GRAPHS AND COMPLEXES OF LATTICES

SAM HUGHES

Abstract. We study lattices acting on CAT(0) spaces via their commensurated subgroups. To do this we introduce the notions of a graph of lattices and a complex of lattices giving graph and complex of group splittings of CAT(0) lattices. Using this framework we characterise irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices by C^*-simplicity and the failure of virtual fibring and biautomaticity. We construct non-residually finite uniform lattices acting on arbitrary products of right angled buildings and non-biautomatic lattices acting on the product of $\mathbb{E}^n$ and a right-angled building. We investigate the residual finiteness, L^2-cohomology, and C^*-simplicity of CAT(0) lattices more generally. Along the way we prove that many right angled Artin groups with rank 2 centre are not quasi-isometrically rigid.

Contents

Date: 7th May, 2021.

1. Introduction

Let H be a locally compact group with Haar measure μ. A discrete subgroup $\Gamma \leqslant H$ is a *lattice* if the covolume $\mu(H/\Gamma)$ is finite. We say the lattice *uniform* is H/Γ is cocompact and *non-uniform otherwise*. We say a lattice Γ in a product $H_1 \times H_2$ is *weakly irreducible* if the projection of Γ to each factor is non-discrete, otherwise we say Γ is *reducible*. Given a pair of locally compact groups H_1 and H_2 there are a number of basic questions one can ask:

(Q1) Does $H_1 \times H_2$ contain weakly irreducible lattices?
(Q2) What are the generic properties of a weakly irreducible lattice?

In the classical setting of lattices in semisimple Lie groups and linear algebraic groups over local fields these questions are well studied. Indeed, there are deep theorems such as the Margulis normal subgroup theorem, super-rigidity theorem, and the arithmeticity theorem [Mar91].

The non-classical setting is more complicated and was initiated by studing lattices in the full automorphism group of a locally-finite polyhedral complex. A striking example of the non-classical setting is given by the work of Burger and Mozes [BM97; BM00a; BM00b]. The authors constructed torsion-free simple groups which could be realised as cocompact irreducible lattices in a product of automorphism groups of locally-finite trees.

Thus, one should find a class of spaces which contain the exciting phenomena to be found in products of polyhedral complexes whilst enjoying a strong geometric grounding. The answer was to be found in the notion of non-positive curvature or CAT(0) spaces. The theory encompasses symmetric spaces, non-positively curved manifolds, Euclidean and hyperbolic buildings, and more [BH99]. The reader is referred to [BH99] for a comprehensive introduction to the theory.

Assumption 1.1. *Throughout this paper, all actions of groups on graphs or polyhedral complexes are assumed to be without inversion. That is, each element of a group fixes pointwise each cell it preserves.*

A systematic study of the full isometry groups of CAT(0) spaces and their lattices was undertaken by Caprace and Monod [CM09b; CM09a; CM19]. The authors showed in [CM09b, Theorem 1.6], that under mild hypotheses on a CAT(0) space X, there is finite index subgroup of $H \leqslant \mathrm{Isom}(X)$ which splits as

$$H \cong \mathrm{Isom}(\mathbb{E}^n) \times S_1 \times \cdots \times S_p \times D_1 \times \cdots \times D_q, \tag{1}$$

for some $n, p, q \geqslant 0$, where each S_i is an almost connected simple Lie group with trivial centre and each D_j is a totally disconnected irreducible group with trivial amenable radical. Moreover by [CM09b, Addendum 1.8], X itself splits as

$$X = \mathbb{E}^n \times X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \tag{2}$$

where each X_i is an irreducible symmetric space of non-compact type and each Y_j is an irreducible minimal CAT(0)-space.

Taking these decompositions as a starting point motivates a new approach towards CAT(0) groups, that is, understanding the lattices in each of the factors individually and then how the factors interact. The later question is the central goal of this paper: To provide a combinatorial framework for studying lattices in products of irreducible CAT(0) spaces and deduce properties of the weakly irreducible lattices. To this end we introduce the notion of a *graph of lattices* (Definition 3.2) with fixed locally-finite Bass-Serre $\mathcal{T}$ (we will also assume that the tree is unimodular and its automorphism group is non-discrete, these are essentially non-degeneracy conditions so that there are tree lattices). Note that in the case of a product of two trees a similar construction was considered by Benakli and Glasner [BG02].

Roughly a graph of lattices is a graph of groups such that all local groups are finite-by-commensurable-H-lattices equipped with a morphism to H. We use this to study lattices in the product of $T := \mathrm{Aut}(\mathcal{T})$ and closed subgroups H of the isometry group of a fairly generic CAT(0) space. We prove a structure theorem for $(H \times T)$-lattices. That is, we show every $(H \times T)$-lattice gives rise to a graph of H-lattices and conversely, we give necessary and sufficient conditions for a graph of H-lattices to be an $(H \times T)$-lattice.

Theorem A (Theorem 3.3)**.** *Let X be a finite dimensional proper* $\mathrm{CAT}(0)$ *space and let* $H = \mathrm{Isom}(X)$ *contain a uniform lattice. Let $(A, \mathcal{A}, \psi)$ be a graph of H-lattices with locally-finite unimodular non-discrete Bass-Serre tree $\mathcal{T}$, and fundamental group Γ. Suppose* $T = \mathrm{Aut}(\mathcal{T})$ *admits a uniform lattice.*

(1) Assume A is finite. If for each local group A_σ the kernel $\mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$, then Γ is a uniform $(H \times T)$-lattice and hence a $\mathrm{CAT}(0)$ *group. Conversely, if Λ is a uniform $(H \times T)$-lattice, then Λ splits as a finite graph of uniform H-lattices with Bass-Serre tree $\mathcal{T}$.*

(2) Under the same hypotheses as (1), Γ is quasi-isometric to $X \times \mathcal{T}$.

(3) Assume X is a $\mathrm{CAT}(0)$ *polyhedral complex. Let μ be the normalised Haar measure on H. If for each local group A_σ the kernel $K_\sigma = \mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$ and the sum $\sum_{\sigma \in VA} \mu(A_\sigma)/|K_\sigma|$ converges, then Γ is a $(H \times T)$-lattice. Conversely, if Λ is a $(H \times T)$-lattice, then Λ splits as a graph of H-lattices with Bass-Serre tree $\mathcal{T}$.*

We also introduce an analogous construction we call a *complex of lattices* (Definition 6.1) by replacing the tree with a $\mathrm{CAT}(0)$ polyhedral complex and then prove an analogous structure theorem (Theorem 6.2). In the process we deduce some consequences about commensurated subgroups of $\mathrm{CAT}(0)$ groups.

We study of various properties of $(H \times T)$-lattices providing answers to (Q2). In Section 4.1 we investigate the L^2-Betti numbers of $(H \times T)$-lattices and some closely related groups. We also compute the rational homological dimension of S-arithmetic lattices in characteristic $p > 0$ (Theorem 4.5). The author expects this latter result is well known however he could not find a reference in the literature. We investigate C^*-simplicity (Section 4.2), virtual fibring (Section 4.3) and autostackability (Section 4.4) of $(H \times T)$-lattices in terms of the properties of H-lattices. We will give the necessary background for each property in the relevant section.

In Section 5 we detail a number of constructions and examples of $(H \times T)$-lattices using elementary Bass-Serre theory answering (Q1). The constructions are reminiscent of the "universal covering trick" of Burger and Mozes [BM00a] and so we provide a comparison in Section 5.3.

Until Leary and Minasyan's examples of $\mathrm{CAT}(0)$ but not virtually biautomatic groups in [LM19] there were no known examples of lattices where the projection to $\mathrm{Isom}(\mathbb{E}^n)$ is non-discrete. In light of this we begin a study of weakly irreducible lattices with non-trivial de Rham factor. We adapt the biautomaticity criterion given in [LM19, Theorem 1.2] to apply to arbitrary $\mathrm{CAT}(0)$ lattices in the presence of a de Rham factor (Theorem 7.7).

For T the automorphism group of a locally-finite tree we give constructions of many more $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices. We then prove the following characterisation of uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices eliciting a number of generic properties of such lattices eleciting a strong answer to (Q2). Note the following theorem is optimal in the sense that irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices are always non-residually finite and not virtually biautomatic, however, there also exist non-residually finite reducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices for $n \geqslant 3$ (this can be seen by taking the direct product of an irreducible $(\mathrm{Isom}(\mathbb{E}^2) \times T)$-lattice with $\mathbb{Z}^{n-2}$, then applying Theorem 7.7).

Theorem B (Theorem 7.13)**.** *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. The following are equivalent:*

(1) Γ is a weakly irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice;
(2) Γ is irreducible as an abstract group;
(3) Γ acts on $\mathcal{T}$ faithfully;
(4) Γ does not virtually fibre;
(5) Γ is C^*-simple;
(6) and if $n = 2$, Γ is non-residually finite and not virtually biautomatic.

In Section 8 we adapt a construction of Horbez and Huang [HH20] to extend actions from a regular tree to the universal cover of a Salvetti complex $\widetilde{S}_L$ with defining graph L. In particular, from a graph of lattices, one obtains a complex of lattices. With a mild hypothesis on the graph L, we use this construction to obtain weakly irreducible non-biautomatic uniform lattices acting on $\widetilde{S}_L \times \mathbb{E}^n$ for $n \geqslant 2$ (Example 11) answering (Q1). We also deduce a consequence about quasi-isometric rigidity of right angled Artin groups with centre containing $\mathbb{Z}^2$.

Corollary C (Example 11 and Corollary 8.5)**.** *Let L be a finite simplicial graph on vertices $\{v_1, \ldots, v_m\}$. Suppose $\langle v_1, \ldots, v_5\rangle = F_5 < A_L$ is a free subgroup and that $\{v_1, \ldots, v_5\} \subseteq \mathrm{Aut}(L) \cdot v_1$. If A_L is irreducible, then there exists a weakly irreducible uniform lattice in $\mathrm{Aut}(\widetilde{S}_L) \times \mathrm{Isom}(\mathbb{E}^n)$ which is not virtually biautomatic nor residually finite. In particular, $A_L \times \mathbb{Z}^2$ is not quasi-isometrically rigid.*

In [Tho06], Thomas constructs a functor from graphs of groups covered by a fixed biregular tree $\mathcal{T}$ to complexes of groups covered by a fixed "sufficiently symmetric" right-angled building X with parameters determined by the valences of $\mathcal{T}$. We will give the relevant definitions in Section 9.1. In Theorem 9.4 we show that Thomas' functor theorem takes a graph of lattices to a complex of lattices and in particular $(H \times T)$-lattices to $(H \times A)$-lattices, where $T = \mathrm{Aut}(\mathcal{T})$, $A = \mathrm{Aut}(X)$, and H is a closed subgroup of the isometry group of a $\mathrm{CAT}(0)$ space (under mild hypothesis). As consequences we construct more $\mathrm{CAT}(0)$ groups which are not virtually biautomatic (Corollary 9.5) and both uniform and non-uniform weakly irreducible lattices in products of fairly arbitrary hyperbolic and Euclidean buildings (Corollary 9.9) answering (Q1). We highlight one special case here:

Corollary D (Special case of Corollary 9.5)**.** *Let X be the right-angled building of a regular m-gon of uniform thickness $10n$ and let $A = \mathrm{Aut}(X)$. For each $n \geqslant 2$ there exists a weakly irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice which is not virtually biautomatic nor residually finite. In particular, if Y is irreducible, then the direct product of a uniform A-lattice with $\mathbb{Z}^2$ is not quasi-isometrically rigid.*

1.1. **Structure of the paper.** In Section 2 we give the relevant background on lattices acting on CAT(0) spaces. In Section 3 we give the relevant background on graphs of groups, define graphs of lattices, and prove the structure theorem (Theorem 3.3). In Section 4 we investigate L^2-cohomology, C^*-simplicity, virtual fibring, and autostackability of $(H \times T)$-lattices. We also compute the rational homological dimension of group schemes over function fields in positive characteristic. In Section 5 we provide a number of constructions and explicit examples of $(H \times T)$-lattices. In Section 6 we give the relevant background on complexes of groups, define complexes of lattices, and prove the structure theorem (Theorem 6.2). In Section 7 we study CAT(0) lattices acting on spaces with non trivial de Rham factor. We prove the non-biautomaticity criterion for general CAT(0) groups and prove the characterisation of $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices. In Section 8 we adapt the construction of Horbez and Huang. In Section 9 we give the relevant background on right-angled buildings and Thomas' functor theorem. We then prove our functor theorem (Theorem 9.4) and deduce a number of consequences. Finally, in Section 10 we record a few questions and conjectures.

Acknowledgements. This paper contains material from the author's PhD thesis. The author would like to thank his PhD supervisor Ian Leary for his guidance and support. The author would like to thank Motiejus Valiunas for sharing his preprint [Val21] and Ashot Minasyan for helpful comments on an earlier draft of this paper. Additionally, the author would like to thank Naomi Andrew, Pierre Emmanuel-Caprace, Mark Hagen, Susan Hermiller, Jingyin Huang, and Harry Petyt for helpful correspondence and conversations.

2. Preliminaries

2.1. **Lattices and covolumes.** Let H be a locally compact topological group with right invariant Haar measure μ. A discrete subgroup $\Gamma \leqslant H$ is a *lattice* if the covolume $\mu(H/\Gamma)$ is finite. A lattice is *uniform* if H/Γ is compact and *non-uniform* otherwise. Let S be a right H-set such that for all $s \in S$, the stabilisers H_s are compact and open, then if $\Gamma \leqslant H$ is discrete the stabilisers are finite.

Let X be a locally finite, connected, simply connected simplicial complex. The group $H = \mathrm{Aut}(X)$ of simplicial automorphisms of X naturally has the structure of a locally compact topological group, where the topology is given by uniform convergence on compacta.

Theorem 2.1 (Serre's covolume formula [Ser71]). *Let X be a locally finite simply-connected simplicial complex. Let $\Gamma \leqslant H$ be a lattice with fundamental domain Δ, then there is a nomalisation of the Harr measure μ on H, depending only on X, such that for each discrete subgroup $\Gamma < H$ we have*

$$\mu(H/\Gamma) = \mathrm{Vol}(X/\Gamma) := \sum_{v \in \Delta^{(0)}} \frac{1}{|\Gamma_v|}. \qquad \square$$

Note that T the automorphism group of a locally finite tree $\mathcal{T}$ admits lattices if and only if the group T is unimodular (that is the left and right Haar measures coincide). In this case we say $\mathcal{T}$ is unimodular.

2.2. **Non-positive curvature.** We will be primarily interested in lattices in the isometry groups of CAT(0) spaces, we will call these groups CAT(0) *lattices* (note that a uniform CAT(0) lattice is a CAT(0) group). We begin by recording several facts about the structure and isometry groups of general CAT(0) spaces. The definitions and results here are largely due to Caprace and Monod [CM09b; CM09a; CM19].

An isometric action of a group H on a CAT(0) space X is *minimal* if there is no non-empty H-invariant closed convex subset $X' \subset X$, the space X is *minimal* if $\mathrm{Isom}(X)$ acts minimally on X. Note that by [CM09b, Proposition 1.5], if X is cocompact and geodesically complete, then it is minimal. The *amenable radical* of a locally compact group H is the largest amenable normal subgroup. We can now state Caprace and Monod's group and space decomposition theorems mentioned in the introduction.

Theorem 2.2. [CM09b, Theorem 1.6] *Let X be a proper* CAT(0) *space with finite dimensional Tits' boundary and assume* $\mathrm{Isom}(X)$ *has no global fixed point in ∂X. There is a canonical closed, convex,* $\mathrm{Isom}(X)$*-stable subset $X' \subseteq X$ such that $G = \mathrm{Isom}(X')$ has a finite index, open, characteristic subgroup $H \trianglelefteq G$ that admits a canonical decomposition*

$$H \cong \mathrm{Isom}(\mathbb{E}^n) \times S_1 \times \cdots \times S_p \times D_1 \times \cdots \times D_q,$$

for some $n, p, q \geqslant 0$, where each S_i is an almost connected simple Lie group with trivial centre and each D_j is a totally disconnected irreducible group with trivial amenable radical. $\square$

Theorem 2.3. [CM09b, Addendum 1.8] *Let X' and H be as above, then*

$$X' \cong \mathbb{E}^n \times X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q$$

where each X_i is an irreducible symmetric space and each Y_j is an irreducible minimal CAT(0)*-space.* $\square$

2.3. **Irreducibility.** Let $X = X_1 \times \cdots \times X_n$ be a product of irreducible proper CAT(0) spaces and let Γ be a lattice in $H = H_1 \times \cdots \times H_n := \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_n)$, with each H_i non-discrete and acting minimally. There are several possible notions of irreducibility for a lattice in H, moreover, in the general setting of CAT(0) groups, they are not necessarily equivalent. In the interest of clarity, we recount each of these and summarise their implications, we follow the treatment in [CM12; CL19].

(Irr1) For every $\Sigma \subset \{1, \ldots, n\}$, the projection $\pi_\Sigma : \Gamma \to H_\Sigma$ has dense image. Here we say Γ is *topologically irreducible* or an *irreducible lattice.*

(Irr2) The projection to each factor H_i is injective.

(Irr3) For every $\Sigma \subset \{1, \ldots, n\}$, the projection $\pi_\Sigma : \Gamma \to H_\Sigma$ has non-discrete image. Here we say Γ is *weakly irreducible* or a *weakly irreducible lattice.*

(Irr4) Γ has no finite index subgroup which splits as a direct product of two infinite subgroups. Here we say Γ is *algebraically irreducible.*

Firstly, if each H_i is a centre-free semisimple algebraic group without compact factors then each of the definitions are equivalent [Mar91]. When each H_i is a non-discrete, compactly generated, tdlc group, then [CL19, Theorem H] summarises all possible implications. Returning to the setting described above we have that **(Irr2)** $\Rightarrow$ **(Irr3)** $\Rightarrow$ **(Irr4)** and if Γ is finitely generated, then by Theorem 2.4 we have **(Irr4)** $\Rightarrow$ **(Irr3)**. Note that in general **(Irr4)** $\Rightarrow$ **(Irr2)** fails, unless Γ is residually finite. The following theorem from [CM09a] shows the equivalence of **(Irr3)** and **(Irr4)** for many CAT(0) lattices.

Theorem 2.4. [CM09a, Theorem 4.2] *Let X be a proper* CAT(0) *space, $H < \mathrm{Isom}(X)$ a closed subgroup acting cocompactly on X, and $\Gamma < H$ a finitely generated lattice.*

(1) If Γ is irreducible as an abstract group, then for for finite index subgroup $\Gamma_0 < \Gamma$ and any Γ_0-equivariant splitting $X = X_1 \times X_2$ with X_1 and X_2 non-compact, the projection of Γ_0 to both $\mathrm{Isom}(X_1)$ *and* $\mathrm{Isom}(X_2)$ *is non-discrete.*

(2) If in addition the H-action is minimal, then the converse holds. □

Finally, we restate a result of Caprace-Monod which we can use as criterion to determine non-residual finiteness of lattices in products.

Theorem 2.5. [CM09a, Theorem 4.10] *Let X be a proper* CAT(0) *space such that $G = \mathrm{Isom}(X)$ acts cocompactly and minimally. Let $\Gamma < \mathrm{Isom}(X)$ be a finitely generated algebraically irreducible lattice. Let $\Gamma' = \Gamma \cap H$, where H is given in Theorem 2.2. If the projection of Γ' to an irreducible factor of X has non-trivial kernel, then Γ is not residually finite.* □

3. Graphs of lattices

In this section we will review Bass-Serre theory, graphs of spaces and tree lattices. These tools will be fundamental to us in the following chapters. We will then define a graph of lattices and prove the structure theorem for $(H \times T)$-lattices.

3.1. **Graphs of groups.** We shall state some of the definitions and results of Bass-Serre theory. In particular, the action will be on the right. We follow the treatment of Bass [Bas93]. Throughout a graph $A = (VA, EA, \iota, \tau)$ should be understood as it is defined by Serre [Ser03], with edges in oriented pairs indicated by $\overline{e}$, and maps $\iota(e)$ and $\tau(e)$ from each edge to its initial and terminal vertices. We will, however, often talk about the geometric realisation of a graph as a metric space. In this case the graph should be assumed to be simplicial (possibly after subdividing) and should have exactly one undirected edge e for each pair $(e, \overline{e})$. We will often not distinguish between the combinatorial and metric notions.

A *graph of groups* $(A, \mathcal{A})$ consists of a graph A together with some extra data $\mathcal{A} = (V\mathcal{A}, E\mathcal{A}, \Phi\mathcal{A})$. This data consists of *vertex groups* $A_v \in V\mathcal{A}$ for each vertex v, *edge groups* $A_e = A_{\overline{e}} \in E\mathcal{A}$ for each (oriented) edge e, and monomorphisms $(\alpha_e : A_e \to A_{\iota(e)}) \in \Phi$ for every oriented edge in A. We will often refer to the vertex and edge groups as *local groups* and the monomorphisms as *structure maps*.

The *path group* $\pi(\mathcal{A})$ has generators the vertex groups A_v and elements t_e for each edge $e \in EA$ along with the relations:

$$\left\{ \begin{array}{c} \text{The relations in the groups } A_v, \\ t_{\overline{e}} = t_e^{-1}, \\ t_e \alpha_{\overline{e}}(g) t_e^{-1} = \alpha_e(g) \text{ for all } e \in EA \text{ and } g \in A_e = A_{\overline{e}}. \end{array} \right\}$$

We will often abuse notation and write $\mathcal{A}$ for a graph of groups. The *fundamental group of a graph of groups* can be defined in two ways. Firstly, considering reduced loops based at a vertex v in the graph of groups, in this case the fundamental group is denoted $\pi_1(\mathcal{A}, v)$ (see [Bas93, Definition 1.15]). Secondly, with respect to a maximal or spanning tree of the graph. Let X be a spanning tree for A, we define $\pi_1(\mathcal{A}, X)$ to be the group generated by the vertex groups A_v and elements t_e for each edge $e \in EA$ with the relations:

$$\left\{ \begin{array}{c} \text{The relations in the groups } A_v, \\ t_{\overline{e}} = t_e^{-1} \text{ for each (oriented) edge } e, \\ t_e \alpha_{\overline{e}}(g) t_e^{-1} = \alpha_e(g) \text{ for all } g \in A_e, \\ t_e = 1 \text{ if } e \text{ is an edge in } X. \end{array} \right\}$$

Note that the definitions are independent of the choice of basepoint v and spanning tree X and both definitions yield isomorphic groups so we can talk about *the fundamental group* of $\mathcal{A}$, denoted $\pi_1(\mathcal{A})$.

Let G be the fundamental group corresponding to the spanning tree X. For every vertex v and edge e, A_v and A_e can be identified with their images in G. We define a tree with vertices the disjoint union of all coset spaces G/A_v and edges the disjoint union of all coset spaces G/A_e respectively. We call this graph the *Bass-Serre tree* of $\mathcal{A}$ and note that the action of G admits X as a fundamental domain.

Given a group G acting on a tree $\mathcal{T}$, there is a *quotient graph of groups* formed by taking the quotient graph from the action and assigning edge and vertex groups as the stabilisers of a representative of each orbit. Edge monomorphisms are then the inclusions, after conjugating appropriately if incompatible representatives were chosen.

Theorem 3.1. [Bas93] *Up to isomorphism of the structures concerned, the processes of constructing the quotient graph of groups, and of constructing the fundamental group and Bass-Serre tree are mutually inverse.* □

Let $(A, \mathcal{A})$ and $(B, \mathcal{B})$ be graphs of groups. A *morphism of graphs of groups* $\phi : (A, \mathcal{A}) \to (B, \mathcal{B})$ consists of:

(1) A graph morphism $f : A \to B$.
(2) Homomorphisms of local groups $\phi_v : A_v \to B_{f(v)}$ and $\phi_e = \phi_{\overline{e}} : A_e \to B_{f(e)}$.
(3) Elements $\gamma_v \in \pi_1(\mathcal{B}, f(v))$ for each $v \in VA$ and $\gamma_e \in \pi(\mathcal{B})$ for each $e \in EA$ such that if $v = i(e)$ then
   - $\delta_e := \gamma_v^{-1}\gamma_e \in \mathcal{B}_{f(v)}$;
   - $\phi_a \circ \alpha_e = \mathrm{Ad}(\delta_e) \circ \alpha_{f(e)} \circ \phi_e$.

3.2. **A Structure theorem.** In this section we will define a graph of lattices and prove the structure theorem for $(H \times T)$-lattices.

Definition 3.2 (Graph of lattices). Let H be a locally compact group with Haar measure μ. A *graph of H-lattices* $(A, \mathcal{A}, \psi)$ is a graph of groups $(A, \mathcal{A})$ equipped with a morphism $\psi : \mathcal{A} \to H$ such that:

(1) Each local group $A_\sigma \in \mathcal{A}$ is covirtually an H-lattice and the image $\psi(A_\sigma)$ is an H-lattice;
(2) The local groups are commensurable in $\Gamma = \pi_1(\mathcal{A})$ and their images are commensurable in H;
(3) For each $e \in EA$ the element t_e of the path group $\pi(\mathcal{A})$ is mapped under ψ to an element of $\mathrm{Comm}_H(\psi_e(A_e))$.

Theorem 3.3 (The Structure Theorem - Theorem A). *Let X be a finite dimensional proper* CAT(0) *space and let $H = \mathrm{Isom}(X)$ contain a uniform lattice. Let $(A, \mathcal{A}, \psi)$ be a graph of H-lattices with locally-finite unimodular non-discrete Bass-Serre tree $\mathcal{T}$, and fundamental group Γ. Suppose $T = \mathrm{Aut}(\mathcal{T})$ admits a uniform lattice.*

(1) Assume A is finite. If for each local group A_σ the kernel $\mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$, then Γ is a uniform $(H \times T)$-lattice and hence a CAT(0) *group. Conversely, if Λ is a uniform $(H \times T)$-lattice, then Λ splits as a finite graph of uniform H-lattices with Bass-Serre tree $\mathcal{T}$.*
(2) Under the same hypotheses as (1), Γ is quasi-isometric to $X \times \mathcal{T}$.

(3) *Assume X is a* CAT(0) *polyhedral complex. Let μ be the normalised Haar measure on H. If for each local group A_σ the kernel $K_\sigma = \mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$ and the sum $\sum_{\sigma\in VA}\mu(A_\sigma)/|K_\sigma|$ converges, then Γ is a $(H\times T)$-lattice. Conversely, if Λ is a $(H \times T)$-lattice, then Λ splits as a graph of H-lattices with Bass-Serre tree $\mathcal{T}$.*

We will divert the majority of the proof to the proof of Theorem 6.2 due to the similarity of the theorem statement and arguments involved in the proof. The minor difference arises from the fact that the category of graphs of groups is not equivalent to the category of 1-complexes of groups (see [Tho06, Proposition 2.1]) due to the difference in morphisms. We highlight the key differences below.

Proof. We first prove (1). The "if direction" is the same as Theorem 6.2(1). For the converse note that an $(H\times T)$-lattice Γ splits as a graph of groups $(A,\mathcal{A})$ by the fundamental theorem of Bass-Serre theory and the projection to H induces a morphism $\pi_H:\mathcal{A}\to H$. The same argument as Theorem 6.2(1) implies that the local groups are commensurable covirtually commensurable H-lattices. In particular, the images of the elements $t_e\in\pi(\mathcal{A})$ for $e\in EA$ are contained in $\mathrm{Comm}_H(\pi_H(A_\sigma))$ for every local group A_σ. ◆

We now prove (2). By (1) Γ acts properly discontinuously cocompactly on $X\times\mathcal{T}$. The result follows from the Švarc-Milnor Lemma [BH99, p. I.8.19]. ◆

The proof of (3) is almost identical to (1) we will highlight the differences. Since X is a CAT(0) polyhedral complex, it follows that $X\times\mathcal{T}$ is. Now, we may apply Serre's Covolume Formula to $\Gamma=\pi_1(\mathcal{A})$. Let Δ be a fundamental domain for Γ acting on $X\times\mathcal{T}$, then the covolume of Γ may be computed as

$$\sum_{\sigma\in\Delta^0}\frac{1}{|\Gamma_\sigma|}=\sum_{\sigma\in\pi_{\mathcal{T}}(\Delta^0)}\sum_{\tau\in\pi_{\mathcal{T}}^{-1}(\sigma)}\frac{1}{|\Gamma_\tau|}=\sum_{\sigma\in\pi_{\mathcal{T}}(\Delta^0)}\frac{1}{|K_\sigma|}\sum_{\tau\in\pi_{\mathcal{T}}^{-1}(\sigma)}\frac{|K_\sigma|}{|\Gamma_\tau|}=\sum_{\sigma\in\pi_{\mathcal{T}}(\Delta^0)}\frac{\mu(\pi_H(\Gamma_\sigma))}{|K_\sigma|}.$$

Since $\pi_{\mathcal{T}}(\Delta^0)$ can be identified with VA and the later sum converges by assumption, it follows as before that Γ acts faithfully properly discontinuously and isometrically with finite covolume on $X\times Y$. For the converse we proceed as in Theorem 6.2(3). ◆ □

3.3. **Reducible lattices.** Let X be a proper minimal CAT(0) space and $H=\mathrm{Isom}(X)$. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T=\mathrm{Aut}(\mathcal{T})$. We will now characterise reducible uniform $(H\times T)$-lattices by both their projections to H and T, and by the separability of the vertex stabilisers in the projection to T. Moreover, if H is linear, we will show that all such lattices are linear, and thus, residually finite. We say that a subgroup $\Lambda\leqslant\Gamma$ is *separable* if it is the intersection of finite-index subgroups of Γ, *virtually normal* if Λ contains a finite index subgroup N such that $N\trianglelefteqslant\Gamma$, and *weakly separable* if it is the intersection of virtually normal subgroups of Γ.

Proposition 3.4. *Let X be a proper minimal* CAT(0) *space and $H=\mathrm{Isom}(X)$. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and let $T=\mathrm{Aut}(\mathcal{T})$. Let Γ be a* *uniform $(H \times T)$-lattice equipped with projections π_H and π_T to H and T respectively, then the following are equivalent:*

(1) *$\pi_H(\Gamma)$ is an H-lattice;*
(2) *$\pi_T(\Gamma)$ is a T-lattice;*
(3) *For every vertex $v \in \mathcal{T}$, the projection of the vertex stabiliser $\pi_T(\Gamma_v)$ is separable in $\pi_T(\Gamma)$;*
(4) *There is a vertex $v \in \mathcal{T}$ such that the projection of the vertex stabiliser $\pi_T(\Gamma_v)$ is weakly separable in $\pi_T(\Gamma)$;*
(5) *Γ is a reducible $(H \times T)$-lattice.*

Proof. First, we will show that (1) implies (2), our proof for this case largely follows [BM00b, Proposition 1.2]. Assume $\pi_H(\Gamma)$ is an H-lattice, then $\Gamma \cdot T$ is closed and so $\Gamma \cap T$ is a uniform T-lattice. Now, $\pi_T(\Gamma)$ normalises $\Gamma \cap T$ and hence by [BM00a, p. 1.3.6] is discrete. Thus, $\pi_T(\Gamma)$ is discrete and so is a lattice in T.

Next, we will show that (2) implies (1). Assume $\pi_T(\Gamma)$ is a lattice in T and consider the kernel K of the action of Γ on $\mathcal{T}$. We will show that K is a finite index subgroup of $\pi_H(\Gamma)$. Assume that K has infinite index, then $\pi_H(\Gamma)/K \leqslant \pi_T(\Gamma)$ is an infinite subgroup of the vertex stabiliser, a profinite group, and so cannot be discrete. Thus, K has finite index in $\pi_H(\Gamma)$. Since K acts trivially on $\mathcal{T}$ we see that $K = \Gamma \cap H$. Since $\Gamma \cdot H$ is closed it follows K is an H-lattice. Thus, $\pi_H(\Gamma)$ is virtually a lattice in H and therefore an H-lattice.

Clearly, (5) implies (1) and (2). We will now prove that (1) and (2) imply (5). By the previous paragraph we have $K \trianglelefteq \pi_H(\Gamma)$ finite index. Let $\Gamma_T = \{\gamma \mid (e, \gamma) \in \Gamma\}$, we want to show that Γ_T is a uniform T-lattice. Since all uniform T-lattices are commensurable Γ_T will be a finite index subgroup of $\pi_T(\Gamma)$. By the first paragraph we see Γ_T is a uniform lattice. Thus, $K \times \Gamma_T$ is a finite index subgroup of Γ and so Γ is reducible.

Now, evidently (3) implies (4). To see that (4) implies (5) we apply [Cap+19, Corollary 30] to $\pi_T(\Gamma)$, noting that a cocompact action on a leafless tree does not preserve any subtree, in particular, $\pi_T(\Gamma)$ is discrete. Finally, we show that (5) implies (3). Observe that $\pi_T(\Gamma)$ is a virtually free T-lattice which splits as a finite graph of finite groups. Since $\pi_T(\Gamma)$ is a finite graph of finite groups, the vertex stabilisers are separable subgroups. □

One immediate consequence of the theorem is that we can determine whether a lattice is irreducible simply by considering the projections to either H or T. Also, note that if H is the automorphism group of a unimodular leafless tree then we recover [BM00b, Proposition 1.2] and and [Cap+19, Corollary 32].

We also have the following observations about the linearity and residual finiteness of reducible lattices.

Proposition 3.5. *With the same notation as before, assume H is linear (or lattices in H are residually finite). If Γ is a uniform reducible $(H \times T)$-lattice, then Γ is linear (resp. residually finite).*

Proof. If Γ is reducible, then Γ is virtually a direct product of a linear (resp. residually finite) group with a virtually free group. In particular, Γ is virtually a direct product of linear (resp. residually finite) groups and therefore linear (resp. residually finite). $\square$

Corollary 3.6. *With H and T as before, assume H is linear. If Γ is a finitely generated uniform $(H \times T)$-lattice, then exactly one of the following holds:*

(1) Γ is reducible and therefore linear (hence residually finite);
(2) Γ is irreducible and linear (hence residually finite);
(3) Γ is irreducible and non-residually finite.

Moreover, if H is a connected centre-free semisimple linear algebraic group without compact factors and Γ is irreducible and linear, then Γ is arithmetic and just-infinite.

Proof. The first case follows from the previous proposition. Now, assume Γ is irreducible and $\pi_H(\Gamma)$ is injective, then π_H is a faithful linear representation of Γ and we are in the second case. Since Γ is linear, π_T must be injective otherwise Γ would contradict Theorem 2.5. Now, if either of π_T or π_H are not injective, then by Theorem 2.5 we see that Γ is not residually finite. Note that π_T not being injective necessarily implies that π_H is not injective because otherwise Γ would admit a faithful linear representation, contradicting non-residual finiteness. To prove the moreover note that Γ is just-infinite follows from the Bader-Shalom Normal Subgroup Theorem [BS06] applied to the closure of Γ in $H \times T$. The arithmeticity of Γ follows from [BFS19]. $\square$

Let $vb_p(\Gamma)$ denote the *pth virtual Betti number* of Γ which is defined to be the maximum of the pth Betti number over all finite index subgroups of Γ, or ∞ if the set is unbounded.

Proposition 3.7. *With H and T as before, assume H is a connected centre-free semisimple linear algebraic group without compact factors. Let Γ be a finitely generated uniform irreducible $(H \times T)$-lattice. If $vb_1(\Gamma) > 0$, then Γ is not residually finite. In particular, if $b_1(\mathcal{T}/\Gamma) > 0$, then Γ is not residually finite.*

Proof. Since Γ is irreducible, by the previous corollary, either Γ is linear and just-infinite, or Γ is not residually finite. Now, if the virtual Betti number of Γ is greater than zero, then a finite index subgroup Γ' of Γ admits $\mathbb{Z}$ as a quotient and so cannot be just infinite. Hence, Γ' is not residually finite and so neither is Γ.

The quotient space $\mathcal{T}/\Gamma$ gives rise to a graph of groups splitting of Γ with Bass-Serre tree $\mathcal{T}$. An easy application of the Mayer-Vietoris sequence applied to $\mathcal{T}$ shows that $b_1(\Gamma) \geqslant b_1(\mathcal{T}/\Gamma)$. $\square$

4. Properties of $(H \times T)$-lattices

In this section we will investigate the L^2-cohomology, C^*-simplicity, virtual fibring, and autostackability of $(H \times T)$-lattices in terms of properties of H-lattices. We remark that in each case the proofs are relatively elementary but depend in an essential way on the structure theorem (Theorem 3.3).

4.1. **L^2–cohomology and dimension.** Let Γ be a group. Both Γ and the complex group algebra $\mathbb{C}\Gamma$ act by left multiplication on the Hilbert space $\ell^2\Gamma$ of square-summable sequences. The *group von Neumann algebra* $\mathcal{N}\Gamma$ is the ring of Γ-equivariant bounded operators on $\ell^2 G$. The regular elements of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}\Gamma$ can be identified with the *ring of affiliated operators* $\mathcal{U}\Gamma$.

There are inclusions $\mathbb{C}\Gamma \subseteq \mathcal{N}\Gamma \subseteq \ell^2\Gamma \subseteq \mathcal{U}\Gamma$ and it is also known that $\mathcal{U}\Gamma$ is a self-injective ring which is flat over $\mathcal{N}\Gamma$. For more details concerning these constructions we refer the reader to [Lüc02] and especially to Theorem 8.22 of Section 8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [Lüc02, Section 8.3].

Let Y be a Γ-CW complex as defined in [Lüc02, Definition 1.25]. The ℓ^2-homology groups of Y are defined to be the equivariant homology groups $H_i^\Gamma(Y;\mathcal{U}\Gamma)$, and we have

$$b_i^{(2)}(Y) = \dim_{\mathcal{U}G} H_i^\Gamma(Y;\mathcal{U}\Gamma).$$

The ℓ^2-Betti numbers of a group Γ are then defined to be the ℓ^2-Betti numbers of $E\Gamma$. By [Lüc02, Theorem 6.54(8)], the zeroth ℓ^2-Betti number of Γ is equal to $1/|\Gamma|$ where $1/|\Gamma|$ is defined to be zero if Γ is infinite. Moreover, if Γ is finite then $b_n^{(2)}(G) = 0$ for $n \geqslant 1$.

In this section we will compute the L^2-Betti numbers of $(H \times T)$-lattices for a very general choice of H and T. Our primary tool will be Gaboriau's invariance of L^2-Betti numbers under measure equivalence.

Two countable groups Γ and Λ are said to be *measure equivalent* if there exist commuting, measure-preserving, free actions of Γ and Λ on some infinite Lebesgue measure space (Ω, m), such that the action of each of the groups Γ and Λ admits a finite measure fundamental domain. The key examples of measure equivalent groups are lattices in the same locally-compact group [Gro93].

Theorem 4.1. *Let H be a unimodular locally compact group with lattices and $\mathcal{T}$ be a locally-finite unimodular tree with automorphism group T. Assume H-lattices do not have two consecutive non-zero L^2-Betti numbers. Let Γ be an $(H \times T)$-lattice and let V and E be a representative set of orbits of vertices and edges respectively for the action of Γ on $\mathcal{T}$. We have*

$$b_n^{(2)}(\Gamma) = \sum_{e \in E} b_{n-1}^{(2)}(\Gamma_e) - \sum_{v \in V} b_{n-1}^{(2)}(\Gamma_v).$$

Proof. Let Λ be a reducible $(H \times T)$-lattice and assume Λ splits as $L \times F_n$ where L is an H-lattice. Using the Künneth formula we see that the L^2-Betti numbers of Λ are non-vanishing in the dimensions precisely 1 higher than the non-vanishing L^2-Betti numbers of L. Both Λ and Γ are measure equivalent, since they both lattices in $(H \times T)$. By Gaboriau's theorem on the invariance of L^2-Betti numbers under measure equivalence [Gab02, Theorem 6.3], the L^2-Betti numbers of Γ are non-vanishing in the same degrees as Λ.

Now, we apply the Γ-equivariant cohomology Mayer-Vietoris ([Bro94, Chapter VII.9]) sequence with $\mathcal{U}\Gamma$ coefficients to the filtration of $E\Gamma$ given by the cell structure of the Bass-Serre tree $\mathcal{T}$. Since the vertex and edge stabilisers of the action on $\mathcal{T}$ do not have two sequential non-zero L^2-Betti numbers, neither does Γ. Thus, the sequence degenerates into short exact sequences

$$0 \to \bigoplus_{e \in E} H^n_\Gamma(\Gamma_e; \mathcal{U}\Gamma) \to \bigoplus_{v \in V} H^n_\Gamma(\Gamma_v; \mathcal{U}\Gamma) \to H^{n+1}_\Gamma(\Gamma; \mathcal{U}\Gamma) \to 0$$

and the result follows from the additivity of von Neumann dimension. □

As an immediate corollary we recover the following well known result.

Corollary 4.2. *Let Γ be a tree lattice, then all L^2-Betti numbers of Γ vanish, except*

$$b^{(2)}_1(\Gamma) = \sum_{e \in E} \frac{1}{|\Gamma_e|} - \sum_{v \in V} \frac{1}{|\Gamma_v|}.$$

The assumption of not having two sequential non-zero L^2-Betti numbers turns out to not be very restrictive as [Lüc02, Theorem 5.12] and [PST18, Theorem 1.6] demonstrate. For arbitrary CAT(0) lattices, the presence of the de Rham factor causes the L^2-Betti numbers to vanish.

Proposition 4.3. *Let X be a proper* CAT(0) *space with non-trivial de Rham factor and $H \leqslant \mathrm{Isom}(X)$ be a closed subgroup acting minimally and cocompactly. If Γ is an H-lattice, then the L^2-Betti numbers of Γ vanish.*

Proof. By [CM19, Theorem 2(i)] Γ has a commensurated free abelian subgroup A and so $b^{(2)}_p(A) = 0$ for all $p \geqslant 0$. Now, we apply [BFS14, Corollary 1.4]. □

Remark 4.4. More generally, let X be a proper CAT(0) space with canonical closed convex $\mathrm{Isom}(X)$-stable subset $X' \subseteq X$ such that $X' = M \times X_1 \times \cdots \times X_n$, where M is a symmetric space of non-compact type and each X_i is irreducible and minimal. Assume $\mathrm{rank}_{\mathbb{C}}(\mathrm{Isom}(M)) - \mathrm{rank}_{\mathbb{C}}(\mathrm{Isom}(M)) = 0$, let $H \leqslant \mathrm{Isom}(X')$ be a closed subgroup acting cocompactly and minimally and let Γ be an H-lattice. By measure rigidity and repeat applications of the Künneth theorem we have $b^{(2)}_p(\Gamma) = 0$ for $p < \frac{1}{2}\dim(M) + \sum_{i=1}^n b_i$, where b_i is the smallest dimension such that an $\mathrm{Isom}(X_i)$-lattice has a non-vanishing L^2-Betti number. In particular, if either the L^2-cohomology of an $\mathrm{Isom}(X_i)$-lattice vanishes or $\text{f-rk}(M) > 0$ (see [Lüc02, Theorem 5.12]), then the L^2-cohomology of Γ vanishes.

4.1.1. *Rational homological dimension of group schemes over function fields.* Let k be the function field of an irreducible projective smooth curve C defined over a finite field $\mathbb{F}_q$. Let S be a finite non-empty set of (closed) points of C. Let $\mathcal{O}_S$ be the ring of rational functions whose poles lie in S. For each $p \in S$ there is a discrete valuation ν_x of k such that $\nu_p(f)$ is the order of vanishing of f at p. The valuation ring $\mathcal{O}_p$ is the ring of functions that do not have a pole at p, that is

$$\mathcal{O}_S = \bigcap_{p \notin S} \mathcal{O}_p.$$

Let $\bar{k}$ denote the algebraic closure of k. Let $\mathbf{G}$ be an affine group scheme defined over $\bar{k}$ such that $\mathbf{G}(\bar{k})$ is almost simple. For each $p \in S$ there is a completion k_p of k and the group $\mathbf{G}(k_p)$ acts on the Bruhat-Tit's building X_p. Thus, we may embed $\mathbf{G}(\mathcal{O}_S)$ into the product $\prod_{p \in S} \mathbf{G}_p$ as an arithmetic lattice.

In [Gan12] it is shown that $\mathrm{cd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \prod_{p \in S} \dim X_p$. In light of this Ian Leary asked the author what is $\mathrm{hd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S))$?

Theorem 4.5. *Let $\mathbf{G}$ be a simple simply connected Chevalley group. Let k and $\mathcal{O}_S$ be as above, then*

$$\mathrm{hd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \mathrm{cd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \prod_{p \in S} \dim X_p.$$

Proof. We first note that the group $\Gamma := \mathbf{G}(\mathcal{O}_S)$ is measure equivalent to the product $\prod_{p \in S} \mathbf{G}(\mathbb{F}_q[t_p])$ for some suitably chosen $t_p \in \mathcal{O}_p$. By [PST18, Theorem 1.6] the group $\mathbf{G}(\mathbb{F}_q[t_p])$ has one non-vanishing L^2-Betti number in dimension $\dim(X_p)$. Hence, by the Künneth formula $\mathbf{G}(\mathbb{F}_q[t_p])$ has one non-vanishing L^2-Betti number in dimension $d = \prod_{p \in S} \dim X_p$ Thus, by Gaboriau's theorem [Gab02], the group Γ has exactly one non-vanishing L^2-Betti number in dimension d. It follows that $\mathrm{hd}_{\mathbb{Q}}(\Gamma) \geqslant d$. The reverse inequality follows from the fact that Γ acts properly on the d-dimensional space $\prod_{p \in S} \dim X_p$. $\square$

4.2. **C^*–simplicity.** Let Γ be a discrete group. The *reduced C^*-algebra* of Γ, denoted $C^*_r(\Gamma)$, is the norm closure of the algebra of bounded operators on $\ell^2(\Gamma)$ by the left regular representation of Γ. We say Γ is *C^*-simple* if $C^*_r(\Gamma)$ has exactly two norm-closed two-sided ideals 0 and $C^*_r(\Gamma)$ itself. A C^*-simple group Γ enjoys a number of properties including having trivial amenable radical, the infinite conjugacy class (icc) property, the unique trace property [Bre+17, Theorem 1.3], and having a free action on its Furstenberg boundary $\partial_F \Gamma$ [KK17].

In 1975 Powers proved that the free group F_2 is C^*-simple [Pow75]. Since this result it has been a major open problem to classify C^*-simple groups, we refer the reader to [Har07] for a general survey and [Bre+17] for a number of recent developments. In the setting of CAT(0) groups there is a characterisation of C^*-simple CAT(0) cubical groups

[KS16] and of linear groups [Bre+17, Theorem 1.6]. In this section we will consider the C^*-simplicity of $(H \times T)$-lattices.

The C^*-simplicity of graphs of groups has been considered before [HP11], however, the methods developed there are not applicable to $(H \times T)$-lattices because the vertex and edge groups are all commensurable. Instead, we will apply the machinery developed in [Bre+17] to prove the C^*-simplicity of $(H \times T)$-lattices via properties of either H or the action on $\mathcal{T}$.

Let Γ be a group. We say a subgroup H is *normalish* if for every $n \geqslant 1$ and $t_1 \dots, t_n$ the intersection $\bigcap_{i=1}^n H^{t_i}$ is infinite.

Proposition 4.6. *Let Γ be the fundamental group of a (possibly infinite) graph of finite groups with leafless Bass-Serre tree $\mathcal{T}$ not quasi-isometric to $\mathbb{R}$. If Γ is infinite, not virtually cyclic and acts faithfully on $\mathcal{T}$, then Γ is C^*-simple.*

Proof. As Γ is not finite or virtually cyclic Γ has a positive (possibly infinite) first L^2-Betti number. Indeed, the chain complex of the Bass-Serre tree $C_*(\mathcal{T};\mathcal{U}\Gamma)$, which is concentrated in dimension 0 and 1, may be used to compute the L^2-homology. As Γ is infinite the boundary map is surjective and so the L^2-homology is concentrated in degree 1. We may pair each orbit of 0-cells v with an orbit of 0-cells e contained in the boundary of e, in each case the dimension of the $\mathcal{U}\Gamma$-module is $1/|\Gamma_v|$ or $1/|\Gamma_e|$, and $1/|\Gamma_e| - 1/|\Gamma_v| \geqslant 0$. Since Γ is non-trivial and not virtually cyclic some of these inequalities must be strict. In particular, we conclude Γ has a (possibly infinite) non-trivial first L^2-Betti number equal to the sum of these partial sums plus extra terms $1/|\Gamma_e|$ for any orbit of edges not accounted for. Since Γ has a trivial amenable radical and a non-trivial L^2-Betti number we may apply [Bre+17, Theorem 6.5] to deduce that Γ is C^*-simple.

Alternatively, we first note that any normalish subgroup of Γ contains a free subgroup since Γ is a faithful graph of finite groups and is not virtually cyclic. Now, we apply [Bre+17, Theorem 6.2] to deduce that Γ is C^*-simple. □

The following theorem and corollary give a partial answer to two questions of de la Harpe [Har07] and consider the more general case of an arbitrary graph of groups. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. The theorem implies the following lattices are C^*-simple:

- H is a semisimple Lie group with trivial centre and Γ is a graph of S-arithmetic lattices. This new whenever Γ is not residually finite. To see this, apply (1) and (2a);
- Γ is a lattice in a product of trees. To see this, apply (2c);
- Γ is the fundamental group of a graph of lattices where each vertex and edge group acts on the universal cover of a Salvetti complex corresponding to a right-angled Artin group with trivial centre. To see this, apply (1) and (2a) to [Bre+17, Theorem 1.6];

- H is the automorphism group of an affine building with no irreducible factor isometric to $\mathbb{E}^n$ and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (2a);
- H is the automorphism group of a hyperbolic building and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (2a);
- H is a product of the above and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (2a);
- $\mathrm{Isom}(\mathbb{E}^n)$ and Γ is an irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. Note this characterises irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices and will follow from (2b) (see Theorem 7.13).

The results in this list are new whenever the $(H \times T)$-lattices in question are not cubical or linear groups.

Theorem 4.7. *Let $X = X_1 \times \cdots \times X_k$ be a product of proper minimal cocompact* CAT(0)*-spaces each not isometric to $\mathbb{R}$ and let $H = \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_k)$ act without fixed point at infinity. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. Let $n \geqslant 0$ and $\Gamma < \mathrm{Isom}(\mathbb{E}^n) \times H \times T$ be a finitely generated lattice.*

(1) Assume Γ is reducible and $n = 0$, then Γ is C^*-simple if and only if $\Gamma \cap H$ is C^*-simple, and Γ has the icc property.

(2) Assume Γ is weakly irreducible. If one of the following holds:

(a) H-lattices have no normalish amenable subgroups;

(b) $\mathrm{Ker}(\pi_T)$ is trivial and $\mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)\times H})$ is infinite;

(c) H-lattices have a non-zero L^2-Betti number and trivial amenable radical;

then Γ is C^*-simple.

Proof. In the reducible case Γ virtually splits as $F_n \times \Gamma_H$. The result follows from the following three observations [Har07, Proposition 19 (i, iii, iv)], a direct product of two C^*-simple groups is C^*-simple, finite index subgroups of C^*-simple groups are simple, and a virtually C^*-simple group is C^*-simple if and only if it satisfies the icc property.

Now, assume Γ is irreducible. We will show that (2a) implies C^*-simplicity. Since Γ is finitely generated $\mathcal{G} = \mathcal{T}/\Gamma$ is finite. We first show that any amenable normalish subgroup N of Γ must fix a vertex of $\mathcal{T}$. Let $g \in \Gamma$ act as a hyperbolic element on $\mathcal{T}$, choose any other element $h \in \Gamma$ acting hyperbolically on $\mathcal{T}$ with an axis not equal to g, then any normalish subgroup N containing g contains the free group $\langle g, h\rangle$ and so cannot be amenable. Thus, N fixes a vertex of $\mathcal{T}$. Now, by Theorem 3.3 every vertex and edge stabiliser of Γ is a finite-by-H-lattice group. Since by assumption H-lattices do not contain any normalish amenable subgroups, neither does Γ. It remains to verify that Γ has no finite normal subgroups, but Γ has trivial amenable radical by [CM09a, Corollary 2.7]. In particular the result now follows from [Bre+17, Theorem 6.2].

We next prove (2b) implies C^*-simplicity. Let $K = \mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)\times H})$, we have that Γ is an extension of K by $\pi_{\mathrm{Isom}(\mathbb{E}^n)\times H)}(\Gamma)$. Now, K is a (possibly infinite) graph of finite groups acting faithfully on $\mathcal{T}$. Indeed, restricting $\pi := \pi_{\mathrm{Isom}(\mathbb{E}^n)\times H}$ to a vertex stabiliser $\Gamma_v < \Gamma$

of the action on $\mathcal{T}$, by Theorem 3.3 we see $\mathrm{Ker}(\pi|_{\Gamma_v})$ is finite. Every finite subgroup of Γ, and hence K, is conjugate to a finite subgroup of some vertex stabiliser. Thus, the graph of groups decomposition is given by $\mathcal{T}/K$.

We claim K is not virtually infinite cyclic. Indeed, if K was virtually cyclic, then there exists a commensurated infinite cyclic subgroup $Z < K < \Gamma$. By [CM19, Theorem 2(i)] Z acts properly on $\mathbb{E}^n$ in the decomposition of X. But $Z < K$, a contradiction.

It follows the group K is C^*-simple by Proposition 4.6. Because $\mathrm{Ker}(\pi_T)$ is trivial, every element acts non-trivially on $\mathcal{T}$ and so the centraliser $C_\Gamma(K)$ is trivial. Now, we apply [Bre+17, Theorem 1.4] to prove the result.

Finally, we will prove (2c) implies C^*-simplicity. We apply the cohomology Γ-equivariant Mayer-Vietoris sequence with $\mathcal{U}\Gamma$ coefficients arising from filtering $E\Gamma$ by the Bass-Serre tree [Bro94, Chapter VII.9]. Since $\mathcal{T}$ is not a quasi-line there is a vertex v connected to an edge e such that the stabilisers satisfy $|\Gamma_v : \Gamma_e| \geqslant 3$, thus the L^2-Betti numbers of Γ_e are at least 3 times the L^2-Betti number of Γ_v. Now, additivity of von Neumann dimension over exact sequences and a simple counting argument implies every $(H \times T)$-lattice must have a non-trivial L^2-Betti number. Alternatively, we note that every $(H \times T)$-lattice is measure equivalent to $L \times F_r$ where L is an H-lattice and F_r is a free group. Now, an application of the Kunneth formula yields that $L \times F_r$ has a non-trivial L^2-Betti number and so by Gaboriau's theorem [Gab02, Theorem 6.3] so does every $(H \times T)$-lattice. By [CM09a, Corollary 2.7] every $(H \times T)$-lattice has trivial amenable radical, the result follows from [Bre+17, Theorem 6.5]. □

A near identical proof to that of 2a yields the following corollary.

Corollary 4.8. *Let Γ be the fundamental group of a finite graph of groups. Assume, that for each edge and vertex that are incident that the intersection of the corresponding edge group and the vertex group does not contain either a normalish amenable subgroup or a non-trivial finite normal subgroup. If Γ is irreducible as an abstract group, then Γ is C^*-simple.*

4.3. **Fibring.** Recall that a group Γ is said to *algebraically fibre* if there is a non-trivial homomorphism $\phi : \Gamma \to \mathbb{Z}$ such that $\mathrm{Ker}(\phi)$ is finitely generated. If Γ has a finite index subgroup which algebraically fibres, then we say Γ *virtually fibres.*

Fix a finite generating set S for Γ. A character $0 \neq \phi \in H^1(\Gamma; \mathbb{R}) = \mathrm{Hom}(\Gamma, \mathbb{R})$ lies in the *first Bieri-Neumann-Strebel-Renz (BNSR) invariant* $\Sigma^1(\Gamma)$ if and only if the full subgraph of $\mathrm{Cay}(\Gamma, S)$ spanned by $\{g \in \Gamma \mid \phi(g) \geqslant 0\}$ is connected. The relevance of the BNSR invariant is due to the following classical theorem of Bieri-Neumann-Strebel.

Theorem 4.9. [BNS87, Theorem B1] *Let Γ be a finitely generated group and let $\phi : \Gamma \to \mathbb{Z}$ be non-trivial, then $\mathrm{Ker}(\phi)$ is finitely generated if and only if $\{\phi, -\phi\} \subseteq \Sigma^1(\Gamma)$.* □

Theorem 4.10. *Let X be a finite dimensional proper* $\mathrm{CAT}(0)$ *space and let $H = \mathrm{Isom}(X)$ be cocompact and minimal. Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Suppose $H^1(L;\mathbb{R}) = 0$ for all H-lattices L, then every $(H \times T)$-lattice Γ does not virtually fibre.*

Proof. Let Γ be an $(H \times T)$-lattice, then Γ splits as a graph of H-lattices A. In particular, every vertex and edge group is finite-by-H-lattice and so has trivial first cohomology. Now, we apply the Mayer-Vietoris sequence of the graph of groups decomposition (see [Bro94, Chapter VI.9]) to obtain an exact sequence

$$0 \longrightarrow H^0(\Gamma) \longrightarrow \bigoplus_{v\in VA} H^0(\Gamma_v) \longrightarrow \bigoplus_{e\in EA} H^0(\Gamma_e) \longrightarrow H^1(\Gamma) \longrightarrow 0.$$

Where the ending 0 is due to the fact $\bigoplus_{v\in VA} H^1(\Gamma_v) = 0$. It follows that $H^1(\Gamma;\mathbb{R}) = H^1(\mathcal{T}/\Gamma;\mathbb{R})$.

Claim: Γ splits as a reduced graph of groups and is not an ascending HNN extension.

We may assume the graph of groups is reduced by contracting any edges with a trivial amalgam $L *_L K$. Note that these contractions do not change the vertex and edge stabilisers, but may change the Bass-Serre tree (the tree will still not be quasi-isometric to $\mathbb{R}$ since there are necessarily other vertices of degree at least 3).

Now for Γ to be an ascending HNN-extension A must consist of a single vertex and edge. Let t be the stable letter of Γ, then t acts as an isometry on X. In particular, by considering covolumes of H-lattices acting on X, the two embeddings of the edge group Γ_e into the vertex group Γ_v must have the same index. Now, since $\mathcal{T}$ is not a quasi-line, these embeddings must have index at least 2 yielding the claim. ◆

Now, $H^1(\Gamma;\mathbb{R}) = \mathrm{Hom}(\Gamma,\mathbb{R})$ and so every character $\phi \in \mathrm{Hom}(\Gamma,\mathbb{R})$ vanishes on every vertex and edge group of the graph of groups decomposition A. Moreover, we may assume A is reduced by contracting any edges of the from $B *_C C$. Thus, we may apply [CL16, Proposition 2.5] to deduce $\phi \notin \Sigma(\Gamma)$. As this is true for every $(H \times T)$-lattice, it follows Γ does not virtually fibre. □

4.4. **Autostackability.** In this section we will discuss *autostackability* of $(H\times T)$-lattices in terms of H-lattices. The property was introduced by Brittenham, Hermiller and Holt in [BHH14] to simultaneously generalise automatic groups and groups with finite rewriting systems - we will not define the property here since our proofs do not require the definition and are elementary. The class of autostackable groups is broad, including all automatic groups, 3-manifold groups [BHS18], Thompson's group F [Cor+20], the Baumslag-Gersten group [HM18], and some groups not of type FP_3 [BHJ16]. In spite of this, it appears to be unknown if every group with solvable word problem is autostackable. Moreover, autostackability properties of the class of CAT(0) groups have largely gone unstudied. In light of Leary and Minasyan's examples of CAT(0) groups which are not biautomatic [LM19] it would be desirable to determine the autostackability properties of these and related groups.

Theorem 4.11. *Let X be a finite dimensional proper* $\mathrm{CAT}(0)$ *space and $H = \mathrm{Isom}(X)$. Let $\mathcal{T}$ be a locally finite unimodular tree and let $T = \mathrm{Aut}(\mathcal{T})$. If uniform H-lattices are (auto)stackable, then uniform $(H \times T)$-lattices are (auto)stackable. Moreover, if X is* $\mathrm{CAT}(0)$ *polyhedral complex and finitely presented H-lattices are (auto)stackable, then finitely presented $(H \times T)$-lattices are (auto)stackable.*

Proof. In either case, by Theorem 3.3 we see Γ splits as a graph of H-lattices. In particular, every local group is a commensurable finite-by-H-lattice. Now, by [BHJ16, Theorem 3.3] (auto)stackable groups are closed under extension, so we see the local groups are (auto)stackable. By [BHS18, Proposition 4.2] (see also [BHJ16, Theorem 3.4]), a group is (auto)stackable with respect to any finite index subgroup. Finally, [BHS18, Theorem 3.5] states that the fundamental group of a graph of groups whose vertex groups are (auto)stackable with respect to the edge groups is (auto)stackable. In particular, Γ is (auto)stackable. □

The following corollary follows by induction on the number of trees n with the base case given by the previous theorem. The inductive step is given by applying previous theorem to deduce the result holds for n trees after assuming the result holds for $n-1$ trees. As an example the corollary applies whenever X is $\mathrm{CAT}(-1)$.

Corollary 4.12. *Let X and H be as above. Let $\prod_{i=1}^{n} \mathcal{T}_i$ be a product of trees and let $T = \prod_{i=1}^{n} \mathrm{Aut}(\mathcal{T}_i)$. If uniform H-lattices are (auto)stackable, then uniform $(H \times T)$-lattices are (auto)stackable. Moreover, if X is* $\mathrm{CAT}(0)$ *polyhedral complex and finitely presented H-lattices are (auto)stackable, then finitely presented $(H \times T)$-lattices are (auto)stackable.*

In Theorem 7.13 we will prove that all irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices are not virtually biautomatic, generalising the result of Leary and Minasyan [LM19]. However, the following corollary proves that all of these lattices are in fact (auto)stackable.

Corollary 4.13. *Uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices are (auto)stackable. In particular, the Leary-Minasyan groups are (auto)stackable.*

Proof. A free abelian group is automatic and hence (auto)stackable. As (auto)stackability is closed under finite extensions it follows $\mathrm{Isom}(\mathbb{E}^n)$-lattices are (auto)stackable. Now, we apply the previous theorem. □

5. Constructions and examples

In this section we will detail a number of constructions and explicit examples of lattices in products of CAT(0) spaces and trees.

5.1. **Residual finiteness and amalgams.** For each symmetric space X of non-compact type with associated Lie group H we will construct infinitely many non-residually finite irreducible $(H \times T)$-lattices, where T is the automorphism group of an appropriate Bass-Serre tree. More generally the construction applies whenever there are upper bounded chains in the poset $(\mathrm{Lat}(H), \leqslant)$.

Theorem 5.1. *Let X be a* CAT(0) *space, let $H = \mathrm{Isom}(X)$ act cocompactly and minimally. Let A, B be commensurable uniform H-lattices such that $A \neq B$. Let $C \leqslant_{f.i.} A \cap B$ and $\Gamma = A *_C B$. Let $\mathcal{T}$ be the Bass-Serre tree of Γ and $T = \mathrm{Aut}(\mathcal{T})$. Assume $\mathcal{T}$ is unimodular, then Γ is a $(H \times T)$-lattice. Moreover,*

(1) If $\langle A, B\rangle < H$ is not an H-lattice, then Γ is an irreducible $(H \times T)$-lattice.

(2) If Γ is irreducible and C is a proper subgroup of $A \cap B$, then Γ is not residually finite.

Proof. The fact that Γ is a lattice follows from Theorem 3.3. Now, (1) follows from Theorem 3.4, since if $\langle A, B\rangle$ is not a lattice, then $\pi_H(\Gamma)$ is not a lattice and so Γ is not reducible and hence irreducible. To prove (2), consider an element γ in $(A \cap B) - C$ and words γ_a and γ_b representing γ in the generating sets of copies of A and B in Γ. Since, $\gamma_a\gamma_b^{-1}$ is not contained in the copy of C in Γ, the element acts non-trivially on $\mathcal{T}$, and so is non-trivial. However, $\pi_H(\gamma_a) = \pi_H(\gamma_b)$, so $\pi_H(\gamma_a\gamma_b^{-1}) = 1_H$. But Γ is irreducible and $\pi_H(\Gamma)$ has a non-trivial kernel so we can apply Caprace and Monod's criteria (Theorem 2.5). $\square$

The following lemma is immediate, but combined with the previous theorem, it implies that we can construct non-residually finite groups out of uniform lattices in each Lie group corresponding to a symmetric space of non-compact type.

Lemma 5.2. *Let H be a locally compact group with Haar measure μ. If there exists a bound ϵ on the minimal μ-covolume of lattices in H and the set of possible covolumes of H-lattices is discrete, then the poset* $\mathrm{Lat}(H)$ *has maximal elements.*

Example 1. Let X be a symmetric space of non-compact type and H the associated Lie group. Let A and B be commensurable maximal H-lattices such that $A \neq B$. Let C be a finite index proper subgroup of $A \cap B$, then $\Gamma = A *_C B$ is a non-residually finite $(H \times T)$-lattice. Such examples exist by considering arithmetic lattices Γ in H. Indeed, Margulis' commensurator criterion states that $\mathrm{Comm}_H(\Gamma)$ is dense in H and so there exist lattices commensurable to Γ which are not contained in Γ.

In the more general setting of CAT(0)-spaces we have the following corollary.

Corollary 5.3. *Assume $\Gamma = A *_C B$ is a uniform $(H \times T)$-lattice such that $A \neq B$ and neither $A < B$ nor $B < A$. If A or B is the upper bound of a chain in $(Lat, \leqslant)$, then Γ is irreducible. Moreover, if C is a proper subgroup of $A \cap B$, then Γ is non-residually finite.*

Proof. Assume without loss of generality that A is the upper bound, then $\langle A, B\rangle$ cannot be a lattice because it would contain A, contradicting the maximality of A. Thus, we can apply Theorem 5.1. $\square$

Example 2 (Change of tree)**.** Given an edge transitive but not vertex transitive irreducible $(H \times T_{k,\ell})$-lattice Γ one may construct a non-residually finite irreducible $(H \times T_{mk,n\ell})$-lattice for all $m, n \geqslant 2$ as follows:

Firstly, note Γ splits as a graph of H-lattices. Indeed, $\Gamma = A *_C B$ where A, B and C are covirtually H-lattices. We may assume that A stabilises a vertex of valence k and B stabilises a vertex of valence ℓ. Let N_A and N_B be finite groups of order m and n respectively and pick split extensions $\widetilde{A} = N_A \rtimes A$ and $\widetilde{B} = N_B \rtimes B$. We may construct a graph of lattices by considering the graph of groups corresponding to $\widetilde{A} *_C \widetilde{B}$. The representations of $\widetilde{A}$ and $\widetilde{B}$ are the given by the composites $\widetilde{A} \twoheadrightarrow A \to H$ and $\widetilde{B} \twoheadrightarrow B \to H$. The resulting fundamental group $\widetilde{\Gamma}$ acts on the $(mk, n\ell)$-regular tree, the lattice is irreducible and non-residually finite by Theorem 5.1.

This technique gives the following partial solution to the problem of realising lattices in every possible tree for H a rank one real Lie group with trivial centre.

Example 3. Let $H = \mathbf{H}(\mathbb{R})$ be a rank one real Lie group with trivial centre and $H_p = \mathbf{H}(\mathbb{Q}_p)$ denote the same group scheme over the p-adic numbers for some prime p. Let X be the rank-one symmetric space associated to H. The Bruhat-Tits' building for H_p is a tree of valence given by some function f of the prime p. In particular, there is an edge transitive but not vertex transitive S-arithmetic lattice acting on $X \times \mathcal{T}_{f(p)}$. By the previous example we may construct irreducible non-residually finite lattices acting on $X \times \mathcal{T}_{mf(p),nf(p)}$ for all $m, n \geqslant 2$.

These groups are C^*-simple by Theorem 4.7, austostackable by Theorem 4.11, and if X is $2n$-dimensional, then the groups have a non-trivial L^2-Betti number in dimension $n+1$ by Theorem 4.1. If X is odd-dimensional, then the L^2-cohomology vanishes.

Concretely, in the case of $H = \mathrm{PSL}_2(\mathbb{R})$, the function f is given by $f(p) = p+1$, so we obtain irreducible lattices acting on the $(m(p+1), n(p+1))$-regular tree for all primes p and integers $m, n \geqslant 2$.

5.2. **Vertex transitive lattices.** In this section we will detail some constructions for lattices in a product of a CAT(0) space and a tree such that the lattices act vertex transitively on the tree.

Proposition 5.4. *Let $L \leqslant H$ be groups and $t \in \mathrm{Comm}_H(L)$, then there exist finite-index subgroups $J, K \leqslant L$ such that $J^t = K$*

Proof. By definition $K = L \cap L^t$ has finite index in L. Now, set $J = K^{t^{-1}}$, this clearly also has finite index in L. $\square$

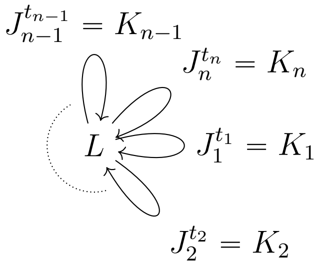

FIGURE 1. A single vertex graph of groups.

Let X be a metric space and let $H = \mathrm{Isom}(X)$. Let L be a H-lattice and let $t_1, \ldots, t_n \in \mathrm{Comm}_H(L)$. Assume that t_i conjugates a finite-index subgroup $J_i \leqslant L$ to a finite-index subgroup $K_i \leqslant L$ (existence of H_i and K_i is given in the next proposition). In light of Proposition 5.4, whilst slightly abusing notation, we can construct a single vertex graph of groups $\mathcal{G}$ where all of the edges are loops (Figure 1). We now define $\Gamma = \mathcal{G}(L, \{(J_1, t_1), \ldots, (J_n, t_n)\}) := \pi_1(\mathcal{G})$. We can associate to Γ the Bass-Serre tree $\mathcal{T}$ of the graph of groups $\mathcal{G}$. Note that $\mathcal{T}$ is an infinite, locally finite, $(\sum_{i=1}^n |\Gamma : J_i| + |\Gamma : K_i|)$-regular, simplicial tree.

Lemma 5.5. *Let Γ be a lattice in a rank-one Lie group H with symmetric space X of non-compact type. Let t be an infinite order elliptic element of H, then*

$$L := \bigcap_{n \in \mathbb{Z}} \Gamma^{t^n}$$

has infinite index in Γ.

Proof. Assume L has finite index, then by Garland and Raghunathan [GR69; GR70], the quotient X/L has finitely many cusps with bounded intersection. Let p be the fixed point of t and consider a Dirichlet domain $\Delta = \Delta_p(L)$ for L at p. Since X/Γ has finitely many cusps, Δ has finitely many sectors (each with bounded intersection) going to infinity. The $\langle t \rangle$-orbit of such a sector is unbounded (indeed it traces out a copy of S^1 in ∂X), but this contradicts Garland-Raghunathan and so we conclude L must have infinite index. $\square$

Theorem 5.6. *Let X be a rank-one symmetric space of non-compact type and let H be the associated Lie group. Let L be an H-lattice, $t_1, \ldots, t_k \in \mathrm{Comm}_H(\Gamma)$ and let $\Gamma := \mathcal{G}(L, \{(J_i, t_i)\})$ with Bass-Serre tree $\mathcal{T}$. Let $T = \mathrm{Aut}(\mathcal{T})$. If $\pi_H\langle t_1, \ldots, t_k \rangle$ contains an infinite order elliptic element t, then Γ is a weakly irreducible $(H \times T)$-lattice.*

Proof. Clearly, the projection of Γ to the group H is not discrete because Γ contains an infinite order elliptic element. Now, the vertex stabilisers of the action of Γ on $\mathcal{T}$ are conjugates of $L < G$ Thus, the kernel of the action is equal to $\mathrm{Core}(\Gamma, L)$, but by the previous lemma this is infinite index in L. It follows that the image of Γ is an infinite subgroup of the vertex stabiliser in T (a compact profinite group) and so cannot have discrete image. □

Example 4. Let H be a non-compact simple Lie group and $\mathcal{O}$ the ring of integers of some number field k. Assume that either $H(\mathcal{O})$ is either an irreducible uniform lattice or rank-one. Now, choose an infinite order elliptic element $t \in \mathrm{Comm}_H(H(\mathcal{O}))$ and construct the group $\Gamma = \mathcal{G}(H(\mathcal{O}), t)$ with Bass-Serre tree $\mathcal{T}$. Let $T = \mathrm{Aut}(\mathcal{T})$. By Theorem 3.3 we conclude that Γ is a lattice in $G = (\prod_{\sigma \in S^\infty} H(K^\sigma) \times T)$. Moreover, if t is irreducible, then Γ is a weakly and algebraically irreducible lattice. To see Γ is weakly irreducible, note that the projection of Γ to any sub-product of G is clearly non-discrete. Now, we apply Theorem 2.4 to see Γ is algebraically irreducible.

In the next example we will present an explicit presentation of a non-residually finite, irreducible, vertex and edge transitive $(\mathrm{PSL}_2(\mathbb{R}) \times T_{60})$-lattice.

Example 5. Consider the following matrices in $\mathrm{SL}_2(\mathbb{R})$ given by

$$a = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{3}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}, \qquad b = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}(\sqrt{2}-1) \\ \frac{1}{2}(-3\sqrt{2}-3) & \frac{1}{2} \end{bmatrix},$$

$$c = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}(\sqrt{2}+1) \\ \frac{1}{2}(-3\sqrt{2}+3) & \frac{1}{2} \end{bmatrix}, \qquad t = \begin{bmatrix} \frac{1}{5} & \frac{2}{5}\sqrt{2} \\ -\frac{6}{5}\sqrt{2} & \frac{1}{5} \end{bmatrix}.$$

The projectivisation of the matrices a, b and c in $\mathrm{PSL}_2(\mathbb{R})$ generate a Fuchsian group of signature $[0; 2, 2, 3, 3]$ with presentation $L = \langle a, b, c \mid a^2 = b^3 = c^3 = (c^{-1}ab^{-1})^2 = 1\rangle$. The conjugate of L by the infinite order elliptic element t in $\mathrm{PSL}_2(\mathbb{R})$ yields an isometric Fuchsian group $L^t = \langle \alpha, \beta, \gamma\rangle$. The intersection is generated by

$$\begin{aligned} K =&\langle acb^{-1}a,\ cac^{-1},\ b^{-1}acb^{-1},\ c^{-1}bca,\ bcabc^{-1},\ b^{-1}cbc^{-1}ab^{-1},\ b^{-1}c^{-1}bc^{-1}b^{-1},\ c^{-1}acab^{-1} \\ &ababc^{-1}ba,\ abacb^{-1}ab^{-1},\ babac^{-1}b^{-1}c^{-1},\ babcac^{-1}b,\ b^{-1}cabc^{-1}b^{-1}c^{-1}\rangle. \end{aligned}$$

We also find that K is index 30 and has signature $[5; 2, 2, 2, 2]$. Since K is contained in L, to complete our construction we simply need to find $J := t^{-1}(K)$, which will also be contained in Γ. A lengthy calculation yields

$$\begin{aligned} J =&\langle c^{-1}abab^{-1},\ b^{-1}ab,\ cab^{-1}c,\ acb^{-1}abac,\ cabac^{-1}ab,\ c^{-1}acab^{-1}a,\ babac^{-1}bac^{-1}b^{-1}, \\ &ac^{-1}ba,\ bacb^{-1}acb^{-1}c^{-1}ab^{-1}ac^{-1},\ bac^{-1}bc^{-1},\ cbc^{-1}ac^{-1}ab,\ cb^{-1}abcac, \\ &c^{-1}ac^{-1}ac^{-1}b^{-1}acb^{-1}\rangle. \end{aligned}$$

The group $\Gamma = \langle a, b, c, t \mid a^2 = b^3 = c^3 = (c^{-1}ab^{-1})^2 = 1,\ J^t = K\rangle$ is a non-residually finite irreducible lattice in $\mathrm{PSL}_2(\mathbb{R}) \times T_{60}$. By Theorem 4.1 the only non-vanishing L^2-Betti number of Γ is in dimension 2 and is equal to $-\frac{1}{3} - (-10) = \frac{29}{3}$. By Theorem 4.7 Γ is C^*-simple, by Theorem 4.11 Γ is autostackable, and by the same argument as in the proof of Theorem 4.10, Γ does not algebraically fibre. Moreover, if Γ has first virtual Betti number equal to 1, then Γ does not virtually fibre.

Example 6 (Mixed products)**.** Consider a uniform weakly irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})\times T_{60}$ constructed as a single vertex graph of groups $\mathcal{G}(\Gamma, t)$, assume that the stable letter t acts on $\mathbb{R}\mathbf{H}^2$ as an infinite order elliptic rotation. Similarly, consider a uniform weakly irreducible lattice in $\mathrm{Isom}(\mathbb{E}^2)\times T_{10}$ constructed as a single vertex graph of groups $\mathcal{G}(\mathbb{Z}^2, s)$, assume that the stable letter s acts on $\mathbb{E}^2$ as an infinite order elliptic rotation (such examples were considered by Leary and Minasyan in [LM19]).

We will now construct a uniform lattice in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{Isom}(\mathbb{E}^2) \times T_{300}$. Let $\Lambda := \mathcal{G}(\Gamma \times \mathbb{Z}^2, r)$, where r acts as t on $\mathbb{R}\mathbf{H}^2$ and as s on $\mathbb{E}^2$. We claim the projections to each sub-product of the factors are non-discrete and so Λ is not commensurable with any reducible lattice. Thus, by Theorem 3.4, Λ is an weakly irreducible lattice.

To prove the claim we investigate each projection in turn. Clearly, the projections to $\mathrm{PSL}_2(\mathbb{R})$, $\mathrm{Isom}(\mathbb{E}^2)$ and $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{Isom}(\mathbb{E}^2)$ are non-discrete. Moreover, by Theorem 5.6 or [LM19, Theorem 7.5] it is easy to see the projection to T_{300} is non-discrete. In fact more is true, the projection is faithful. In light of this it is easy to see the projections to $\mathrm{PSL}_2(\mathbb{R}) \times T_{300}$ and $\mathrm{Isom}(\mathbb{E}^2) \times T_{300}$ are non-discrete.

Note that the choices of the ambient groups $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{Isom}(\mathbb{E}^2)$ were arbitrary. Indeed, the reader can pick any combination of symmetric spaces of non-compact (and Euclidean) type, or any irreducible proper minimal CAT(0) space which contains lattices which have a non-discrete commensurator and construct a weakly irreducible lattice in the product of the automorphism group of the Bass-Serre tree and the associated real simple Lie groups (and $\mathrm{Isom}(\mathbb{E}^n)$) and the isometry group of the CAT(0) space. This is markedly different to the arithmetic setting where the Lie groups must be isogenous.

Example 7 (Non-uniform lattices in products of trees)**.** Fix a prime p. Consider the linear algebraic group $H = \mathrm{PSL}_2(\mathbb{F}_p((t)))$ and the non-uniform lattice $L = \mathrm{PSL}_2(\mathbb{F}_p[t]) < H$. The Bruhat-Tits' building for H is a $(p+1)$-regular tree $\mathcal{T}$ and L acts with finite covolume and fundamental domain an infinite ray. Let $t \in \mathrm{Comm}_H(L)$ be infinite order and elliptic. By Proposition 5.4 there exist finite index subgroups $J, K \leqslant L$ such that $J^t = K$. Let $n \geqslant 1$ and consider the HNN-extension Γ of L^n over finite index subgroups J^n and K^n where each copy of J is mapped to the corresponding copy of K by t. The group Γ is non-uniform lattice acting on $\mathcal{T}_{p+1}^n \times \mathcal{T}_{2k^n}$ where $k = |L : J|$. Moreover, it is easy to see that Γ is a weakly irreducible lattice.

More generally by [AR09] non-uniform tree lattices of 'Nagao type' have a dense commensurator in the full automorphism group of the universal covering tree. The construction can be easily adapted to this setting.

5.3. **The universal covering trick.** In this section we will compare the notion of a graph of lattices with the "universal covering trick" of Burger–Mozes [BM00a, Section 1.8] and generalised by Caprace–Monod [CM09a, Section 6.C]. In particular, we will show how in many cases one can obtain a graph of lattices from the universal covering trick. We take the opportunity to point out that many of the groups constructed in the previous sections cannot be obtained from universal covering trick.

Example 8 (The universal covering trick)**.** Let A be the geometric realisation of a locally finite graph (not reduced to a single point) and let $Q < \mathrm{Isom}(A)$ be a vertex transitive closed subgroup. Let C be an infinite profinite group acting level transitively on a locally finite rooted tree $\mathcal{T}_0$. Let B be the 1-skeleton of the square complex $A \times \mathcal{T}_0$ and let $\mathcal{T}$ be the universal cover. Define D to be the extension $1 \rightarrowtail \pi_1(B) \rightarrowtail D \twoheadrightarrow C \times Q \twoheadrightarrow 1$. By [CM09a, Proposition 6.8], there exists a CAT(0) space Y such that $D \rightarrowtail \mathrm{Isom}(Y)$ is a closed subgroup, and D acts cocompactly and minimally without fixed point at infinity.

The classical situation where this is applied is as follows: Let Q be a product of p-adic Lie groups, H be a product of real Lie groups and $\Gamma < H \times Q$ to be an S-arithmetic irreducible lattice. Let A be the 1-skeleton of the Bruhat-Tit's building for X, let $\mathcal{T}$ be the universal cover of A and let $T = \mathrm{Aut}(\mathcal{T})$. Now, Γ lifts to a weakly irreducible lattice $\widetilde{\Gamma} < H \times Q \times T$ and the corresponding graph of lattices is obtained by considering the graph A/Γ equipped with local groups given by the stabilisers of the action of Γ on A.

6. Complexes of lattices

In this section we will introduce the notion of a complex of H-lattices. We will then prove a structure theorem analogous to Theorem 3.3 for these complexes of H-lattices.

6.1. **Complexes of groups.** The definitions in this section are adapted from [Tho06, Section 1.4] and [Hae91; Hae92]. Throughout this section if X is a polyhedral complex then X' is its first barycentric subdivision. This is a simplicial complex with vertices VX' and edges EX'. Each $e \in EX'$ corresponds to cells $\tau \subset \sigma$ of X and so we may orient them from σ to τ. We will write $i(e) = \sigma$ and $t(e) = \tau$. We say two edges e and f of X' are *composable* if $i(e) = t(f)$, in which case there exists an edge $g = ef$ of X' such that $i(c) = i(e)$ and $t(c) = t(f)$, and $e,\ f$ and g form the boundary of a 2-simplex in X. We denote the set of composable edges by E^2X'.

A *complex of groups* $G(X) = (G_\sigma, \psi_e, g_{e,f})$ over a polyhedral complex X is given by the following data:

(1) For each vertex σ of VX', a group G_σ called the *local group at* σ.
(2) For each edge e of EX', a monomorphism $\psi_e : G_{i(e)} \to G_{t(e)}$ called the *structure map*.
(3) For each pair of composable edges e and f, an element $g_{e,f} \in G_{t(e)}$ called the *twisting element*. We require these elements to satisfy the following conditions:
   (a) For $(e,f) \in EX'$, we have $\mathrm{Ad}(g_{e,f})\psi_{ef} = \psi_e\psi_f$.

(b) For each triple of composable edges a, b and c we have a *cocycle condition* $\psi_a(g_{b,a}) = g_{c,b}g_{cb,a}$.

We say $G(X)$ is *simple* if each of the twisting elements $g_{e,f}$ are the identity.

Some complexes of groups arise from actions on polyhedral complexes. Let G be a group acting without inversions on a polyhedral complex Y. Let $X = Y/G$ with natural projection $p : Y \to X$. For each $\sigma \in VX'$, choose a lift $\widetilde{\sigma} \in VY'$ such that $p\widetilde{\sigma} = \sigma$. The local group G_σ is the stabiliser of $\widetilde{\sigma}$ in G, and the structure maps and twisting elements are given by further choices. The resulting complex of groups $G(X)$ is unique up to isomorphism. A complex of groups isomorphic to a complex of groups arising from a group action is called *developable.*

Let $G(X)$ be a complex of groups over a polyhedral complex X. Let T be a maximal tree in the 1-skeleton of X' and fix a basepoint σ in T. The *fundamental group* of $G(X)$, denoted $\pi_1(G(X), \sigma_0)$, is generated by the set

$$\coprod_{\sigma \in VX'} G_\sigma \coprod \{e^+, e^- : e \in EX'\}$$

subject to the relations

$$\left\{ \begin{array}{l} \text{the relations in the groups } G_\sigma, \\ (e^+)^{-1} = e^- \text{ and } (e^-)^{-1} = e^+, \\ e^+ f^+ = g_{e,f}(ef)^+, \ \forall (e,f) \in E^2X', \\ \psi_e(g) = e^+ g e^-, \ \forall g \in G_{i(e)}, \\ e^+ = 1, \ \forall e \in T. \end{array} \right\}$$

If $G(X)$ is developable, then it has a *universal cover* $\widetilde{G(X)}$. This is a simply connected polyhedral complex, equipped with an action of $G = \pi_1(G(X), \sigma_0)$ such that the complex of groups given by $\widetilde{G(X)}/G$ is isomorphic to $G(X)$.

Let $G(X) = (G_\sigma, \psi_e)$ and $H(Y) = (H_\tau, \psi_f)$ be complexes of groups over polyhedral complexes X and Y. Let $f : X' \to Y'$ be a simplicial map sending vertices to vertices and edges to edges. A *morphism* $\Phi : G(X) \to H(Y)$ over f consists of:

(1) A homomorphism $\phi_\sigma : G_\sigma \to H_{f(\sigma)}$ for each $\sigma \in VX'$.
(2) For each $e \in EX'$ an element $g_e \in H_{t(f(e))}$ such that
  (a) $\mathrm{Ad}(g_e)\psi_{f(e)}\phi_{i(e)} = \phi_{t(e)}\psi_e$;
  (b) For all $(a, b) \in E^2X'$ we have $\phi_{t(a)}(g_{a,b})g_{ab} = g_e\psi_{f(a)}(g_b)g_{f(a),f(b)}$.

6.2. **Complexes of lattices.** In this section we introduce complexes of lattices in analogy with the graphs of lattices we defined previously.

Definition 6.1 (Complex of lattices)**.** Let H be a locally compact group with Haar measure μ. A *complex of H-lattices* $(G(X), \psi)$ is a developable complex of groups equipped with a morphism ψ to H such that:

(1) For each $\sigma \in VX'$, the local group G_σ is covirtually an H-lattice and the image $\psi(G_\sigma)$ is an H-lattice;
(2) The local groups are commensurable in $\Gamma = \pi_1(G(X), \sigma)$ and their images are commensurable in H.
(3) For each $e \in EX'$, the elements e^+ and e^- in Γ are mapped to elements of $\mathrm{Comm}_H(\psi(G_\sigma))$.

The analogous structure theorem is given as follows.

Theorem 6.2. *Let X be a finite dimensional proper $\mathrm{CAT}(0)$ space and let $H = \mathrm{Isom}(X)$ contain a uniform lattice. Let $(G(Z), \psi)$ be a complex of H-lattices over a polyhedral complex Z, with universal cover Y, and fundamental group Γ. Suppose $A = \mathrm{Aut}(Y)$ admits a uniform lattice.*

(1) Assume Z is finite and Y is a $\mathrm{CAT}(0)$ space. If for each local group G_σ the kernel $\mathrm{Ker}(\psi|_{G_\sigma})$ acts faithfully on Y, then Γ is a uniform $(H \times A)$-lattice and hence a $\mathrm{CAT}(0)$ group. Conversely, if Λ is a uniform $(H \times A)$-lattice, then Λ splits as a finite complex of uniform H-lattices with universal cover Y.
(2) Under the same hypotheses as (1), Γ is quasi-isometric to $X \times Y$.
(3) Assume X is a $\mathrm{CAT}(0)$ polyhedral complex and Y is a $\mathrm{CAT}(0)$ space. Let μ be the normalised Haar measure on H. If for each local group G_σ the kernel $K_\sigma = \mathrm{Ker}(\psi|_{G_\sigma})$ acts faithfully on Y and the sum $\sum_{\sigma \in VZ} \mu(G_\sigma)/|K_\sigma|$ converges, then Γ is a $(H \times A)$-lattice. Conversely, if Λ is a $(H \times A)$-lattice, then Λ splits as a finite complex of H-lattices with universal cover Y.

Note that by definition we are assuming all complexes of lattices are developable complexes of groups.

Proof. We first prove (1). The fundamental group Γ of $G(Z)$ acts on the universal cover Y and on X via the homomorphism $\psi : \Gamma \to H$. The action on the product space $X \times Y$ is properly discontinuous cocompact and by isometries. The kernel of the action is contained in the intersection $\bigcap_{\sigma \in Z'} \mathrm{Ker}(\psi|_{G_\sigma})$. But this acts faithfully on Y, thus, the action is faithful. It follows Γ is an $(H \times A)$-lattice.

We now prove the converse. Assume Γ is an $(H \times A)$-lattice, and note that the action of Γ on Y yields a developable complex of groups $G(Z) = (\Gamma_\sigma, \psi_a, g_{a,b})$ with spanning tree T and equipped with a homomorphism $\pi_H : \Gamma \to H$. It suffices to show the local groups corresponding to the vertices of Z are covirtually H-lattices. Indeed, for an edge $e \in EZ'$, if the index $|\Gamma_{t(e)} : \psi_e(\Gamma_{i(e)})|$ is infinite, then the universal cover of $G(Z)$ would not be locally finite. It follows that all of the local groups are commensurable and hence, commensurable in H. Consequently, the elements e^+ and e^- for all $e \in E^2Z'/T$ in Γ must commensurate the local groups.

Let $\sigma \in Y$ be a vertex and consider the stabiliser $\Gamma_\sigma < \Gamma$ for the action on $X \times Y$. Suppose Γ_σ does not act cocompactly on $X \times \sigma$, then there is no compact set whose Γ_σ translates cover $X \times \sigma$. Let D be a non-compact set whose Γ_σ-translates cover $X \times \sigma$, but there is a compact set C whose Γ translates cover $X \times Y$. We may arrange our subsets such that $C' = C \cap (X \times \sigma) \subseteq D$. In particular, there are elements $g_i \in \Gamma/\Gamma_\sigma$ whose translates of C' cover D. But some of these elements fix must $X \times \sigma$ yielding a contradiction. Hence, Γ_σ is cocompact.

It is clear that $\mathrm{Ker}(\Gamma_\sigma \to H)$ is finite. Otherwise Γ would act with infinite point stabilisers on $X \times Y$ contradicting the discreteness of Γ. It remains to show that the projection $\overline{\Gamma}_\sigma$ of Γ_σ to H is discrete. Assume that $\overline{\Gamma}_\sigma$ is not discrete, then there does not exists a neighbourhood N of $1 \in H$ such that $N \cap \overline{\Gamma}_\sigma = \{1\}$. But this immediately implies there does not exist a neighbourbood N' of $1 \in H \times A$ such that $N' \cap \Gamma = \{1\}$ which contradicts the discreteness of Γ. It follows Γ_σ is covirtually an H-lattice.

The final step is to show the elements e^+ and e^- for each $e \in EX'$ are mapped to elements of $\mathrm{Comm}_H(\pi_H(\Gamma_\sigma))$. But this is immediate since the local groups map to H with finite kernel, the elements e^+ and e^- commensurate the local groups, and so must still preserve the appropriate conjugation relations in the map to H. ◆

We now prove (2). By (1), Γ acts properly discontinuously cocompactly on $X \times Y$. The result follows from the Švarc-Milnor Lemma [BH99, p. I.8.19]. ◆

The proof of (3) is almost identical to 1 we will highlight the differences. Since X is a CAT(0) polyhedral complex, it follows that $X \times Y$ is. Now, we may apply Serre's Covolume Formula to Γ. Let Δ be a fundamental domain for Γ acting on $X \times Y$, then the covolume of Γ may be computed as

$$\sum_{\sigma \in \Delta^0} \frac{1}{|\Gamma_\sigma|} = \sum_{\sigma \in \pi_Y(\Delta^0)} \sum_{\tau \in \pi_Y^{-1}(\sigma)} \frac{1}{|\Gamma_\tau|} = \sum_{\sigma \in \pi_Y(\Delta^0)} \frac{1}{|K_\sigma|} \sum_{\tau \in \pi_Y^{-1}(\sigma)} \frac{|K_\sigma|}{|\Gamma_\tau|} = \sum_{\sigma \in \pi_Y(\Delta^0)} \frac{\mu(\pi_X(\Gamma_\sigma))}{|K_\sigma|}.$$

Since $\pi_Y(\Delta^0)$ can be identified with Z and the later sum converges by assumption, it follows as before that Γ acts faithfully properly discontinuously and isometrically with finite covolume on $X \times Y$. For the converse the only adjustment required is that the compact sets C and C' in the proof of (1) should be replaced with ones of finite covolume. The remainder of the proof is identical. ◆ □

6.3. **Properties: L^2-cohomology and C^*-simplicity.** In this section we will prove a result on L^2-cohomology in the spirit of Theorem 4.1 and a result on C^*-simplicity in the spirit of Theorem 4.7 for $(H \times A)$-lattices.

Theorem 6.3. *Let H be a unimodular locally compact group with lattices and X be a locally-finite* CAT(0) *polyhedral complex with cocompact minimal automorphism group A. Assume any two non-zero L^2-Betti numbers of an H-lattice are in dimensions separated by at least* $\dim(X)$ *and that A-lattices have at most one non-vanishing L^2-Betti number* *in dimension k. Let Γ be an $(H \times A)$-lattice and $\Delta^{(p)}$ be a representative set of p-cells for the action of Γ on X. We have*

$$b_n^{(2)}(\Gamma) = \sum_{p=0}^{\dim(X)} \sum_{\sigma \in \Delta^{(p)}} (-1)^p b_{n-k}^{(2)}(\Gamma_\sigma).$$

Proof. The proof is essentially identical to Theorem 4.1, except now we use a G-equivariant spectral sequence [Bro94, Chapter VII.7] applied to the filtration of X by skeleta with $\mathcal{U}\Gamma$ coefficients. The assumption that any two non-zero L^2-Betti numbers of an H-lattice are in dimensions separated by at least $\dim(X)$ forces any higher differentials to be 0. In particular, the E^2-page equals the E^∞ page of the spectral sequence. Moreover, the E^2-page is computed by using the same measure equivalence argument as in Theorem 4.1. $\square$

The proof of the following theorem is essentially the same measure equivalence and Künneth formula argument as in Theorem 4.7(2c).

Theorem 6.4. *Let $X = X_1 \times \cdots \times X_k$ be a product of proper minimal cocompact* CAT(0)*-spaces each not isometric to $\mathbb{R}$ and let $H = \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_k)$ act without fixed point at infinity. Let Y be a locally-finite* CAT(0) *polyhedral complex not quasi-isometric to $\mathbb{E}^n$ and let $A = \mathrm{Aut}(Y)$ act without fixed point at infinity. Let $\Gamma < H \times T$ be a finitely generated weakly irreducible lattice. If both H- and A-lattices have a non-zero L^2-Betti number and trivial amenable radical, then Γ is C^*-simple.*

7. Lattices with non-trivial de-Rahm factor

In this section we will characterize irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices. We will also strengthen the virtual biautomaticity criterion for a Leary-Minasyan group [LM19, Theorem 8.5] to arbitrary CAT(0)-lattices. Along the way we will prove a number of results about $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices. To this end we will examine the projections $\pi_{\mathrm{Isom}(\mathbb{E}^n)}$ and $\pi_{\mathrm{O}(n)}$ more closely.

Lemma 7.1. *Let X be a proper* CAT(0)*-space, let $H = \mathrm{Isom}(X)$, and let Γ be a finitely generated $(\mathrm{Isom}(\mathbb{E}^n) \times H)$-lattice. If the projection $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is not discrete, then $\pi_{\mathrm{O}(n)}(\Gamma)$ contains an element of infinite order.*

Proof. For $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ to be not discrete at least one of the following must be true:

(1) $\pi_{\mathrm{O}(n)}(\Gamma)$ is not discrete and thus, contains an element of infinite order.
(2) There exists a sequence of elements $\mathbf{g}_i \in \mathbb{R}^n$ such that $\mathbf{g}_i \to \mathbf{0}$ as $i \to \infty$.

If the first case holds we are done, so assume it does not. After passing to a subsequence we may assume that each $\mathbf{g}_i$ is not some power or root of any other $\mathbf{g}_j$ and so $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma) \cap \mathbb{R}^n$ contains an infinitely generated abelian subgroup A. Since we have assumed the first case does not hold $\pi_{\mathrm{O}(n)}(\Gamma)$ is a finite group F and we have a short exact sequence

$$\{1\} \to A \to \pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma) \to F \to \{1\}.$$

But this implies $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is an infinitely generated quotient of the finitely generated group Γ, a contradiction. Hence, $\pi_{\mathrm{O}(n)}(\Gamma)$ contains an element of infinite order. □

The following propositions give criteria for irreducibility in terms of the action of $\pi_{\mathrm{O}(n)}(\Gamma)$ on $\mathbb{R}^n$.

Proposition 7.2. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice, then Γ is weakly and algebraically irreducible if and only if $\pi_{\mathrm{O}(n)}(\Gamma)$ is not virtually contained in some $\mathrm{O}(n-1)$. In particular, if Γ is weakly irreducible, then no finite index subgroup of $\pi_{\mathrm{O}(n)}(\Gamma)$ fixes a 1-dimensional subspace of $\mathbb{R}^n$.*

The analogous result for $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices is as follows. We will prove both results simultaneously.

Proposition 7.3. *Let X be an irreducible locally finite $\mathrm{CAT}(0)$ polyhedral complex and let $A = \mathrm{Aut}(X)$ act cocompactly and minimally. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice, then Γ is weakly and algebraically irreducible if and only if $\pi_{\mathrm{O}(n)}(\Gamma)$ is not virtually contained in some $\mathrm{O}(n-1)$. In particular, if Γ is weakly irreducible, then no finite index subgroup of $\pi_{\mathrm{O}(n)}(\Gamma)$ fixes a 1-dimensional subspace of $\mathbb{R}^n$.*

Proof of Proposition 7.2 and 7.3. Suppose Γ is reducible then Γ has a virtually normal $\mathbb{Z}$ subgroup. Clearly, $\pi_{O(n)}(\Gamma)$ virtually centralises this subgroup and so $\pi_{\mathrm{O}(n)}(\Gamma)$ must be virtually contained in some $\mathrm{O}(n-1)$.

Conversely, suppose $\pi_{\mathrm{O}(n)}(\Gamma)$ is virtually contained in some $\mathrm{O}(n-1)$. Passing to the corresponding finite index subgroup Λ we see that the action of Λ preserves two subspaces of $\mathbb{R}^n$. One isomorphic to $\mathbb{R}^{n-1}$ and one isomorphic to $R \cong \mathbb{R}$. Now, Λ splits as a graph of lattices in which every vertex and edge group has an infinite order generator which acts freely cocompactly on R and stabilises the subspace $R^{\perp}$ setwise via $\pi_{\mathrm{Isom}(\mathbb{E}^n)}$. The infinite cyclic groups intersect in some infinite cyclic subgroup $Z < \Lambda$. The stable letters of Λ must virtually centralise Z since otherwise they would map R into $R^{\perp}$. Thus, Z is virtually normal in Λ and hence Γ. By [CM19, Theorem 2(ii)] Γ is reducible. □

The following corollary is immediate.

Corollary 7.4. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^2) \times T)$-lattice, then Γ is an irreducible lattice if and only if $\pi_{\mathrm{O}(2)}(\Gamma)$ contains an element of infinite order.*

The following propositions give criteria for irreducibility in terms of the action of Γ on $\mathcal{T}$.

Proposition 7.5. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. Then Γ is weakly and algebraically irreducible if and only if Γ acts on $\mathcal{T}$ faithfully.*

The analogous result for $(\mathrm{Isom}(\mathbb{E}^n)\times A)$-lattices is as follows. We will prove both results simultaneously.

Proposition 7.6. *Let X be an irreducible locally finite* CAT(0) *polyhedral complex and let $A=\mathrm{Aut}(X)$ act cocompactly and minimally. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n)\times A)$-lattice, then Γ is weakly and algebraically irreducible if and only if Γ acts on X faithfully.*

Proof of Proposition 7.5 and 7.6. Assume Γ is irreducible. By [CM19, Corollary 3], Γ has finite amenable radical B. Such a non-trivial element $g\in B$ stabilises a vertex of the Bass-Serre tree $\mathcal{T}$ (resp. complex X). Now, either g has infinitely many conjugates which contradicts the finiteness of B, or g stabilises the whole of $\mathcal{T}$ (resp. X) and so is contained in $\Gamma\cap\mathrm{Isom}(\mathbb{E}^n)$. By Lemma 7.1 and Proposition 7.2 (Proposition 7.3) there is an infinite order element in $\pi_{O(n)}(\Gamma)$ and hence an infinite order element in $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ which does not commute with g. But now the normal closure of g in Γ must contained infinitely many conjugates of g. Hence, B is infinite, a contradiction. Thus, B must be trivial.

The converse in the tree case follows from Proposition 3.4. If Γ acts on X faithfully, then the projection $\pi_A(\Gamma)$ is non-discrete. By Theorem 2.4 it suffices to show $P=\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is non-discrete. Suppose P is discrete, then there is a finite index subgroup of P isomorphic to $Z=\mathbb{Z}^n$. But this is a virtually normal free abelian subgroup, so by [CM19, Theorem 2(ii)], Γ is reducible and so there is a finite index subgroup of Z which acts trivially on X, a contradiction. Thus, P is non-discrete and so Γ is weakly irreducible and by Theorem 2.4 algebraically irreducible. ☐

As an brief application we will construct (virtually) torsion-free irreducible $(\mathrm{Isom}(\mathbb{E}^n)\times T_{10})$-lattices.

Example 9. Recall the Leary-Minasyan group $\mathrm{LM}(A)$ where A is the matrix corresponding to the Pythagorean triple $(3,4,5)$ which acts on $\mathbb{E}^2\times\mathcal{T}_{10}$. (Note that these groups were classified up to isomorphism by Valiunas [Val20].) By [LM19], this has presentation

$$\mathrm{LM}(A)=\langle a,b,t\mid [a,b],\ ta^2b^{-1}t^{-1}=a^2b,\ tab^2t^{-1}=a^{-1}b^2\rangle.$$

Using this group we will construct a virtually torsion-free irreducible $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice where T is the automorphism group of the $10n$-regular tree for all $n\geqslant 3$.

Let $\mathbb{Z}^n=\langle a_0,\dots,a_{n-1}\rangle$ and let $F=\langle f\rangle$ be a cyclic group of order n acting on L by cyclically permuting the a_i. Let $L=\mathbb{Z}^n\rtimes F$, this is a crystallographic group and so embeds into $\mathrm{Isom}(\mathbb{E}^n)$. Now, consider the $(n\times n)$-matrix B given by

$$B=\begin{bmatrix} A & \mathbf{0}\\ \mathbf{0} & I_{n-2}\end{bmatrix}.$$

We define Γ_n to be the HNN extension of L by the matrix B, the Bass-Serre tree of this HNN extension will be regular of valence $10n$. This has generators $a_0,\dots,a_{n-1},f,t$ and relations

$$f^n = 1,\ [a_i, a_j] = 1,\ f a_i f^{-1} = a_{i+1 \pmod n},\ [a_2, t] = 1,\ \ldots,\ [a_{n-1}, t] = 1,$$
$$t a_0^2 a_1^{-1} t^{-1} = a_0^2 a_1,\ t a_0 a_1^2 t^{-1} = a_0^{-1} a_1^2,$$

where $i, j \in \{0, \ldots, n-1\}$. Here the first three sets of relation come from L, the relations $[a_i, t] = 1$ for $i \geqslant 2$ come from the fact B fixes $\{a_2, \ldots, a_{n-1}\}$ point-wise, and the last two relations arise from the action of B on $\langle a_0, a_1\rangle$. Now, let $a := a_0$, then we may write Γ_n as

$$\Gamma_n = \langle a, f, t \mid f^n = 1,\ t a^2 a^{-f} t^{-1} = a^2 a^f,\ t a (a^2)^f t^{-1} = a^{-1}(a^2)^f,\ [a^{f^i}, a^{f^j}] = 1\rangle$$

for $i, j \in \{0, \ldots, n-1\}$. Thus, Γ_n is a 3 generator, $\frac{1}{2}n(n-1) + 3$ relator group.

To see Γ_n is irreducible note that $\pi_{\mathrm{O}(n)}(\Gamma)$ is not virtually contained in some $\mathrm{O}(n-1) < \mathrm{O}(n)$. Indeed, consider the subgroup generated by the $\pi_{\mathrm{O}(n)}(f)$-orbit of $\pi_{\mathrm{O}(n)}(t)$. To show Γ_n is virtually torsion-free note that every torsion element of Γ_n has non-trivial image in $\pi_{\mathrm{O}(n)}(\Gamma_n)$. This is generated by the images of f and t and so is a finitely generated linear group and hence has a finite index torsion-free subgroup P_n. The preimage of P_n in Γ_n is torsion-free.

7.1. **Biautomaticity.** In this section we give a condition to determine the failure of biautomaticity for a CAT(0) group in the presence of a non-trivial de Rham factor.

For the rest of this section we fix the following notation and terminology, the treatment roughly follows [LM19, Section 2] and [Eps+92, Section 2.3, 2.5]. Let $\mathcal{A}$ be a finite set and let Γ be a group with a map $\mu : \mathcal{A} \to \Gamma$. We say that Γ is *generated by* $\mathcal{A}$ if the unique extension of μ to the homomorphism from the free monoid $\mathcal{A}^*$ to Γ is surjective. We will call elements of $\mathcal{A}^*$ *words* and for any $w \in \mathcal{A}^*$, if $\mu(w) = g$ for some $g \in \Gamma$, we will say *w represents g*. We will always assume $\mathcal{A}$ is closed under inversion, that is, there is an involution $i : \mathcal{A} \to \mathcal{A}$ such that $\mu(i(a)) = \mu(a)^{-1}$, in this case we will denote $i(a)$ as a^{-1}. Any *subset* $\mathcal{L} \subseteq \mathcal{A}^*$ will be called a *language over* $\mathcal{A}$.

An *automatic structure* for a group Γ is a pair $(\mathcal{A}, \mathcal{L})$, where $\mathcal{A}$ is a finite generating set of Γ equipped with a map $\mu : \mathcal{A} \to \Gamma$ and closed under inversion, and $\mathcal{A} \subseteq \mathcal{A}^*$ is a language satisfying three conditions. Firstly, $\mu(\mathcal{L}) = \Gamma$, secondly $\mathcal{L}$ is a *regular language*, that is, it is accepted by some finite state automaton, and thirdly, it satisfies a fellow traveller property (which we will not make precise here). We say $(\mathcal{A}, \mathcal{L})$ is *biautomatic structure* if both $(\mathcal{A}, \mathcal{L})$ and $(\mathcal{A}, \mathcal{L}^{-1})$ are automatic structures. A group Γ is said to be *automatic (resp. biautomatic)* if it admits an automatic (resp. biautomatic) structure.

A (bi)automatic structure is *finite-to-one* if $|\mu^{-1}(g) \cap \mathcal{A}| < \infty$ for all $g \in \Gamma$. As noted in [LM19, Page 8] by [Eps+92, Theorem 2.5.1] it may be assumed that all (bi)automatic structures are finite-to-one. So without loss of generality we will make this assumption and we will also suppose that all the automata in this paper have no dead states.

A subgroup $H < \Gamma$ is *$\mathcal{L}$-quasiconvex* if there exists $\kappa \geqslant 0$ such that for any path p in the Cayley graph of Γ with respect to $\mathcal{A}$, starting at 1_Γ, ending at some $h \in H$, and labelled by a word $w \in \mathcal{L}$, then every vertex of p lies in the κ-neighbourhood of H. The main examples of $\mathcal{L}$-quasiconvex subgroups are centralisers of finite subsets as proved in [GS91, Proposition 4.3] and [Eps+92, Theorem 8.3.1 and Corollary 8.3.5].

Theorem 7.7. *Let $X = \prod_{i=1}^m X_i$ be a product of proper irreducible* CAT(0) *spaces each not isometric to $\mathbb{E}$ and $H < \mathrm{Isom}(X)$ be a closed subgroup acting minimally and cocompactly on X. Let $n \geqslant 2$ and let Γ be an $(\mathrm{Isom}(\mathbb{E}^n) \times H)$-lattice. If the projection $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is not discrete, then Γ is not virtually biautomatic.*

Proof. Assume $(\mathcal{B}, \mathcal{L})$ is a biautomatic structure on Γ. By [CM19, Theorem 2(i)] there exists a commensurated free abelian subgroup $A \leqslant \Gamma$ acting properly on $\mathbb{E}^n$ of rank n.

Claim: *There is a finite index subgroup of A that is $\mathcal{L}$-quasiconvex.*

By the Flat Torus Theorem the rank of a maximal abelian subgroup of Γ is bounded by the rank of a maximal flat in $X \times \mathbb{E}^n$. Let F be such a flat acted on by A. Fix a set of generators S_A for A and a set of generators S containing S_A for the maximal abelian subgroup containing A stabilising F.

We may split X into a product $Y_1 \times Y_2$ where A acts trivially on Y_1 and non-trivially on Y_2. For $j = 1, 2$ let $K_j = \mathrm{Isom}(Y_j) \cap H$. Since, A acts trivially on Y_1 it follows A and $\Gamma \cap K_1$ commute. Now, Γ splits as a complex of $(\mathrm{Isom}(\mathbb{E}^n) \times K_1)$-lattices. In particular, A is a subgroup of a vertex group Γ_v, which is covirtually virtually isomorphic to $A \times K_v$, where K_v is a lattice in K_1. Define S_K to be a set of generators for K_v and for each $s \in S_K$ let $s' \in K_v$ be some element which does no commute with s. Define a set $S'_K = \{s, s' \colon s \in S_K\}$ and note that it is finite.

Let $N = \mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)})$. For each irreducible factor Z_j for $j = 1, \dots, \ell$ of Y_2 choose some element $g_j \in N < \Gamma$ which acts non-trivially on Z_j. Note the kernel N is non-empty since otherwise Γ would be a finitely generated linear group and hence residually finite, contradicting [CM09b, Theorem 2(iv)]. Now, we can choose such an element so that it centralises a finite index subgroup of A. Indeed, we may choose $g_j \in \langle\langle A \rangle\rangle \cap N$. Since A is commensurated g_j centralises $A^{g_j} \cap A$ a finite index subgroup of A. For each g_j pick another element g'_j which centralises a finite index subgroup of A and does not commute with g_j. Let $S_{Y_2} = \{g_j, g'_j \colon j = 1 \dots, \ell\}$ and note that it is finite. Let $A' = \left(\bigcap_{g \in S_{Y_2}} A^g\right) \cap A$, since this is the intersection of finitely many commensurable subgroups A' is a finite index subgroup of A. By construction A' is the centraliser of the finite set $S'_K \cup S_{Y_2} \cup S_A$. Thus, by [GS91, Proposition 4.3], A' is $\mathcal{L}$-quasiconvex. ◆

Now, by Lemma 7.1 there exists an element in $\bar{t} \in \pi_{O(n)}(\Gamma)$ with infinite order, let t denote a preimage of $\bar{t}$ in Γ. By [LM19, Corollary 5.4], there is a finite index subgroup $\Gamma^0 \trianglelefteqslant \Gamma$ such that every finitely generated subgroup of Γ^0 centralises a finite index subgroup of A. After passing to a suitable power we may assume $t^k \in \Gamma^0$. But $\langle t^k \rangle$ does not centralise a finite index subgroup of A, a contradiction. Hence, there is no biautomatic structure on Γ. Since the hypotheses on Γ pass to finite index subgroups, it follows Γ is not virtually biautomatic. □

The following corollary characterises the biautomaticity of $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattices.

Corollary 7.8. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{E}$ and let $T=\mathrm{Aut}(\mathcal{T})$. Let $n\geqslant 2$ and let Γ be a $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice. Then, Γ is virtually biautomatic if and only if Γ is uniform and the projection $\pi_{\mathrm{O}(n)}(\Gamma)$ is finite.*

Proof. Note that a non-uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice is not finitely generated and hence, not virtually biautomatic. Indeed, it must split as a graph of groups with infinitely many vertices since $\mathrm{Isom}(\mathbb{E}^n)$ does not have any non-uniform lattices. Thus, we may assume Γ is uniform. Now, if Γ is virtually biautomatic then by Theorem 7.7 $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is discrete and hence $\pi_{\mathrm{O}(n)}(\Gamma)$ is finite. Conversely, if $\pi_{\mathrm{O}(n)}(\Gamma)$ is finite then Γ virtually splits as $\mathbb{Z}^n\times F_r$ which is biautomatic. □

Example 10. The group Γ_n for each $n\geqslant 2$ constructed in Example 9 is an irreducible $(\mathrm{Isom}(\mathbb{E}^n)\times T_{10n})$-lattice that is not virtually biautomatic.

Remark 7.9. In light of M. Valiunas' result [Val21, Theorem 1.2] Theorem 7.7 can be strengthened to state that Γ does not embed into any biautomatic group. It may also be possible to simplify the proof using their result.

7.2. **Fibring.** In this section we characterise irreducible $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattices as those which do not virtually fibre. This result is new even for Leary-Minasyan groups.

Theorem 7.10. *Let $\mathcal{T}$ be a locally-finite leafless unimodular tree, not isometric to $\mathbb{R}$, and let $T=\mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice, then Γ virtually algebraically fibres if and only if Γ is reducible.*

Proof. If Γ is reducible, then Γ virtually splits as $\mathbb{Z}\times\Gamma'$, in which case Γ virtually fibres.

We will now prove every irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice does not algebraically fibre, this will prove the theorem since a finite index subgroup of an irreducible lattice is an irreducible lattice. Now, suppose Γ is an irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice. By Theorem 3.3, the group Γ splits as a graph of $\mathrm{Isom}(\mathbb{E}^n)$-lattices, and so is the fundamental group of a graph of groups with vertex and edge stabilisers finite-by-$\mathrm{Isom}(\mathbb{E}^n)$-lattices. By the same argument as in the claim of the proof of Theorem 4.10 we may assume Γ is a reduced graph of groups which does not split as an ascending HNN-extension.

Now, $H^1(\Gamma;\mathbb{Z})\otimes\mathbb{R}\cong H^1(\Gamma;\mathbb{Z})$ and by Proposition 7.11, for every character $\phi\in H^1(\Gamma;\mathbb{R})$ we see that ϕ restricted to a vertex or edge group is zero. Since Γ is the fundamental group of a reduced graph of groups, is not an ascending HNN extension, and ϕ vanishes on every edge group, we may apply [CL16, Proposition 2.5] to deduce that $\phi \notin \Sigma(\Gamma)$. Hence, Γ does not algebraically fibre. $\square$

Proposition 7.11. *Let $\mathcal{T}$ be a locally-finite leafless unimodular tree, not isometric to $\mathbb{R}$, let $T = \mathrm{Aut}(\mathcal{T})$, and let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. If Γ is irreducible, then $H^1(\Gamma; \mathbb{Z}) = H^1(\mathcal{T}/\Gamma; \mathbb{Z})$.*

The analogous result for $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices is as follows. We will prove both results simultaneously.

Proposition 7.12. *Let X be an irreducible locally finite $\mathrm{CAT}(0)$ polyhedral complex and let $A = \mathrm{Aut}(X)$ act cocompactly and minimally, and let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice. If Γ is algebraically irreducible, $H^1(\Gamma; \mathbb{Z}) = H^1(X/\Gamma; \mathbb{Z})$.*

Proof of Proposition 7.11 and 7.12. Let $\phi \in H^1(\Gamma; \mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z})$. Suppose ϕ is non-zero on some local group L, then after passing to a finite index subgroup the restriction of ϕ is non-zero on some subgroup isomorphic to $\mathbb{Z}^n$. In particular, ϕ defines a codimension 1 subgroup of $\mathbb{Z}^n$ contained in $\mathrm{Ker}(\phi)$. Moreover, after passing to a further finite index subgroup $L' \cong \mathbb{Z}^n$, by commensurability of the local groups, there is codimension 1 subgroup $K \cong \mathbb{Z}^{n-1}$ of L' which is contained in every local group. Now, the flat $\mathbb{R} \otimes K$ is an $(n-1)$-dimensional flat stabilised by $P = \pi_{\mathrm{O}(n)}(\Gamma)$, contradicting Proposition 7.2 (Proposition 7.3). Thus, every local group is contained in $\mathrm{Ker}(\phi)$.

The isomorphism now follows from applying the equivariant spectral sequence to the filtration of $\mathcal{T}$ or X by skeleta (see [Bro94, Chapter VII.7]). The previous paragraph shows that $E_2^{0,1} = 0$, thus $H^1(\Gamma; \mathbb{Z}) = E_2^{1,0} = E_\infty^{1,0} = H^1(X/\Gamma; \mathbb{Z})$. $\square$

7.3. **A characterisation.** We are now ready to prove the characterisation of irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices (Theorem B) from the introduction.

Theorem 7.13 (Theorem B)**.** *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. The following are equivalent:*

(1) Γ is a weakly irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice;
(2) Γ is irreducible as an abstract group;
(3) Γ acts on $\mathcal{T}$ faithfully;
(4) Γ does not virtually fibre;
(5) Γ is C^*-simple;
(6) and if $n = 2$, Γ is non-residually finite and not virtually biautomatic.

Proof. The equivalence of (1) and (2) is given by Theorem 2.4. The equivalence of (1) and (3) is given by Proposition 7.5. The equivalence of (1) and (4) is given by Theorem 7.10.

To see (1) and (3) imply (5), observe that by [CM19, Theorem 2(iv)] Γ is non-residually finite and so $\mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)})$ is infinite. Now, Γ satisfies the conditions of Theorem 4.7(2b) and so Γ is C^*-simple. If Γ is reducible, then Γ virtually splits as $\Lambda = \mathbb{Z}^k \times \Gamma'$ for some $1 \leqslant k \leqslant n$. In particular, Λ is not C^*-simple since Λ has non-trivial amenable radical. It follows that Γ is not C^*-simple. Thus, (5) is equivalent to (1).

Assume $n = 2$ and note, by Corollary 7.4, Γ is irreducible if and only if $\pi_{O(n)}(\Gamma)$ contains an infinite order element. It follows from [CM19, Theorem 2(iv)] that Γ is reducible if and only if Γ is residually finite. The equivalence of (1) and (6) is given by Corollary 7.8. $\square$

8. Products with Salvetti complexes

In this section we will adapt a construction of Horbez and Huang [HH20, Proposition 4.5] to extend actions from trees to Salvetti complexes. Horbez–Huang constructed an example of a non-uniform lattice acting on the universal cover of the Salvetti complex $\widetilde{S}_L$ provided L is not a complete graph. We generalise this to construct a tower of uniform lattices in $\mathrm{Aut}(\widetilde{S}_L)$ and with an additional hypothesis on L non-biautomatic lattices in $\mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Aut}(\widetilde{S}_L)$.

8.1. **Graph and polyhedral products.** Let K be a simplicial complex on the vertex set $[m] := \{1, \ldots, m\}$. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i) \mid i \in [m]\}$ be a collection of CW-pairs. The *polyhedral product* of $(\underline{X}, \underline{A})$ and K, is the space

$$(\underline{X}, \underline{A})^K := \bigcup_{\sigma \in K} \prod_{i=1}^{m} Y_i^\sigma \subseteq \prod_{i=1}^{m} X_i \quad \text{where} \quad Y_i^\sigma = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Let K be a simplicial complex on $[m]$ vertices. Let $\underline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_m\}$ be a set of discrete groups. The *graph product* of $\underline{\Gamma}$ and K, denoted $\underline{\Gamma}^K$ is quotient of the free product $*_{i \in [m]} \Gamma_i$ by the relations $[\gamma_i, \gamma_j] = 1$ for all $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ if i and j are connected by an edge in K. Let $B\underline{\Gamma} = \{B\Gamma_1, \ldots, B\Gamma_n\}$. The graph product $\underline{\Gamma}^K$ is the fundamental group of the polyhedral product $X = (B\underline{\Gamma}, *)^K$. Moreover, if K is a *flag complex*, i.e. every nonempty set of vertices which are pairwise connected by edges spans a simplex, then X is a $K(\underline{\Gamma}^K, 1)$ [Sta15, Theorem 1.1].

If every vertex group in a graph product $\underline{\Gamma}^L$ is $\mathbb{Z}$ then we call the group a *right-angled Artin group (RAAG)* and denote $\underline{\Gamma}^L$ by A_L. In this case we will identify the generating set of A_L with the vertex set VL of L. The polyhedral product $(S^1, *)^L$ is a classifying space for A_L, is referred to as the *Salvetti complex* for A_L and denoted by S_L. We denote the universal cover by $\widetilde{S}_L$.

8.2. **Extending actions over the Salvetti complex.** We will now adapt the construction of Horbez and Huang [HH20, Proposition 4.5] to extend actions from trees to Salvetti complexes and present some applications.

Construction 8.1. *Let L be a finite simplicial graph on vertices $\{v_1, \ldots, v_m\}$ and suppose $\langle v_1, \ldots, v_k \rangle = F_k < A_L$ is a free subgroup. Let Γ be a group acting on $\mathcal{T}_{2k}$ by isometries* *such that the action is label-preserving, then the action of* Γ *on* $\mathcal{T}$ *extends to an action of* $\widetilde{\Gamma}$ *on* $\widetilde{S}_L$ *by isometries. Moreover, if* Γ *is a* T_{2k}*-lattice then* $\widetilde{\Gamma}$ *is an* $\mathrm{Aut}(\widetilde{S}_L)$*-lattice.*

Proof. There is an isometric embedding $\mathcal{T}_{2k} \rightarrowtail \widetilde{S}_L$ with edges labelled by $\mathcal{V} = \{v_1, \dots, v_k\} \subseteq VL$. Define $\phi : A_L \twoheadrightarrow F_k$ by $v \mapsto 1$ unless $v \in \mathcal{V}$ and let $\pi : \widetilde{S}_L \to X$ be the covering space corresponding to $\mathrm{Ker}(\phi)$. Let Γ be a group acting on $\mathcal{T}_{2k}$ preserving the labelling, we want to extend the action of Γ on $\mathcal{T}_{2k}$ to an action on $\widetilde{S}_L$.

We may identify the vertex set of $\mathcal{T}_{2k}$ with the vertex set of X via the embedding of $\mathcal{T}_{2k} \rightarrowtail \widetilde{S}_L$. We orient each edge of $\widetilde{S}_L$ and endow X with the induced labelling and orientation. The 1-skeleton $X^{(1)}$ of X is obtained from $\mathcal{T}_{2k}$ by attaching to each vertex of $\mathcal{T}_{2k}$ a circle for each $v \in VL \backslash \mathcal{V}$.

Since Γ acts by isometries on $\mathcal{T}_{2k}$ label preservingly, it follows Γ acts by isometries on $X^{(1)}$ label preservingly and preserves the orientation of edges in $VL \backslash \mathcal{V}$. It follows the action extends to X. Let $\widetilde{\Gamma}$ be the group of lifts of all automorphisms in Γ, we have a short exact sequence

$$1 \longrightarrow \mathrm{Aut}(\pi) \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1.$$

We have $\widetilde{S}_L/\widetilde{\Gamma} = X/\Gamma$ so there is a bijection between the $\widetilde{\Gamma}$-orbits of $\widetilde{S}_L^{(0)}$ and the Γ-orbits of $\mathcal{T}_{2k}^{(0)}$. For a vertex $v \in X$, each lift of $g \in \mathrm{Stab}_\Gamma(\mathrm{v})$ fixes a unique vertex $\tilde{v} \in \widetilde{S}_L$. In particular, the cardinality of the vertex stabilisers is preserved. It follows from Serre's covolume formula that if Γ was a T_{2k}-lattice, then $\widetilde{\Gamma}$ is an $\mathrm{Aut}(\widetilde{S}_L)$-lattice. □

Proposition 8.2. *There is an ascending tower of lattices in* $T_4 = \mathrm{Aut}(\mathcal{T}_4)$ *with label preserving action.*

Proof. The groups will be index two subgroups of the HNN extensions constructed in [BK90, Example 7.4]. We describe them here for the convenience of the reader. Let $V_r = \{f : \mathbb{Z}_r \to \mathbb{Z}_2 : f \text{ a function}\} \cong \mathbb{Z}_2^r$ and $\alpha_r \in \mathrm{Aut}(V_r)$ by $\alpha_r(f)(i) = f(i+1)$. Let $W_r = \{f \in V_r : f(0) = 1\} \cong \mathbb{Z}_2^{r-1}$ and define Γ_r to be the HNN extension

$$\langle V_r, t \mid f^t = \alpha_r(f)\ \forall f \in W_r \rangle.$$

By [BK90, Proposition 7.6] the group Γ_r acts faithfully on $\mathcal{T}_4$ with quotient a loop (one vertex and one edge) and covolume $1/m^r$. Moreover, if $r|r'$ then $\Gamma_r \leqslant \Gamma_{r'}$ with index $m^{r'-r}$ and so for $r \geqslant 2$, the sequence $(\Gamma_{r^s})_{s\geqslant 1}$ is an infinite ascending chain in $\mathrm{Lat}_{\mathrm{u}}(\mathcal{T}_4)$.

Now, define $\phi : \Gamma_r \to \mathbb{Z}_2$ by $\phi(V_r) = 0$ and $\phi(t) = 1$. The kernel Λ_r is an index two subgroup which satisfies the same properties as Γ_r except now the quotient has fundamental domain the first barycentric subdivision of a loop (two vertices and two edges) and covolume $2/m^r$. □

Corollary 8.3. *Let L be a finite flag complex which is not a full simplex, then the automorphism group* $\mathrm{Aut}(\widetilde{S}_L)$ *of the universal cover of the Salvetti complex contains a tower of uniform lattices.*

Proof. Fix $r \geqslant 2$. We apply Construction 8.1 to the lattices Λ_{r^s} for $s \geqslant 1$ in the preceding proposition and obtain a sequence of lattices $\widetilde{\Lambda}_{r^s}$ in $\mathrm{Aut}(\widetilde{S}_L)$. The group $\widetilde{\Lambda}_{r^s}$ has two orbits of vertices, each stabilised by a group of order m^{r^s}, it follows from Serre's Covolume Formula that $\widetilde{\Lambda}_{r^s}$ has covolume equal to $2/m^{r^s}$. It remains to show that the inclusions $\Lambda_{r^s} \rightarrowtail \Lambda_{r^{s'}}$ induce inclusions $\widetilde{\Lambda}_{r^s} \rightarrowtail \widetilde{\Lambda}_{r^{s'}}$ for $s' < s$. Consider the covering space $\pi : \widetilde{S}_L \to X$ where X is as in Construction 8.1. Note that X and hence $\mathrm{Aut}(\pi)$ does not depend on r or s since each group acts with the same fundamental domain. In particular, as $\Lambda_{r^s} < \Lambda_{r^{s'}}$ we have $\widetilde{\Lambda}_{r^s} < \widetilde{\Lambda}_{r^{s'}}$ for $s < s'$. □

Theorem 8.4. *Let L be a finite simplicial graph on vertices $\mathcal{V} = \{v_1, \dots, v_m\}$ and suppose $\langle v_1, \dots, v_k \rangle = F_k < A_L$ is a free subgroup and that $\{v_1, \dots, v_k\} \subseteq \mathrm{Aut}(L) \cdot v_1$. Let X be a proper* $\mathrm{CAT}(0)$ *space and assume $H < \mathrm{Isom}(X)$ acts cocompactly and minimally.*

1. *Let Γ be a group acting on $\mathcal{T}_{2k}$ by isometries, then the action of Γ on $\mathcal{T}$ extends to an action of $\widetilde{\Gamma}$ on $\widetilde{S}_L$ by isometries.*
2. *If Γ is a uniform lattice in $H \times T_{2k}$, then $\widetilde{\Gamma}$ is a uniform lattice in $H \times \mathrm{Aut}(\widetilde{S}_L)$.*
3. *If in addition X is a* $\mathrm{CAT}(0)$ *polyhedral complex and Γ is an $(H \times T_{2k})$-lattice, then $\widetilde{\Gamma}$ is an $(H \times \mathrm{Aut}(\widetilde{S}_L))$-lattice.*
4. *If the projection of Γ to H (resp. T_{2k}) is non-discrete, then so is the projection of $\widetilde{\Gamma}$ to H (resp. $\mathrm{Aut}(\widetilde{S}_L)$).*

Proof. The proof of (1) is identical to Construction 8.1 except now we do not require the action to be label preserving on $\mathcal{T}_{2k}$. Indeed, the assumption that $\{v_1, \dots, v_k\} \subseteq \mathrm{Aut}(L) \cdot v_1$ implies there is an isometry of $\widetilde{S}_L$ that permutes the edges around any vertex of $\mathcal{T}_{2k}$ and so we can extend any action on $\mathcal{T}_{2k}$ to $\widetilde{S}_L$. ◆

The proof of (2) follows from taking the diagonal embedding $\widetilde{\Gamma} \rightarrowtail H \times \mathrm{Aut}(\widetilde{S}_L)$ and then noting that the quotient $(\widetilde{S}_L \times X)/\widetilde{\Gamma}$ is compact and that cardinality of each of the vertex stabilisers is finite. ◆

We prove (3) in the same manner, noting the covolume on the product space is finite by Serre's Covolume Formula. ◆

The images of the projections of Γ and $\widetilde{\Gamma}$ to H coincide. Since any element of Γ which acts non-trivially on $\mathcal{T}_{2k}$ lifts to an element acting non-trivially on $\widetilde{S}_L$, the non-discreteness of $\pi_{T_{2k}}(\Gamma)$ implies the non-discreteness of $\pi_{\mathrm{Aut}(\widetilde{S}_L)}(\widetilde{\Gamma})$. This proves (4). ◆ □

Example 11. Applying the previous theorem to the Leary-Minasyan group $\mathrm{LM}(A)$ which acts irreducibly on the product of a 10-regular tree and $\mathbb{E}^2$ we obtain a lattice Γ in $\mathrm{Isom}(\mathbb{E}^2) \times \mathrm{Aut}(\widetilde{S}_L)$. Moreover, the projection to either factor is non-discrete. Thus, if $\widetilde{S}_L$

is irreducible, then Γ is algebraically irreducible by [CM09a]. By Theorem 7.7 the group Γ is not virtually biautomatic.

Recall that a group Γ is *quasi-isometrically rigid* if every group quasi-isometric to Γ is virtually isomorphic to Γ. The quasi-isometric rigidity of right angled Artin groups has received a lot of attention recently (see for instance [Hua18] and the references therein). The following corollary is immediate and appears to be new if L has no induced 4-cycle and A_L is not a free group.

Corollary 8.5 (Corollary C)**.** *Let L be a finite simplicial graph on vertices $\{v_1, \dots, v_m\}$. Suppose $\langle v_1, \dots, v_5\rangle = F_5 < A_L$ is a free subgroup and that $\{v_1, \dots, v_5\} \subseteq \mathrm{Aut}(L)\cdot v_1$. If A_L is irreducible, then there exists a weakly irreducible uniform lattice in $\mathrm{Aut}(\widetilde{S}_L) \times \mathrm{Isom}(\mathbb{E}^n)$ which is not virtually biautomatic nor residually finite. In particular, $A_L \times \mathbb{Z}^2$ is not quasi-isometrically rigid.*

Proof. The group Γ constructed in Example 11 is algebraically irreducible, non-residually finite, and quasi-isometric to $A_L \times \mathbb{Z}^2$. Both properties are virtual isomorphism invariants but $A_L \times \mathbb{Z}^2$ is algebraically reducible and residually finite. In particular, $A_L \times \mathbb{Z}^2$ is quasi-isometric to Γ but not virtually isomorphic to Γ and so cannot be quasi-isometrically rigid. □

Remark 8.6. It seems likely that one could take a polyhedral product of locally CAT(0) cube complexes over a flag complex and then repeat the above constructions to obtain towers of lattices in the automorphism group of the universal cover and more weakly irreducible lattices in mixed products.

9. From trees to right-angled buildings

In this section will show that the functors introduced by A. Thomas in [Tho06] take graphs of H-lattices with a fixed Bass-Serre tree to complexes of H-lattices whose development is a "sufficiently symmetric" right-angled building (we will make this precise later). Finally, we will combine these tools to construct a number of examples. In particular, non-residually finite $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices where A is the automorphism group of a sufficiently symmetric right-angled building, and non-residually finite algebraically irreducible lattices in products of arbitrarily many isometric and non-isometric sufficiently symmetric right-angled buildings.

9.1. **Right angled buildings.** Let (W, I) be a right-angled Coxeter system. Let N be the finite nerve of (W, I) and P' be the simplicial cone on N' with vertex x_0. A *right-angled building* of type (W, I) is a polyhedral complex X equipped with a maximal family of subcomplexes called *apartments*. Such an apartment is isometric to the Davis complex for (W, I) and the copies of P' in X are called *chambers*. Moreover, the apartments and chambers satisfy the axioms for a Bruhat–Tits building.

Let $\mathcal{S}$ denote the set of $J \subseteq I$ such that $W_J \leqslant W$ is finite. Note that $W_\varnothing = \{1\}$ so $\varnothing \in \mathcal{S}$. For each $i \in I$, the vertex P' of *type* $\{i\}$ will be called an *i-vertex*, and the union of the simplices of P' which contains the i-vertex but not x_0 will be called the *i-face* There is a one-to-one correspondence between the vertices of P' and the types $J \in \mathcal{S}$.

Let X be a right-angled building. A vertex of X has a type $J \in \mathcal{S}$ induced by the types of P'. For $i \in I$ an *$\{i\}$-residue* of X is the connected subcomplex consisting of all chambers which meet in a given i-face. The cardinality of the $\{i\}$-residue is the number of copies of P' in it.

Theorem 9.1 ([HP03])**.** *Let (W, I) be a right-angled Coxeter system and $\{q_i : i \in I\}$ a set of integers such that $q_i \geqslant 2$, then up to isometry there exists a unique building X of type (W, I) such that for each $i \in I$ the $\{i\}$-residue of X has cardinality q_i.*

If (W, I) is generated by reflections in an n-dimensional right-angled hyperbolic polygon P, then P' is the barycentric subdivision of P. Moreover, the apartments of X are isometric to $\mathbb{R}\mathbf{H}^n$. In this case we call X a *hyperbolic building.* We remark that a right-angled building can be expressed as the universal cover of a polyhedral product, however, we will not use this observation elsewhere.

Remark 9.2. Let (W, I) be a right-angled Coxeter system with parameters $\{q_i\}$ and nerve N. Let E_i be a set of size q_i and let CE_i denote the simplicial cone on E_i, denote the collections of these by $\underline{E}$ and $C\underline{E}$ respectively. The right-angled building of type (W, I) with parameters $\{q_i\}$ is the universal cover of the polyhedral product $(C\underline{E}, \underline{E})^N$.

9.2. **A functor theorem.** In this section we will recap a functorial construction of A. Thomas which takes graphs of groups with a given universal covering tree to complexes of groups with development a right-angled building. We will then show that this functor takes graphs of lattices to complexes of lattices and deduce some consequences.

Let X be a right-angled building of type (W, I) and parameters $\{q_i\}$ with chamber P'. Suppose $m_{i_1,i_2} = \infty$ and define the following two symmetry conditions due to Thomas [Tho06]:

(T1) There exists a bijection g on I such that $m_{i,j} = m_{g(i),g(j)}$ for all $i, j \in I$, and $g(i_1) = i_2$.

(T2) There exists a bijection $h : \{i \in I : m_{i_1,i} < \infty\} \to \{i \in I : m_{i_2,i} < \infty\}$ such that $m_{i,j} = m_{h(i),h(j)}$ for all i, j in the domain, $h(i_1) = i_2$, and for all i in the domain $q_i = q_{h(i)}$.

We include the construction adapted from [Tho06] for completeness and for utility in the proofs of the new results which will follow. An example of the construction for a graph of groups consisting of a single edge is given in Figure 2

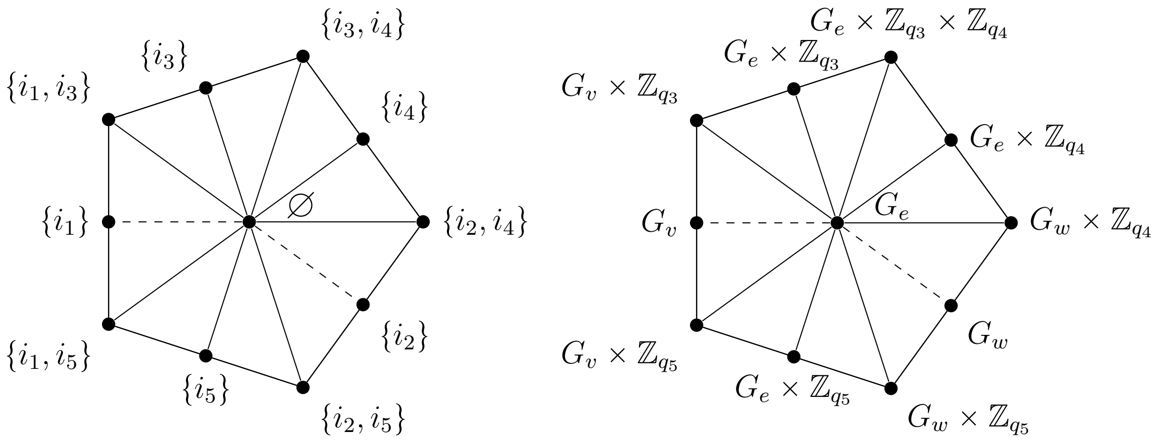

FIGURE 2. The left pentagon shows a labelling of the types $J \in \mathcal{S}$. The right pentagon shows the local groups after applying Thomas' functor to a graph of groups with a single edge. In both pentagons the dashed line shows the embedding of the graph. If the graph of groups has a single vertex, then $G_v = G_w$, $q_1 = q_2$, $q_3 = q_4$, the edge $(\{i_1, i_5\}, \{i_1\})$ is glued to $(\{i_2, i_5\}, \{i_2\})$, and the edge $(\{i_1, i_3\}, \{i_1\})$ is glued to $(\{i_2, i_4\}, \{i_2\})$.

Construction 9.3 (Thomas' Functor [Tho06])**.** *Let X be a right-angled building of type (W, I) and parameters $\{q_i\}$. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h. Then there is functor $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(X)$ preserving faithfulness and coverings.*

We will construct F as a composite $F_2 \circ F_1$. We first define $F_1 : \mathcal{G} \to \mathcal{C}_1$. Let $(A, \mathcal{A})$ be a graph of groups and $|A|$ the geometric realisation of A. We will construct a complex of groups $F_1(A)$ over $|A|$. For the objects we have:

- The local groups at the vertices of $|A|$ are the vertex groups of $\mathcal{A}$.
- For all $e \in EA$ let $\sigma_e = \sigma_{\overline{e}}$ be the vertex of the barycentric subdivision $|A|'$ at the midpoint of e.
- The local group at σ_e in $F_1(A)$ is $A_e = A_{\overline{e}}$.
- A monomorphism $\alpha_e : A_e \to A_{i(e)}$ in A induces the same monomorphism in $F_1(A)$.

Let $\phi : A \to B$ be a morphism of graphs of groups over a map of graphs f, note that by [Tho06, Proposition 2.1] F_1 is not injective on morphisms. We define $F_1(\phi)$ as follows:

- The map f induces a polyhedral map $f' : |A|' \to |B|'$ so we will define $F_1(\phi) : F_1(A) \to F_1(B)$ over f.
- Now take the morphisms on the local groups to be the same as for ϕ.

Let $\mathcal{C}(\mathcal{T}) = \mathrm{Im}(F_1(\mathcal{G}(\mathcal{T})))$ and $G(Y) \in \mathcal{C}(\mathcal{T})$. Now, we will define $F_2 : \mathcal{C}(\mathcal{T}) \to \mathcal{C}(X)$ as follows:

- We first embed Y' into a canonically constructed polyhedral complex $F_2(Y)$. For each $e \in EY$ let P'_e be a copy of P' and identify the midpoint of e with the cone vertex x_0 of P'_e.

- If Y is 2-colourable with colours i_1 and i_2 (from the valences of the Bass-Serre tree if $q_{i_1} \neq q_{i_2}$), then we identify the vertex of e of type i_j with the i_j-vertex of P'_e.
- Suppose Y is not 2-colourable. If $e \in EY$ is not a loop in Y then identify one vertex of e with the i_1-vertex of P'_e and the other with the i_2-vertex. If e forms a loop then we attach P'_e/h (where h is the isometry from the assumption) and identify the vertex of e to the image of the i_1- and i_2-vertices of in P'_e/h.
- Glue together, either by preserving type on the i_1- and i_2-faces or by the isometry h, the faces of the the P'_e and P'_e/h whose centres correspond to the same vertex of Y. Let $F_2(Y)$ denote the resulting polyhedral complex.
- Note that $Y' \rightarrowtail F_2(Y)$ and that each vertex of $F_2(Y)$ has a unique type $J \in \mathcal{S}$ or two types J and $h(J)$ where $i_1 \in J \in \mathcal{S}$ and h is the isometry from the assumption.
- Fix the local groups and structure maps induced by the embedding of Y' in $F(Y)$. For each $i \in I$ let $G_i = \mathbb{Z}_{q_i}$ and for $J \subseteq I$ let $G_J = \prod_{j \in J} G_j$. For each $e \in EY$ let G_e be the local group at the midpoint of e.
- Let $J \in \mathcal{S}$ such that neither i_1 or i_2 are in J. The local group at a vertex of type J is $G_e \times G_J$. The structure maps between such local groups are the natural inclusions.
- Let $J \in \mathcal{S}$ and suppose $i_k \in J$ for one of $k = 1$ or $k = 2$. Since $m_{i_1,i_2} = \infty$ both i_1 and i_2 cannot be in J. Let F_e be the i_k-face of P'_e or the glued face of P'_e/h. The vertex of type J in P'_e or P'_e/h is contained in F_e. Let v be the vertex of Y identified with the centre of F_e and let G_v be the local group at v in $G(Y)$
- The local group at the vertex of type J is $G_v \times G_{J \setminus \{i_k\}}$. For each $J' \subset J$ with $i_k \in J'$ the structure map $G_v \times G_{J' \setminus \{i_k\}} \rightarrowtail G_v \times G_{J \setminus \{i_k\}}$ is the natural inclusion. For each $J' \subset J$ with $i_k \notin J'$ the structure map $G_e \times G_{J'} \rightarrowtail G_v \times G_{J \setminus \{i_k\}}$ is the product of the structure map $G_e \rightarrowtail G_v$ in $G(Y)$ and the natural inclusion.

Now, let $\phi : G(Y) \to H(Z)$ be a morphism in $\mathcal{C}(\mathcal{T})$ over a non-degenerate polyhedral map $f : Y \to Z$. We will define $F_2(\phi)$ as follows:

- If Y and Z are two colourable f extends to a polyhedral map $F_2(f) : F_2(Y) \to F_2(Z)$. Otherwise we use (T1) to construct $F_2(f)$.
- If $\tau \in VF(Y)$ then $G_\tau = G_\sigma \times G_J$ where σ is a vertex of Y'. The homomorphism of local groups $G_\sigma \times G_J \to H_{f(\sigma)} \times G_J$ is ϕ_σ on the first factor and the identity on the other factors.
- Let $a \in EF(Y)$. If ψ_a, the structure map along $a \in F_2(G(Y))$, has a structure map ψ_b from $G(Y)$ as its first factor, put $F_2(\phi)(b) = \phi(a)$. Otherwise set $F_2(\phi)(b) = 1$.

We will now show the functor takes graphs of lattices to complexes of lattices and deduce a number of consequences. Recall for a locally compact group H that Lat(H) denotes the (po)set of H-lattices and $\mathrm{Lat_u}$(H) denotes the (po)set of uniform H-lattices.

Theorem 9.4. *Let Y be a right-angled building of type (W, I) and parameters $\{q_i\}$ and let $A = \mathrm{Aut}(Y)$. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree and let $T = \mathrm{Aut}(\mathcal{T})$. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h, and let $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(Y)$ be Thomas' functor. Let X be a finite dimensional proper $\mathrm{CAT}(0)$ space and assume $H = \mathrm{Isom}(X)$ contains a cocompact lattice. The following conclusions hold:*

(1) If $G(\mathcal{T})$ is a graph of H-lattices, then $F(G(\mathcal{T}))$ is a complex of H-lattices.
(2) F induces an inclusion of sets $\mathrm{Lat_u(H \times T)} \rightarrowtail \mathrm{Lat_u(H \times A)}$.
(3) If Y is a $\mathrm{CAT}(0)$ polyhedral complex then F induces an inclusion of sets $\mathrm{Lat(H \times T)} \rightarrowtail \mathrm{Lat(H \times A)}$.

Let Γ be a uniform $(H \times T)$-lattice and let $F\Gamma$ be the corresponding $(H \times A)$-lattice.

(4) $\pi_T(\Gamma)$ is discrete if and only if $\pi_A(F\Gamma)$ is discrete. Moreover, $\pi_H(\Gamma) = \pi_H(F\Gamma)$.
(5) If Γ satisfies any of {algebraically irreducible, non-residually finite, not virtually torsion free}, then so does $F\Gamma$.

Proof. We first prove (1). We will first verify the conditions on the local groups and then construct a morphism to H. Let $(B, \mathcal{B}, \psi)$ be a graph of H-lattices and consider the image $L(Z)$ of $\mathcal{B}$ under F. Here $Z = F(B)$. Each local group in $L(Z)$ is of the form $G_\sigma \times G_J$ where G_σ is a local group in $\mathcal{B}$ and G_J is a finite product of finite cyclic groups. We have a morphism $\psi : \mathcal{B} \to H$ such that the image of each local group G_σ is an H-lattice and the restriction to G_σ has finite kernel. Thus, by construction the local groups in $L(Z)$ are commensurable in $\pi_1(L(Z))$. We define $F(\psi_\sigma)$ to be the composite $\psi|_{G_\sigma} \circ \pi_\sigma : G_\sigma \times G_J \twoheadrightarrow G_\sigma \to \psi(G_\sigma)$, thus commensurability of the images in H is immediate.

We will now deal with the edges. Note the twisting elements in $L(Z)$ are all trivial and the complex of groups H has all structure maps the identity. Let the structure maps in $L(Z)$ be denoted by λ_a for $a \in EZ'$ and the structure maps in $\mathcal{B}$ by α_e for $e \in EB$. The family of elements $(t_e)_{e \in EB}$ in the path group $\pi(\mathcal{B})$ are mapped under ψ to elements of $\mathrm{Comm}_H(\psi(G_\sigma))$ where G_σ is some local group. Now, let $a \in EZ'$, then by construction a either corresponds to a subdivision of an edge a in EB in which case we define $(F\psi)(a) = \psi(a)$. Or, a corresponds to a inclusion of local groups $G_\sigma \times G_{J'} \to G_\sigma \times G_J$, in which case we define $(F\psi)(a) = 1_H$.

It remains to verify the two edge axioms for a morphism. For each $a \in EZ'$ corresponding to the subdivision of an edge a in EB we have

$$\mathrm{Ad}((F\psi)(a)) \circ F(\psi_{i(a)}) = \mathrm{Ad}(\psi(a)) \circ \psi_{i(a)} \circ \pi_a = \psi_{t(a)} \circ \alpha_a \circ \pi_a = F(\psi_{t(a)}) \circ F(\alpha_a),$$

where π_a is the surjection $G_a \times G_J \twoheadrightarrow G_a$. For any other edge $a \in EZ'$ we have

$$\mathrm{Ad}((F\psi)(a)) \circ F(\psi_{i(a)}) = F(\psi_{i(a)}) \text{ and } F(\psi_{t(a)}) \circ \lambda_a = F(\psi_{i(a)}).$$

Finally, the other condition that $(F\psi)(ab) = (F\psi)(a)(F\psi)(b)$ for $(a,b) \in E^2Z'$ is verified trivially. Thus, $F(\mathcal{B}) = L(Z)$ is a complex of H-lattices. ◆

We will next prove (2). Let Γ be an $(H \times T)$-lattice. By Theorem 3.3, Γ splits as graph of H-lattices $\mathcal{B}$. Thus, by (1) we obtain a complex of H-lattices $F(\mathcal{B})$ with fundamental group Λ. By Theorem 6.2(1) it suffices to show that for each local group G_σ in $F(\mathcal{B})$ the kernel $K_\sigma = \mathrm{Ker}(\pi_H|_{FG_\sigma})$ acts faithfully on X. Now, K_σ is a direct product of $L_\sigma = \mathrm{Ker}(\pi_H|_{G_\sigma})$ with a direct product of cyclic groups G_J, where G_σ is a local group in $\mathcal{B}$. By construction G_J acts faithfully on X and by Theorem 3.3, K_σ acts faithfully on $\mathcal{T}$ whose automorphism group embeds into A. In particular, K_σ acts faithfully on X. ◆

We will next prove (3). We construct a complex of lattices as in the previous case. The proof for (3) is now identical once we have verified that covolume condition in Theorem 6.2(3). Let c denote the covolume of an $(H \times T)$-lattice Γ with associated graph of lattices $(B, \mathcal{B})$, this is given by the formula $c = \sum_{\sigma \in VA} \mu(\Gamma_\sigma) < \infty$. Now, every vertex of the complex $Z = F(B)$ has local group isomorphic to a finite extension of some Γ_σ. In particular we may bound $\sum_{\sigma \in Z} \mu(\Gamma_\sigma)$ by $\ell \times c$ where ℓ is the number of vertices in the finite Coxeter nerve of X. ◆

The proof of (4) follows from the proof of (1). ◆

The proof of (5) follows from either applying Theorem 2.4 to (4) (algebraically irreducible) or the fact $\Gamma \rightarrowtail F\Gamma$ and the properties of residual finiteness and virtual torsion-freeness are subgroup closed. ◆ □

9.3. **Examples and applications.** In this section we will detail some sample examples and applications of the functor theorem.

We can obtain a number of examples by applying Thomas' functor to any irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. This will give a non-biautomatic group acting properly discontinuously cocompactly on $\mathbb{E}^n \times X$ where X is a sufficiently symmetric right-angled building. More precisely, we have the following corollary:

Corollary 9.5 (General version of Corollary D)**.** *Let Y be a right-angled building of type (W, I) and parameters $\{q_i\}$ and let $A = \mathrm{Aut}(Y)$. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree and let $T = \mathrm{Aut}(\mathcal{T})$. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h and let $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(Y)$ be Thomas' functor. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice and suppose $\pi_{\mathrm{O}(n)}(\Gamma)$ is infinite, then $F\Gamma$ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice which is not virtually biautomatic nor residually finite. In particular, if Y is irreducible, then the direct product of a uniform A-lattice with $\mathbb{Z}^2$ is not quasi-isometrically rigid.*

Proof. By Theorem 9.4 $F\Gamma$ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice with a non-discrete projection to $\mathrm{O}(n)$. That $F\Gamma$ is not virtually biautomatic then follows from Theorem 7.7. The failure of quasi-isometric rigidity follows from the fact that the direct product of a uniform A lattice with $\mathbb{Z}^2$ is reducible, whereas, the weakly irreducible lattice is algebraically irreducible by Theorem 2.4 and so does not virtually split as a direct product of two infinite groups. In particular, the groups cannot by virtually isomorphic. □

Example 12. Let $\Gamma = \mathrm{LM}(A)$ where A is the matrix corresponding to the Pythagorean triple $(3,4,5)$. Recall the group acts on $\mathbb{E}^2 \times \mathcal{T}_{10}$. Let X be the right angled building whose Coxeter nerve is the regular pentagon and whose parameters are given by $q_1 = q_2 = 10$, $q_3 = q_4 = k$, and $q_5 = \ell$. Let A be the automorphism group of X and consider $F\Gamma$ the image of Γ under Thomas' functor F as in Figure 2. By Theorem 9.4, the group $F\Gamma$ is a non-residually finite $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice with non-discrete projections to both factors and is irreducible as an abstract group. Moreover, by the previous corollary, $F\Gamma$ is not virtually biautomatic.

We will now construct a presentation for $\Lambda_{k,\ell} := F\Gamma$. The group has generators a, b, x_3, x_4, x_5, t and relations

$$x_3^k = x_4^k = x_5^\ell = 1,\ [a,b],\ [a,x_3],\ [a,x_4],\ [a,x_5],\ [b,x_3],\ [b,x_4],\ [b,x_5],\ [x_3,x_4],$$
$$ta^2b^{-1}t^{-1} = a^2b,\ tab^2t^{-1} = a^{-1}b^2,\ tx_3t^{-1} = x_4, [t,x_5].$$

The following proposition shows the group is virtually torsion-free.

Proposition 9.6. *The group $\Lambda_{2,2}$ in Example 12 is virtually torsionfree. This is witnessed by the index 16 subgroup*

$$\Delta := \langle a,\ b,\ x_3tx_4t^{-1},\ x_3x_4t^{-2},\ (x_5x_3)^2,\ (x_5x_4)^2,\ t^{-1}x_3x_4t^{-1},\ (tx_5x_4t^{-1})^2\rangle.$$

Proof. The quotient $\Lambda_{2,2}/\Delta$ is isomorphic to $D_4 \times \mathbb{Z}_2$ which has order 16. By construction every torsion element of $\Lambda_{2,2}$ is conjugate to some power of x_3, x_4, x_5 or x_3x_4. Indeed, every torsion element is contained in a vertex or edge stabiliser of the action on the pentagonal building and acts trivially on $\mathbb{E}^2$. Each of these elements is mapped to a non-trivial element of $D_4 \times \mathbb{Z}_2$. In particular, the kernel Δ is torsion-free. □

Corollary 9.7. *The group Δ admits a presentation with 8 generators*

$$a,\ b,\ y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6$$

and 20 relations

$$[a,b],\ [a,y_4],\ [a,y_3],\ [b,y_3],\ [b,y_4],$$
$$a^{-2}b^{-1}y_6a^2by_6^{-1},$$
$$a^{-1}y_1ba^2y_1^{-1}b^{-1}a^{-1},$$
$$ba^{-1}by_1^{-1}b^{-2}ay_1,$$
$$y_6y_2^{-1}b^{-1}y_2y_6^{-1}y_2^{-1}by_2,$$
$$y_2y_6^{-1}y_2^{-1}a^{-1}y_2y_6y_2^{-1}a,$$
$$y_2^{-1}ab^{-1}ay_2y_5^{-1}a^{-1}ba^{-1}y_5,$$

$$y_5a^{-2}b^{-1}y_5^{-1}y_3y_5a^2by_5^{-1}y_3^{-1},$$
$$y_4y_5^{-1}y_3^{-1}y_5y_1^{-1}y_2y_6y_2^{-1}y_4^{-1}y_3y_1y_6^{-1},$$
$$y_5^{-1}ab^{-3}y_5y_6y_5^{-1}b^2a^{-1}by_5y_6^{-1},$$
$$y_5^{-1}ba^{-1}b^2y_5y_4y_5^{-1}b^{-1}ab^{-2}y_5y_4^{-1},$$
$$y_5^{-1}ab^{-3}y_5b^{-1}a^{-2}y_5^{-1}b^2a^{-1}by_5a^2b,$$
$$b^{-1}a^{-3}b^{-1}a^{-2}y_5^{-1}b^{-1}ab^{-3}a^2y_5ba^2b^{-3},$$
$$y_2^{-1}baby_2y_5^{-1}b^{-1}ab^{-2}y_5b^{-1}a^{-2}b^{-2}ay_5^{-1}b^{-1}ab^{-2}y_5b^{-1}a^{-2},$$
$$ay_5a^4ba^2b^2y_5^{-1}b^2a^{-1}b^3a^{-1}b^3a^{-1}by_5a^4b^2y_5^{-1}b^2,$$
$$y_3y_5y_4^{-1}y_6aba^3by_5^{-1}b^3a^{-1}y_5a^2by_5^{-1}b^2a^{-1}by_5y_6^{-1}y_4y_5^{-1}y_3^{-1}b^2ay_5ba^4by_5^{-1}b^2a^{-1}b$$

and the abelianization of Δ is isomorphic to $\mathbb{Z}_8^2 \oplus \mathbb{Z}^6$.

Remark 9.8. It follows immediately from the presentation of Δ that it and hence $\Lambda_{2,2}$ contain a subgroup isomorphic to $\mathbb{Z}^3$. For example $\langle a, b, y_3\rangle$ or $\langle a, b, y_4\rangle$. Note that this coincides with the dimension of a maximal flat in $X \times \mathbb{E}^2$. Since both groups have a commensurated abelian subgroup their L^2-cohomology vanishes (see Proposition 4.3).

Example 13. Let $n \geqslant 2$ and let Γ_n be the irreducible lattice constructed in Example 9 acting on $\mathbb{E}^n \times \mathcal{T}_{10n}$. Let X be a right angled building satisfying (T1) and (T2) with automorphism group A and parameters $\{q_i\}$ all equal to $10n$. Applying Thomas' functor and Theorem 9.4 to Γ_n we obtain a non-residually finite $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice with non-discrete projections to both factors. Moreover by Corollary 9.5, Γ_n is not virtually biautomatic.

We will now show the existence of non-residually finite lattices in arbitrary products of sufficiently symmetric isometric and non-isometric right-angled buildings. We note that Bourdon's "hyperbolization of Euclidean buildings" [Bou00, Section 1.5.2] can be used to construct weakly irreducible uniform lattices in products of hyperbolic buildings. We will provide a number of examples to show that the groups we construct here are distinct.

Corollary 9.9. *Let Γ be a weakly irreducible lattice in product of trees $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n$ such that $\mathcal{T}_k$ is (t_{k_1}, t_{k_2})-biregular. Let $X_1 \times \cdots \times X_n$ be a product of irreducible right angled buildings satisfying (T1) and (T2). Suppose X_k is of type (W_k, I_k), has parameters $\{t_{k_1}, t_{k_2}, q_{k_3}, \dots, q_{k_{n_k}}\}$ where $m_{k_{i_1}, k_{i_2}} = \infty$ and $A_k = \mathrm{Aut}(X_k)$. The lattice $\Lambda = F^n\Gamma$ obtained by applying Thomas' functor n times (once for each tree $\mathcal{T}_k$ corresponding to the building X_k) is a lattice in $A_1 \times \cdots \times A_n$, is weakly and algebraically irreducible, and is non-residually finite.*

Proof. Let $T_k = \mathrm{Aut}(\mathcal{T}_k)$. The result follows from applying Theorem 9.4 n times as follows. Consider Γ as a graph of $(T_2 \times \cdots \times T_n)$-lattices and apply F to obtain a $(A_1 \times T_2 \times \cdots \times T_n)$-lattice with the desired properties (non-residual finiteness follows from the fact that the projection to $T_2 \times \cdots \times T_n$ has a non-trivial kernel). Now, consider $F\Gamma$ as a graph of $(A_1 \times T_3 \times \dots T_n)$-lattices and proceed by induction on the index k. $\square$

Examples 14. We will detail three examples:

(1) In [RSV19, Theorem 2.27, Theorem 3.15] the authors construct infinite series of explicit examples of irreducible S-arithmetic quaternionic lattices acting simply transitively on the vertices of products of $n \geqslant 1$ trees of constant valency, in each case we may apply Theorem 9.9 to obtain algebraically and weakly irreducible non-residually finite uniform lattices acting on a product of n buildings. It is unclear whether these groups are related to the groups constructed by Bourdon's hyperbolization.

(2) In [BM00b; BM97] Burger and Mozes construct for each pair of sufficiently large even integers (m, n) a finitely presented simple group as a uniform lattices in a product of trees $\mathcal{T}_m \times \mathcal{T}_n$ (for more examples see [Rat07b; Rat07a; Rad20]). Applying Theorem 9.9, we obtain uniform non-residually finite algebraically and weakly irreducible lattices acting on a product of buildings $X_1 \times X_2$ each satisfying (T1) and (T2) with X_1 having some parameters equal to m and X_2 having some parameters equal to n.

(3) Applying Theorem 9.9 to the non-uniform lattices in products of arbitrarily many trees constructed in Example 7 yields weakly irreducible non-uniform lattices in products of arbitrarily many sufficiently symmetric right-angled buildings.

10. Some questions

In this section we will raise a conjecture and some questions left open by this paper. In light of the results in Section 4.4 showing that many CAT(0) groups are autostackable (in particular the Leary-Minasyan groups) we raise the following conjecture:

Conjecture 10.1. *Every* CAT(0) *group is autostackable.*

In every example of an $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice known to the author, the lattice is virtually torsion-free.

Question 10.2. *Are there non-virtually torsion-free* $(\mathrm{Isom}(\mathbb{E}^n) \times T)$*-lattices?*

Since it is possible to characterise $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice in terms of C^*-simplicity and virtual fibring, it would be interesting to recover the characterisation for complexes of $\mathrm{Isom}(\mathbb{E}^n)$-lattices.

Question 10.3. *Are the weakly irreducible non-biautomatic groups constructed in Section 8 and Section 9 C^*-simple? Do they virtually fibre?*

More generally we ask:

Question 10.4. *When is a* CAT(0) *lattice* C^**-simple?*

The characterisation of weakly irreducible $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattices (Theorem B) suggests the following question:

Question 10.5. *Can* C^**-simplicity and virtual fibring of a Leary-Minasyan group* $\mathrm{LM}(A)$ *be determined by properties of the matrix* A*?*

Finally, we remark that in [Hug21] the constructions in this paper were used by the author to construct an example of a hierarchically hyperbolic group which is not virtually torsion-free.

References

[AR09] Peter Abramenko and Bertrand Rémy. "Commensurators of some non-uniform tree lattices and Moufang twin trees". In: *Essays in geometric group theory.* Vol. 9. Ramanujan Math. Soc. Lect. Notes Ser. Ramanujan Math. Soc., Mysore, 2009, pp. 79–104.

[BFS14] Uri Bader, Alex Furman, and Roman Sauer. "Weak notions of normality and vanishing up to rank in L^2-cohomology". In: *Int. Math. Res. Not. IMRN* 12 (2014), pp. 3177–3189. ISSN: 1073-7928. DOI: 10.1093/imrn/rnt029.

[BFS19] Uri Bader, Alex Furman, and Roman Sauer. "An adelic arithmeticity theorem for lattices in products". In: *Math. Z.* 293.3-4 (2019), pp. 1181–1199. ISSN: 0025-5874. DOI: 10.1007/s00209-019-02241-9.

[BS06] Uri Bader and Yehuda Shalom. "Factor and normal subgroup theorems for lattices in products of groups". In: *Invent. Math.* 163.2 (2006), pp. 415–454. ISSN: 0020-9910. DOI: 10.1007/s00222-005-0469-5.

[Bas93] Hyman Bass. "Covering theory for graphs of groups". In: *J. Pure Appl. Algebra* 89.1-2 (1993), pp. 3–47. ISSN: 0022-4049. DOI: 10.1016/0022-4049(93)90085-8.

[BK90] Hyman Bass and Ravi Kulkarni. "Uniform tree lattices". In: *J. Amer. Math. Soc.* 3.4 (1990), pp. 843–902. ISSN: 0894-0347. DOI: 10.2307/1990905.

[BG02] Nadia Benakli and Yair Glasner. "Automorphism groups of trees acting locally with affine permutations". In: *Geom. Dedicata* 89 (2002), pp. 1–24. ISSN: 0046-5755. DOI: 10.1023/A:1014290316936.

[BNS87] Robert Bieri, Walter D. Neumann, and Ralph Strebel. "A geometric invariant of discrete groups". In: *Invent. Math.* 90.3 (1987), pp. 451–477. ISSN: 0020-9910. DOI: 10.1007/BF01389175. URL: https://doi.org/10.1007/BF01389175.

[Bou00] Marc Bourdon. "Sur les immeubles fuchsiens et leur type de quasi-isométrie". In: *Ergodic Theory Dynam. Systems* 20.2 (2000), pp. 343–364. ISSN: 0143-3857. DOI: 10.1017/S014338570000016X. URL: https://doi.org/10.1017/S014338570000016X.

[Bre+17] Emmanuel Breuillard et al. "C^*-simplicity and the unique trace property for discrete groups". In: *Publ. Math. Inst. Hautes Études Sci.* 126 (2017), pp. 35–71. ISSN: 0073-8301. DOI: 10.1007/s10240-017-0091-2.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature.* Vol. 319. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999, pp. xxii+643. ISBN: 3-540-64324-9. DOI: 10.1007/978-3-662-12494-9.

[BHH14] Mark Brittenham, Susan Hermiller, and Derek Holt. "Algorithms and topology of Cayley graphs for groups". In: *J. Algebra* 415 (2014), pp. 112–136. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2014.06.001.

[BHJ16] Mark Brittenham, Susan Hermiller, and Ashley Johnson. "Homology and closure properties of autostackable groups". In: *J. Algebra* 452 (2016), pp. 596–617. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2015.11.053.

[BHS18] Mark Brittenham, Susan Hermiller, and Tim Susse. "Geometry of the word problem for 3-manifold groups". In: *J. Algebra* 499 (2018), pp. 111–150. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2017.12.001.

[Bro94] Kenneth S. Brown. *Cohomology of groups.* Vol. 87. Graduate Texts in Mathematics. Corrected reprint of the 1982 original. Springer-Verlag, New York, 1994, pp. x+306. ISBN: 0-387-90688-6.

[BM97] Marc Burger and Shahar Mozes. "Finitely presented simple groups and products of trees". In: *C. R. Acad. Sci. Paris Sér. I Math.* 324.7 (1997), pp. 747–752. ISSN: 0764-4442. DOI: 10.1016/S0764-4442(97)86938-8.

[BM00a] Marc Burger and Shahar Mozes. "Groups acting on trees: from local to global structure". In: *Inst. Hautes Études Sci. Publ. Math.* 92 (2000), 113–150 (2001). ISSN: 0073-8301.

[BM00b] Marc Burger and Shahar Mozes. "Lattices in product of trees". In: *Inst. Hautes Études Sci. Publ. Math.* 92 (2000), 151–194 (2001). ISSN: 0073-8301.

[CL19] Pierre-Emmanuel Caprace and Adrien Le Boudec. "Bounding the covolume of lattices in products". In: *Compos. Math.* 155.12 (2019), pp. 2296–2333. ISSN: 0010-437X. DOI: 10.1112/s0010437x19007644.

[CM09a] Pierre-Emmanuel Caprace and Nicolas Monod. "Isometry groups of non-positively curved spaces: discrete subgroups". In: *J. Topol.* 2.4 (2009), pp. 701–746. ISSN: 1753-8416. DOI: 10.1112/jtopol/jtp027.

[CM09b] Pierre-Emmanuel Caprace and Nicolas Monod. "Isometry groups of non-positively curved spaces: structure theory". In: *J. Topol.* 2.4 (2009), pp. 661–700. ISSN: 1753-8416. DOI: 10.1112/jtopol/jtp026.

[CM12] Pierre-Emmanuel Caprace and Nicolas Monod. "A lattice in more than two Kac-Moody groups is arithmetic". In: *Israel J. Math.* 190 (2012), pp. 413–444. ISSN: 0021-2172. DOI: 10.1007/s11856-012-0006-3.

[CM19] Pierre-Emmanuel Caprace and Nicolas Monod. *Erratum and addenda to "Isometry groups of non-positively curved spaces: discrete subgroups".* 2019. arXiv: 1908.10216 [math.GR].

[Cap+19] Pierre-Emmanuel Caprace et al. "On the residual and profinite closures of commensurated subgroups". In: *Mathematical Proceedings of the Cambridge Philosophical Society* (2019), pp. 1–22. DOI: 10.1017/S0305004119000264.

[CL16] Christopher H. Cashen and Gilbert Levitt. "Mapping tori of free group automorphisms, and the Bieri-Neumann-Strebel invariant of graphs of groups". In: *J. Group Theory* 19.2 (2016), pp. 191–216. ISSN: 1433-5883. DOI: 10.1515/jgth-2015-0038.

[Cor+20] Nathan Corwin et al. "Autostackability of Thompson's group F". In: *J. Algebra* 545 (2020), pp. 111–134. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2019.04.004.

[Eps+92] David B. A. Epstein et al. *Word processing in groups.* Jones and Bartlett Publishers, Boston, MA, 1992, pp. xii+330. ISBN: 0-86720-244-0.

[Gab02] Damien Gaboriau. "Invariants l^2 de relations d'équivalence et de groupes". In: *Publ. Math. Inst. Hautes Études Sci.* 95 (2002), pp. 93–150. ISSN: 0073-8301. DOI: 10.1007/s102400200002.

[Gan12] Giovanni Gandini. "Bounding the homological finiteness length". In: *Bull. Lond. Math. Soc.* 44.6 (2012), pp. 1209–1214. ISSN: 0024-6093. DOI: 10.1112/blms/bds047.

[GR69] H. Garland and M. S. Raghunathan. "Fundamental domains for lattices in rank one semisimple Lie groups". In: *Proc. Nat. Acad. Sci. U.S.A.* 62 (1969), pp. 309–313. ISSN: 0027-8424. DOI: 10.1073/pnas.62.2.309.

[GR70] H. Garland and M. S. Raghunathan. "Fundamental domains for lattices in (R-)rank 1 semisimple Lie groups". In: *Ann. of Math. (2)* 92 (1970), pp. 279–326. ISSN: 0003-486X. DOI: 10.2307/1970838.

[GS91] S. M. Gersten and H. B. Short. "Rational subgroups of biautomatic groups". In: *Ann. of Math. (2)* 134.1 (1991), pp. 125–158. ISSN: 0003-486X. DOI: 10.2307/2944334.

[Gro93] M. Gromov. "Asymptotic invariants of infinite groups". In: *Geometric group theory, Vol. 2 (Sussex, 1991).* Vol. 182. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[Hae91] André Haefliger. "Complexes of groups and orbihedra". In: *Group theory from a geometrical viewpoint (Trieste, 1990).* World Sci. Publ., River Edge, NJ, 1991, pp. 504–540.

[Hae92] André Haefliger. "Extension of complexes of groups". In: *Ann. Inst. Fourier (Grenoble)* 42.1-2 (1992), pp. 275–311. ISSN: 0373-0956.

[HP03] Frédéric Haglund and Frédéric Paulin. "Constructions arborescentes d'immeubles". In: *Math. Ann.* 325.1 (2003), pp. 137–164. ISSN: 0025-5831. DOI: 10.1007/s00208-002-0373-x. URL: https://doi.org/10.1007/s00208-002-0373-x.

[Har07] Pierre de la Harpe. "On simplicity of reduced C^*-algebras of groups". In: *Bull. Lond. Math. Soc.* 39.1 (2007), pp. 1–26. ISSN: 0024-6093. DOI: 10.1112/blms/bdl014.

[HP11] Pierre de la Harpe and Jean-Philippe Préaux. "C^*-simple groups: amalgamated free products, HNN extensions, and fundamental groups of 3-manifolds". In: *J. Topol. Anal.* 3.4 (2011), pp. 451–489. ISSN: 1793-5253. DOI: 10.1142/S1793525311000659.

[HM18] Susan Hermiller and Conchita Martínez-Pérez. "HNN extensions and stackable groups". In: *Groups Geom. Dyn.* 12.3 (2018), pp. 1123–1158. ISSN: 1661-7207. DOI: 10.4171/GGD/467.

[HH20] Camille Horbez and Jingyin Huang. *Measure equivalence classification of transvection-free right-angled Artin groups.* 2020.

[Hua18] Jingyin Huang. "Commensurability of groups quasi-isometric to RAAGs". In: *Invent. Math.* 213.3 (2018), pp. 1179–1247. ISSN: 0020-9910. DOI: 10.1007/s00222-018-0803-3.

[Hug21] Sam Hughes. *Hierarchically hyperbolic groups, products of CAT(-1) spaces, and virtual torsion-freeness*. 2021. arXiv: 2105.02847 [math.GR].

[KK17] Mehrdad Kalantar and Matthew Kennedy. "Boundaries of reduced C^*-algebras of discrete groups". In: *J. Reine Angew. Math.* 727 (2017), pp. 247–267. ISSN: 0075-4102. DOI: 10.1515/crelle-2014-0111.

[KS16] Aditi Kar and Michah Sageev. "Ping pong on CAT(0) cube complexes". In: *Comment. Math. Helv.* 91.3 (2016), pp. 543–561. ISSN: 0010-2571. DOI: 10.4171/CMH/395.

[LM19] Ian J. Leary and Ashot Minasyan. *Commensurating HNN-extensions: non-positive curvature and biautomaticity*. 2019. arXiv: 1907.03515 [math.GR].

[Lüc02] Wolfgang Lück. *L^2-invariants: theory and applications to geometry and K-theory*. Vol. 44. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2002, pp. xvi+595. ISBN: 3-540-43566-2. DOI: 10.1007/978-3-662-04687-6.

[Mar91] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*. Vol. 17. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1991, pp. x+388. ISBN: 3-540-12179-X. DOI: 10.1007/978-3-642-51445-6.

[PST18] Henrik Densing Petersen, Roman Sauer, and Andreas Thom. "L^2-Betti numbers of totally disconnected groups and their approximation by Betti numbers of lattices". In: *J. Topol.* 11.1 (2018), pp. 257–282. ISSN: 1753-8416. DOI: 10.1112/topo.12056.

[Pow75] Robert T. Powers. "Simplicity of the C^*-algebra associated with the free group on two generators". In: *Duke Math. J.* 42 (1975), pp. 151–156. ISSN: 0012-7094.

[Rad20] Nicolas Radu. "New simple lattices in products of trees and their projections". In: *Canad. J. Math.* 72.6 (2020). With an appendix by Pierre-Emmanuel Caprace, pp. 1624–1690. ISSN: 0008-414X. DOI: 10.4153/s0008414x19000506.

[Rat07a] Diego Rattaggi. "A finitely presented torsion-free simple group". In: *J. Group Theory* 10.3 (2007), pp. 363–371. ISSN: 1433-5883. DOI: 10.1515/JGT.2007.028.

[Rat07b] Diego Rattaggi. "Three amalgams with remarkable normal subgroup structures". In: *J. Pure Appl. Algebra* 210.2 (2007), pp. 537–541. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2006.10.008.

[RSV19] Nithi Rungtanapirom, Jakob Stix, and Alina Vdovina. "Infinite series of quaternionic 1-vertex cube complexes, the doubling construction, and explicit cubical Ramanujan complexes". In: *Internat. J. Algebra Comput.* 29.6 (2019), pp. 951–1007. ISSN: 0218-1967. DOI: 10.1142/S0218196719500371.

[Ser71] Jean-Pierre Serre. "Cohomologie des groupes discrets". In: *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*. 1971, 77–169. Ann. of Math. Studies, No. 70.

[Ser03] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. Springer-Verlag, Berlin, 2003, pp. x+142. ISBN: 3-540-44237-5.

[Sta15] Mentor Stafa. "On the fundamental group of certain polyhedral products". In: *J. Pure Appl. Algebra* 219.6 (2015), pp. 2279–2299. ISSN: 0022-4049. DOI: 10.1016/j.jpaa.2014.09.001.

[Tho06] Anne Thomas. "Lattices acting on right-angled buildings". In: *Algebr. Geom. Topol.* 6 (2006), pp. 1215–1238. ISSN: 1472-2747. DOI: 10.2140/agt.2006.6.1215.

[Val20] Motiejus Valiunas. *Isomorphism classification of Leary-Minasyan groups*. 2020. arXiv: 2011.08143 [math.GR].

[Val21] Motiejus Valiunas. *Leary-Minasyan subgroups of biautomatic groups*. 2021. arXiv: 2104.13688 [math.GR].