Discussion 8: CTFT, DTFT, DFT!

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1 DFT

Definition 1 *The frequency coefficients of a finite discrete time signal, thought of as a signal, are called the DFT.*

DFT: For a signal $x(n) = x(0), x(1), \ldots, x(p-1)$ of duration p signals,

$$x(n) = \frac{1}{p}\sum_{m=0}^{p-1} X_m' e^{im\omega_0 n}, \qquad \text{where } \omega_0 = \frac{2\pi}{p} \qquad \text{(Synthesis)} \tag{1}$$

$$X_m' = \sum_{k=0}^{p-1} x(k) e^{-i\omega_0 mk} \qquad \text{(Analysis)} \tag{2}$$

The DFT can be thought of as sampled versions of the DTFT.

2 CTFT

The CTFT is a lot like the DTFT. In particular for a periodic signal, recall the FS representation:

$$x(t) = \sum_{m=-\infty}^{\infty} X_m e^{im\omega_0 t}, \qquad \text{where } \omega_0 = \frac{2\pi}{p} \qquad \text{(p is period)} \tag{3}$$

Where X_m's are FS coefficients for periodic signal $x_p(t)$. Letting $p \to \infty$ as we did before for the DTFT and replacing sums with integrals in continuous time, we will get precisely

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \tag{4}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \tag{5}$$

Unlike the DTFT, the CTFT is not 2π periodic.

3 Connecting them all together

Time domain properties	Finite/Periodic	Infinite Duration	
Continuous Time	FS	CTFT	Infinite Duration
Discrete Time	DFS/DFT	DTFT	Finite/Periodic
	Discrete Spectrum	Continuous Spectrum	Frequency domain properties

4 CTFT Transform Pairs

Here is a list of common Fourier Transform pairs:

$$
\begin{aligned}
x(t) &\overset{\mathcal{CTFT}}{\longleftrightarrow} X(\omega) \\
x(-t) &\overset{\mathcal{CTFT}}{\longleftrightarrow} X(-\omega) \\
\delta(t) &\overset{\mathcal{CTFT}}{\longleftrightarrow} 1 \\
\delta(t-\tau) &\overset{\mathcal{CTFT}}{\longleftrightarrow} e^{-i\omega\tau} \\
x(n-\tau) &\overset{\mathcal{CTFT}}{\longleftrightarrow} X(\omega)e^{-i\omega\tau} \\
x(t)e^{i\omega_0 t} &\overset{\mathcal{CTFT}}{\longleftrightarrow} X(\omega-\omega_0) \\
\alpha x_1(t)+\beta x_2(t) &\overset{\mathcal{CTFT}}{\longleftrightarrow} \alpha X_1(\omega)+\beta X_2(\omega) \\
x_1(t)x_2(t) &\overset{\mathcal{CTFT}}{\longleftrightarrow} \frac{1}{2\pi}X_1(\omega)*X_2(\omega) \\
x_1(t)*x_2(t) &\overset{\mathcal{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega) \\
x(t)=e^{-at}u(t)) \text{ where } a>0 &\overset{\mathcal{CTFT}}{\longleftrightarrow} X(\omega)=\frac{1}{a+j\omega} \\
x(at) &\overset{\mathcal{CTFT}}{\longleftrightarrow} X(\omega)=\frac{1}{|a|}X(\frac{\omega}{a}) \\
x(t)=\begin{cases}1 & |t|\le\frac{1}{2}\\ 0 & \text{otherwise}\end{cases} &\overset{\mathcal{CTFT}}{\longleftrightarrow} X(\omega)=\frac{\sin(\omega/2)}{\omega/2} \\
x(t)=\frac{\sin(Wt)}{\pi t} &\overset{\mathcal{CTFT}}{\longleftrightarrow} X(\omega)=\begin{cases}1 & |\omega|\le W\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

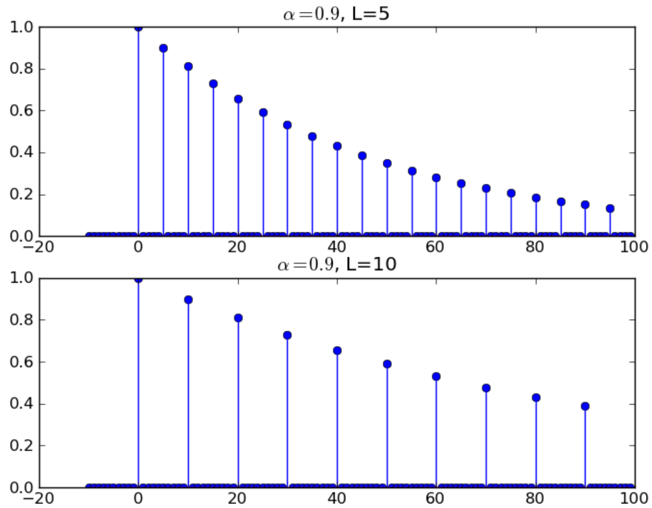
5 Working with systems in time and frequency

Suppose we are given an input signal $x(n)$ and system $H(\omega)$. The output signal is $y(n)$, or in frequency domain $Y(\omega)$:

$$x(n) = \begin{cases} \alpha^{n/L}u(n) & \mod(n, L) = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq \alpha < 1$

What would this look like?



It is basically stretching out the signal. Recall

$$\begin{array}{rcl} x(n/L) & \overset{\mathcal{FT}}{\longleftrightarrow} & X(\omega L) \\ x(t) = a^n u(n) \text{ where } |a| < 1 & \overset{\mathcal{FT}}{\longleftrightarrow} & X(\omega) = \dfrac{1}{1 - e^{-i\omega}} \end{array}$$

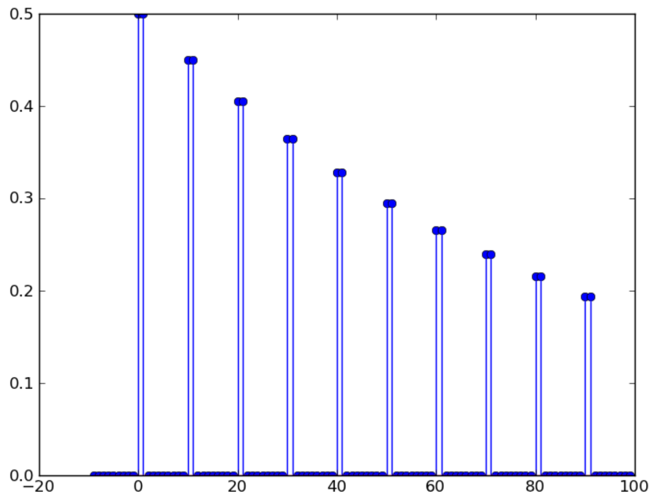
The DTFT of our signal is

$$X(\omega) = \frac{1}{1 - e^{-i\omega L}}$$

If we had an LTI system with impulse response $h(n) = \frac{1}{2}(\delta(n) + \delta(n-1))$ What would be our output?

$$\begin{aligned} y(n) &= (x \star h)(n) \\ &= \sum_{k=-\infty}^{+\infty} x(k)h(n-k) \\ &= \frac{1}{2}(x(n) + x(n-1)) \\ &= \begin{cases} \frac{1}{2}\alpha^{(n/L)} & \mod(n, L) = 0 \\ \frac{1}{2}\alpha^{(n/L)} & \mod(n-1, L) = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

For $L = 10$ and $\alpha = 0.9$, $y(n)$ would look like



What would be $Y(\omega)$?
Given our $h(n)$, our frequency response $H(\omega) = \frac{1}{2}(1 + e^{-i\omega})$. Recall our DTFT transform pairs.
Since $Y\omega) = H(\omega)X(\omega)$,

$$Y(\omega) = \left(\frac{1}{2}(1 + e^{-i\omega})\right)\left(\frac{1}{1 - e^{-i\omega L}}\right)$$

$$= \frac{1}{2}\frac{1}{1 - \alpha e^{-i\omega L}} + \frac{1}{2}e^{-i\omega} \cdot \frac{1}{1 - \alpha e^{-i\omega L}}$$

Notice this is the same as if we took the DTFT of $y(n) = \frac{1}{2}(x(n) + x(n-1))$.

6 Slight Differences between CTFT and DTFT transform pairs

Notice how in discrete time, we have

$$x(an) \overset{\mathcal{FT}}{\longleftrightarrow} X(\omega) = X(\frac{\omega}{a})$$

and in continuous time, we have

$$x(at) \overset{\mathcal{FT}}{\longleftrightarrow} X(\omega) = \frac{1}{|a|}X(\frac{\omega}{a})$$

Not every transform pair is identical in both cases. Because of unique properties in discrete-time vs continuous-time, such as in the conservation of energy, there may be slight differences in the transform pairs.

7 DFT Transform Pairs

Some common transform pairs:

$$\begin{aligned} x(n) &\overset{\mathcal{DFT}}{\longleftrightarrow} X_k \\ x(n) = \delta(n) &\overset{\mathcal{DFT}}{\longleftrightarrow} 1 \\ x(n) = 1 &\overset{\mathcal{DFT}}{\longleftrightarrow} X_k = p\delta(k) \\ x(n) = \delta(n-m) &\overset{\mathcal{DFT}}{\longleftrightarrow} X_k = e^{-i\frac{2\pi}{p}m} \end{aligned}$$

What if $(n) = e^{i\frac{2\pi}{p}mn}$ where m is an integer $0 \leq m < p$? Let's apply the DFT to this signal.

$$X_k = \sum_{n=0}^{p-1} e^{i2\frac{\pi}{p}mn} e^{-i\frac{2\pi}{p}kn} = \sum_{n=0}^{p-1} (e^{-i\frac{2\pi}{p}(k-m)})^n$$

$$X_k = \frac{1 - e^{-i2\pi(k-m)}}{1 - e^{-i\frac{2\pi}{p}(k-m)}}$$

If $k \neq m$,

$$X_k = \frac{1 - e^0}{1 - e^{-i\frac{2\pi}{p}(k-m)}} = 0$$

If $k = m$, we need to use L'hospital's rule because we have $\frac{0}{0}$,

$$X_k = \frac{-i2\pi e^{-i2\pi(0)}}{\frac{-i2\pi}{p} e^{\frac{-i2\pi}{p}(0)}} = p$$

Therefore,

$$x(n) = e^{i\frac{2\pi}{p}mn} \overset{\mathcal{DFT}}{\longleftrightarrow} X_k = \begin{cases} p, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}$$

8 Extra Problems

(1) Calculate the convolution of the signals $e^{-\alpha t}u(t)$ and $e^{-\beta t}u(t)$ for the following cases:
a) $\alpha \neq \beta$
b) $\alpha = \beta$
What is the CTFT of the result?

Solution:
We have

$$(e^{-\alpha t}u(t)) * (e^{-\beta t}u(t)) = \int_{-\infty}^{\infty} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau) \quad (1)$$

$$\stackrel{(a)}{=} \int_{0}^{t} e^{-\alpha\tau}u(\tau)e^{-\beta(t-\tau)}u(t-\tau) \quad (2)$$

$$(3)$$

where equality (a) follows from the following fact:

$$u(\tau)u(t-\tau) = \begin{cases} 1 & \text{if } 0 \leq \tau \leq t \\ 0 & \text{else.} \end{cases} \quad (4)$$

Note that $u(\tau)u(t-\tau) \equiv 0$ if $t < 0$.
(Case 1, $\alpha \neq \beta$).

$$(e^{-\alpha t}u(t)) * (e^{-\beta t}u(t)) = \begin{cases} e^{-\beta t}\int_0^t e^{(-\alpha+\beta)\tau}d\tau & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (5)$$

$$= \begin{cases} \frac{1}{\beta-\alpha}\left(e^{-\alpha t} - e^{-\beta t}\right) & t \geq 0 \\ 0 & t < 0. \end{cases} \quad (6)$$

(Case 2, $\alpha = \beta$).

$$(e^{-\alpha t}u(t)) * (e^{-\beta t}u(t)) = \begin{cases} e^{-\beta t}\int_0^t 1d\tau & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (7)$$

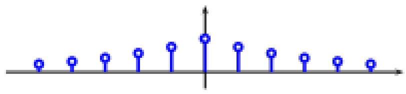
$$= \begin{cases} te^{-\beta t} & t \geq 0 \\ 0 & t < 0. \end{cases} \quad (8)$$

Since convolution in the time domain is equivalent to multiplication in the frequency domain,

$$Y(\omega) = \frac{1}{\alpha + j\omega}\frac{1}{\beta + j\omega}$$

(2) What is the DTFT of the following signal:

2) $x[n] = a^{|n|}$, $|a| < 1$



Solution:

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \underbrace{\sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}}_{\sum_{n=1}^{\infty} a^n e^{j\omega n} = \sum_{n=1}^{\infty} a^n e^{-j(-\omega)n} = \sum_{n=0}^{\infty} a^n e^{-j(-\omega)n} - 1} && (11)\\
&= \frac{1}{1 - ae^{-j\omega}} + \frac{1}{1 - ae^{j\omega}} - 1 && (12)\\
&= \frac{2 - a(e^{j\omega} + e^{-j\omega})}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2} - 1 && (13)\\
&= \frac{2 - 2a\cos\omega}{1 - 2a\cos\omega + a^2} - 1 = \frac{1 - a^2}{1 + a^2 - 2a\cos\omega} && (14)
\end{aligned}
$$

Note: $X(e^{j\omega}) = X(\omega)$