Discussion 6: Impulse Response, Convolution, and Feedback

EE 20 Spring 2014
Sam Kim

1 Impulse Response to Frequency Response

Recall, for an LTI system H, its frequency domain representation can be described by the frequency response $H(\omega)$ and its time domain representation can be described by the impulse response $h(n)$. On the frequency domain xside, if the input is $x(n) = e^{i\omega n}$, then the output is $y(n) = H(\omega)e^{i\omega n}$. At the same time on the time domain,

$$\begin{aligned}
y(n) &= (h * x)(n) \\
&= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= \sum_{k=-\infty}^{\infty} h(k)e^{i\omega(n-k)} \\
&= \left(\sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k}\right) e^{i\omega n}.
\end{aligned}$$

Since k is merely a dummy variable for the summation, our result is a function only in ω,

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-i\omega k},$$

where $h(n)$ is the impulse response of a LTI system. For a simple sanity check, note that there is no n term in the expression for $H(\omega)$; the frequency response of a system is completely independent of time. With this equation, the two are related by something called the **Fourier Transform** (which will be discussed later)!

2 Convolution

The delta function $\delta(n)$ is so fundamental to this course because it is considered the identity function in signals and systems. What does this mean? When you convolve $x(n)$ with $\delta(n)$, you get itself back, $x(n)$.

$$(\delta * x)(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

$$= ... + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + ...$$

$$= x(n)$$

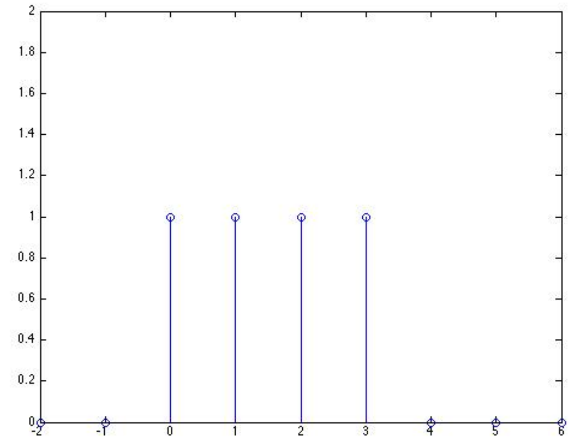
Similarly if you convolve $x(n)$ with $\delta(n-N)$, you get $x(n-N)$.

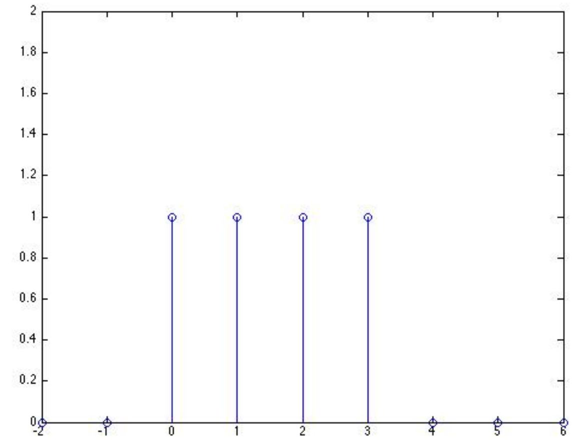
Since when we put into $\delta(n)$ into an LTI system and output $h(n)$, when we convolve $x(n)$ with $h(n)$, we get an output $y(n)$.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

*Remember that convolution is associative.

Let's observe what happens when you convolve the two following signals.





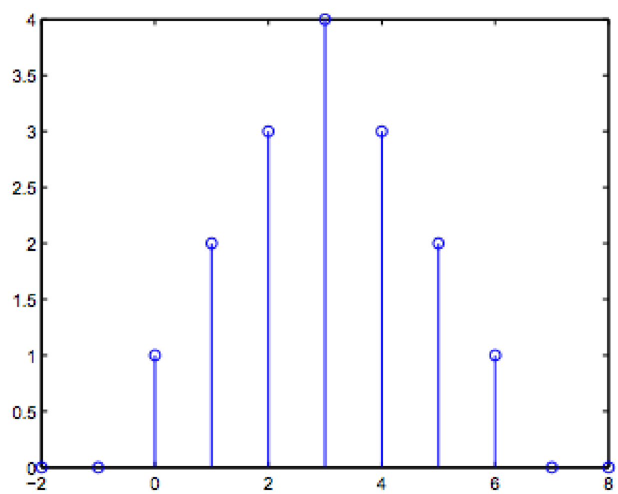
Let's treat the first one as the signal $x(n)$ and the other as the impulse response $h(n)$. The impulse response $h(n)$ can be expressed as

$$h(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)$$

Convolving the two, we get

$$(h * x)(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= x(n) + x(n-1) + x(n-2) + x(n-3)$$

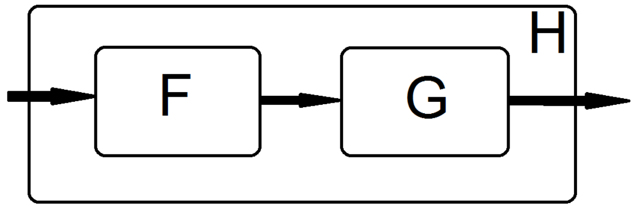


You can also apply the flip and shift method which will produce the same result.

How many nonzero components are there? 7
In general, the number of nonzero components will be (length of $x(n)$) + (length of $h(n)$) - 1.

3 Cascade System

Here is an example of a cascade system.



Let the frequency response of system F be $F(\omega)$ and its impulse response be $f(n)$. Similarly for G, Let the frequency response of system G be $G(\omega)$ and its impulse response be $g(n)$.
Let,

$$f(n) = \frac{1}{2}(\delta(n) + \delta(n-1))$$

$$g(n) = \frac{1}{2}(\delta(n) - \delta(n-1))$$

Therefore,

$$F(\omega) = \frac{1}{2} + \frac{1}{2}e^{-i\omega}$$

$$G(\omega) = \frac{1}{2} - \frac{1}{2}e^{-i\omega}$$

Are F and G causal? Yes! Both of their impulse responses are 0 for all $n < 0$.
What are the frequency response $H(\omega)$ and the impulse response $h(n)$ of the overall system H?

$$\begin{aligned} h(n) &= (f * g)(n) \\ &= \frac{1}{4}(\delta(n) - \delta(n-2)) \\ H(\omega) &= F(\omega) * G(\omega) \\ &= \frac{1}{4} - \frac{1}{4}e^{-i2\omega} \end{aligned}$$

Remember that this system is associative so

$$H(\omega) = F(\omega) * G(\omega) = G(\omega) * F(\omega)$$

$$h(n) = (f * g)(n) = (g * f)(n)$$

What is the $|H(\omega)|$? More frequency response practice!
Remember, there are multiple ways to approach this

1. Geometric approach. Plot the vectors in the complex plane and see how the ω affects the magnitude

2. Using Euler's identity. Convert the complex exponentials to cartesian form and using pythagorean theorem $\sqrt{(\text{real part})^2 + (\text{image part})^2}$

$$H(\omega) = \frac{1}{4} - \frac{1}{4}(\cos(2\omega) - i\sin(2\omega))$$

$$|H(\omega)| = \sqrt{(\frac{1}{4} - \frac{1}{4}\cos(2\omega))^2 + (-\frac{1}{4}\sin(2\omega)^2)}$$

$$= \sqrt{\frac{1}{16}(1 - 2\cos(2\omega) + \cos(2\omega)^2 + \sin(2\omega)^2)}$$
$$= \frac{1}{4}\sqrt{(2 - 2\cos(2\omega))}$$
$$= \frac{1}{4}\sqrt{4\sin(\omega)^2}$$
$$= \frac{1}{2}\sin(\omega)$$

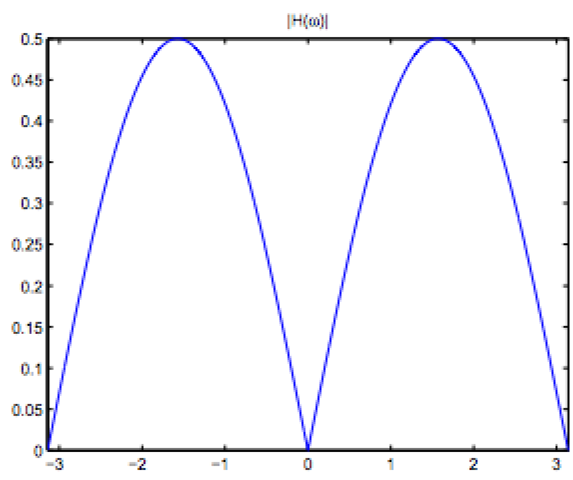
3. Finding the square of the magnitude $|H(\omega)| = H(\omega)H^*(\omega)$ and square rooting it

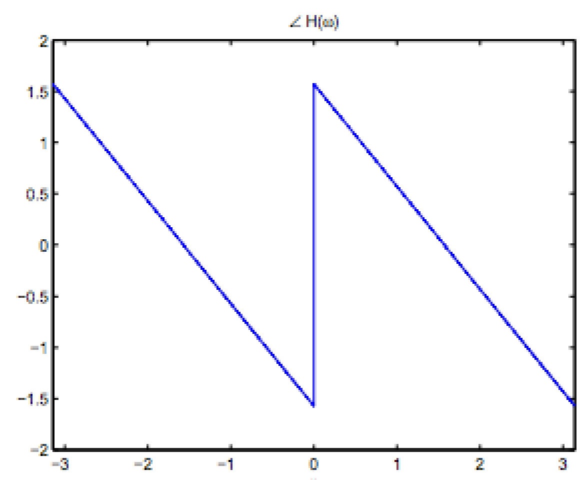
$$|H(\omega)| = (\frac{1}{4} - \frac{1}{4}e^{-i2\omega}) * (\frac{1}{4} - \frac{1}{4}e^{i2\omega})$$
$$= \frac{1}{16}(1 - e^{i\omega} - e^{-i\omega} + 1)$$
$$= \frac{1}{16}(2 - 2\cos(\omega))$$
$$= \frac{1}{4}\sin(\omega)^2$$
$$|H(\omega)| = \frac{1}{2}\sin(\omega)$$

4. half-angle trick

$$H(\omega) = \frac{1}{4} - \frac{1}{4}e^{-i2\omega}$$
$$= (\frac{1}{4}e^{i\omega} - \frac{1}{4}e^{-i\omega})e^{-i\omega}$$
$$= \frac{1}{2}\sin(\omega)e^{-i\omega}$$
$$H(\omega) = \frac{1}{2}\sin(\omega)$$

Be wary of this method however. It may seem the simplest, but it may not work for every frequency response.

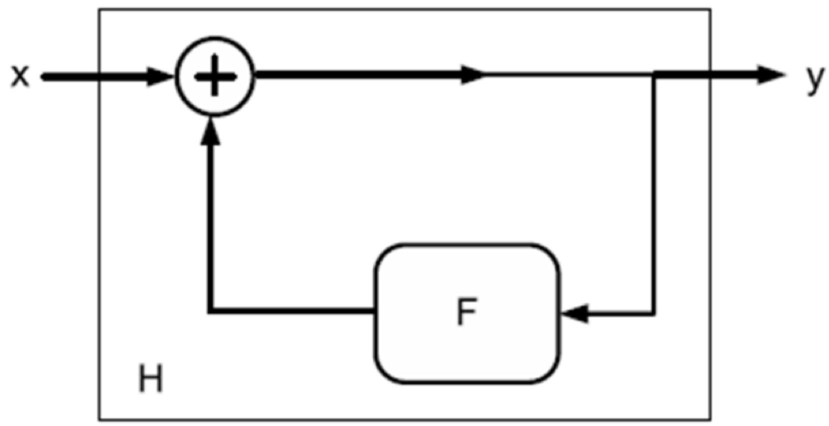




Notice how F is a low pass filter, G is a high pass filter, and H is a band-pass filter.

4 Feedback

Practice with Feedback!
Consider the following system where $F(\omega) = \frac{1}{4}e^{-i2\omega}$



To find the frequency response of this system, we input $e^{i\omega t}$ into the system.

$$e^{i\omega t} + F(\omega)H(\omega)e^{i\omega t} = H(\omega)e^{i\omega t}$$

$*$Really understand how I derived this equation.

Much more difficult systems may occur in future problems and midterms

$$H(\omega) = \frac{1}{1 - F(\omega)}$$

$$H(\omega) = \frac{1}{1 - \frac{1}{4}e^{-i2\omega}}$$

What would be the LCCDE that corresponds to this system?
Recall from our previous relationship between $H(\omega)$ and $h(n)$,
$\delta(n)$ corresponds to 1
$h(n)$ corresponds to $H(\omega)$
$h(n-1)$ corresponds to $H(\omega)e^{-i\omega}$
$h(n-N)$ corresponds to $H(\omega)e^{-iN\omega}$
Again, this relationship is described by the Fourier Transform which will be discussed soon.
If we can find the impulse response from the frequency response, then we can determine the LCCDE.

$$H(\omega) = \frac{1}{1 - \frac{1}{4}e^{-i2\omega}}$$

$$H(\omega) - \frac{1}{4}e^{-i2\omega}H(\omega) = 1$$

$$h(n) - \frac{1}{4}h(n-2) = \delta(n)$$

Therefore, the LCCDE is

$$y(n) - \frac{1}{4}y(n-2) = x(n)$$

Remember, feedback systems have infinite impulse responses, IIR. What is this impulse response? Assuming the system is causal, we know that $h(n) = 0$ for $n < 0$.

$$h(0) = 1$$

$$h(1) = 0$$
$$h(2) = (0.25)$$
$$h(3) = 0$$
$$h(4) = (0.25)(0.25)$$
$$.$$
$$.$$
$$.$$

Noticing a pattern, the impulse response $h(n)$ is

$$\begin{cases} (0.25)^{n/2}, & \text{if } n \text{ is even and } n \geq 0 \\ 0, & \text{if } n \text{ is odd } orn < 0 \end{cases}$$