out}(F_N, \mathcal{C})$ which satisfies Property (CF). The geometric dimension of H is bounded by*

$$\underline{\text{gd}}(H) \leqslant 2N - 2 - |\mathcal{F}_\mathcal{C}|.$$

Proof. Since H satisfies Property (CF), it fixes $\mathcal{C}$ elementwise. Hence H fixes $\mathcal{F}_\mathcal{C}$ elementwise. Thus, we have a natural homomorphism

$$1 \to K \to H \to \prod_{[A] \in \mathcal{F}_\mathcal{C}^{\geqslant 2}} \text{Out}(A)$$

whose kernel K is a subgroup of $\text{Out}(F_N, \mathcal{F}_\mathcal{C}^{(t)})$. By Lemma 6.4, the image of H is torsion free. By Proposition 6.3, the geometric dimension of H is bounded by

$$\underline{\text{gd}}(H) \leqslant \underline{\text{gd}}(\text{Out}(F_N, \mathcal{F}_\mathcal{C}^{(t)})) + \sum_{[A] \in \mathcal{F}_\mathcal{C}^{\geqslant 2}} \underline{\text{gd}}(\text{Out}(A)).$$

By [CV86], for every $[A] \in \mathcal{F}_\mathcal{C}^{\geqslant 2}$, we have $\underline{\text{gd}}(\text{Out}(A)) = 2\text{rk}(A) - 3$. Combining this with Theorem 6.2, we obtain

$$\underline{\text{gd}}(H) \leqslant 2N + \sum_{[A] \in \mathcal{F}^{\geqslant 2}} (2 \cdot \text{rk}(A) - 2 \cdot \text{rk}(A) + 2 - 3) - 2 - |\mathcal{F}_\mathcal{C}^{\mathbb{Z}}|.$$

Therefore, we have

$$\begin{aligned} \underline{\text{gd}}(H) &\leqslant 2N - 2 - |\mathcal{F}_\mathcal{C}^{\geqslant 2}| - |\mathcal{F}_\mathcal{C}^{\mathbb{Z}}| \\ &= 2N - 2 - |\mathcal{F}_\mathcal{C}|. \qquad \square \end{aligned}$$

We need another theorem due to Shenitzer [She55] and Swarup [Swa86] (see also the work of Stallings [Sta91] and Bestvina-Feighn [BF94, Lemma 4.1]) in order to understand the proper geometric dimension of the Weyl group of a Dehn twist.

Lemma 6.6. [She55, Swa86] *Let T be an F_N-tree whose edge stabilisers are infinite cyclic. There exists an oriented edge $e_+ \in E^+(F_N \backslash T)$ with origin $v \in V(F_N \backslash T)$ which satisfies:*

(1) the group G_v splits as $G_v = A * G_{e_+}$ for some nontrivial subgroup $A \subseteq G_v$;

(2) for every oriented edge $e'_+ \in E^+(F_N \backslash T)$ with origin v and distinct from e_+, some conjugate of $G_{e'_+}$ is contained in A.

We can now prove the key lemma in order to bound the geometric dimension of the Weyl group of a Dehn twist. Recall the definition of $T_\mathcal{A}$ for a Dehn twist $\phi \in \mathrm{IA}_N(3)$. If $v \in V(F_N \backslash T_\mathcal{A})$, Recall the definition of $\mathrm{IA}_v(3)$ in Theorem 2.12.

Lemma 6.7. *Let $N \geqslant 2$ and let $\phi \in \mathrm{IA}_N(3)$ be a Dehn twist. Let $T_\mathcal{A}$ be the associated JSJ tree. Then*

$$\sum_{v \in V(F_N \backslash T_\mathcal{A})} \underline{\mathrm{gd}}(\mathrm{IA}_v(3)) \leqslant 2N - 3 - |E(F_N \backslash T_\mathcal{A})|.$$

Proof. In order to simplify the notations, let $E = E(F_N \backslash T_\mathcal{A})$ and let $V = V(F_N \backslash T_\mathcal{A})$. For every $v \in V$, let $\mathcal{F}_v = \mathcal{F}_{\mathrm{Inc}_v}|_{G_v}$ and let $\mathcal{F} = \coprod_{v \in V} \mathcal{F}_v$. Note that for every $v \in V$, the free factor system $\mathcal{F}_v$ is a free factor system of G_v and not of F_N.

We claim that $|\mathcal{F}| \geqslant |E| + 1$. The proof is by induction on $|E| \geqslant 1$.

Suppose that $|E| = 1$ and let e be the (unoriented) edge of $F_N \backslash T_\mathcal{A}$. If e has two distinct endpoints v and w, then, since the stabiliser of e is nontrivial, we have $|\mathcal{F}_v|, |\mathcal{F}_w| = 1$ and $|\mathcal{F}| \geqslant 2$.

Suppose now that e is a loop based at v. By Lemma 6.6, one of the orientation of e, say e_+, is such that the group G_v splits as $G_v = A * G_{e_+}$ where A is a nontrivial subgroup of G_v. Since e is a loop, we have $\mathrm{Inc}_v = \{[G_{e_+}], [G_{e_-}]\} = \{[G_{e_+}], [tG_{e_+}t^{-1}]\}$, where $t \in F_N - G_v$ and $tG_et^{-1} \subseteq A$. Since $G_v = A * G_{e_+}$, we have $\mathcal{F} = \mathcal{F}_v = \{[A'], [G_{e_+}]\}$, where A' is the smallest free factor of A containing tG_et^{-1}. Thus, we have $|\mathcal{F}| \geqslant 2$. This proves the base case.

Suppose that $|E| \geqslant 2$, let $e \in E$, let $v \in V$ be adjacent to e. We also assume that, once chosen the orientation e_+ of e such that v is the origin of e_+, the oriented edge e_+ is the one given by Lemma 6.6. Let w be the other endpoint of e_+ (which is possibly equal to v). Let T' be the tree obtained from $T_\mathcal{A}$ by collapsing the orbit of the edge e_+. Let $E' = E(F_N \backslash T')$ and let $V' = V(F_N \backslash T')$. For every $x \in V'$, let $\mathcal{F}'_x = \mathcal{F}_{\mathrm{Inc}_x}|_{G_x}$ and let $\mathcal{F}' = \coprod_{x \in V'} \mathcal{F}_x$. For simplicity, if $v = w$, we set $\mathrm{Inc}_w = \varnothing$. By induction, we have $|\mathcal{F}'| \geqslant |E'| + 1 = |E|$.

For every $x \in V$, let $\bar{x}$ be the image of x in $F_N \backslash T'$. Note that, for every $x \in V - \{v, w\}$, we have $\mathcal{F}'_{\bar{x}} = \mathcal{F}_x$.

Let $A \subseteq G_v$ be such that $G_v = A * G_{e_+}$. By Lemma 6.6, for every oriented edge $e'_+ \in E^+$ with origin v and distinct from e_+, a conjugate of $G_{e'_+}$ is contained in A. Thus, we have $\mathrm{Inc}_{\bar{v}} \subseteq (\mathrm{Inc}_v - \{[G_{e_+}]\}) \cup \mathrm{Inc}_w$. Hence we see that

$$|\mathcal{F}_{\bar{v}}| \leqslant |\mathcal{F}_{\mathrm{Inc}_v - \{[G_{e_+}]\}}|_{G_v}| + |\mathcal{F}_w|.$$

Since $\mathcal{F}_{\mathrm{Inc}_v - \{[G_{e_+}]\}}|_{G_v} \leqslant \{[A]\}$ by Lemma 6.6, we also have

$$\mathcal{F}_v = \mathcal{F}_{\mathrm{Inc}_v - \{[G_{e_+}]\}}|_{G_v} \coprod \{[G_{e_+}]\},$$

so that $|\mathcal{F}_v| = |\mathcal{F}_{\mathrm{Inc}_v - \{[G_{e_+}]\}}|_{G_v}| + 1$.

Combining all the above remarks, we see that

$$|\mathcal{F}| \geqslant \sum_{x \in V - \{v,w\}} |\mathcal{F}'_{\overline{x}}| + |\mathcal{F}_v| + |\mathcal{F}_w|$$

$$\geqslant \sum_{x \in V - \{v,w\}} |\mathcal{F}'_{\overline{x}}| + |\mathcal{F}_w| + |\mathcal{F}_{\mathrm{Inc}_v - \{[G_{e_+}]\}}|G_v| + 1$$

$$\geqslant |\mathcal{F}'| + 1 \geqslant |E| + 1.$$

This proves the claim.

By for instance [CL99, Fact 7.3], we have

$$N - 1 = \sum_{v \in V} (\mathrm{rk}(G_v) - 1).$$

Note that $C(\phi)$ satisfies Property (CF) by Theorem 2.2. By Lemma 6.4, for every $v \in V$, the group $\mathrm{IA}_v(3)$ also satisfies Property (CF). By Corollary 6.5, for every $v \in V$, we have

$$\underline{\mathrm{gd}}(\mathrm{IA}_v(3)) \leqslant 2\mathrm{rk}(G_v) - 2 - |\mathcal{F}_v|.$$

Thus, we see that

$$\sum_{v \in V(F_N \backslash T_{\mathcal{A}})} \underline{\mathrm{gd}}(\mathrm{IA}_v(3)) \leqslant \sum_{v \in V(F_N \backslash T_{\mathcal{A}})} (2 \cdot \mathrm{rk}(G_v) - 2 - |\mathcal{F}_v|)$$

$$= 2 \left(\sum_{v \in V(F_N \backslash T_{\mathcal{A}})} (\mathrm{rk}(G_v) - 1) \right) - |\mathcal{F}|$$

$$= 2(N - 1) - |\mathcal{F}|$$

$$\leqslant 2N - 3 - |E|;$$

where the last inequality follows from the above claim. $\qquad\square$

Proof of Proposition 6.1. Let $\phi \in \mathrm{IA}_N(3)$ be a Dehn twist. Let $T_{\mathcal{A}}$ be the JSJ tree associated with ϕ. By Theorem 2.12, the group $C(\phi)$ fits in a short exact sequence

$$1 \to K' \to C(\phi) \to \prod_{v \in V(F_N \backslash T_{\mathcal{A}})} \mathrm{IA}_v(3) \to 1,$$

where K' is a free abelian group of rank $|E(F_N \backslash T_{\mathcal{A}})|$ and, for every $v \in V(F_N \backslash T_{\mathcal{A}})$, the group $\mathrm{IA}_v(3)$ is a finite index subgroup of $\mathrm{Out}(G_v, \mathrm{Inc}_v)$.

Moreover, the outer automorphism ϕ is in the kernel of this homomorphism. Thus, the Weyl group $W(\phi)$ of ϕ fits in a short exact sequence

$$1 \to K \to W(\phi) \to \prod_{v \in V(F_N \backslash T_{\mathcal{A}})} \mathrm{IA}_v(3) \to 1,$$

where K is an abelian group of rank $|E(F_N \backslash T_{\mathcal{A}})| - 1$. By Theorem 2.2 (2) and Lemma 6.4, the group $\mathrm{IA}_v(3)$ is torsion free. Thus, by Proposition 6.3 and Lemma 6.7, we have

$$\underline{\mathrm{gd}}(W(\phi)) \leqslant \underline{\mathrm{gd}}(K) + \sum_{v \in V(F_N \backslash T_{\mathcal{A}})} \underline{\mathrm{gd}}(\mathrm{IA}_v(3))$$

$$\leqslant 2N - 3 - |E(F_N \backslash T_{\mathcal{A}})| + |E(F_N \backslash T_{\mathcal{A}})| - 1$$

$$\leqslant 2N - 4,$$

which concludes the proof. $\qquad\square$

7. THE VIRTUALLY CYCLIC DIMENSION OF $\mathrm{IA}_N(3)$

In this section we prove Theorem A. Before proceeding to the proof we need some preliminary results.

Let G be a group. Denote by $\mathcal{VC}_\infty$ the collection of infinite virtually cyclic subgroups of G. Consider $[\mathcal{VC}_\infty]$ the set of commensuration classes of $\mathcal{VC}_\infty$. Let I be a set of representatives of conjugacy classes in $[\mathcal{VC}_\infty]$.

Theorem 7.1. *Let G be a group. Let I be defined as above, and assume that for each $H \in I$ we have $N_G[H] = N_G(H)$. For each $H \in I$, choose models for $\underline{E}N_G(H)$ and $\underline{E}W_G(H)$, where $W_G(H) = N_G(H)/H$. Now consider the homotopy G-pushout:*

$$
\begin{array}{ccc}
\coprod_{H\in I} G \times_{N_G(H)} \underline{E}N_G(H) & \xrightarrow{\;\;i\;\;} & \underline{E}G \\
{\scriptstyle \coprod_{H\in I} Id_G \times_{N_G(H)} f_H}\Big\downarrow & & \Big\downarrow \\
\coprod_{H\in I} G \times_{N_G(H)} \underline{E}W_G(H) & \longrightarrow & X
\end{array}
$$

where $\underline{E}W_G(H)$ is viewed as an $N_G(H)$-CW-complex by restricting with the projection $N_G(H) \to W_G(H)$, the maps starting from the left upper corner are cellular and one of them is an inclusion of G-CW-complexes. Then X is a model for $\underline{E}G$.

Proof. Let $H \in I$. Consider the quotient projection $p : N_G(H) \to W_G(H)$. Let $\mathcal{F}in^*$ be the family of subgroups of $N_G(H)$ that have finite image in $W_G(H)$. Let $\mathcal{F}[H]$ be the family of all finite subgroups of $N_G[H]$ and all virtually cyclic subgroups of $N_G[H]$ that are commensurable with H. Note that $\mathcal{F}in^* = \mathcal{F}[H]$, and as a consequence every model for $\underline{E}W_G(H)$ is a model for $E_{\mathcal{F}[H]}N_G(H)$. Now the statement follows from [LW12, Theorem 2.3]. $\qquad\square$

An immediate corollary of Proposition 6.3 is the following.

Corollary 7.2. *Under the hypothesis of Theorem 7.1, we have the following*

$$\underline{\mathrm{gd}}(G) \leqslant \max\{\underline{\mathrm{gd}}(G) + 1, \underline{\mathrm{gd}}(W_G(H)) | H \in I\}.$$

Proposition 7.3. *Let H be an infinite cyclic subgroup of $\mathrm{IA}_3(3)$, then $\underline{\mathrm{gd}}(W(H)) = \underline{\mathrm{gd}}(\mathrm{IA}_3(3)) - 1 = 2$.*

Proof. Denote $G = \mathrm{IA}_3(3)$, and recall that G is torsion free. Then by Theorem 5.1 we have four possibilities for $C(H)$ (and so does for $W(H)$), let us work out each of them using the same numeration as in the theorem.

Case 1. The centraliser $C(H)$ is $\mathbb{Z}^r$ with $r \leqslant 3$. Thus $W(C)$ is an abelian group of rank at most 2. Hence $\mathbb{R}^{r-1}$ is a model for $\underline{E}W(H)$, in particular this model has dimension at most 2. $\qquad\blacklozenge$

Case 2. The centraliser $C(H)$ is isomorphic to $F \times \mathbb{Z}$ where F is a finitely generated free group and H lies inside the $\mathbb{Z}$ under the isomorphism. The Weyl group $W(H)$ has the form $F \times C$ where C is a finite cyclic group. Since proper classifying space models respect direct products we have $\underline{E}W(H) = \underline{E}F \times \underline{E}C = \underline{E}F$ and the latter can be taken to be a tree. $\qquad\blacklozenge$

Case 3. The centraliser $C(H)$ is isomorphic to a direct product $K \times \mathbb{Z}$ where K is a finite index subgroup of a direct product of two finitely generated free groups and H lies in the $\mathbb{Z}$ factor under the isomorphism. In this case $C(H) \leqslant F_1 \times F_2 \times \mathbb{Z}$, hence the Weyl group $W(H)$ embeds into $F_1 \times F_2 \times C$ with C a finite cyclic group. Thus it is enough to find a model for $\underline{E}(F_1 \times F_2 \times C)$ since by restriction it will be also a model for $\underline{E}W(H)$. As in the previous item such a model can be taken to be of the form $T_1 \times T_2$ with T_1 and T_2 are trees. Thus we have a model for $\underline{E}W(H)$ of dimension 2.   ◆

Case 4. The group H is generated by a Dehn twist. By Proposition 6.1, we have a model for $\underline{E}W(H)$ of dimension 2.   ◆

We have exhausted all possible cases for $W(H)$, completing the proof.   □

Proposition 7.4. *Let $N \geqslant 3$. Consider a maximal infinite cyclic subgroup H of $G = \mathrm{IA}_N(3)$. Then $\underline{\mathrm{gd}}(W_G(H)) \leqslant \underline{\mathrm{gd}}(G) - 1 = 2N - 4$.*

Proof. Let ϕ generate H. We proceed by induction. Our base case $N = 3$ was already proved in Proposition 7.3.

Assume now that for every $N' < N$ the statement is true. Let $\mathcal{F} = \{[A_1], \ldots, [A_\ell]\}$ be the free factor system given by Corollary 4.8. For every $i \in \{1, \ldots, \ell\}$, identify $\mathrm{IA}(A, 3)$ with $\mathrm{IA}_{n_i}(3)$ where $\mathrm{rk}(A) = n_i$. Note that, for every $i \in \{1, \ldots, \ell\}$, we have $\underline{\mathrm{gd}}(\mathrm{IA}_{n_i}(3)) = 2n_i - 3$ if $n_i \geqslant 2$ and 0 otherwise. Now, we will prove the statement for N, by exhausting all cases for $W(H)$ described in Corollary 4.8. We use the notation from that theorem without further explanation.

Case 1. ϕ is a Dehn twist. This is already proven in Proposition 6.1.   ◆

Case 2. The group $W(\phi)$ is isomorphic to a subgroup of $\mathrm{Out}(A) \times \mathrm{Out}(B)$, where $A, B \subseteq F_N$ are such that $\mathrm{rk}(A) + \mathrm{rk}(B) = N + 1$ and $\mathrm{rk}(A), \mathrm{rk}(B) \leqslant N - 1$.

We have $\underline{\mathrm{gd}}(\mathrm{Out}(A)) = 2 \cdot \mathrm{rk}(A) - 3$ and $\underline{\mathrm{gd}}(\mathrm{Out}(B)) = 2 \cdot \mathrm{rk}(B) - 3$ (both A and B are non-abelian). So

$$\begin{aligned}\underline{\mathrm{gd}}(W(\phi)) &\leqslant 2(\mathrm{rk}(A) + \mathrm{rk}(B)) - 6 \\ &= 2(N + 1) - 6 \\ &= 2N - 4\end{aligned}$$

as required.   ◆

Case 3. The group $W(\phi)$ is isomorphic to a subgroup of $\mathrm{Out}(A * \langle s^t \rangle, [s], [s^t])$, where $A \subseteq F_N$, $\mathrm{rk}(A) = N - 1$, $s \in A$ and t is a basis element of F_N. Let $\mathcal{C} = \{[s], [s^t]\}$ and note that $|\mathcal{F}_\mathcal{C}| = 2$ since $s \in A$ (see the notations in Corollary 6.5). By Corollary 6.5, a model for $\underline{E}\,\mathrm{Out}(A * \langle s^t \rangle, [s], [s^t])$ has dimension

$$2\mathrm{rk}(A * \langle s^t \rangle) - 2 - |\mathcal{F}_\mathcal{C}| = 2N - 4.$$

A model for $\underline{E}W(\phi)$ is given by $\underline{E}\,\mathrm{Out}(A * \langle s^t \rangle, [s], [s^t])$. Thus,

$$\underline{\mathrm{gd}}(W(\phi)) \leqslant 2N - 4$$

as required.   ◆

Case 4. The group $W(\phi)$ is isomorphic to a subgroup of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$, where $A, B \subseteq F_N$ are nontrivial subgroups such that $F_N = A * B$.

If $\mathrm{rk}(A) = 1$, then $\mathrm{Aut}(A)$ is finite and so $\underline{\mathrm{gd}}(\mathrm{Aut}(A)) = 0$. Otherwise, a model X for $\underline{E}\,\mathrm{Aut}(A)$ is given by an $\mathrm{Aut}(A)$-fibration $X \to \underline{E}\,\mathrm{Out}(A)$. Here, the stabilisers of the $\mathrm{Aut}(A)$-action on $\underline{E}\,\mathrm{Out}(A)$ are virtually free, being extensions of A by finite subgroups of $\mathrm{Out}(A)$. It follows from Proposition 6.3 that $\underline{\mathrm{gd}}(\mathrm{Aut}(A)) = 2\mathrm{rk}(A) - 2$. An identical argument gives $\underline{\mathrm{gd}}(\mathrm{Aut}(B)) \leqslant 2\mathrm{rk}(B) - 2$. Now, a model for for $\underline{E}W(\phi)$ is given by $\underline{E}\,\mathrm{Aut}(A) \times \underline{E}\,\mathrm{Aut}(B)$. So,

$$\underline{\mathrm{gd}}(W(\phi)) \leqslant 2(\mathrm{rk}(A) + \mathrm{rk}(B)) - 4$$
$$= 2N - 4$$

as required.   ◆

Case 5. The Weyl group $W(\phi)$ fits into an exact sequence

$$1 \to K \to W(\phi) \to H_\infty/\rho_\infty(\langle\phi\rangle) \times H_T,$$

where K is a finite index subgroup of a direct product of two finitely generated free groups.

In this case $\mathcal{F} = \{A_1, A_2\}$, A_1 is never trivial, and A_2 might be trivial or not, see Remark 4.4. We subdivide into two subcases depending on whether A_2 is trivial or not.

Assume A_2 is trivial. Then $\mathcal{F} = \mathcal{F}_\infty$, $\mathcal{F}_T$ is empty, and $n_1 \leqslant N-1$. Hence, we have the following exact sequence

$$1 \to K \to W(\phi) \to C(\langle\phi|_{A_1}\rangle)/\rho_\infty(\langle\phi\rangle) = W_{\mathrm{IA}(A_1,3)}(\rho_\infty(\langle\phi\rangle)).$$

Now, by hypothesis ϕ is not a proper power. This implies that $W_{\mathrm{IA}(A_1,3)}(\rho_\infty(\langle\phi\rangle))$ is torsion free. Now by Proposition 6.3 and the induction hypothesis we get

$$\underline{\mathrm{gd}}(W(\phi)) \leqslant 2 + \underline{\mathrm{gd}}(W_{IA(A_1,3)}(\rho_\infty(\langle\phi\rangle)))$$
$$\leqslant 2 + 2n_1 - 4$$
$$\leqslant 2 + 2(N-1) - 4 = 2N - 4.$$

Now, assume A_2 is not trivial. We distinguish two cases depending on $\mathcal{F}_\infty$. If $\mathcal{F}_\infty = \{A_1\}$, then we have the exact sequence

$$1 \to K \to W(\phi) \to W_{IA(A_1,3)}(\rho_\infty(\langle\phi\rangle)) \times \mathrm{IA}(A_2, 3).$$

By Proposition 6.3 and the induction hypothesis we get

$$\underline{\mathrm{gd}}(W(\phi)) \leqslant 2 + \underline{\mathrm{gd}}(W_{IA(A_1,3)}(\rho_\infty(\langle\phi\rangle))) + \underline{\mathrm{gd}}(\mathrm{IA}(A_2,3))$$
$$\leqslant 2 + 2n_1 - 4 + 2n_2 - 3$$
$$\leqslant 2(n_1 + n_2) - 5 \leqslant 2N - 4,$$

where the last inequality follows from the fact that $\sum n_i \leqslant N$, see Remark 4.4. Finally, if $\mathcal{F}_\infty = \{A_1, A_2\}$, then we have the exact sequence

$$1 \to K \to W(\phi) \to (C(\phi|_{A_1}) \times C(\phi|_{A_2}))/\rho_\infty(\langle\phi\rangle) =: Q.$$

On the other hand, taking the quotient of $C(\phi|_{A_1}) \times C(\phi|_{A_2})$ by $\langle\phi|_{A_1}, \phi|_{A_2}\rangle \cong \mathbb{Z}^2$, we conclude Q fits in the following exact sequence

$$1 \to \mathbb{Z} \to Q \to W(\phi|_{A_1}) \times W(\phi|_{A_2}).$$

Hence, we get

$$\underline{\mathrm{gd}}(Q) \leqslant 1 + \underline{\mathrm{gd}}(W(\phi|_{A_1})) + \underline{\mathrm{gd}}(W(\phi|_{A_2}))$$
$$\leqslant 1 + 2n_1 - 4 + n_2 - 4$$
$$\leqslant 2N - 7$$

and therefore

$$\underline{\mathrm{gd}}(W(\phi) \leqslant 2 + 2N - 7 = 2N - 5$$

as required.

Case 6. The Weyl group $W(\phi)$ fits into an exact sequence

$$1 \to K \to W(\phi) \to \prod_{[A] \in \mathcal{F}} \mathrm{IA}(A, 3)$$

where K is the direct product of a finitely generated free (maybe cyclic or trivial) group and a finite group. Moreover, we have $|\mathcal{F}| \leqslant 2$ by Theorem 4.1 (2). Write $K = K' \times T$ where K' is a (possibly trivial) free group and T is the finite group. Thus, we may rewrite the quotient as $\mathrm{IA}_{n_1}(3) \times \mathrm{IA}_{n_2}(3)$ such that each $n_i \leqslant N - 1$ and $n_1 + n_2 \leqslant N$ (see Remark 4.4). We may now build a model X for $\underline{E}W(\phi)$ as a G-fibration

$$\underline{E}K' \to X \to E\,\mathrm{IA}_{n_1}(3) \times E\,\mathrm{IA}_{n_2}(3).$$

Since K' is free we have $\underline{\mathrm{gd}}(K') = 1$ and on the other end of the fibration we have $\underline{\mathrm{gd}}(\mathrm{IA}_{n_i}(3)) = 2n_i - 3$ if $n_i \geqslant 2$ and 0 otherwise. If both $n_1, n_2 \geqslant 2$, then

$$\underline{\mathrm{gd}}(W(\phi)) \leqslant 1 + \sum_{i=1}^{2}(2n_i - 3) \leqslant 1 + 2N - 6 \leqslant 2N - 4.$$

If say $n_2 \leqslant 1$, then $2 \leqslant n_1 \leqslant N - 1$ and

$$\underline{\mathrm{gd}}(W(\phi)) \leqslant 1 + 2(N-1) - 3 \leqslant 1 + 2N - 5 \leqslant 2N - 4$$

as required.   ◆

We have exhausted all possible cases for $W(\phi)$ completing the proof.   □

Theorem A. *Let $N \geqslant 1$. Then, $\underline{\underline{\mathrm{gd}}}(\mathrm{IA}_N(3)) = 2N - 2$.*

Proof. The statement is clear for $N = 1$. For $N = 2$ follows from the fact that $\mathrm{IA}_2(N)$ is nonabelian free. Let $N \leqslant 3$. By [CV86] $\mathrm{Out}(F_N)$ admits a model for $\underline{E}\,\mathrm{Out}(F_N)$ of dimension $2N - 3$, thus $\underline{\mathrm{gd}}(\mathrm{IA}_N(3)) \leqslant \underline{\mathrm{gd}}(\mathrm{Out}(F_N)) \leqslant 2N - 3$. From the latter inequality, Corollary 7.2 and Proposition 7.4 we conclude $\underline{\underline{\mathrm{gd}}}(\mathrm{IA}_N(3)) \leqslant 2N - 2$. On the other hand, by [CV86] there is a copy of $\overline{\mathbb{Z}}^{2N-3}$ inside $\mathrm{IA}_N(3)$. By [CFH] we have $\underline{\underline{\mathrm{gd}}}(\mathbb{Z}^{2N-3}) = 2N - 2$. Therefore $2N - 2 \leqslant \underline{\underline{\mathrm{gd}}}(\mathrm{IA}_N(3))$. This concludes the proof.   □

Corollary B. *Let $N \geqslant 1$. Then, $\underline{\underline{\mathrm{gd}}}(\mathrm{Out}(F_N))$ is finite.*

Proof. This is trivial for $N = 1$. Suppose now $N = 2$. By the Kaloujnine–Krasner Theorem we may embed $\mathrm{Out}(F_N)$ into the wreath product $G = \mathrm{IA}_N(3) \wr \mathrm{GL}_N(3)$. A model for $\underline{E}G$ is given by $\prod_{|\mathrm{GL}_N(3)|} \underline{E}\,\mathrm{IA}_N(3)$ and this space has dimension

$$|\mathrm{GL}_N(3)| \cdot \underline{\underline{\mathrm{gd}}}(\mathrm{IA}_N(3)) = 3^{\frac{1}{2}N(N-1)} \cdot (2N - 2) \cdot \prod_{i=1}^{N-1}(3^{N-i} - 1). \qquad □$$

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