Discussion 11: More Sampling and Anti-Aliasing Filter

EE 20 Spring 2014
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0.1 Sampling Overview

Let's review the steps when we sample a signal. Consider the following signal $x(t)$: Now, if we sample this

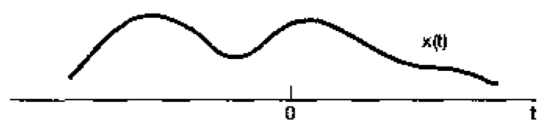


Figure 0.1: Continuous-time signal x(t)

signal at a rate of $f_s = 1/T$, then each sample of our sampled signal $x_p(t)$ is spaced evenly by T: How we

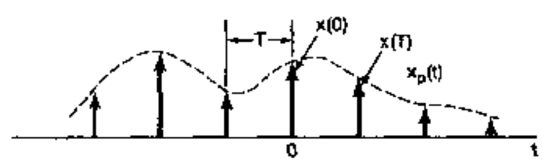


Figure 0.2: Sampled version of x(t)

actually do this is multiply this by an impulse train defined by $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ Therefore,

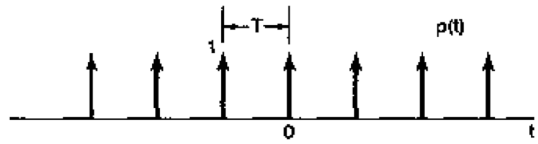


Figure 0.3: Impulse train p(t)

$$x_p(t) = x(t)p(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Note: this step involving the impulse train was skipped in lecture. Instead, we skip directly to stating $y(n) = x(nT)$, which is basically compressing our $x_p(t)$ by a factor of T. But the impulse train is necessary for the derivation. Please follow the lecture notes on how to derive the spectrum, $Y(\omega)$, of $y(n)$. The

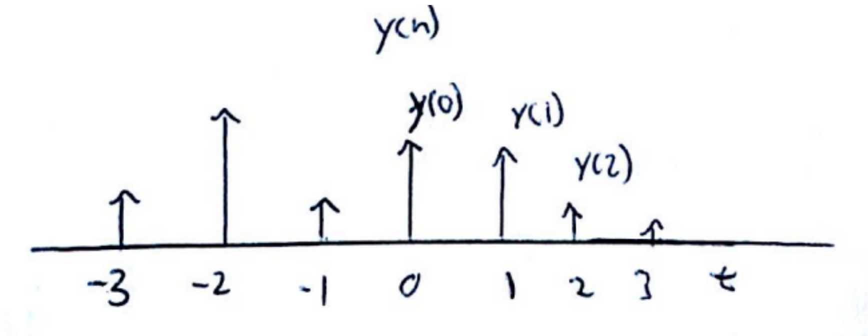


Figure 0.4: Impulse train p(t)

derivation is not trivial and won't be tested, but it is very important for understanding how sampling works. Let's explore the frequency domain of sampling and try to arrive to the same DTFT as derived in lecture:

$$Y(\omega) = \sum_{k=-\infty}^{\infty} f_s X(f_s(\omega - 2\pi k))$$

So we have our sampled signal $x_p(t) = x(t)p(t)$. From the multiplication property, we know that

$$X_p(\omega) = \frac{1}{2\pi}(X(\omega) * P(\omega))$$

Let's assume that $X(\omega)$ looks like this Now the question is, what is $P(\omega)$, the FT of the impulse train? Let's

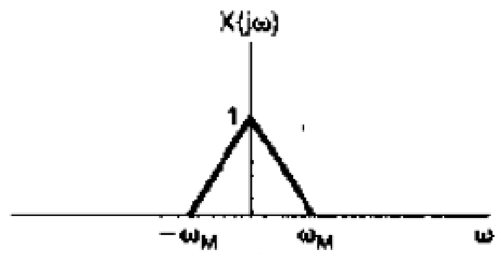


Figure 0.5: CTFT of x(t)

calculate the fourier coefficients, X_k, of our impulse train $p(t)$.

$$\begin{aligned} X_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} p(t)e^{-ik\frac{2\pi}{T}t}dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-ik\frac{2\pi}{T}t}dt \text{ since we are integrating over the interval } [-T/2, T/2] \\ &= \frac{1}{T} \end{aligned}$$

Thus, we can write $p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}e^{ik\frac{2\pi}{T}t}$. Now taking the CTFT of our impulse train newly defined and using the frequency shift property, (Recall $e^{ik\frac{2\pi}{T}t} \xleftrightarrow{\mathcal{FT}} 2\pi\delta(\omega - k\frac{2\pi}{T})$), we get

$$P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{T})$$

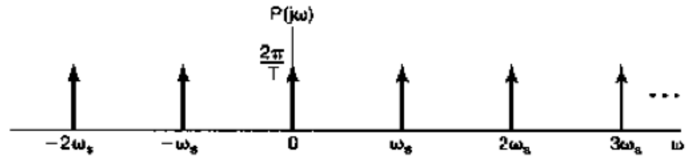


Figure 0.6: Fourier Transform of impulse train.

Note: $\omega_s = 2\pi f_s$ Now we can convolve $\frac{1}{2\pi}(X(\omega) * P(\omega))$

$$X_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\frac{2\pi}{T})$$

Note: $\omega_s = 2\pi f_s$

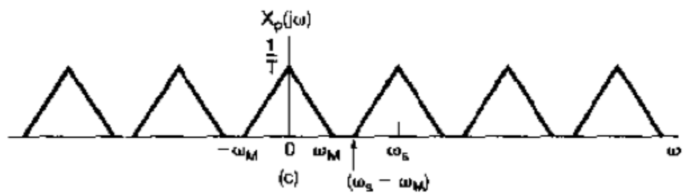


Figure 0.7: Fourier Transform of sampled signal

which creates repeated replicas centered at a multiple of $2\pi f_s$. Remember when we sample, all frequencies $f + Nf_s$ map to the frequency f, which explains the repetition.
Now since we know $y(n) = x(nT)$, we can find the $Y(\omega)$ simply by using the time expansion property $x(nT) \stackrel{\mathcal{FT}}{\longleftrightarrow} X(n/T)$

$$\begin{aligned} Y(\omega) &= X(\omega/T) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega/T - k\frac{2\pi}{T}) \\ &= \sum_{k=-\infty}^{\infty} f_s X(f_s(\omega - 2\pi k)) \end{aligned}$$

which causes each triangle to now be compressed by a factor of $f_s = 1/T$ and centered at a multiple of 2π,

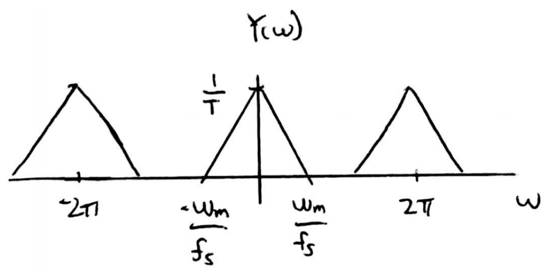
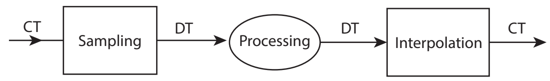


Figure 0.8: DTFT of y(n)

which is a property of the DTFT.

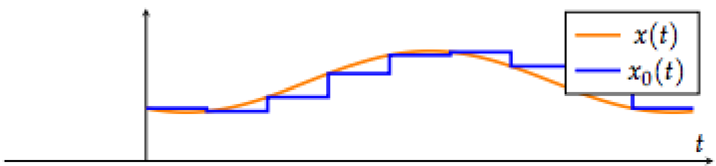
We have finally arrived to the second block!

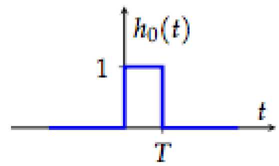


0.2 Interpolation

To finally recover our original signal, we can interpolate the sampled signal. There are three types ways you will learn on how to interpolate the sampled signal:

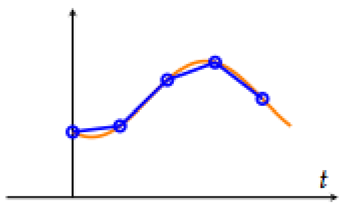
(1) Zero-order hold interpolation

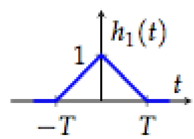




$$H_0(\omega) = e^{i\omega T/2}\frac{\sin(\omega T/2)}{\omega/2}$$

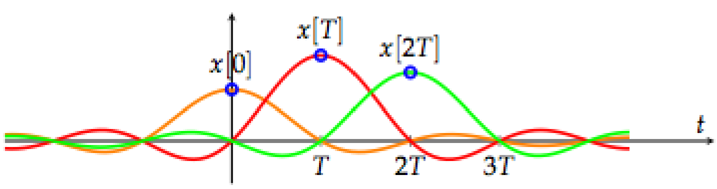
(2) First-order hold interpolation

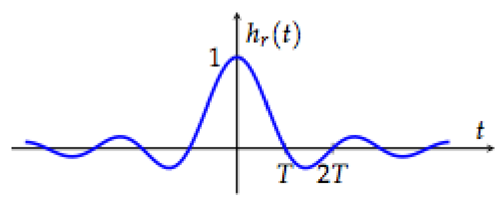




$$H_1(\omega) = \frac{1}{T}\left(\frac{\sin(\omega T/2)}{\omega/2}\right)^2$$

(3) Ideal interpolation

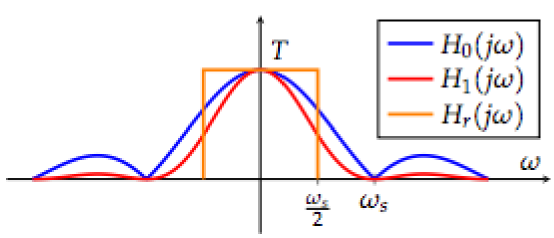




$$\begin{aligned} h_r(t) &= \text{sinc}(\pi f_s t) = \text{sinc}(\pi/Tt) \\ &= \frac{\sin(\pi/Tt)}{\pi/Tt} \end{aligned}$$

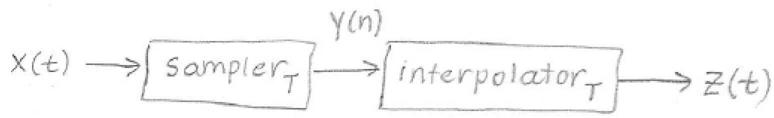
$$H_r(\omega) = \begin{cases} T & |\omega| < \pi/T \\ 0 & otherwise \end{cases}$$

The Fourier transforms of all of the interpolators looks like:



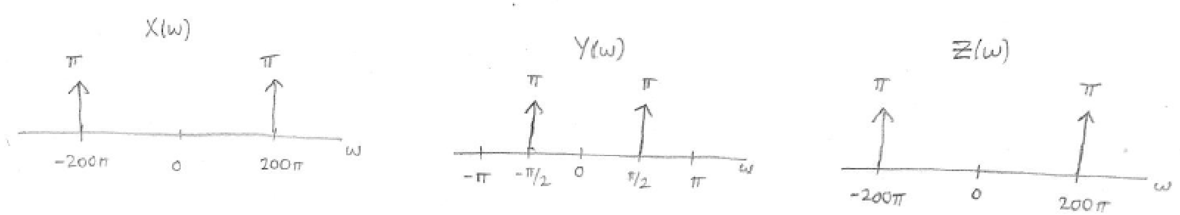
Try to derive each of these on your own.

0.3 Discussion Question

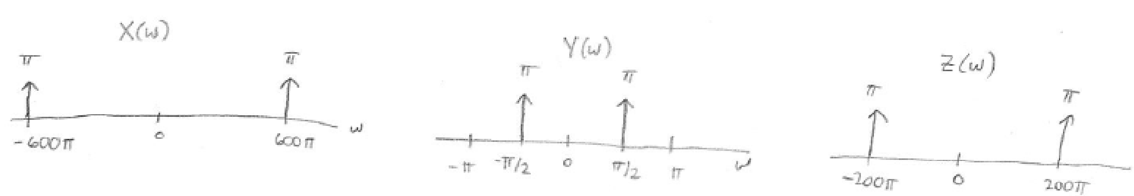


Assume the interpolator is ideal, and that $T = \frac{1}{400}$. Determine and draw $x(t)$, $y(n)$, $z(t)$, $X(\omega)$, $Y(\omega)$, and $Z(\omega)$ when

- $x(t) = cos(2\pi 100t)$
  Here, we have $y(n) = x(nT) = cos(\frac{\pi}{2}n)$. Using the result from problem 1c, $z(t) = cos(2\pi\frac{1}{4T}t) = cos(2\pi 100t)$. Note that $x(t)$ and $z(t)$ are the same, since we have sampled above the Nyquist rate.

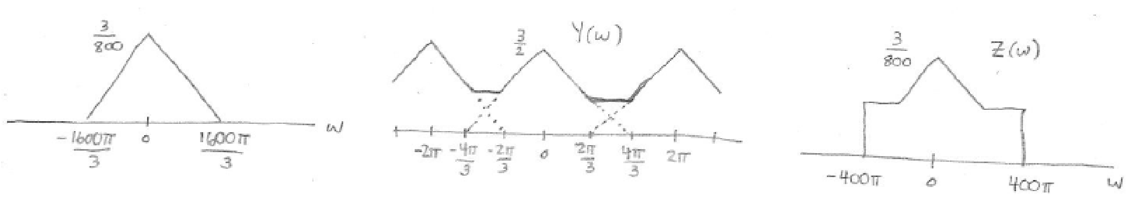


- $x(t) = cos(2\pi 300t)$
  Here, we have $y(n) = x(nT) = cos(\frac{3\pi}{2}n) = cos(-\frac{3\pi}{2}n) = cos(\frac{\pi}{2}n)$. Again, $z(t) = cos(2\pi 100t)$. Note that $x(t)$ and $z(t)$ are *not* the same, since we have sampled below the Nyquist rate. The frequency $300Hz$ masqueraded as $2\pi 400Hz - 2\pi 300Hz = 2\pi 100Hz$.



- $x(t) = sinc^2(\frac{800\pi}{3}t)$
  This is much easier done in the frequency domain. Recall from the previous discussion that the Fourier transform of $sinc^2(\frac{\pi t}{T})$ is a triangle with height T and base extending from $-\frac{2\pi}{T}$ to $\frac{2\pi}{T}$. We can then plot $X(\omega)$, $Y(\omega)$, and $Z(\omega)$ as shown below.

What are two things we can do to avoid aliasing?
We can either increase our sampling rate or apply a low pass filter to $x(t)$. The latter is called an anti-aliasing filter.



0.4 Anti-Aliasing Filters

In real life, most signals are not band limited. Therefore, in order to reconstruct $x(t)$, our sampling rate had to be infinity. However, since this is not practical, we may use an anti-aliasing filter to control the amount of error we have.

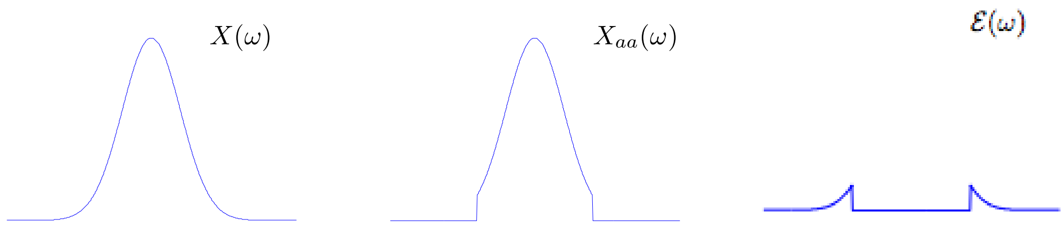


Figure 0.9: Example $X(\omega)$, the filtered $X_{aa}(\omega)$, and the error $\mathcal{E}(\omega)$.

By Parseval's identity, there exists a relationship between the error in time and corruption in frequency

$$\int_{\mathbb{R}} |\varepsilon(t)|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} |\mathcal{E}(\omega)|^2 d\omega = \delta.$$

This gives control over the amount of corruption of the signal. If we have a higher f_s, then the anti-aliasing filter can have a wider band and lessen the effects of aliasing.

However, there is a tradeoff. A higher f_s reduces the information lost but is more expensive. A lower f_s increases the information loss and is cheaper computationally. Regardless, the anti-aliasing filter gives us the ability to control the energy of $E(\omega)$, the information lost.