Discussion 4

Time Invariance

How do we determine if a system is TI?

*Note Time Invariance and Linearity describe a system, not a signal

Let's ask ourselves, does $y(t-\tau) = F(x(t-\tau))$?

If I shift $x(t)$ by τ, then I should expect the same amount of shift in $y(t)$.

Time Invariance

Steps on how to determine if a system is TI

1. Calculate $y(t-\tau)$

Time Invariance

Steps on how to determine if a system is TI

1. Calculate $y(t-\tau)$
2. Let $x'(t) = x(t-\tau)$; Then calculate output $y'(t) = F(x'(t))$;

Time Invariance

Steps on how to determine if a system is TI

1. Calculate $y(t-\tau)$
2. Let $x'(t) = x(t-\tau)$; Then calculate output $y'(t) = F(x'(t))$;
3. Does $y(t-\tau) = y'(t)$? If so, it is **Time Invariant**

Time Invariance

Examples of Time Invariant systems:

- $y(t) = x(t-1)$ delay
- $y[n] = 1/2x[n] + 1/4x[n-1] + 1/8x[n-2] + \ldots$
- $y[n] = \frac{1}{2}(x[n] + x[n-1])$ Low-pass filters
- $y(t) = d/dt[x(t)]$ differentiator
- $y(t) = x^2(t)$

Time Invariance

Examples of systems that are not time invariant:

- y(t) = x(t^2)
- y[n] = x[n/N] if n mod N = 0; 0 otherwise (upsampler)
- y[n] = $\frac{1}{T}\int_0^t x(\tau)d\tau$ integrator
- y(t) = x(-t)

Sines and Cosines

How to relate sin and cos with complex exponentials

cos(x) = ½(e^(ix) + e^(-ix))

sin(x) = 1/2i(e^(ix) - e^(-ix))

Sines and Cosines

How to relate sin and cos with complex exponentials

cos(x) = ½(e^(ix) + e^(-ix))
sin(x) = 1/2i(e^(ix) - e^(-ix))

Why? Look at the geometry of the complex plane!
(*go to board)

Frequency Response

Why use complex exponentials for LTI systems?

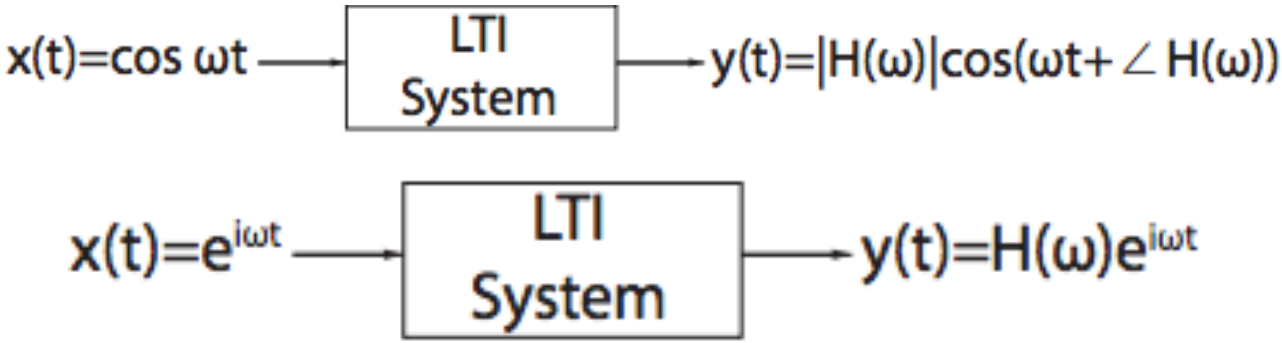
Frequency Response

Why use complex exponentials for LTI systems?

It makes the math easier!

Also by our previous relationship between e^(iwt) and cos (wt), knowing what an LTI system does to a complex exponential means we know what it does to cos(wt)

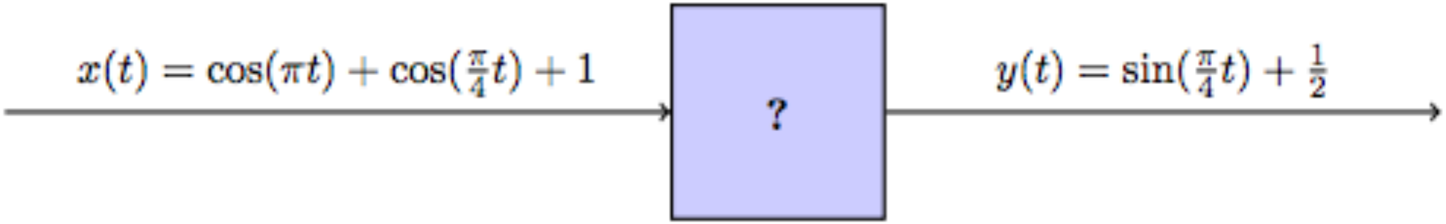
Frequency Response



Every LTI system has a unique frequency response.

Frequency Response

What is the frequency response of this system?



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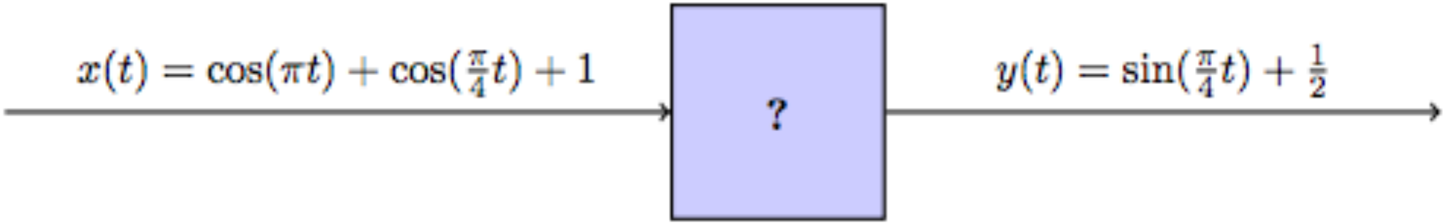
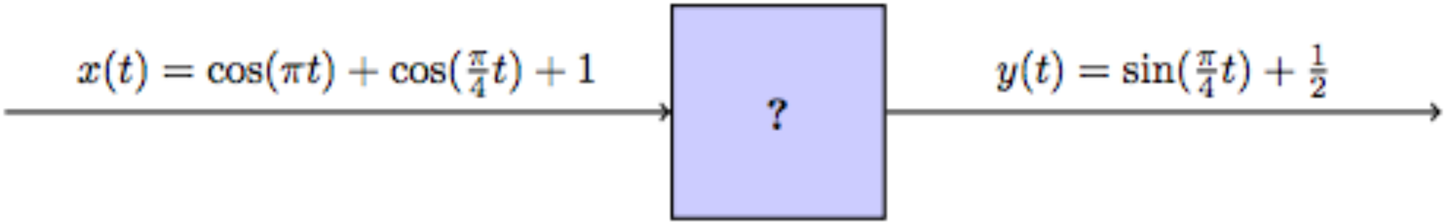


Let’s first solve this in terms of cosines

Frequency Response

What is the frequency response of this system?



The frequency response H(ω) can be uniquely determined by A(ω) and Θ(ω)

Frequency Response

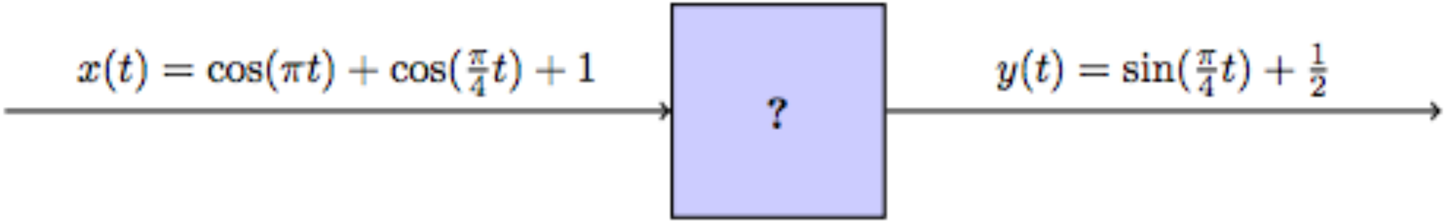
Theorem:

For an LTI system, if the input signal is a sinusoid, $x(t) = \cos(\omega_0 t)$, the output signal will be a sinusoid of the same frequency, $y(t) = A(\omega_0)\cos(\omega_0 t + \Theta(\omega_0))$.

$A(\omega_0)$ is the amplitude gain, $\Theta(\omega_0)$ is the phase change, and both may depend on ω_0, the frequency of the input.

Frequency Response

What is the frequency response of this system?



Inputs: $x(t) = x1(t) + x2(t) + x3(t)$

$x1(t) = \cos(\pi t)$; $y(t) = A(\pi)\cos(\pi t + \Theta(\pi))$; $A(\pi) = 0$;

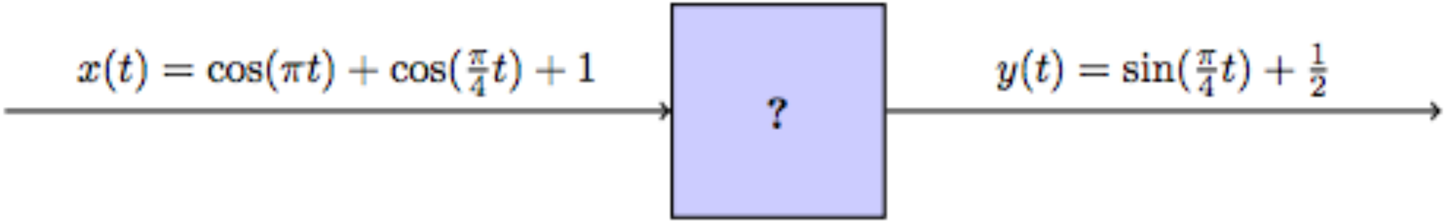
$x2(t) = \cos(\pi/4t)$; $y(t) = A(\pi/4)\cos(\pi/4t + \Theta(\pi/4)) = \sin(\pi/4t)$;

$A(\pi/4) = 1$; $\Theta(\pi/4) = -\pi/2$

$x3(t) = \cos(0t) = 1$; $y(t) = ½$; $A(0) = ½$;

Frequency Response

What is the frequency response of this system?



From the given input-output pair, we can determine 4 facts about $H(\omega)$:

$A(\pi) = 0$;

$A(\pi/4) = 1$;

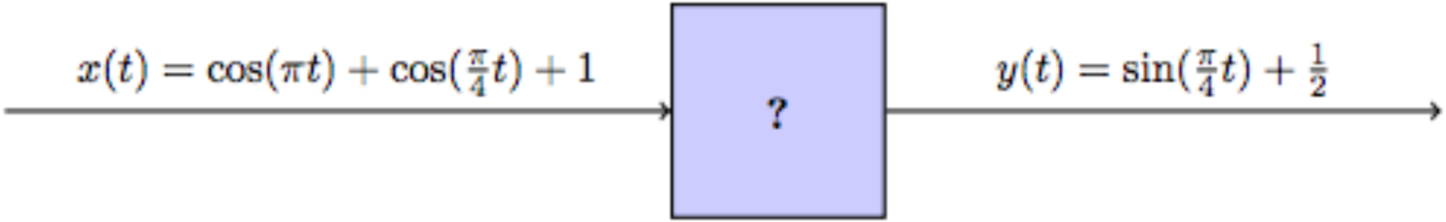
$\Theta(\pi/4) = -\pi/2$;

$A(0) = ½$;

*This does not describe the whole frequency response however

Frequency Response

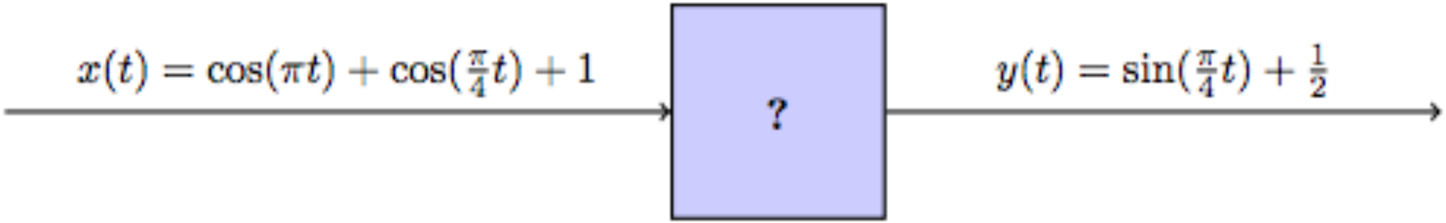
What is the frequency response of this system?



Now let’s first solve this in terms of complex exponentials

Frequency Response

What is the frequency response of this system?

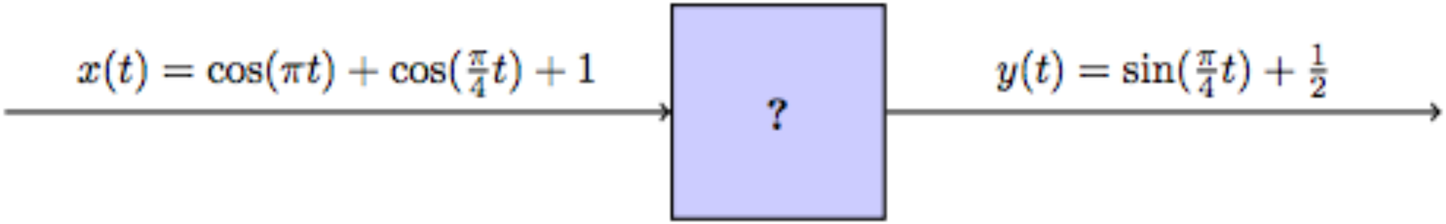


In terms of complex exponentials,

x(t) = ½(e^(iπt) + e^(-iπt)) + 1/2(e^(iπ/4t) + e^(-iπ/4t)) + e^(i0t);

Frequency Response

What is the frequency response of this system?



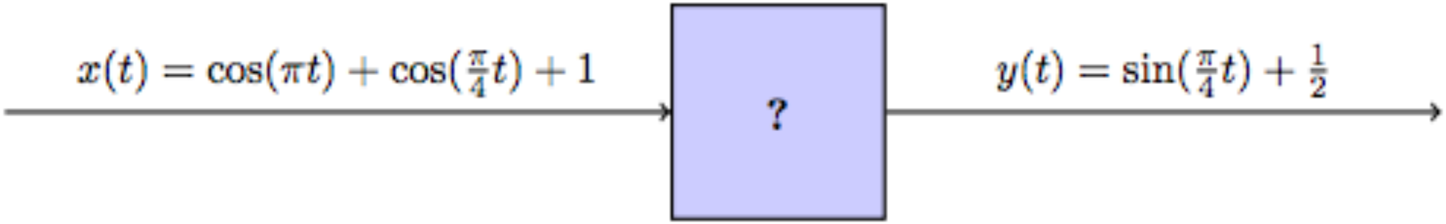
In terms of complex exponentials

$x(t) = \frac{1}{2}(e^{(i\pi t)} + e^{(-i\pi t)}) + 1/2(e^{(i\pi/4t)} + e^{(-i\pi/4t)}) + e^{(i0t)};$

$y(t) = 1/2i(e^{(i\pi/4t)} - e^{(-i\pi/4t)}) + 1/2e^{(i0t)};$

Frequency Response

What is the frequency response of this system?

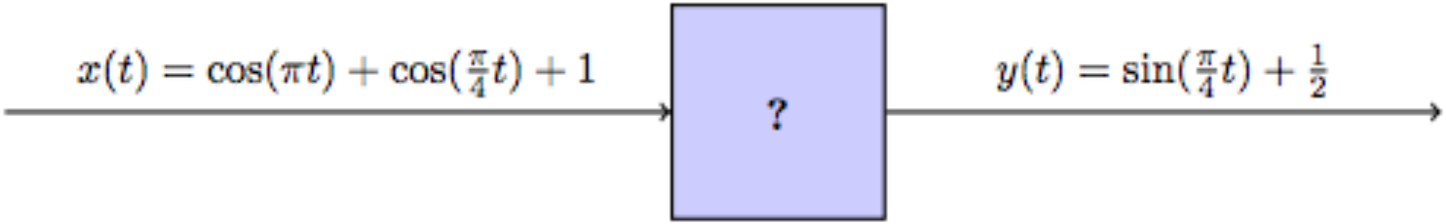


y(t) = 1/2i(e^(iπ/4t) - e^(-iπ/4t)) + 1/2e^(i0t);

y(t) = H(π/4)½e^(iπ/4t) + H(-π/4)½e^(-iπ/4t) + H(0)e^(i0t);

Frequency Response

What is the frequency response of this system?



What do we know about the frequency response H(ω) so far?

H(π/4) = 1/i = e^(-iπ/2);

H(-π/4) = -1/i = e^(iπ/2);

H(π) = H(-π) = 0;

H(0) = ½;

Frequency Response

Complex conjugate symmetry

Notice that $H(-\pi/4) = H^*(\pi/4)$

In general, if the signal is real ($x^*(t) = x(t)$), then

$$H(-\omega) = H^*(\omega)$$

Frequency Response

Magnitude:

If the frequency response is $H(\omega)$, then

$$|H(\omega)| = |H^*(\omega)|$$

Also,

$$|H(\omega)|^2 = H(\omega)H^*(\omega)$$

Which may be helpful when solving for the magnitude of the frequency response.