IRREDUCIBLE LATTICES FIBRING OVER THE CIRCLE

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ABSTRACT. We investigate the Bieri–Neumann–Strebel–Renz (BNSR) invariants of irreducible uniform lattices in the product of $\mathrm{Isom}(\mathbb{E}^n)$ and $\mathrm{Aut}(\mathcal{T})$ or $\mathrm{Aut}(\widetilde{S}_L)$, where $\mathcal{T}$ is locally finite tree and $\widetilde{S}_L$ is the universal cover of the Salvetti complex of the right-angled Artin group on the graph L. In the case of a tree we show that vanishing of the BNSR invariants for all finite-index subgroups of a given uniform lattice is equivalent to irreducibility. In the case of the Salvetti complex we construct irreducible uniform lattices whose BNSR invariants are related to those of certain right-angled Artin groups. These appear to be the first examples of irreducible lattices in a non-trivial product admitting characters with arbitrary finiteness properties.

1. INTRODUCTION

Let H be a locally compact group with Haar measure μ. A *lattice* Γ in H is a discrete subgroup such that H/Γ has finite measure. We say Γ is *uniform* if H/Γ is compact. Roughly speaking, a lattice Γ in a product $G \times H$ is *irreducible* if the projections of Γ to G and H are non-discrete and Γ does not virtually split as a direct product of two infinite groups, otherwise we say Γ is *reducible* (we will give the precise definition in Section 2.2). A celebrated application of Margulis's normal subgroup theorem [Mar78] connects, in the case of lattices in semsimple Lie groups, irreducibility with vanishing of the first cohomology group.

Theorem 1.1 (Margulis)**.** *Let Γ be a lattice in semisimple Lie group with finite centre. If $H^1(\Gamma) \neq 0$, then Γ is reducible.*

We will now broaden our scope to lattices in products of isometry groups of irreducible minimal CAT(0) spaces. Here a CAT(0) space X is *irreducible* if X

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does not split as a direct product of two unbounded spaces and is *minimal* if there is no $\mathrm{Isom}(X)$-invariant closed convex non-empty subspace $X' \subset X$. In this later case we say that $\mathrm{Isom}(X)$ acts *minimally.* The reader can consult [BH99] for a comprehensive introduction to the theory of CAT(0) spaces and [CM09b; CM09a; CM19] for a structure theory of the spaces and their isometry groups.

In this more general setting the universal covering trick of Burger–Mozes shows that a generalisation of Theorem 1.1 even to lattices in products of trees and symmetric spaces fails (see [BM00]). However, if the first cohomology group is non-zero we are able to deploy secondary invariants introduced in [BNS87; BR88] called *BNSR* or Σ-*invariants* $\Sigma^n(\Gamma)$ and $\Sigma^n(\Gamma;\mathbb{Z})$ which measure how far a first cohomology class is from a fibration $B\Gamma \to S^1$ of finite CW complexes.

A first cohomology class φ and its inverse $-\varphi$ are in $\Sigma^n(\Gamma)$ (resp. $\Sigma^n(\Gamma;\mathbb{Z})$) if and only if φ is F_n-fibred (resp. FP_n-fibred). Here, φ is F_n-fibred (resp. FP_n-fibred) if $\mathrm{Ker}(\varphi)$ is type F_n, that is, there exists a model for $K(\mathrm{Ker}(\varphi), 1)$ with finite n-skeleton (resp. type FP_n, that is, there exists a projective resolution $P_* \to \mathbb{Z}$ over $\mathbb{Z}[\mathrm{Ker}(\varphi)]$ such that for each $i \leqslant n$ the module P_i is a finitely generated $\mathbb{Z}[\mathrm{Ker}(\varphi)]$-module). Motivated by this we ask the following question and answer it in several cases.

Question 1.2. *Let Γ be a uniform lattice in a product $X_1 \times X_2$ of proper minimal* CAT(0) *spaces. If $\Sigma^n(\Gamma)$ or $\Sigma^n(\Gamma;\mathbb{Z})$ is non-empty for some $n \geqslant 1$, then is Γ necessarily reducible?*

There are plenty of irreducible CAT(0) groups which virtually fibre - we will explain how these either give positive answers to Question 1.2 or are not within its remit. In the seminal work of Bestvina and Brady [BB97] the authors show that there exist characters of right angled Artin groups (RAAGs) which FP_2-fibre but not F_2-fibre. We mention here that every RAAG is either a direct product of two infinite subgroups or is a lattice in a single irreducible CAT(0) space. Generalisations to obtain uncountably many (quasi-isometry classes of) groups of type FP have been considered by Leary [Lea18a] (Kropholler–Leary–Soroko [KLS20]) and Brown–Leary [BL20]. For right angled Coxeter groups (RACGs) there is work of Jankiewicz–Norin–Wise [JNW21] where the authors algebraically fibre certain finite index subgroups and work of Schesler–Zaremsky [SZ21] where the authors take a probabilistic viewpoint. As in the case of RAAGs every RACG is either a direct product of two infinite subgroups or is a lattice in a single irreducible CAT(0) space.

A deep theorem of Agol states that hyperbolic 3-manifolds virtually fibre [Ago13]. We briefly mention that this result has been generalised to the setting of RFRS groups by Kielak [Kie20] and improved further by Fisher [Fis21]. In higher dimensions a number of hyperbolic n-manifolds have been algebraically fibred in the work of Battista, Isenrich, Italiano, Martelli, Migliorini, and Py [BM21; IMM21b; IMP21]. We highlight the paper of Italiano–Martelli–Migliorini [IMM21a] where the authors fibre a hyperbolic 5-manifold over S^1. Of course in every case each group is a lattice in a single irreducible CAT(0) space.

In the case of a uniform lattice in the product of a locally-finite tree and a Euclidean space we give a positive answer to Question 1.2. The existence of irreducible lattices was demonstrated by Leary and Minasyan - where they construct the first examples of CAT(0) but not biautomatic groups [LM19]; a rough classification of such lattices was obtained by the author in [Hug21b]. Note that the following theorem is new even for Leary–Minasyan groups. Later, we will show that upon replacing the tree with a Salvetti complex there are irreducible lattices whose BNSR-invariants are non-empty for all n.

Theorem A (Theorem 3.6)**.** *Let $\mathcal{T}$ be a locally-finite leafless unimodular tree, not isometric to $\mathbb{R}$, and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice, then Γ virtually F_1-fibres if and only if Γ virtually F_∞-fibres if and only if Γ is reducible.*

A group Γ *virtually fibres* if there exists a finite-index subgroup $\Gamma' \leqslant \Gamma$ and a character $\varphi \in H^1(\Gamma';\mathbb{R})$ such that $\mathrm{Ker}(\varphi)$ is of type F, that is, there exists a finite model for $K(\mathrm{Ker}(\varphi), 1)$.

Corollary B (Corollary 3.8)**.** *With notation as in Theorem A, suppose $n = 2$. Then, Γ virtually fibres if and only if Γ is reducible.*

The main obstruction to extending the previous corollary to higher dimensional Euclidean spaces (i.e. $n \geqslant 3$) is that we do not know if every $(\mathrm{Isom}(\mathbb{E}^{n-1})\times T)$-lattice is virtually torsion-free (see [Hug21b, Question 9.1]).

Let $\mathcal{CW}$ denote the category of CW complexes. Let L be a flag complex on the vertex set $[m] := \{1, \dots, m\}$ and let $\mathcal{S}_L$ denote the category with objects simplices of L and morphisms inclusions of simplices. Define a functor $D\colon \mathcal{S}_L \to \mathcal{CW}$ by

$$D(\sigma) = \prod_{i\in[m]} Y_i \quad \text{where} \quad Y_i = \begin{cases} S^1 & i \in \sigma, \\ * & i \notin \sigma. \end{cases}$$

The *Salvetti complex* S_L *on* L is the colimit of the diagram D, that is, $S_L := \mathrm{colim}_{\sigma\in L} D(\sigma) = \bigcup_{\sigma\in L} D(\sigma)$. The fundamental group $A_L := \pi_1(S_L)$ is the *right-angled Artin group* (RAAG) on L. This has universal cover $\widetilde{S}_L$ which is the quintessential example of a CAT(0) cube complex. We will denote the isometry group of $\widetilde{S}_L$ by H_L and endow it with the topology given by uniform convergence on compacta. We say a RAAG is *irreducible* if it does not split as the direct product of two infinite subgroups.

Let G act on some object X, recall that the invariants of the G-action on X are denoted by X^G.

Theorem C. *Let $m \geqslant 3$. Let K be a pointed flag complex on $[m]$, and let $L = \bigvee_{i=1}^5 K$. If A_L is irreducible, then there exists an irreducible uniform* $(\mathrm{Isom}(\mathbb{E}^2) \times H_L)$*-lattice Γ_L and explicit bijections*

$$\Sigma^n(\Gamma_L) \leftrightarrow \Sigma^n(A_L)^{\mathbb{Z}/5} \quad \textit{and} \quad \Sigma^n(\Gamma_L;\mathbb{Z}) \leftrightarrow \Sigma^n(A_L)^{\mathbb{Z}/5},$$

where the $\mathbb{Z}/5$ action is the action induced by cyclically permuting the five copies of K about the basepoint.

The previous theorem is easy to apply because the BNSR invariants of RAAGs are known [BB97; MMV98; BG99]. We reproduce the result here for the convenience of the reader.

Let L be a flag complex with RAAG A_L. Each vertex of L corresponds to a standard generator of A_L. Given a character $\psi\colon A_L \to \mathbb{R}$, let $L^\dagger$ denote the full subcomplex of L spanned by vertices v such that $\psi(v) = 0$, and let L^* denote the full subcomplex of L spanned by vertices v such that $\psi(v) \neq 0$

Theorem 1.3 (Bestvina–Brady, Meier–Meinert–VanWyk, Bux–Gonzalez)**.** *Let L be a flag complex. The following are equivalent:*

(1) $\varphi \in \Sigma^{n+1}(A_L;\mathbb{Z})$, resp. $\varphi \in \Sigma^{n+1}(A_L)$.
(2) For every (possibly empty) dead simplex $\sigma \in L^\dagger$ the living link $\mathrm{Lk}_{L^*}(\sigma) := L^* \cap \mathrm{Lk}_L(\sigma)$ is $(n-\dim(\sigma)-1)$-acyclic, resp. L^* is, additionally, n-connected.

Example D. Let K be a pointed flag triangulation of a disc D^2 such that $K^{(1)}$ has diameter at least 3 and let $L = \bigvee_{i=1}^5 K$ where the wedge is over the chosen basepoints. There is an obvious $\Psi := \mathbb{Z}/5$ action on L which cyclically permutes the copies of K whilst fixing the basepoint. By Theorem 1.3, the character $\hat{\varphi}$ sending every generator of A_K to 1 is F_∞-fibred and is clearly Ψ-invariant. This induces a character $\varphi \in \Sigma^\infty(\Gamma_L)$ and so we see Γ_L is F_∞-fibred. In fact, Γ_L is topologically fibred as $B\operatorname{Ker}(\hat{\varphi}) \to B\Gamma_L \to S^1$ where each space is homotopic to a finite CW complex. Indeed, $\mathrm{cd}_{\mathbb{Z}}(\Gamma_L) < \infty$ so $\operatorname{Ker}(\hat{\varphi})$ is type F.

Corollary E. *Question 1.2 has a negative answer.*

There has been considerable interest in constructing groups of type FP_2 which are not finitely presented [BB97; Lea18a; Lea18b; KLS20; BL20; Kro21]. In light of this we note one special case of the construction.

Remark 1.4. Let $n \geqslant 1$ and L be an n-acyclic flag complex such that $\pi_1(L) \neq \{1\}$; then we obtain characters which FP_{n+1}-fibre but do not F_2-fibre á la Bestvina and Brady.

Suppose L is not connected, then the BNSR invariants of A_L vanish. We suspect this behaviour holds for all H_L-lattices and all irreducible $(\operatorname{Isom}(\mathbb{E}^n) \times H_L)$-lattices.

Conjecture 1.5. *If L is not connected, then the BNSR invariants vanish for every irreducible uniform $(\operatorname{Isom}(\mathbb{E}^n) \times H_L)$-lattice.*

More generally, we ask:

Question 1.6. *Let L be a flag complex and let Γ be an irreducible uniform $(\operatorname{Isom}(\mathbb{E}^n)\times H_L)$-lattice. Can the BNSR invariants of Γ be determined in terms of $H^1(\widetilde{S}_L/\Gamma)$ and the BNSR invariants of A_L?*

Note that the appearance of $H^1(\widetilde{S}_L/\Gamma)$ is directly related to Proposition 3.2 where we prove for any such lattice $H^1(\Gamma) \cong H^1(\widetilde{S}_L/\Gamma)$.

The author suspects that a positive answer to a variation on Question 1.2 may hold for lattices in products of trees.

Conjecture 1.7. *Let Γ be a uniform lattice in a product $\prod_{i=1}^n T_{k_i}$ of automorphism groups of locally finite trees $\mathcal{T}_{k_i}$. If $\Sigma^1(\Gamma) \neq \varnothing$, then Γ is reducible. Moreover, if $\Sigma^{n-1}(\Gamma) \neq \varnothing$, then Γ is virtually a direct product of n free groups.*

Conjecture 1.8. *Let Γ be a uniform lattice in a product $\mathcal{T}_{k_1} \times \mathcal{T}_{k_2}$ of locally finite trees. Then, Γ is irreducible if and only if Γ is not virtually F_1-fibred.*

Structure of the paper. In Section 2 we give the relevant background on lattices in CAT(0) spaces. In Section 3 we prove Theorem A and Corollary B. We also prove for uniform irreducible $(\operatorname{Isom}(\mathbb{E}^n) \times A)$-lattices, where A is the automorphism group of a CAT(0) polyhedral complex X acting cocompactly and minimally on X, that $H^1(\Gamma) \cong H^1(X/\Gamma)$. In Section 4 we detail the constructions of the groups Γ_L and in Section 5 we prove Theorem C.

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Bibliographical note. This is the second part of a longer paper contained in the author's PhD thesis which was split at the request of a referee (see [Hug21a, Paper 5]). The first part of this longer paper can be found in [Hug21b]. Note that some of the results here are not contained in the author's PhD thesis. Also note that a number of group presentations and results regarding residual finiteness, ℓ^2-Betti numbers, and autostackability will only exist in the thesis version. Finally, we remark the existence of the other companion paper [Hug21c] where the author constructs a lattice (and in fact a hierarchically hyperbolic group) in a product of trees which is not virtually torsion-free.

2. Preliminaries

2.1. **Lattices.** Let H be a locally compact topological group with right invariant Haar measure μ. A discrete subgroup $\Gamma \leqslant H$ is a *lattice* if the covolume $\mu(H/\Gamma)$ is finite. A lattice is *uniform* if H/Γ is compact and *non-uniform* otherwise. Let S be a right H-set such that for all $s \in S$, the stabilisers H_s are compact and open. Then, if $\Gamma \leqslant H$ is discrete, the stabilisers of Γ acting on S are finite.

Let X be a locally finite, connected, simply connected simplicial complex. The group $H = \mathrm{Aut}(X)$ of simplicial automorphisms of X naturally has the structure of a locally compact topological group, where the topology is given by uniform convergence on compacta.

Note that T, the automorphism group of a locally-finite tree $\mathcal{T}$, admits lattices if and only if the group T is unimodular (that is, the left and right Haar measures coincide). In this case we say $\mathcal{T}$ is *unimodular*. We say a tree $\mathcal{T}$ is *leafless* if it has no vertices of valence one.

2.2. **Irreducibility.** Two notions of irreducibility will feature in this paper; for uniform CAT(0) lattices they are equivalent due to a theorem of Caprace–Monod. See [Hug21b, Section 2.3] for an extended discussion concerning these definitions.

Let $X = \mathbb{E}^n \times X_1 \times \cdots \times X_m$ be a product of irreducible proper CAT(0) spaces with each X_i not quasi-isometric to $\mathbb{E}^1$ and let Γ be a lattice in $H = H_0 \times H_1 \times \cdots \times H_m :=$ $\mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_m)$, such that for each $i \geqslant 1$ the group H_i is non-discrete, cocompact, and acting minimally on X_i. Suppose $n = 0$, then we say Γ is *weakly irreducible* if the projection of Γ to each proper subproduct $H_I := \prod_{i \in I} H_i$ for $I \subset \{1, \dots, m\}$ is non-discrete.

Now, suppose Γ is a uniform lattice. If $n = 1$, then Γ is always reducible by [CM19]. If $n \geqslant 2$, then we observe that Γ is contained in $\prod_{j=1}^{\ell} \mathrm{Isom}(\mathbb{E}^{k_j}) \times \prod_{i=1}^{m} H_i$ where $\ell \geqslant 1$, $\sum_{j=1}^{\ell} k_j = n$, and each k_j is minimal (so ℓ is maximal amongst all choices of orthonormal bases for $\mathbb{R}^n$). Denote each $\mathrm{Isom}(\mathbb{E}^{k_j})$ by E_j and the corresponding orthogonal group by O_j. Then for Γ to be *weakly irreducible* we require that each $k_j \geqslant 2$, and that the projection $\pi_{I,J}$ of Γ to each proper subproduct, $G_{I,J} := \prod_{j \in J} O_j \times \prod_{i \in I} H_i$ for $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, \ell\}$, of H is non-discrete (here at least one of I or J is a proper subset).

We say Γ is *algebraically irreducible* if Γ has no finite index subgroup splitting as the direct product of two infinite groups.

For every lattice we consider in this paper the two definitions will be equivalent by [CM09a, Theorem 4.2]; so we will simply refer to a lattice as *irreducible* or *reducible*.

2.3. **Graphs and complexes of lattices.** Let Γ be a group and $K, L \leqslant \Gamma$ be subgroups. If $L \cap K$ has finite index in L and K then we say L and K are *commensurable*. The *commensurator* of L in Γ is the subgroup

$$\mathrm{Comm}_\Gamma(L) := \{g \in \Gamma \mid L^g \cap L \text{ has finite index in } L \text{ and } L^g\}.$$

If $\mathrm{Comm}_\Gamma(L) = \Gamma$ then we say L is *commensurated.*

Rather than recall the definitions and machinery from [Hug21b] we will use it as a black box. The key result for us is the following:

Theorem 2.1. [Hug21b, Corollary B] *Let $X = X_1 \times X_2$ be a proper cocompact minimal* CAT(0) *space and $H = \mathrm{Isom}(X_1) \times \mathrm{Isom}(X_2)$. Suppose X_1 is a* CAT(0) *polyhedral complex. Then, for any uniform H-lattice Γ, the cell stabilisers of X_1 in Γ are commensurated, commensurable, and isomorphic to finite-by-*$\{\mathrm{Isom}(X_2)$*-lattices*$\}$*.*

In our situation we will take X_1 to be a locally finite tree, or the universal cover of a Salvetti complex for a right-angled Artin group, and $X_2 = \mathbb{E}^n$. The quotient space X_1/Γ is endowed with a natural graph or complex of groups structure. In the language of [Hug21b] we call this data a *graph* or *complex of* $\mathrm{Isom}(\mathbb{E}^n)$*-lattices*. Thus, every uniform H-lattice (where $H = \mathrm{Aut}(X_1) \times \mathrm{Isom}(\mathbb{E}^n)$) splits as a graph or complex of commensurable finite-by-$\{n$-crystallographic$\}$ groups.

2.4. **Leary–Minasyan groups.** The following groups were introduced in [LM19] by Leary and Minasyan as a class of groups containing the first examples of CAT(0) but not biautomatic groups; they were classified up to isomorphism by Valiunas [Val20]. In fact, they are not subgroups of any biautomatic group [Val21]. Let $n \geqslant 0$, let $A \in \mathrm{GL}_n(\mathbb{Q})$, and let $L \leqslant \mathbb{Z}^n \cap A^{-1}(\mathbb{Z}^n)$ be a finite index subgroup. The group $\mathrm{LM}(A, L)$ is defined by the presentation

$$\langle x_1, \dots, x_n, t \mid [x_i, x_j] = 1 \text{ for } 1 \leqslant i < j \leqslant n, t\mathbf{x}^{\mathbf{v}}t^{-1} = \mathbf{x}^{A\mathbf{v}} \text{ for } \mathbf{v} \in L\rangle,$$

where we write $\mathbf{x}^{\mathbf{w}} := x_1^{w_1} \cdots x_n^{w_n}$ for $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n$. If L is the largest subgroup of $\mathbb{Z}^n$ such that AL is also a subgroup of $\mathbb{Z}^n$, then we denote $\mathrm{LM}(A, L)$ by $\mathrm{LM}(A)$. We refer to the groups $\mathrm{LM}(A, L)$ and $\mathrm{LM}(A)$ as *Leary–Minasyan groups*. The groups clearly split as HNN extensions $\mathbb{Z}^n{*_L}$. The groups are CAT(0) if and only if A is conjugate to an orthogonal matrix in $\mathrm{GL}_n(\mathbb{R})$ [LM19, Theorem 7.2].

As a concrete example, take

$$A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \text{ and } L = \left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle \text{ so } AL = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\rangle.$$

Note that L is index 5 in $\mathbb{Z}^2$ and so must be a maximal subgroup. It follows that

$$\mathrm{LM}(A, L) = \mathrm{LM}(A) = \langle a, b, t \mid [a, b],\ ta^2b^{-1}t^{-1} = a^2b,\ tab^2t^{-1} = a^{-1}b^2\rangle. \tag{1}$$

Theorem 2.2. [LM19, Theorem 7.5] *Suppose that A has infinite order and is conjugate in* $\mathrm{GL}_n(\mathbb{R})$ *to an orthogonal matrix. Then,* $\mathrm{LM}(A, L)$ *is a uniform lattice in* $\mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Aut}(\mathcal{T})$ *whose projections to the factors are not discrete. In particular, if A is an irreducible matrix, then* $\mathrm{LM}(A, L)$ *is an irreducible lattice.*

We will detail the action on $\mathbb{E}^2$ in the case of the Leary–Minasyan group (1). The group $\mathrm{LM}(A)$ has a representation π to $\mathrm{Isom}(\mathbb{E}^n)$ given by $\pi(a) = [1, 0]^T$, $\pi(b) = [0, 1]^T$, and $\pi(t) = A$. The matrix A is a rotation by the irrational number $\cos^{-1}(3/5)$ and so has infinite order. In particular, $\mathrm{LM}(A)$ is irreducible.

3. Fibring lattices in a product of a tree and a Euclidean space

In this section we characterise irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices as those which do not virtually F_1-fibre (Theorem 3.6). Note that this result is new even for Leary-Minasyan groups. Before we prove the theorem, we will collect some propositions.

Proposition 3.1. *Let $\mathcal{T}$ be a locally-finite leafless unimodular tree, not isometric to $\mathbb{R}$, let $T = \mathrm{Aut}(\mathcal{T})$, and let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. If Γ is irreducible, then $H^1(\Gamma;\mathbb{Z}) \cong H^1(\mathcal{T}/\Gamma;\mathbb{Z})$.*

The analogous result for $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices is as follows. We will prove both results simultaneously.

Proposition 3.2. *Let X be an irreducible locally finite $\mathrm{CAT}(0)$ polyhedral complex, let $A = \mathrm{Aut}(X)$ act cocompactly and minimally, and let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice. If Γ is algebraically irreducible, then $H^1(\Gamma;\mathbb{Z}) \cong H^1(X/\Gamma;\mathbb{Z})$.*

Proof of Proposition 3.1 and 3.2. Abusing notation denote both $\mathcal{T}$ and X by X. Let $\varphi \in H^1(\Gamma;\mathbb{Z}) = \mathrm{Hom}(\Gamma,\mathbb{Z})$, $P := \pi_{\mathrm{O}(n)}(\Gamma)$, and $N := \mathrm{Ker}(\pi_{\mathrm{O}(n)}) \lhd \Gamma$. For the remainder of the proof an omission of coefficients in a (co)homology functor should be taken to mean coefficients with the trivial module $\mathbb{Z}$.

Claim 3.3. *φ is P-invariant.*

Proof of claim: The group Γ is an extension $N \cdot P$ so we may have the following cohomological inflation-restriction sequence

$$0 \longrightarrow H^1(P) \longrightarrow H^1(\Gamma) \longrightarrow H^1(N)^P \xrightarrow{d^2} H^2(P) \longrightarrow H^2(\Gamma).$$

Thus, $H^1(\Gamma) \cong H^1(P) \oplus \mathrm{Ker}(H^1(N)^P \xrightarrow{d^2} H^2(P))$; the extension is split because Γ is type F_∞ and so $H^1(P)$ is a finitely generated free abelian group. Clearly, from this splitting φ is P-invariant. ◆

Claim 3.4. *Let L be a cell stabiliser in the action of Γ on X. Then, $\varphi|_L = 0$.*

Proof of claim: Suppose for contradiction φ is non-zero on some cell stabiliser L of the Γ action on X. Then, after passing to a finite index subgroup of L, the restriction of φ is non-zero on some subgroup isomorphic to $\mathbb{Z}^n$. In particular, φ defines a codimension 1 subgroup K of $\mathbb{Z}^n$ contained in $\mathrm{Ker}(\phi)$. Let $F := \mathbb{R} \otimes K \subset X \times \mathbb{E}^n$ be the $(n-1)$-dimensional flat given by the flat torus theorem. In particular, since φ is P-invariant (Claim 3.3), the flat $F' := \pi_{\mathbb{E}^n}(F) \cong \mathbb{E}^{n-1}$ is stabilised by P.

Thus, a finite-index subgroup of P fixes the one-dimensional subspace $F^\perp$ and so fixes a point in $\partial(\mathbb{E}^n \times X)$. It follows that Γ is reducible by [CM19, Theorem 2] a contradiction. Thus, $\varphi|_L = 0$. ◆

Let $\Sigma^{(p)}$ be a representative set of orbits of p-cells for the action of Γ on X. The isomorphism will follow from a computation using the Γ-equivariant spectral sequence applied to the filtration of X by skeleta (see [Bro94, Chapter VII.7]). This spectral sequence takes the form

$$E_1^{p,q} := \bigoplus_{\sigma\in\Sigma^{(p)}} H^q(\Gamma_\sigma) \Rightarrow H^{p+q}(\Gamma).$$

Since we are only interested in computing $H^1(\Gamma)$, the relevant part of the E_1-page is given by:

$$\begin{array}{c|ccccc}
p & & & & & \\
1 & \bigoplus_{\sigma\in\Sigma^{(0)}} H^1(\Gamma_\sigma) & \xrightarrow{d_1^{0,1}} & \bigoplus_{\sigma\in\Sigma^{(1)}} H^1(\Gamma_\sigma) & & \\
0 & \bigoplus_{\sigma\in\Sigma^{(0)}} H^0(\Gamma_\sigma) & \xrightarrow{d_1^{0,0}} & \bigoplus_{\sigma\in\Sigma^{(1)}} H^0(\Gamma_\sigma) & \xrightarrow{d_1^{1,0}} & \bigoplus_{\sigma\in\Sigma^{(2)}} H^0(\Gamma_\sigma) \\
\hline
 & 0 & & 1 & & 2 \quad q
\end{array}$$

Using the description of d_1 given in [Bro94, Chapter VII.8] it is easy to see that $E_2^{p,0} \cong H^p(X/\Gamma)$. Now, the group $E_\infty^{0,1}$ is the image of the sum of restrictions

$$\bigoplus_{\sigma\in\Sigma^{(0)}} \operatorname{res}^\Gamma_{\Gamma_\sigma} : H^1(\Gamma) \to \bigoplus_{\sigma\in\Sigma^{(0)}} H^1(\Gamma_\sigma)$$

and so must be 0 by Claim 2. Also note for dimensional reasons $E_2^{0,0} = E_\infty^{0,0}$, $E_2^{1,0} = E_\infty^{1,0}$, $E_3^{0,1} = E_\infty^{0,1}$ and $E_3^{2,0} = E_\infty^{2,0}$. Thus, the relevant part of the E_∞-page is given by:

$$\begin{array}{c|ccc}
p & & & \\
1 & 0 & E_\infty^{1,1} & \\
0 & \mathbb{Z} & H^1(X/\Gamma) & E_\infty^{2,0} \\
\hline
 & 0 & 1 & 2 \quad q
\end{array}$$

and so the desired isomorphism $H^1(\Gamma) \cong H^1(X/\Gamma)$ follows. □

We say a graph of groups $\mathcal{G}$ is *reduced*, if given an edge e with distinct end points v_1, v_2, the inclusions $\Gamma_e \rightarrowtail \Gamma_{v_i}$ are proper. We say that a graph of groups $\mathcal{G}$ is not an ascending HNN-extension if it is not an HNN-extension (it has more than one edge or more than one vertex), or it is an HNN-extension but both Γ_e and $\Gamma_{\bar{e}}$ are proper subgroups of Γ_v.

We will need the following proposition of Cashen–Levitt [CL16, Proposition 2.5].

Proposition 3.5 (Cashen–Levitt)**.** *Let Γ be the fundamental group of a finite reduced graph of groups with Γ finitely generated. Assume that Γ is not an ascending HNN-extension. If $\varphi \in \Sigma^1(\Gamma)$, then φ is non-trivial on every edge group.*

We are now ready to prove Theorem A from the introduction.

Theorem 3.6 (Theorem A)**.** *Let $\mathcal{T}$ be a locally-finite leafless unimodular tree, not isometric to $\mathbb{R}$, and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice, then Γ virtually F_1-fibres if and only if Γ virtually F_∞-fibres if and only if Γ is reducible.*

Proof. If Γ is reducible, then Γ virtually splits as $\mathbb{Z} \times \Gamma'$, where Γ' is a CAT(0) group. Hence, Γ' is type F_∞. In particular, Γ virtually F_∞-fibres.

We will now prove every irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice does not algebraically fibre, and this will prove the theorem since a finite index subgroup of an irreducible lattice is an irreducible lattice. Now, suppose Γ is an irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. By Theorem 2.1, the group Γ splits as a graph of $\mathrm{Isom}(\mathbb{E}^n)$-lattices, and so is the fundamental group of a graph of groups with vertex and edge stabilisers finite-by-$\{\mathrm{Isom}(\mathbb{E}^n)$-lattices$\}$.

Claim 3.7. Γ *splits as a reduced graph of groups and is not an ascending HNN extension.*

Proof of Claim: We may assume the graph of groups is reduced by contracting any edges with a trivial amalgam $L *_L K$. Note that these contractions do not change the vertex and edge stabilisers, but may change the Bass-Serre tree (the tree will still not be quasi-isometric to $\mathbb{R}$ since there are necessarily other vertices of degree at least 3).

Now for Γ to be an ascending HNN-extension the graph $\mathcal{T}/\Gamma$ must consist of a single vertex and edge. Let t be the stable letter of Γ, then t acts as an isometry on $\mathcal{T} \times \mathbb{E}^n$ and so preserves covolume of stabilisers in Γ acting on $\mathbb{E}^n$. Now, covolume is multiplicative when passing to covers. In particular, under the projection $\pi_{\mathrm{Isom}(\mathbb{E}^n)}$, the two embeddings of the projection of the edge group Γ_e into the projection of the vertex group Γ_v must have the same index. Now, if $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(t)$ (virtually) centralised $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma_v)$, then Γ would clearly be reducible. Thus, the two embeddings of $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma_e)$ into the vertex group $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma_v)$ must both have index at least 2, yielding the claim. ◆

Now, $H^1(\Gamma;\mathbb{Z})\otimes\mathbb{R}\cong H^1(\Gamma;\mathbb{R})$ and by Proposition 3.1 (see Claim 3.4), for every character $\phi\in H^1(\Gamma;\mathbb{R})$ we see that ϕ restricted to a vertex or edge group is zero. Since Γ is the fundamental group of a reduced graph of groups, is not an ascending HNN extension, and ϕ vanishes on every edge group, we may apply Proposition 3.5 to deduce that $\phi\notin\Sigma^1(\Gamma)$. Hence, Γ does not (virtually) F_1-fibre. □

Corollary 3.8 (Corollary B)**.** *With notation as in Theorem 3.6 suppose $n = 2$. Then Γ virtually fibres if and only if Γ is reducible.*

Proof. This follows from Theorem 3.6 and the fact that every reducible uniform lattice in $\mathrm{Isom}(\mathbb{E}^2)\times T$ is virtually $F_m\times\mathbb{Z}^2$ for some $m\geqslant 2$. □

4. Uniform lattices in Salvetti complexes and Euclidean spaces

We will summarise and specialise the construction in [Hug21b, Theorem 7.4] for our purposes.

Let K be a pointed flag complex with at least 3 vertices and let $L=\bigvee_{i=1}^5 K$. Let $\mathrm{LM}(A)$ denote the group with presentation (1) and let $\mathcal{T}_{10}$ denote the (10-regular) Bass-Serre tree of $\mathrm{LM}(A)$.

Mark a vertex in K distinct from the basepoint and denote the set of five copies of this vertex in L by V. Note that the induced subgraph on V is five disjoint points so the corresponding RAAG is free of rank 5. In particular, we may denote this subgroup of A_L by A_V unambiguously.

Consider $\pi\colon A_L\twoheadrightarrow A_V$ given by $v\mapsto v$ if $v\in V$ and $v\mapsto 1$ otherwise. This has kernel $\mathrm{Ker}(\pi)$ and covering space $\widetilde{S}_L\to X$. We may identify the vertex set of $\mathcal{T}_{10}$ with the vertex set of X via the embedding of $\mathcal{T}_{10}\rightarrowtail X$ given by 'unwrapping' the $\bigvee_{i=1}^5 S^1\subseteq S_L$ corresponding to the vertices $v\in V$. The 1-skeleton $X^{(1)}$ of X is obtained from $\mathcal{T}_{10}$ by attaching to each vertex of $\mathcal{T}_{10}$ a circle for each $v\in VL\backslash\mathcal{V}$.

Now, $\mathrm{LM}(A)$ acts by isometries on $\mathcal{T}_{10}$, moreover, the local action of vertex stabiliser is $\mathbb{Z}/5$ which lifts to $\mathrm{Isom}(X^{(1)})$ and in fact to $\mathrm{Isom}(X)$ and H_L. It follows $\mathrm{LM}(A)$ acts by isometries on X. Let Γ_L be the group of lifts of all automorphisms in $\mathrm{LM}(A)$, we have a short exact sequence

$$1\longrightarrow \mathrm{Aut}(\pi)\longrightarrow \Gamma_L\longrightarrow \mathrm{LM}(A)\longrightarrow 1.$$

Proposition 4.1. [Hug21b, Theorem 7.4] *Let K be a pointed flag complex with at least 3 vertices and let $L = \bigvee_{i=1}^{5} K$. If A_L is irreducible, then group Γ_L is a uniform irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times H_L)$-lattice.*

5. Computing the BNSR-invariants

The goal of this section is to prove Theorem C.

Theorem 5.1 (Theorem C)**.** *Let $m \geqslant 3$. Let K be a pointed flag complex on $[m]$, and let $L = \bigvee_{i=1}^{5} K$. If A_L is irreducible, then there exists an irreducible uniform $(\mathrm{Isom}(\mathbb{E}^2) \times H_L)$-lattice Γ_L and explicit bijections*

$$\Sigma^n(\Gamma_L) \leftrightarrow \Sigma^n(A_L)^{\mathbb{Z}/5} \quad \textit{and} \quad \Sigma^n(\Gamma_L; \mathbb{Z}) \leftrightarrow \Sigma^n(A_L)^{\mathbb{Z}/5},$$

where the $\mathbb{Z}/5$ action is the action induced by cyclically permuting the five copies of K about the basepoint.

We sketch the argument before going into the details: First, we show that the quotients of $\widetilde{S}_L$ by A_L and Γ_L are related by a $\mathbb{Z}/5$-action. Second, we exhibit a constructive bijection of the character spheres $S(A_L)^{\mathbb{Z}/5}$ and $S(\Gamma_L)$. Next, we will show that corresponding characters induce the same height function on $\widetilde{S}_L$. Finally, we will use the analysis of these height functions due to Bux–Gonzalez [BG99] along with the fact all of our cell stabilisers are finitely generated abelian groups (Theorem 2.1) to show the bijection descends to the BNSR invariants.

Throughout the rest of the section, let K be a pointed flag complex with at least 3 vertices and let $L = \bigvee_{i=1}^{5} K$. Let Γ_L be the group constructed in Section 4 and let $\mathrm{LM}(A)$ denote the group with presentation (1).

5.1. Analysing the quotient spaces.

Let J be a simplicial complex on $[m]$ and let $V \subseteq [m]$. The *double of J over V*, denoted $\mathcal{D}(J, V)$, is the simplicial complex with vertices $[m] \backslash V \cup \{v^+, v^- : v \in V\}$ and simplices described as follows: $[w_1^{\epsilon_1}, \dots, w_n^{\epsilon_n}]$, where $\epsilon_i \in \{+, -, \ \}$, spans an n-simplex in $\mathcal{D}(J, V)$ if and only if $[w_1, \dots, w_n]$ spans an n-simplex in J.

Proposition 5.2. $\widetilde{S}_L/\Gamma_L \cong S_L/(\mathbb{Z}/5) \cong S_K$.

Proof. As in the construction of Γ_L consider $\pi\colon A_L \twoheadrightarrow A_V$. This has kernel $\mathrm{Ker}(\pi)$ and corresponding covering space $\widetilde{S}_L \to X$. The action of Γ_L on X is vertex transitive because we can identify $X^{(0)}$ with $\mathcal{T}_{10}^{(0)}$ and $\mathrm{LM}(A)$ acts on $\mathcal{T}_{10}$ vertex transitively.

The group $\mathrm{LM}(A)$ is generated by a, b, t where a and b commute and stabilise a vertex in $\mathcal{T}_{10}$. Let v denote the vertex of $\mathcal{T}_{10}$ and X stabilised by $\langle a, b\rangle$. We will now describe the action of a and b on $\mathcal{T}_{10}$.

In the action of $\langle a, b\rangle$ on the ball of radius one about $v \in \mathcal{T}_{10}$, the groups $\langle a\rangle$ and $\langle b\rangle$ both act as $\mathbb{Z}/5$ cyclically permuting the edges of the tree in two blocks of 5. On the link of v this amounts to permuting 10 points in two blocks of 5. We will now examine the action of $\mathrm{LM}(A)$ on the covering space X.

The link of a vertex in X is exactly $\mathcal{D}(L, V)$. Indeed, the link of a vertex in $\widetilde{S}_L$ is $\mathcal{D}(L)$ and action of $\mathrm{Ker}(\pi)$ identifies w^+ and w^- for every $w \in [m]\backslash V$. Since $\mathrm{LM}(A)$ acts vertex transitively on X we see the action of $\mathrm{LM}(A)$ identifies each pair of vertices v^+ and v^- in $\mathcal{D}(L, V)$. Moreover, the action of $\langle a, b\rangle$ cyclically permutes the edges of $\mathcal{T} \rightarrowtail X$ adjacent to v in two blocks of 5. In particular, the five copies of K in L are identified. It follows that the quotient $\widetilde{S}_L/\Gamma_L = X/\mathrm{LM}(A)$ is a union of tori and the link of the basepoint is $K = L/(\mathbb{Z}/5)$. In particular, $\widetilde{S}_L/\Gamma_L = S_K$. $\square$

5.2. **A bijection of character spheres.** Recall that the BNSR invariants of a group Γ are subsets of the *character sphere* $S(\Gamma) := (H^1(\Gamma; \mathbb{R})\backslash\{0\})/\mathbb{R}^\times_{>0}$.

Proposition 5.3. *There is a bijection* $S(A_L)^{\mathbb{Z}/5} \leftrightarrow S(\Gamma_L)$.

Proof. Since S_L is a $K(A_L, 1)$ we have that $H^1(A_L; \mathbb{R}) \cong H^1(S_L; \mathbb{R})$. By Proposition 3.2 we have that $H^1(\Gamma_L; \mathbb{R}) \cong H^1(\widetilde{S}_L/\Gamma_L; \mathbb{R})$. By Proposition 5.2, this latter group is isomorphic to $H^1(S_L/\mathbb{Z}/5; \mathbb{R})$. In particular, every $\mathbb{Z}/5$-invariant character of A_L is a character of Γ_L. Similarly, every character φ of Γ_L can be extended to a character $\hat{\varphi}$ of A_L by defining it on each generator of A_L as follows: Each generator g of A_L corresponds to a vertex of L and so a copy of a vertex of K. Because $S_K \cong S_L/(\mathbb{Z}/5)$ and $H^1(\Gamma_L; \mathbb{R}) \cong H^1(S_L/(\mathbb{Z}/5); \mathbb{R})$, it follows that φ is determined by its values on the vertices of K. Define $\hat{\varphi}(g)$ to be the value of its corresponding vertex in K. Then, $\varphi \leftrightarrow \hat{\varphi}$ gives the desired bijection. $\square$

5.3. **Height functions.**

Proposition 5.4. *The characters φ and $\hat{\varphi}$ induce the same height function $\widetilde{S}_L \to \mathbb{R}$.*

Proof. This essentially follows from Proposition 5.3 but we will spell out the details. Let $\varphi \in H^1(\Gamma_L; \mathbb{R})$. The projection $\widetilde{S}_L \to S_K$ given by quotienting out the Γ_L action and the identification $H^1(\Gamma_L; \mathbb{R}) \cong H^1(S_K; \mathbb{R})$ allows us to lift φ to some height function $\phi\colon \widetilde{S}_L \to \mathbb{R}$. Indeed, ϕ is the composite

$$\widetilde{S}_L \longrightarrow \mathbb{R}^{|L^{(0)}|} \longrightarrow \mathbb{R},$$

where the first map is a lift of $\bigvee_{i=1}^{5} S_K = S_L \rightarrowtail \prod_{|L^{(0)}|} S^1$ and the second map is the sum of elements. Similarly, we may lift $\hat{\varphi} \in H^1(A_L;\mathbb{R})$ to a height function $\hat{\phi}\colon \widetilde{S}_L \to \mathbb{R}$. By the bijection given in the proof of Proposition 5.3 it follows the functions ϕ and $\hat{\phi}$ coincide on $\widetilde{S}_L^{(0)}$ and, by extending linearly on cubes, on the whole of $\widetilde{S}_L$. □

5.4. **Completing the computation.** Let Γ be a group and X be a Γ-CW complex. We say X is n-*good* if

(1) X is n-acyclic, i.e. $\widetilde{H}_k(X) = 0$ for $k \leqslant n$;
(2) for $0 \leqslant p \leqslant n$, the stabiliser of Γ_σ of any p-cell σ is of type $\mathrm{FP_{n-p}}$.

A *filtration* of X is a family $\{X_\alpha\}_{\alpha\in I}$ of Γ-invariant subcomplexes such that I is a directed set, $X_\alpha \subseteq X_\beta$ when $\alpha \leqslant \beta$, and $X = \bigcup_\alpha X_\alpha$. The filtration is of *finite n-type* if the X_α/Γ have finite n-skeleton. We say that $\{X_\alpha\}$ is $\widetilde{H}_k$-*essentially trivial* (resp. π_k-*essentially trivial*) if for each α there is $\beta \geqslant \alpha$ such that $\widetilde{H}_k(\ell_{\alpha,\beta}) = 0$ (resp. $\pi_k(\ell_{\alpha,\beta}) = 0$), where $\ell_{\alpha,\beta}\colon X_\alpha \rightarrowtail X_\beta$ is the inclusion.

We will make use of the two criteria due to Brown.

Theorem 5.5. [Bro87] *Let X be an n-good Γ-complex with a filtration $\{X_\alpha\}$ of finite n-type. Then Γ is of type $\mathrm{FP_n}$ if and only if the directed system $\{X_\alpha\}$ is $\widetilde{H}_k$-essentially trivial for all $k < n$.* □

Theorem 5.6. [Bro87] *Let X be a simply connected Γ-complex such that the vertex stabilisers are finitely presented and the edge stabilisers are finitely generated. Let $\{X_\alpha\}$ be a filtration of X of finite 2-type and let $v \in \bigcap X_\alpha$ be a basepoint. If Γ is finitely generated, then Γ is finitely presented if and only if $\{(X_\alpha, v)\}$ is π_1-essentially trivial.* □

Proposition 5.7. *The following holds:*

(1) $\varphi \in \Sigma^n(\Gamma_L;\mathbb{Z})$ if and only if $\hat{\varphi} \in \Sigma^n(A_L;\mathbb{Z})^{\mathbb{Z}/5}$.
(2) $\varphi \in \Sigma^n(\Gamma_L)$ if and only if $\hat{\varphi} \in \Sigma^n(A_L)^{\mathbb{Z}/5}$.

Proof. For $\varphi \in H^1(\Gamma_L;\mathbb{R})$ we obtain a filtration of $\widetilde{S}_L$ by simply using the filtration corresponding to $\hat{\varphi} \in H^1(A_L;\mathbb{R})$. Indeed, the height functions are the same due to Proposition 5.4. The explicit details of this filtration are not needed so we defer the interested reader to [BG99]. The important part for us is that both groups A_L and Γ_L act on $\widetilde{S}_L$ cocompactly and either freely in the first case or with finitely generated virtually abelian stabilisers in the second case (see Theorem 2.1). In fact, in our case they are isomorphic to $\mathbb{Z}^2$ because every stabiliser is conjugate to a stabiliser in LM(A). Both Ker(φ) and Ker($\hat{\varphi}$) acts cocompactly on a level set of the induced height function. It follows that the stabilisers in the action of Ker(φ) on the level set are at worst finitely generated virtually abelian groups and so of type F_∞ (note Ker($\hat{\varphi}$) acts freely). Thus, the hypotheses of Brown's criteria are satisfied for both $\mathrm{Ker}(\hat{\varphi}) < A_L$ and $\mathrm{Ker}(\varphi) < \Gamma_L$ when acting on a level set of the height function $\widetilde{S}_L \to \mathbb{R}$. In particular, the kernels have the same finiteness properties and the result follows. $\square$

The previous proposition, along with the bijection constructed in Proposition 5.3, and Proposition 4.1 clearly implies Theorem C.

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