# HIGHER TOPOLOGICAL COMPLEXITY OF HYPERBOLIC GROUPS

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Abstract. We prove for non-elementary torsion-free hyperbolic groups $\Gamma$ and all $r \geqslant 2$ that the higher topological complexity $\mathsf{TC}_r(\Gamma)$ is equal to $r\cdot\mathrm{cd}(\Gamma)$. In particular, hyperbolic groups satisfy the rationality conjecture on the $\mathsf{TC}$-generating function, giving an affirmative answer to a question of Farber and Oprea. More generally, we consider certain toral relatively hyperbolic groups.

## 1. Introduction

Let $r \geqslant 2$ be an integer. The higher (or sequential) topological complexity $\mathsf{TC}_r(X)$ of a path-connected space $X$ was introduced by Rudyak [Rud10], generalising Farber's topological complexity [Far03]. The motivation for these numerical invariants arises from robotics. They provide a measure of complexity for the motion planning problem in the configuration space $X$ with prescribed initial and final states, as well as $r-2$ consecutive intermediate states. More precisely, consider the path-fibration $p\colon X^{[0,1]} \to X^r$ that maps a path $\omega\colon [0,1] \to X$ to the tuple $(\omega(0), \omega(\frac{1}{r-1}), \dots, \omega(\frac{r-2}{r-1}), \omega(1))$. Then $\mathsf{TC}_r(X)$ is defined as the minimal integer $n$ for which $X^r$ can be covered by $n+1$ many open subsets $U_0, \dots, U_n$ such that $p$ admits a local section over each $U_i$. If no such $n$ exists, one sets $\mathsf{TC}_r(X) := \infty$. Note that $\mathsf{TC}_2(X)$ recovers the usual topological complexity.

Since the higher topological complexities are homotopy invariants, one obtains interesting invariants of groups $\Gamma$ by setting $\mathsf{TC}_r(\Gamma) := \mathsf{TC}_r(K(\Gamma,1))$, where $K(\Gamma,1)$ is an Eilenberg–MacLane space. The topological complexities $\mathsf{TC}_r(\Gamma)$ have been computed for several classes of groups (see e.g. [Far+19; FM20; Dra20] for $r = 2$, [FO19; AGO20; GGY16] for $r \geqslant 2$, and references therein). In a celebrated result of Dranishnikov [Dra20] (see also [FM20]), the topological complexity $\mathsf{TC}_2(\Gamma)$ of groups with cyclic centralisers, such as hyperbolic groups, was shown to equal $\mathrm{cd}(\Gamma\times\Gamma)$. Here cd denotes the cohomological dimension. We generalise this result to all higher topological complexities $\mathsf{TC}_r(\Gamma)$ for $r \geqslant 2$, as well as to a larger class of groups containing certain toral relatively hyperbolic groups. Recall that a collection $\{P_i \mid i \in I\}$ of subgroups of $\Gamma$ is called *malnormal*, if for all $g \in \Gamma$ and $i, j \in I$ we have $gP_ig^{-1} \cap P_j = \{e\}$, unless $i = j$ and $g \in P_i$.

**Theorem A.** *Let $r \geqslant 2$ and let $\Gamma$ be a torsion-free group with $\mathrm{cd}(\Gamma) \geqslant 2$. Suppose that $\Gamma$ admits a malnormal collection of abelian subgroups $\{P_i \mid i \in I\}$ satisfying $\mathrm{cd}(P_i^r) < \mathrm{cd}(\Gamma^r)$ such that the centraliser $C_\Gamma(g)$ is cyclic for every $g \in \Gamma$ that is not conjugate into any of the $P_i$. Then $\mathsf{TC}_r(\Gamma) = \mathrm{cd}(\Gamma^r)$.*

The preceding theorem was obtained for the case $r = 2$ by the second author in [Li21].

For a space $X$, the $\mathsf{TC}$-*generating function* $f_X(t)$ is defined as the formal power series

$$f_X(t) := \sum_{r=1}^{\infty} \mathsf{TC}_{r+1}(X) \cdot t^r.$$

The $\mathsf{TC}$-generating function of a group $\Gamma$ is set to be $f_\Gamma(t) := f_{K(\Gamma,1)}(t)$. Recall that a group $\Gamma$ is said to be *of type F* (or *geometrically finite*) if it admits a finite model for $K(\Gamma,1)$.

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Following [FO19], we say that a finite CW-complex $X$ (resp. a group $\Gamma$ of type $F$) satisfies the *rationality conjecture* if the $\mathsf{TC}$-generating function $f_X(t)$ (resp. $f_\Gamma(t)$) is a rational function of the form $\frac{P(t)}{(1-t)^2}$, where $P(t)$ is an integer polynomial with $P(1) = \mathrm{cat}(X)$ (resp. $P(1) = \mathrm{cd}(\Gamma)$). Here cat denotes the Lusternik–Schnirelmann category. While a counter-example to the rationality conjecture for finite CW-complexes was found in [FKS20], the rationality conjecture for groups of type $F$ remains open. It is known to hold, e.g. for abelian groups of type $F$, right-angled Artin groups, fundamental groups of closed orientable surfaces, and Higman's group (see [FO19, Section 8]). Our result extends the class of groups for which the rationality conjecture holds as follows.

**Corollary B.** *Let $\Gamma$ be a group as in Theorem A. If $\Gamma$ is of type $F$, then*
$$f_\Gamma(t) = \mathrm{cd}(\Gamma)\frac{(2-t)t}{(1-t)^2}.$$
*In particular, the rationality conjecture holds for $\Gamma$.*

As remarked by Farber and Oprea in [FO19, page 159], it is particularly interesting to determine the validity of the rationality conjecture for the class of hyperbolic groups. We answer their question in the affirmative.

**Corollary C.** *The rationality conjecture holds for torsion-free hyperbolic groups.*

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## 2. Background

We recall the notion of a classifying space for a family of subgroups (see e.g. [Lüc05]). A *family $\mathcal{F}$ of subgroups* of a group $G$ is a non-empty set of subgroups that is closed under conjugation and finite intersections. The family consisting only of the trivial subgroup is denoted by $\mathcal{TR}$. The family $\mathcal{F}\langle\mathcal{H}\rangle$ generated by a set of subgroups $\mathcal{H}$ is the smallest family containing $\mathcal{H}$. For a family $\mathcal{F}$ of subgroups of $G$ and a subgroup $H$ of $G$, we denote by $\mathcal{F}|_H$ the family $\{L \cap H \mid L \in \mathcal{F}\}$ of subgroups of $H$. A *classifying space $E_\mathcal{F}G$ for the family $\mathcal{F}$* is a terminal object in the $G$-homotopy category of $G$-CW-complexes with isotropy groups in $\mathcal{F}$. Note that a model for $E_{\mathcal{TR}}G$ is given by $EG$ and in particular, there exists a $G$-map $EG \to E_\mathcal{F}G$ that is unique up to $G$-homotopy.

Let $\mathcal{E} \subset \mathcal{F}$ be two families of subgroups of $G$. We say that $G$ satisfies condition $(M_{\mathcal{E}\subset\mathcal{F}})$ if every element in $\mathcal{F}\backslash\mathcal{E}$ is contained in a unique maximal element $M \in \mathcal{F}\backslash\mathcal{E}$, and that $G$ satisfies condition $(NM)_{\mathcal{E}\subset\mathcal{F}}$ if additionally $M$ equals its normaliser $N_G(M)$. We denote the Weyl group by $W_G(M) := N_G(M)/M$. The following proposition is a special case of a construction due to Lück and Weiermann [LW12, Corollary 2.8] stated in [Li21, Corollary 2.2].

**Theorem 2.1** (Lück–Weiermann)**.** *Let $G$ be a group and $\mathcal{E} \subset \mathcal{F}$ be two families of subgroups. Let $\{M_i \mid i \in I\}$ be a complete set of representatives for the conjugacy classes of maximal elements in $\mathcal{F}\backslash\mathcal{E}$.*

(i) *If $\mathcal{E} = \mathcal{TR}$ and $G$ satisfies condition $(M_{\mathcal{TR}\subset\mathcal{F}})$, then a model for $E_\mathcal{F}G$ is given by the following $G$-pushout*
$$\begin{array}{ccc} \coprod_{i\in I} G \times_{N_G(M_i)} E(N_G(M_i)) & \longrightarrow & EG \\ \downarrow & & \downarrow \\ \coprod_{i\in I} G \times_{N_G(M_i)} E(W_G(M_i)) & \longrightarrow & E_\mathcal{F}G; \end{array}$$

(ii) *If $G$ satisfies conditions $(M_{\mathcal{E}\subset\mathcal{F}})$ and $(NM_{\mathcal{E}\subset\mathcal{F}})$, then a model for $E_{\mathcal{F}}G$ is given by the following $G$-pushout*

$$\begin{array}{ccc}\coprod_{i\in I} G\times_{M_i} E_{\mathcal{E}|_{M_i}}M_i & \longrightarrow & E_{\mathcal{E}}G\\ \big\downarrow & & \big\downarrow\\ \coprod_{i\in I} G/M_i & \longrightarrow & E_{\mathcal{F}}G.\end{array}$$

The *$\mathcal{F}$-restricted orbit category $\mathcal{O}_{\mathcal{F}}G$* has $G$-sets $G/H$ with $H\in\mathcal{F}$ as objects and $G$-maps as morphisms. Let $A$ be an *$\mathcal{O}_{\mathcal{F}}G$-module*, that is a contravariant functor from the orbit category $\mathcal{O}_{\mathcal{F}}G$ to the category of modules. The equivariant cellular cohomology $H^*_G(X;A)$ of a $G$-CW-complex $X$ with isotropy groups in $\mathcal{F}$ is called *Bredon cohomology* [Bre67]. In particular, Bredon cohomology satisfies the Mayer–Vietoris axiom for $G$-pushouts.

The (higher) topological complexity $\mathsf{TC}_r(\Gamma)$ of a group $\Gamma$ for $r\geqslant 2$ can be characterised in terms of classifying spaces for families [FO19, Theorem 3.1], generalising a result of Farber–Grant–Lupton–Oprea [Far+19, Theorem 3.3] for the case $r=2$. Consider $G=\Gamma^r$ and let $\mathcal{D}$ be the family of subgroups that is generated by the diagonal subgroup $\Delta(\Gamma)\subset\Gamma^r$.

**Theorem 2.2** (Farber–Oprea)**.** *Let $\Gamma$ be a group and $r\geqslant 2$. Then $\mathsf{TC}_r(\Gamma)$ equals the infimum of integers $n$ for which the canonical $\Gamma^r$-map*

$$E(\Gamma^r)\to E_{\mathcal{D}}(\Gamma^r)$$

*is $\Gamma^r$-equivariantly homotopic to a $\Gamma^r$-map with values in the $n$-skeleton $E_{\mathcal{D}}(\Gamma^r)^{(n)}$.*

As a consequence [FO19, Theorem 5.1], a lower bound for $\mathsf{TC}_r(\Gamma)$ is given by the supremum of integers $n$ for which the canonical $\Gamma^r$-map $E(\Gamma^r)\to E_{\mathcal{D}}(\Gamma^r)$ induces a non-trivial map in Bredon cohomology

$$H^n_{\Gamma^r}(E_{\mathcal{D}}(\Gamma^r);A)\to H^n_{\Gamma^r}(E(\Gamma^r);A)$$

for some $\mathcal{O}_{\mathcal{D}}(\Gamma^r)$-module $A$.

## 3. Proofs

Fix an integer $r\geqslant 2$. Let $\Gamma$ be a group and $\Delta\colon\Gamma\to\Gamma^r$ be the diagonal map. For $\gamma=(\gamma_1,\dots,\gamma_{r-1})\in\Gamma^{r-1}$ and a subset $S\subset\Gamma$, we define the subgroup $H_{\gamma,S}$ of $\Gamma^r$ as

$$H_{\gamma,S}:=(\gamma_1,\dots,\gamma_{r-1},e)\cdot\Delta(C_\Gamma(S))\cdot(\gamma_1^{-1},\dots,\gamma_{r-1}^{-1},e).$$

Here $C_\Gamma(S)$ denotes the centraliser of $S$ in $\Gamma$. For $b\in\Gamma$, we write $H_{\gamma,b}$ instead of $H_{\gamma,\{b\}}$. Denote the element $\underline{e}:=(e,\dots,e)\in\Gamma^{r-1}$ and note that $H_{\underline{e},e}=\Delta(\Gamma)$. The elementary proof of the following lemma is omitted.

**Lemma 3.1.** *Let $\gamma=(\gamma_1,\dots,\gamma_{r-1}),\delta=(\delta_1,\dots,\delta_{r-1})\in\Gamma^{r-1}$ and $S,T\subset\Gamma$ be subsets. The following hold:*

(i) *For every $g=(g_1,\dots,g_r)\in\Gamma^r$ we have*

$$g\cdot H_{\gamma,S}\cdot g^{-1}=H_{\gamma',S'},$$

*where $\gamma'=(g_1\gamma_1g_r^{-1},\dots,g_{r-1}\gamma_{r-1}g_r^{-1})\in\Gamma^{r-1}$ and $S'=\{g_rsg_r^{-1}\in\Gamma\mid s\in S\}\subset\Gamma$;*

(ii) $H_{\gamma,S}\cap H_{\delta,T}=H_{\gamma,S\cup T\cup\{\delta_1^{-1}\gamma_1,\dots,\delta_{r-1}^{-1}\gamma_{r-1}\}}$;

(iii) $N_{\Gamma^r}(H_{\gamma,S})=\{(\gamma_1k_1h\gamma_1^{-1},\dots,\gamma_{r-1}k_{r-1}h\gamma_{r-1}^{-1},h)\in\Gamma^r\mid h\in N_\Gamma(C_\Gamma(S)),k_1,\dots,k_{r-1}\in C_\Gamma(C_\Gamma(S))\}$.

Let $\mathcal{F}_1 \subset \mathcal{D}$ be the families of subgroups of $\Gamma^r$ defined as

$$\mathcal{D} := \mathcal{F}\langle\{\Delta(\Gamma)\}\rangle;$$
$$\mathcal{F}_1 := \mathcal{F}\langle\{H_{\gamma,b} \mid \gamma \in \Gamma^{r-1}, b \in \Gamma\backslash\{e\}\}\rangle.$$

**Lemma 3.2.** *Let $\Gamma$ be a group as in Theorem A and let $\underline{e} = (e, \dots, e) \in \Gamma^{r-1}$. Then for $n = \operatorname{cd}(\Gamma^r)$ and every $\mathcal{O}_{\mathcal{D}}(\Gamma^r)$-module $A$, we have*

$$H^n_{\Gamma^r}(\Gamma^r \times_{H_{\underline{e},e}} E_{\mathcal{F}_1|_{H_{\underline{e},e}}}(H_{\underline{e},e}); A) = 0.$$

*Proof.* We have that conditions $(M_{\mathcal{TR}\subset\mathcal{F}_1|_{H_{\underline{e},e}}})$ and $(NM_{\mathcal{TR}\subset\mathcal{F}_1|_{H_{\underline{e},e}}})$ hold for the group $H_{\underline{e},e}$. To see this we follow the same argument as in [Li21, Lemma 3.5], using Lemma 3.1 instead of [Li21, Lemma 3.1]. By Theorem 2.1(ii), we obtain an $H_{\underline{e},e}$-pushout

$$\begin{tikzcd}
\coprod_{H_{\underline{e},b}\in\mathcal{M}} H_{\underline{e},e} \times_{H_{\underline{e},b}} E(H_{\underline{e},b}) \arrow[r] \arrow[d] & E(H_{\underline{e},e}) \arrow[d] \\
\coprod_{H_{\underline{e},b}\in\mathcal{M}} H_{\underline{e},e}/H_{\underline{e},b} \arrow[r] & E_{\mathcal{F}_1|_{H_{\underline{e},e}}}(H_{\underline{e},e}),
\end{tikzcd} \tag{1}$$

where $\mathcal{M}$ is a complete set of representatives of conjugacy classes of maximal elements in $\mathcal{F}_1|_{H_{\underline{e},e}}\backslash\mathcal{TR}$. Since $\operatorname{cd}(H_{\underline{e},e}) < n$ and $\operatorname{cd}(H_{\underline{e},b}) < n-1$ for $b \in \Gamma\backslash\{e\}$, the Mayer–Vietoris sequence for $H^*_{H_{\underline{e},e}}(-; A)$ applied to the pushout (1) yields the lemma. $\square$

*Proof of Theorem A.* Throughout the proof let $n = \operatorname{cd}(\Gamma^r)$. We show that the $\Gamma^r$-map $E(\Gamma^r) \to E_{\mathcal{D}}(\Gamma^r)$ induces a surjective map $H^n_{\Gamma^r}(E_{\mathcal{D}}(\Gamma^r); A) \to H^n_{\Gamma^r}(E(\Gamma^r); A)$ for every $\mathcal{O}_{\mathcal{D}}(\Gamma^r)$-module $A$. Then the theorem follows from Theorem 2.2 and the upper bound $\mathsf{TC}_r(\Gamma) \leqslant n$ (see e.g. [FO19, (4)]).

First, observe that condition $(M_{\mathcal{TR}\subset\mathcal{F}_1})$ holds and that moreover, for $\gamma \in \Gamma^{r-1}$ and $b \in \Gamma\backslash\{e\}$ there is an isomorphism $N_{\Gamma^r}(H_{\gamma,b}) \cong C_\Gamma(b)^r$. To see this we follow the same argument as in [Li21, Lemma 3.5], using Lemma 3.1 instead of [Li21, Lemma 3.1]. It follows that for every $\mathcal{O}_{\mathcal{D}}(\Gamma^r)$-module $A$, we have

$$H^n_{\Gamma^r}(\Gamma^r \times_{N_{\Gamma^r}(H_{\gamma,b})} E(N_{\Gamma^r}(H_{\gamma,b})); A) = 0.$$

By Theorem 2.1(i), we obtain a $\Gamma^r$-pushout

$$\begin{tikzcd}
\coprod_{H_{\gamma,b}\in\mathcal{M}} \Gamma^r \times_{N_{\Gamma^r}(H_{\gamma,b})} E(N_{\Gamma^r}(H_{\gamma,b})) \arrow[r] \arrow[d] & E(\Gamma^r) \arrow[d] \\
\coprod_{H_{\gamma,b}\in\mathcal{M}} \Gamma^r \times_{N_{\Gamma^r}(H_{\gamma,b})} E(W_{\Gamma^r}(H_{\gamma,b})) \arrow[r] & E_{\mathcal{F}_1}(\Gamma^r),
\end{tikzcd} \tag{2}$$

where $\mathcal{M}$ is a complete set of representatives of conjugacy classes of maximal elements in $\mathcal{F}_1\backslash\mathcal{TR}$. Applying the Mayer–Vietoris sequence for $H^*_{\Gamma^r}(-; A)$ to the pushout (2) shows that the induced map $H^n_{\Gamma^r}(E_{\mathcal{F}_1}(\Gamma^r); A) \to H^n_{\Gamma^r}(E(\Gamma^r); A)$ is surjective.

Second, conditions $(M_{\mathcal{F}_1\subset\mathcal{D}})$ and $(NM_{\mathcal{F}_1\subset\mathcal{D}})$ hold by the same argument as in [Li21, Lemma 3.2], using Lemma 3.1 instead of [Li21, Lemma 3.1]. By Theorem 2.1(ii), we obtain a $\Gamma^r$-pushout

$$\begin{tikzcd}
\Gamma^r \times_{H_{\underline{e},e}} E_{\mathcal{F}_1|_{H_{\underline{e},e}}}(H_{\underline{e},e}) \arrow[r] \arrow[d] & E_{\mathcal{F}_1}(\Gamma^r) \arrow[d] \\
\Gamma^r/H_{\underline{e},e} \arrow[r] & E_{\mathcal{D}}(\Gamma^r).
\end{tikzcd} \tag{3}$$

Lemma 3.2 and the Mayer–Vietoris sequence for $H^*_{\Gamma^r}(-; A)$ applied to the pushout (3) yield that the map $H^n_{\Gamma^r}(E_{\mathcal{D}}(\Gamma^r); A) \to H^n_{\Gamma^r}(E_{\mathcal{F}_1}(\Gamma^r); A)$ is surjective.

Together, the map $H^n_{\Gamma^r}(E_{\mathcal{D}}(\Gamma^r); A) \to H^n_{\Gamma^r}(E(\Gamma^r); A)$ is surjective for every $\mathcal{O}_{\mathcal{D}}(\Gamma^r)$-module $A$. This finishes the proof. □

*Proof of Corollary B.* For groups $\Gamma$ of type $F$, we have $\mathrm{cd}(\Gamma^r) = r \cdot \mathrm{cd}(\Gamma)$ by [Dra19, Corollary 2.5]. The result now follows from Theorem A. □

*Proof of Corollary C.* The result follows from Corollary B using the fact that torsion-free hyperbolic groups are of type $F$ (see e.g. [BH99, Corollary 3.26 on page 470]). □

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