Discussion 10: Sampling, Nyquist Rate, Aliasing!

EE 20 Spring 2014
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0.1 Sampling

In order to process signals from the real world on a computer and output to the real world, there are three steps or more accurately "ingredients" that need to be followed:

1. Convert the CT signal into DT through sampling
2. Process the discrete time signal (perhaps put it through a filter or compress it)
3. Interpolate to convert the DT signal back into a CT signal

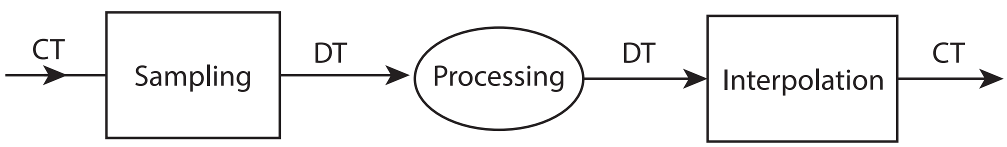


Figure 0.1: System that inputs and outputs CT signals and processes in DT

0.2 Aliasing

Aliasing is something bad can happen if you down sample too much. It means you can't reconstruct the original signal anymore. Hence, it is our worst enemy when it comes to sampling, which answers our question to #3. Examples where we have already seen aliasing are:

1. In the guest lecture by Mike Lustig, if we don't sample fast enough when we produce our MRI, it can distort the image, for example, cause a dark spot to form which could lead to a misdiagnosis.
2. In the airplane video we previously saw, it did not accurately capture the speed of the propeller
3. Image resizing: downsampling a high resolution image caused a distortion of the image

Consider the following signal $x(t)$. If we sample it at a rate of $f_s = 1/T$, we get

$$\text{If } x(t) = \cos 2\pi f t,$$
$$\text{Then } y(n) = x(nT) = \cos 2\pi f T n$$

$$
\begin{aligned}
\text{If } u(t) &= \cos 2\pi(f + Nf_s)t, \\
\text{Then } v(n) &= u(nT) \\
&= \cos 2\pi(f + Nf_s)Tn \\
&= \cos(2\pi fTn + 2\pi Nf_sTn) \\
&= \cos(2\pi fTn + 2\pi Nn) \\
&= \cos 2\pi fTn \\
&= y(n)
\end{aligned}
$$

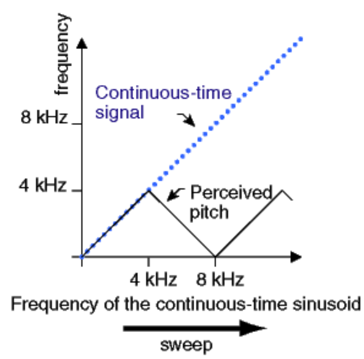


Figure 0.2: Aliasing explained

This high frequency signal got masqueraded as a low frequency signal!

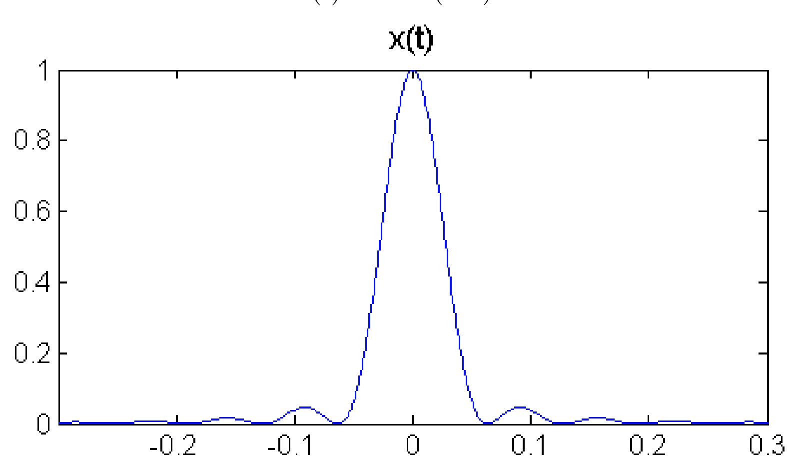
0.3 Aliasing

0.3.1 Sampling Theorem

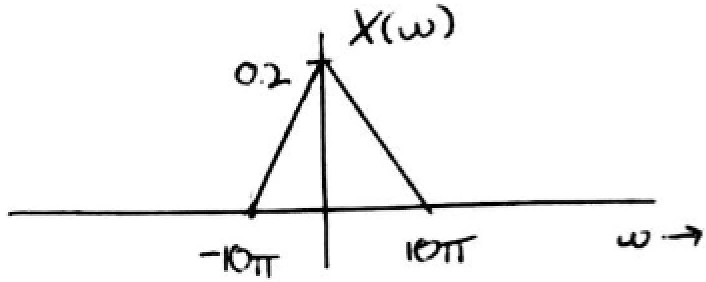
Theorem 0.1 *If $X(\omega)$ is bandlimited to BHz, then any $f_s > 2BHz$ is sufficient to avoid aliasing.*

Let's observe what happens when we sample below, at, and above the Nyquist rate. Consider the following signal:

$$x(t) = \text{sinc}^2(5\pi t)$$



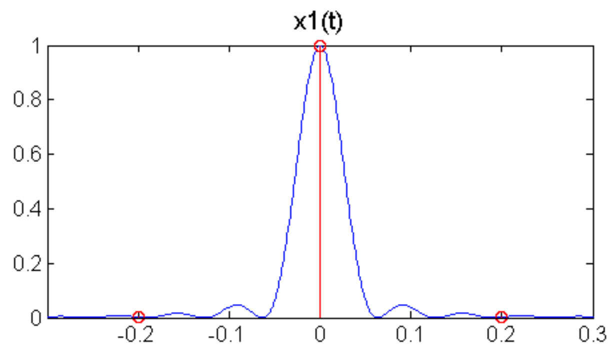
Its spectrum looks like

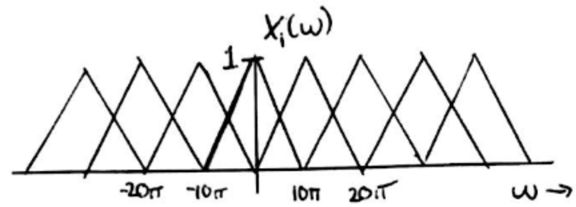


When we sample at a rate of $f_s = 1/T$, the resulting spectrum $y(n) = x(nT)$ in terms of $X(\omega)$ is

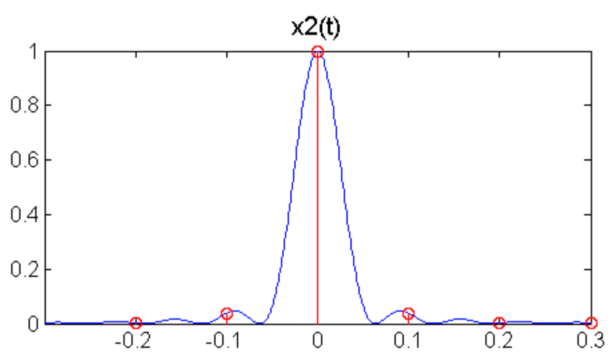
$$Y(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X((\omega - k2\pi f_s))$$

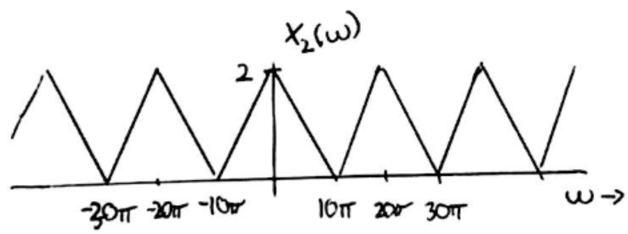
which is basically scaled versions of $X(\omega)$ repeating every $2\pi f_s$.
If we sample at a rate of $f_s = 5Hz$,



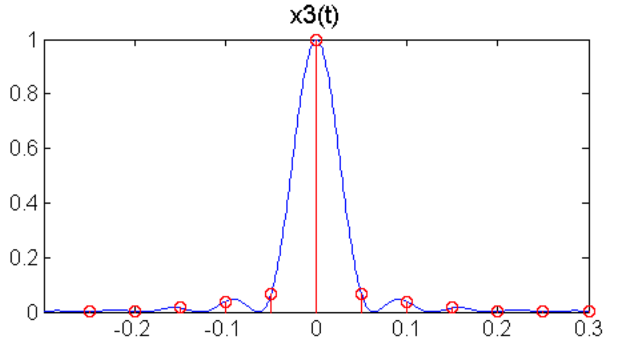


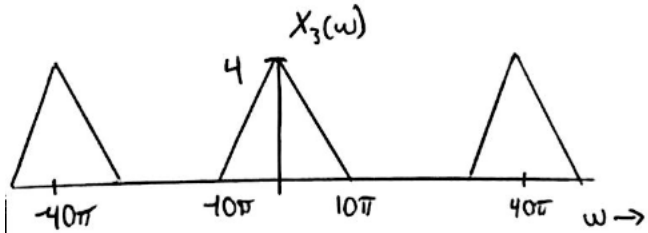
$f_s = 10Hz$





$f_s = 20Hz$

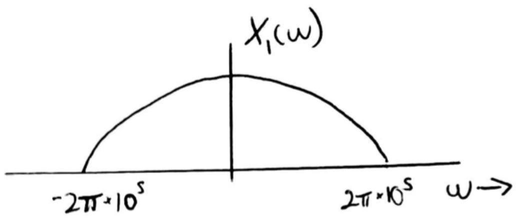


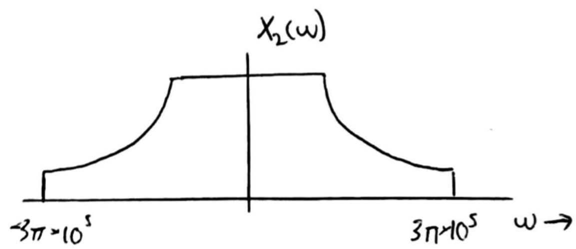


As you can see and observe, we need to sample at least twice the greatest frequency in order to be able to recover the original signal. It is easy to see if we observe that there is no overlap in the spectrum.

0.4 Nyquist Rates

The Fourier spectrums of two arbitrary signals, $x_1(t)$ and $x_2(t)$, are shown below:





Determine the Nyquist sampling rates for the following signals

(a) $\frac{dx_1(t)}{dt}$

Recall

$$\frac{dx_1(t)}{dt} \overset{\mathcal{FT}}{\longleftrightarrow} i\omega X_1(\omega)$$

The spectrum is not expanded or compressed. The Nyquist rate is the same as that of $x_1(t)$. $f_s = 2 * 10^5 Hz$.

(b) $x_2(t)\cos(2\pi ft)$

Multiplying by a cosine will shift the spectrum in both directions centered at $+/-\omega_0$. The resulting spectrum will be 0 for $|f| > f + 3/2 * 10^5 Hz$. Therefore, $f_s = 2f + 3 * 10^5 Hz$.

(c) $x_1^2(t)$

Again,

$$x_1(t)x_1(t) \overset{\mathcal{FT}}{\longleftrightarrow} \frac{1}{2\pi} X_1(\omega) * X_1(\omega)$$

The Nyquist rate is $f_s = 2 * 10^5 Hz$.

(d) $x_1^3(t)$

Using part c), we convolve the spectrum of $x_1^2(t)$ with $X_1(\omega)$. The Nyquist rate is $f_s = 3 * 10^5 Hz$.

(e) $x_1(t)x_2(t)$

Convolving $X_1(\omega)$ with $X_2(\omega)$, the resulting spectrum is 0 for $|\omega| > 2\pi * 10^5 + 3\pi * 10^5$. The Nyquist rate is $f_s = 5 * 10^5 Hz$.