

# Discussion 6: Impulse Response, Convolution, and Feedback

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## 1 Impulse Response to Frequency Response

Recall, for an LTI system  $H$ , its frequency domain representation can be described by the frequency response  $H(\omega)$  and the time domain representation can be described by the impulse response  $h(n)$ . On the frequency side, if the input is  $x(n) = e^{i\omega n}$ , then the output is  $y(n) = H(\omega)e^{i\omega n}$ . At the same time on the time domain,

$$\begin{aligned}y(n) &= (h * x)(n) \\&= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\&= \sum_{k=-\infty}^{\infty} h(k)e^{i\omega(n-k)} \\&= \left( \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k} \right) e^{i\omega n}.\end{aligned}$$

Since  $k$  is merely a dummy variable for the summation, our result is a function only in  $\omega$ ,

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k},$$

where  $h(n)$  is the impulse response of a LTI system. For a simple sanity check, note that there is no  $n$  term in the expression for  $H(\omega)$ ; the frequency response of a system is completely independent of time. With this equation, the two are related by something called the Fourier Transform (which will be discussed later)!

## 2 Convolution

The delta function  $\delta(n)$  can be considered the identity function in signals and systems because when you convolve  $x(n)$  with  $\delta(n)$ , you get  $x(n)$ .

$$\begin{aligned}(\delta * x)(n) &= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \\&= \dots + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \dots \\&= x(n)\end{aligned}$$

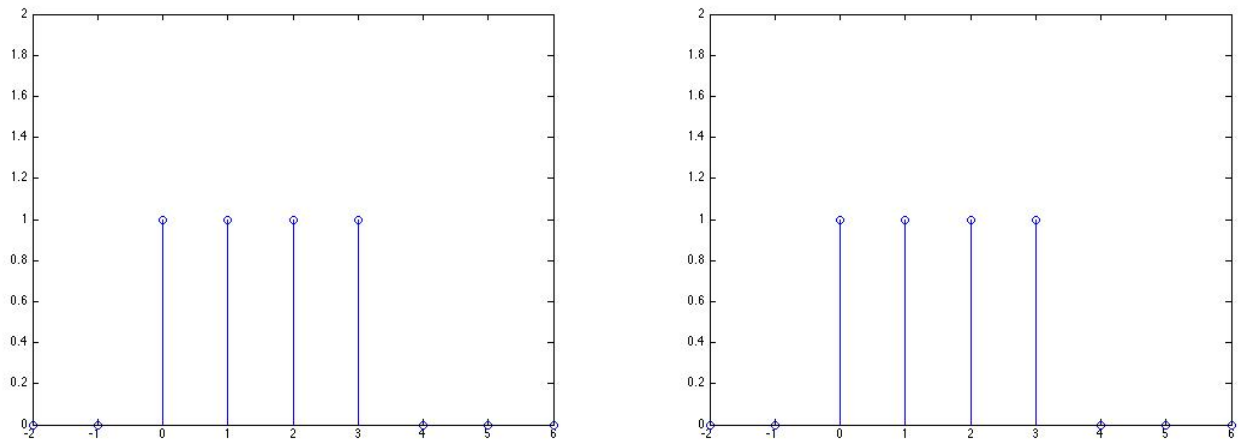
Similarly if you convolve  $x(n)$  with  $\delta(n-N)$ , you get  $x(n-N)$ .

Since when we put into  $\delta(n)$  into an LTI system and output  $h(n)$ , when we convolve  $x(n)$  with  $h(n)$ , we output  $y(n)$ .

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

\*Remember that convolution is associative.

Let's observe what happens when you convolve the two following signals.

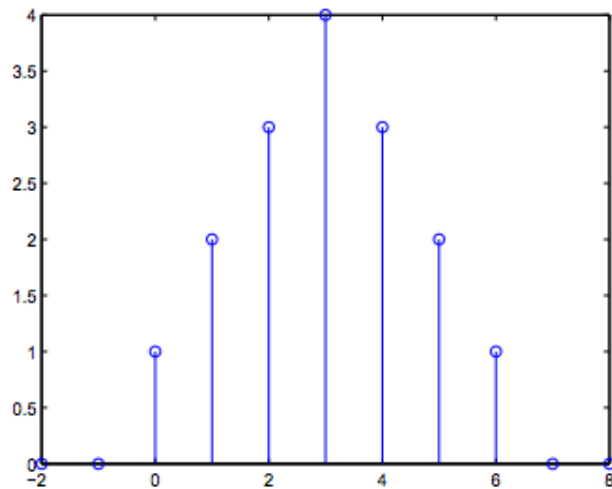


Let's treat the first one as the signal  $x(n)$  and the other as the impulse response  $h(n)$ . The impulse response  $h(n)$  can be expressed as

$$h(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)$$

Convoluting the two, we get

$$\begin{aligned} (h * x)(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= x(n) + x(n-1) + x(n-2) + x(n-3) \end{aligned}$$

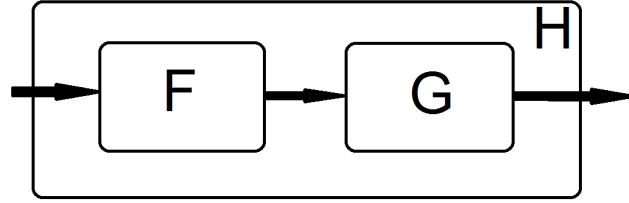


You can also apply the flip and shift method which will produce the same result.

How many nonzero components are there? 7  
In general, the number of nonzero components will be (length of  $x(n)$ ) + (length of  $h(n)$ ) - 1.

### 3 Cascade System

Here is an example of a cascade system.



Let the frequency response of system F be  $F(\omega)$  and its impulse response be  $f(n)$ . Similarly for G, Let the frequency response of system G be  $G(\omega)$  and its impulse response be  $g(n)$ .

Let,

$$f(n) = \frac{1}{2}(\delta(n) + \delta(n-1))$$

$$g(n) = \frac{1}{2}(\delta(n) - \delta(n-1))$$

Therefore,

$$F(\omega) = \frac{1}{2} + \frac{1}{2}e^{-i\omega}$$

$$G(\omega) = \frac{1}{2} - \frac{1}{2}e^{-i\omega}$$

Are F and G causal? Yes! Both of their impulse responses are 0 for all  $n < 0$ .

What are the frequency response  $H(\omega)$  and the impulse response  $h(n)$  of the overall system H?

$$\begin{aligned} h(n) &= (f * g)(n) \\ &= \frac{1}{4}(\delta(n) - \delta(n-2)) \end{aligned}$$

$$\begin{aligned} H(\omega) &= F(\omega) * G(\omega) \\ &= \frac{1}{4} - \frac{1}{4}e^{-i2\omega} \end{aligned}$$

Remember that this system is associative so

$$H(\omega) = F(\omega) * G(\omega) = G(\omega) * F(\omega)$$

$$h(n) = (f * g)(n) = (g * f)(n)$$

What is the  $|H(\omega)|$ ? More frequency response practice!

Remember, there are multiple ways to approach this

1. Geometric approach. Plot the vectors in the complex plane and see how the  $\omega$  affects the magnitude
2. Using Euler's identity. Convert the complex exponentials to cartesian form and using pythagorean theorem  $\sqrt{(\text{real part})^2 + (\text{image part})^2}$

$$H(\omega) = \frac{1}{4} - \frac{1}{4}(\cos(2\omega) - i \sin(2\omega))$$

$$|H(\omega)| = \sqrt{\left(\frac{1}{4} - \frac{1}{4}\cos(2\omega)\right)^2 + \left(-\frac{1}{4}\sin(2\omega)\right)^2}$$

$$\begin{aligned}
&= \sqrt{\frac{1}{16}(1 - 2\cos(2\omega) + \cos(2\omega)^2 + \sin(2\omega)^2)} \\
&= \frac{1}{4}\sqrt{(2 - 2\cos(2\omega))} \\
&= \frac{1}{4}\sqrt{4\sin(\omega)^2} \\
&= \frac{1}{2}\sin(\omega)
\end{aligned}$$

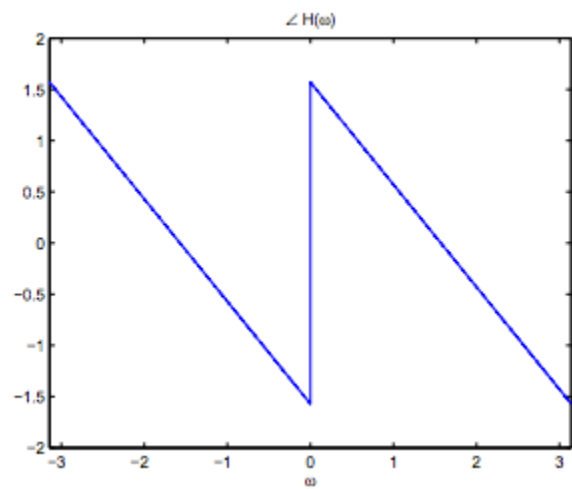
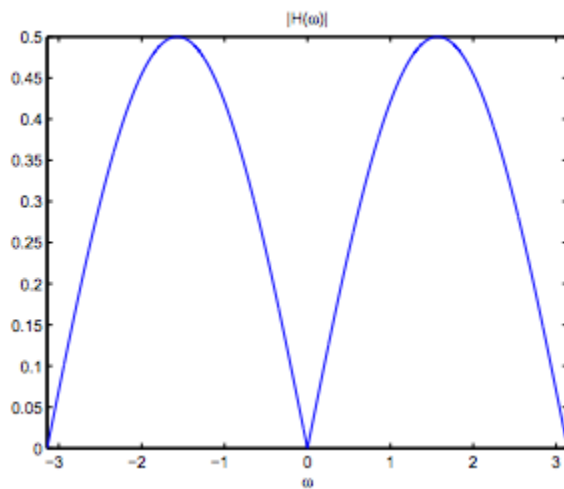
3. Finding the square of the magnitude  $|H(\omega)| = H(\omega)H^*(\omega)$  and square rooting it

$$\begin{aligned}
|H(\omega)| &= \left(\frac{1}{4} - \frac{1}{4}e^{-i2\omega}\right) * \left(\frac{1}{4} - \frac{1}{4}e^{i2\omega}\right) \\
&= \frac{1}{16}(1 - e^{i\omega} - e^{-i\omega} + 1) \\
&= \frac{1}{16}(2 - 2\cos(\omega)) \\
&= \frac{1}{4}\sin(\omega)^2 \\
|H(\omega)| &= \frac{1}{2}\sin(\omega)
\end{aligned}$$

4. half-angle trick

$$\begin{aligned}
H(\omega) &= \frac{1}{4} - \frac{1}{4}e^{-i2\omega} \\
&= \left(\frac{1}{4}e^{i\omega} - \frac{1}{4}e^{-i\omega}\right)e^{-i\omega} \\
&= \frac{1}{2}\sin(\omega)e^{-i\omega} \\
H(\omega) &= \frac{1}{2}\sin(\omega)
\end{aligned}$$

Be wary of this method however. It may seem the simplest, but it may not work for every frequency response.

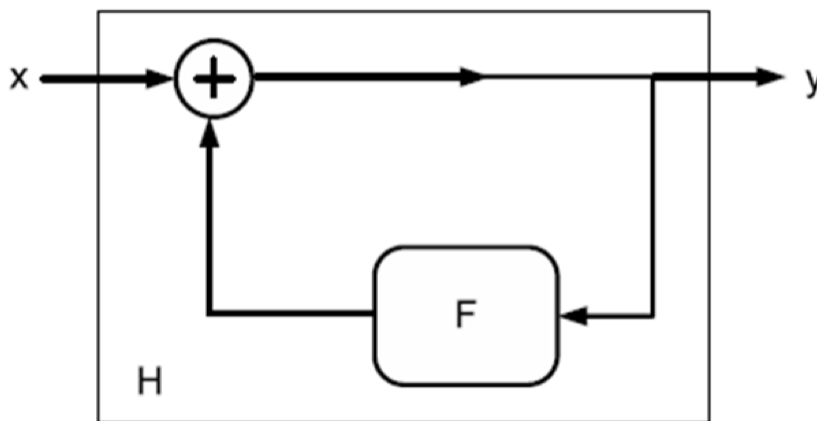


Notice how F is a low pass filter, G is a high pass filter, and H is a band-pass filter.

## 4 Feedback

Practice with Feedback!

Consider the following system where  $F(\omega) = \frac{1}{4}e^{-i2\omega}$



To find the frequency response of this system, we input  $e^{i\omega t}$  into the system.

$$e^{i\omega t} + F(\omega)H(\omega)e^{i\omega t} = H(\omega)e^{i\omega t}$$

\*Really understand how I derived this equation.

Much more difficult systems may occur in future problems and midterms

$$H(\omega) = \frac{1}{1 - F(\omega)}$$

$$H(\omega) = \frac{1}{1 - \frac{1}{4}e^{-i2\omega}}$$

What would be the LCCDE that corresponds to this system?

Recall from our previous relationship between  $H(\omega)$  and  $h(n)$ ,

$\delta(n)$  corresponds to 1

$h(n)$  corresponds to  $H(\omega)$

$h(n-1)$  corresponds to  $H(\omega)e^{-i\omega}$

$h(n-N)$  corresponds to  $H(\omega)e^{-iN\omega}$

Again, this relationship is described by the Fourier Transform which will be discussed soon.

If we can find the impulse response from the frequency response, then we can determine the LCCDE.

$$H(\omega) = \frac{1}{1 - \frac{1}{4}e^{-i2\omega}}$$

$$H(\omega) - \frac{1}{4}e^{-i2\omega}H(\omega) = 1$$

$$h(n) - \frac{1}{4}h(n-2) = \delta(n)$$

Therefore, the LCCDE is

$$y(n) - \frac{1}{4}y(n-2) = x(n)$$

Remember, feedback systems have infinite impulse responses, IIR. What is this impulse response? Assuming the system is causal, we know that  $h(n) = 0$  for  $n < 0$ .

$$h(0) = 1$$

$$h(1) = 0$$

$$h(2) = (0.25)$$

$$h(3) = 0$$

$$h(4) = (0.25)(0.25)$$

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Noticing a pattern, the impulse response  $h(n)$  is

$$\begin{cases} (0.25)^{n/2}, & \text{if } n \text{ is even and } n \geq 0 \\ 0, & \text{if } n \text{ is odd or } n < 0 \end{cases}$$