Lattices in non-positive curvature

Sam Hughes

March 2021

Presentations

- Let $X = \{a_1, \ldots, a_n\}$ be a finite set and let F_X denote the free group on X.
- The free group F_X is the set of all words in $X \cup X^{-1}$ with composition given by concatenation and cancellation.
- A group G is *finitely generated* if there exists a surjection $F_X \to G$.
- A finitely generated group G is *finitely presented* if the kernel of the surjection is finitely normally generated.
- In the later case we may write $G \cong \langle X \mid R \rangle$ where R is a normal generating set for the kernel.
- E.g. $\mathbb{Z}_n = \langle a \mid a^n \rangle$.
- **Boone-Novikov:** The question "does $\langle X \mid R \rangle$ present the trivial group?" is undecidable.

Geometric group theory

- Let $G = \langle X \rangle$ be finitely generated and let $\mathrm{Cay}(G, X)$ be the Cayley graph of G with respect to X.
- This is a metric space by assigning edges length 1.
- **Problem:** Different finite generating sets give different graphs.
- **Solution:** Quasi-isometries.

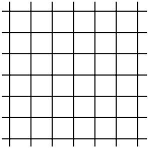
Definition (Quasi-isometry)

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function, we say f is a *quasi-isometry* if there exist constants $L \geq 1$ and $C \geq 0$ such that

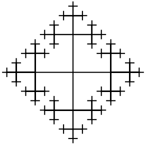
1. $\forall x, y \in X$:
   $L^{-1} d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq L d_X(x, y) + C$.
2. Let $z \in Y$, then $\exists x \in X$ such that $d_Y(z, f(x)) \leq C$.

Examples

$\mathrm{Cay}(\mathbb{Z}^n, X)$ is quasi-isometric to $\mathbb{E}^n$.



$\mathrm{Cay}(F_X, X)$ is quasi-isometric to a tree.



In both cases X is the “obvious” symmetric generating set.

Geometric Group Theory

Proposition

Let X and Y be two finite generating sets for a group G, then $\mathrm{Cay}(G,X)$ and $\mathrm{Cay}(G,Y)$ are quasi-isometric.

Let G act on a metric space X, the action is:

- *cocompact* if there exists a compact set K such that $\bigcup_{g \in G} gK = X$.
- *properly discontinuous* if for every compact set $K \subseteq X$ the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite.

Milnor-Švarc Lemma

Lemma (Milnor-Švarc)

Suppose a finitely generated group G acts properly discontinuously cocompactly by isometries on a metric space X, then G is quasi-isometric to X.

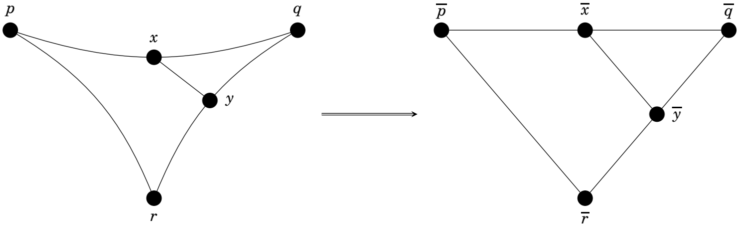
Slogan: Study finitely generated groups by their geometric actions on nice metric spaces.

What if we turn the question on its head and start with a nice space?

$\mathrm{CAT}(0)$ spaces

A geodesic metric space X is $\mathrm{CAT}(0)$ if for every geodesic triangle $P = \triangle(p, q, r) \subseteq X$ there exists a comparison triangle in $\mathbb{E}^2$ with the same side lengths as P such that for each pair of points $x, y \in \partial P$ we have

$$d_X(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$



Properties and examples

$\mathrm{CAT}(0)$ spaces are contractible and have unique geodesics.

Examples

- $\mathbb{E}^n$
- Trees
- Non-compact symmetric spaces (e.g. $\mathbb{R}\mathbf{H}^2$)
- Infinite buildings

Lattices in locally compact groups

Let $H = \mathrm{Isom}(X)$ be a locally compact group with Haar measure μ. A discrete subgroup $\Gamma \leq H$ is:

- a *lattice* if X/Γ has finite covolume;
- a *uniform lattice* if X/Γ is compact.

For a lattice Γ in a product $\prod_{i=1}^{n} H_i$ we say Γ is:

- *irreducible* if the projection to each subproduct of the H_i is non-discrete;
- *reducible* otherwise.

Key point

Uniform lattices in the isometry group of a $\mathrm{CAT}(0)$ space are $\mathrm{CAT}(0)$ groups.

Idea: Study lattices in $\mathrm{Isom}(X)$ where X is a $\mathrm{CAT}(0)$ space.

Questions about lattices

Let X be a metric space and $H = \mathrm{Isom}(X)$ with Haar measure μ.

1. Does H have lattices?
2. What are properties of a generic lattice in H?
3. Do the properties of lattices in H reflect properties of H?
4. Can we classify lattices in H up to isomorphism, commensurability, or quasi-isometry?

Crystallographic groups

- Let $\mathbb{E}^n$ denote Euclidean n-space, then $\mathrm{Isom}(\mathbb{E}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$.
- A lattice $\Gamma < \mathrm{Isom}(\mathbb{E}^n)$ admits a short exact sequence $\{1\} \to \mathbb{Z}^n \to \Gamma \to \pi_{\mathrm{O}(n)}(\Gamma) \to \{1\}$.
- A virtually abelian group Γ admits a short exact sequence $\{1\} \to \mathbb{Z}^n \to \Gamma \to P \to \{1\}$.
- **Zassenhaus:** If G is a finitely generated virtually abelian group, then G acts properly discontinuously cocompactly on $\mathbb{E}^n$ and so is $\mathrm{CAT}(0)$. Moreover, Γ is an $\mathrm{Isom}(\mathbb{E}^n)$-lattice if and only if $P \rightarrowtail \mathrm{GL}_n(\mathbb{R})$.

Lattices in Lie groups

Let $G(\mathbb{R})$ be a non-compact semisimple real Lie group with finite centre.

e.g. $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})^2$.

Let K be a finite extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$ and let $\Gamma = G(\mathcal{O}_K)$

e.g. $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ or $\Lambda = \mathrm{SL}_2(\mathbb{Z})^2$.

For each place s_i of K we have an embedding $\sigma_i : K \rightarrowtail \mathbb{R}$ or $\mathbb{C}$

e.g. $\sqrt{2} \mapsto \pm\sqrt{2} \in \mathbb{R}$.

This induces a diagonal map $G(\mathcal{O}_K) \to G(\mathbb{R})^m \times G(\mathbb{C})^n$.

e.g. $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}]) \rightarrowtail \mathrm{SL}_2(\mathbb{R})^2$.

Lattices in Lie groups (cont.)

Both $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ and $\Lambda = \mathrm{SL}_2(\mathbb{Z})^2$ are lattices in $\mathrm{SL}_2(\mathbb{R})^2$.

Γ is irreducible and Λ is reducible.

Both of them are *arithmetic groups*.

Questions about lattices

Let X be a metric space and $H = \mathrm{Isom}(X)$ with Haar measure μ.

1. Does H have lattices?
2. What are properties of a generic lattice in H?
3. Do the properties of lattices in H reflect properties of H?
4. Can we classify lattices in H up to isomorphism, commensurability, or isometry?

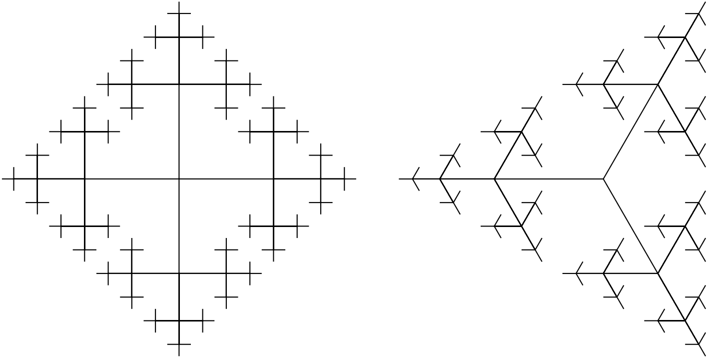
Answers for Lie Groups

Let H be a real semisimple Lie group with trivial centre.

1. H has lattices.
2. In higher rank the lattices are just-infinite [Margulis 1979]. They are finitely presented, linear, residually finite and Hopfian.
3. They have property (T) if H does. Homomorphisms of higher rank lattices induce homomorphisms of H. [Margulis Super Rigidity]
4. In theory in higher rank yes. All lattices are arithmetic [Margulis 1984] so possible using number theoretic techniques.

Tree lattices

Let $\mathcal{T}$ be a regular or biregular infinite tree e.g.



Let $T = \mathrm{Aut}(\mathcal{T})$. We want to study uniform lattices in T, i.e. groups which act cocompactly with finite stabilisers on $\mathcal{T}$.

Graphs of groups

Let $\mathcal{A} = (V\mathcal{A}, E\mathcal{A}, \iota, \tau)$ be a graph equipped with the following data:

- For each vertex v and edge e a group A_v or A_e;
- For each edge e, monomorphisms $\psi_e : A_e \to A_{\iota(e)}$ and $\psi_{\overline{e}} : A_e \to A_{\tau(e)}$

We denote this by A and call it a *graph of groups over $\mathcal{A}$*.

Graphs of groups (cont.)

From a graph of groups A we obtain the *fundamental group*. Let $\mathcal{X}$ be a spanning tree of $\mathcal{A}$ then $\pi_1(A)$ has generators

$$A_v,\ t_e \text{ for } v \in V\mathcal{A},\ e \in E\mathcal{A},$$

and relations

- $t_e = 1$ if $e \in E\mathcal{X}$;
- the relations in the groups A_v;
- $t_e \psi_e(g) t_e^{-1} = \psi_{\overline{e}}(g)$ for all $g \in A_e$.

We also obtain a tree by taking the graph with vertices $\coprod_{v \in V\mathcal{A}} \pi_1(A)/A_v$ and edges $\coprod_{e \in E\mathcal{A}} \pi_1(A)/A_e$.

Bass-Serre Theory

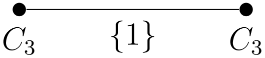
From a group action on a tree we can also construct a graph of groups from the vertex and edge stabilisers in a fundamental domain.

Theorem (Bass-Serre)

The processes of constructing of the Bass-Serre tree from a graph of groups and constructing a graph of groups from a group action on a tree are mutually inverse.

A quick example

- Let $\mathcal{A}$ be a single edge e with two vertices v and w.
- Let $A_v = A_w = C_3$ and $A_e = \{1\}$;



- $\pi_1(A) = \langle a, b \mid a^3, b^3 \rangle = C_3 * C_3$;
- The Bass-Serre tree is the 3-regular tree.
- Every vertex is stabilised by a conjugate of $\langle a \rangle$ or $\langle b \rangle$.

Tree lattices (revisited)

Theorem (Bass-Kulkarni 1990)

Every tree lattice splits as a faithful graph of finite groups, and conversely every faithful graph of finite groups is a tree lattice.

Corollary

Uniform tree lattices are virtually free.

Caprace–Monod's Decomposition Theorem (2009)

Let Y be proper $\mathrm{CAT}(0)$ space and assume $\mathrm{Isom}(Y)$ acts cocompactly and minimally. There exists a $\mathrm{Isom}(Y)$-stable subset X and $H \trianglelefteq_{\mathrm{f.i}} \mathrm{Isom}(X)$ such that:

$$X = \mathbb{R}^n \times M_1 \times \cdots \times M_p \times X_1 \times \cdots \times X_q,$$

and

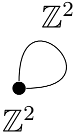
$$H \cong \mathrm{Isom}(\mathbb{E}^n) \times L_1 \times \cdots \times L_p \times D_1 \times \cdots \times D_q,$$

where each M_i is a symmetric space of non-compact type with associated Lie group L_i and X_j is an irreducible $\mathrm{CAT}(0)$ space with totally disconnected isometry group D_j. Moreover, $\mathrm{AR(H)} = \mathrm{Isom(\mathbb{E}^n)}$.

On $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices

- It was thought every $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice was reducible i.e. virtually $\mathbb{Z}^n \times F_m$.
- **Leary-Minasyan 2019:** Constructed groups $\mathrm{LM}(A)$ using a graph with a single vertex and edge which are irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices.
- These groups are not linear, residually finite, or biautomatic answering a 25 year old question about whether every $\mathrm{CAT}(0)$ group is biautomatic.

Leary-Minasyan Groups



Let $A \in \mathrm{O}(2)$ and L_1 be a finite index subgroup of $\mathbb{Z}^2$ and $L_2 = A(L_1)$.

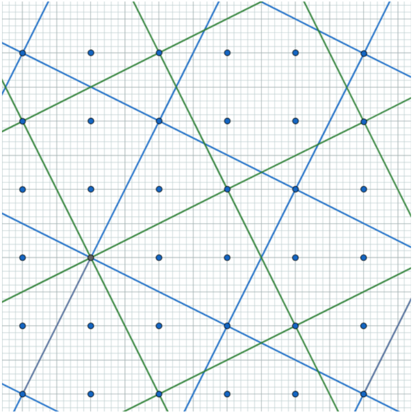
$$\mathrm{LM}(A) = \langle a, b, t \mid [a, b],\ tL_1t^{-1} = L_2 \rangle$$

$$\phi : \mathrm{LM}(A) \to \mathrm{Isom}(\mathbb{E}^n) \quad \text{by} \quad \begin{cases} a \mapsto & [1, 0]^T \\ b \mapsto & [0, 1]^T \\ t \mapsto & A. \end{cases}$$

Leary-Minasyan Groups (cont.)

$$A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix},\ L_1 = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle \text{ and } L_2 = \left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\rangle$$

$$G := \mathrm{LM}(A) = \langle a, b, t \mid [a, b],\ ta^2bt^{-1} = a^2b^{-1},\ tab^2t^{-1} = a^{-1}b^2 \rangle$$



Irreducibility

Theorem (Leary-Minasyan)

G is an irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T_{10})$-lattice.

Here $T_{10} := \mathrm{Aut}(\mathcal{T}_{10})$ the Bass-Serre tree of G.

The Flat Torus Theorem

Theorem

Let L be a free abelian group acting properly by semisimple isometries on $\mathrm{CAT}(0)$ space X. If a group Γ of isometries of X normalizes L, then a finite-index subgroup of Γ centralizes L. If Γ is finitely generated, then ΓL has a finite-index subgroup containing L as a direct factor.

Irreducibility

- Γ acts cocompactly and freely on $\mathbb{E}^n \times \mathcal{T}_{10}$;
- It follows Γ is a lattice in $\mathrm{Isom}(\mathbb{E}^n) \times T_{10}$, where $T_{10} = \mathrm{Aut}(\mathcal{T}_{10})$;
- The element A is an infinite order orthogonal matrix so $\mathrm{LM}(A)$ has a non-discrete projection to $\mathrm{Isom}(\mathbb{E}^n)$;
- Let C denote the intersection of all conjugates of $L = \langle a, b \rangle$;
- Now $C \trianglelefteq \mathrm{LM}(A)$ so by *the flat torus theorem* there exists some $k > 0$ such that t^k centralises L;
- Thus, t^k acts trivially on $C \otimes \mathbb{R}$. Which is nonsensical unless $C = 0$.
- It follows L acts faithfully on $\mathcal{T}_{10}$ as a vertex stabiliser. Hence the projection is non-discrete.

Generic properties?

Theorem (H. 2021)

Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{E}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. The following are equivalent:

1. Γ *is an irreducible* $(\mathrm{Isom}(\mathbb{E}^n) \times T)$*-lattice;*
2. Γ *is irreducible as an abstract group;*
3. Γ *acts on* $\mathcal{T}$ *faithfully;*
4. Γ *does not virtually fibre;*
5. Γ *is* C^**-simple;*
6. *and if* $n = 2$, Γ *is non-residually finite and not virtually biautomatic.*

Thanks for listening!