EE20N Spring 2011
Diagnostic Quiz - Solutions

Take a deep breath, you will not be graded on this quiz. You should complete this quiz **then** look at the provided solutions. Students with solid prerequisites should be able to finish this quiz in about 15 minutes. If you have trouble, you may have a learning curve to catch up with the course, and if so you should contact your lab GSI or the professors.

1. (*0 points*) *Complex numbers*
Let

$$\begin{aligned} x =& 1 - i \\ y =& 2e^{i\frac{\pi}{3}} \end{aligned}$$

where $i = \sqrt{-1}$.

(a) $x - y = ?$

Answer. Addition and subtraction of complex numbers are easiest done in rectangular form.

$$\begin{aligned} y =& 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] \\ =& 1 + i\sqrt{3} \end{aligned}$$

$$\begin{aligned} x - y =& (1 - i) - (1 + i\sqrt{3}) \\ =& -i(1 + \sqrt{3}) \end{aligned}$$

(b) $xy = ?$

Answer. Multiplication and division of complex numbers are easiest done in polar form.

$$x = \sqrt{2}e^{-i\frac{\pi}{4}}$$

$$\begin{aligned} xy =& \sqrt{2}e^{-i\frac{\pi}{4}} \cdot 2e^{i\frac{\pi}{3}} \\ =& 2\sqrt{2}e^{i\left(-\frac{\pi}{4}+\frac{\pi}{3}\right)} \\ =& 2\sqrt{2}e^{i\frac{\pi}{12}} \end{aligned}$$

2. (*0 points*) *Calculus*

$$\int_{-1}^{1}\left[e^{2t} + \sin(t)\right]dt = ?$$

Answer. Because integrals are linear, the integral of a sum is a sum of integrals:

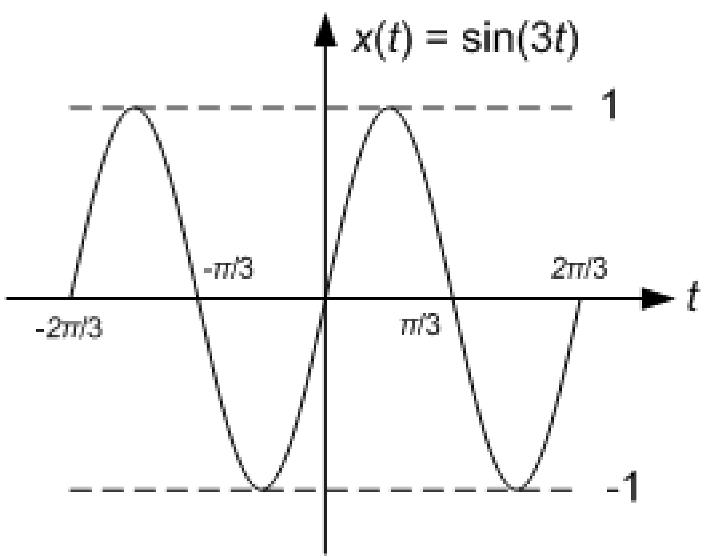
$$\begin{aligned}\int_{-1}^{1}\left[e^{2t}+\sin(t)\right]dt &= \int_{-1}^{1} e^{2t}dt + \int_{-1}^{1}\sin(t)dt \\ &= \left.\frac{e^{2t}}{2}\right|_{t=-1}^{1} - \cos(t)|_{t=-1}^{1} \\ &= \frac{e^2-e^{-2}}{2} - \cos(1)+\cos(-1) \\ &= \frac{e^2-e^{-2}}{2} + 0\end{aligned}$$

Note: Since $\sin(t)$ is an odd function being integrated over symmetrical bounds, as expected it integrates to zero.

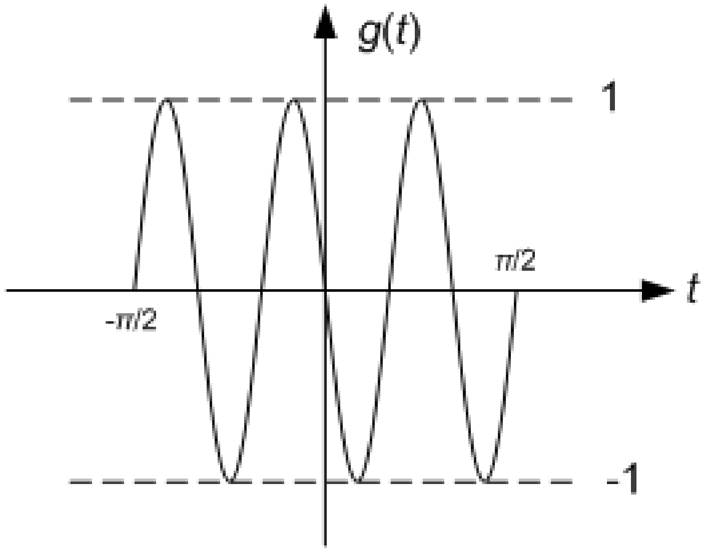
3. (*0 points*) *Composite functions*

If $x(t) = \sin(3t)$, plot $y(t) = x(-2t+2)$, labeling maxima, minima, and zero-crossings.

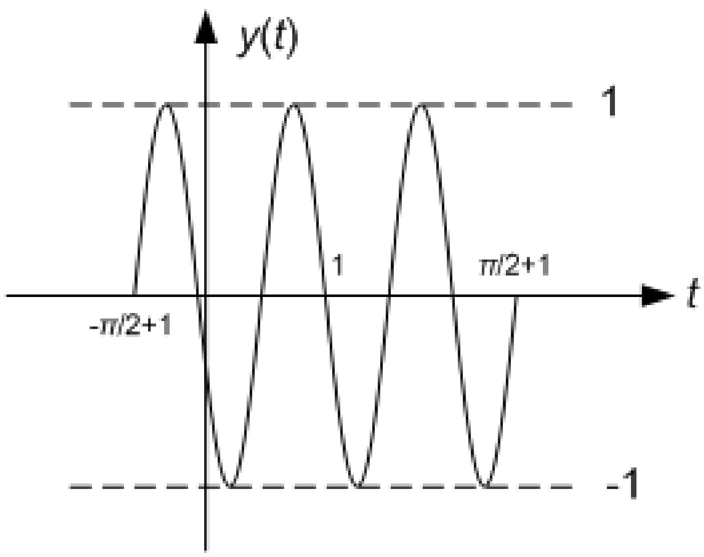
Answer. Note that we can decompose the transformation by $y(t) = x(-2(t-1))$, that is we can think of it as a flip / time-scaling then a shift. To illustrate this, here is the original $x(t)$:



Let $g(t) = x(-2t) = -\sin(6t)$. We flip the plot horizontally and compress it in t by a factor of 2 so the zero-crossing are every $\pi/6$:



Let $y(t) = g(t-1) = x(-2(t-1)) = -\sin(6(t-1))$. We shift the plot 1 to the right:



with zero-crossings at $1 + k\frac{\pi}{6}, k \in \mathbb{Z}$. Students are encouraged to try the same approach with the following: $h(t) = x(t+2), y(t) = h(-2t)$.

4. **(**$\mathit{0\ points}$**)** *Arithmetic with vectors*
Let

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ x_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

If

$$y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = a_1x_1 + a_2x_2 + a_3x_3$$

where a_i are scalars, find a_3. **More advanced:** Observe that x_1, x_2, and x_3 form an orthogonal basis for $\mathbb{R}^3$. Give a simple procedure for finding a_3 that leverages this fact.

Answer. This problem can be rewritten:

$$\begin{aligned} 1 =& 1 \cdot a_1 + 0 \cdot a_2 + 0 \cdot a_3 \\ 1 =& 0 \cdot a_1 + 1 \cdot a_2 - 1 \cdot a_3 \\ 0 =& 0 \cdot a_1 + 1 \cdot a_2 + 1 \cdot a_3 \end{aligned}$$

We don't actually need the 1st equation. By subtracting the 2nd equation from the 3rd, we get:

$$-1 = 2a_3 \Rightarrow a_3 = -\frac{1}{2}$$

However, this approach does not make use of the orthogonality property of $\{x_1, x_2, x_3\}$. Using the linear property of inner products:

$$\begin{aligned} \langle y, x_3 \rangle &= \langle a_1 x_1 + a_2 x_2 + a_3 x_3, x_3 \rangle \\ &= a_1 \langle x_1, x_3 \rangle + a_2 \langle x_2, x_3 \rangle + a_3 \langle x_3, x_3 \rangle \\ &= a_1 \cdot 0 + a_2 \cdot 0 + a_3 \|x_3\|^2 \end{aligned}$$

The above greatly simplifies because $\{x_1, x_2, x_3\}$ form an orthogonal basis. Therefore

$$\begin{aligned} a_3 =& \frac{\langle y, x_3 \rangle}{\langle x_3, x_3 \rangle} \\ =& \frac{-1}{2} \end{aligned}$$

In other words, we can find a_3 by projecting y onto x_3 with the proper normalization because $\{x_1, x_2, x_3\}$ form an orthogonal basis.

Also, this course will heavily involve writing mathematical proofs and short answer solutions, as well as using programming and debugging skills in lab. If you are shaky in the above, we strongly encourage you to closely work with the professors, GSIs, and your study group to develop these skills.