Discussion 6: Impulse Response, Convolution, and Feedback

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1 DTFT Derivation

Recall how

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n}.$$

We can do the same for some other aperiodic signal $x(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n}. \tag{1}$$

Now recall the synthesis and analysis equations for the Fourier Series:

$$\begin{aligned} x(n) &= \sum_{k=-\frac{p}{2}}^{\frac{p}{2}} X_k e^{ik\omega_0 n} && \text{(synthesis)} \\ X_k &= \frac{1}{p}\sum_{n=-\frac{p}{2}}^{\frac{p}{2}} x(n)e^{-ik\omega_0 n} && \text{(analysis)} \end{aligned}$$

NOTE: The X_k's and the functions $X(\omega)$ are different quantities. Just because they both have a capital X in them doesn't mean they are the same thing! Combining the synthesis and analysis equations, we get,

$$x(n) = \sum_{k=-\frac{p}{2}}^{\frac{p}{2}} \left(\frac{1}{p} \sum_{n=-\frac{p}{2}}^{\frac{p}{2}} x(n)e^{-ik\omega_0 n} \right) e^{ik\omega_0 n} \tag{2}$$

where $\omega_0 = \frac{2\pi}{p}$.

Remember that Fourier series analysis is only for periodic signals. If we had a long aperiodic signal, we can basically do Fourier series analysis on the signal with the assumption that the period p gets arbitrarily large. Notice how as p becomes really large, we have the approximation:

$$\sum_{n=-\frac{p}{2}}^{\frac{p}{2}} x(n)e^{-ik\frac{2\pi}{p}n} \approx X\left(k\frac{2\pi}{p}\right).$$

Let's continue with equation 13.1, replacing ω_0 with $\frac{2\pi}{p}$:

$$\begin{aligned}
x(n) &= \sum_{k=-\frac{p}{2}}^{\frac{p}{2}} \left(\frac{1}{p} \sum_{n=-\frac{p}{2}}^{\frac{p}{2}} x(n) e^{-i\left(k\frac{2\pi}{p}\right)n} \right) e^{i\left(k\frac{2\pi}{p}\right)n} \\
&= \frac{1}{2\pi} \sum_{k=-\frac{p}{2}}^{\frac{p}{2}} \left(\sum_{n=-\frac{p}{2}}^{\frac{p}{2}} x(n) e^{-i\left(k\frac{2\pi}{p}\right)n} \right) e^{i\left(k\frac{2\pi}{p}\right)n} \times \frac{2\pi}{p} \\
&\approx \frac{1}{2\pi} \sum_{k=-\frac{p}{2}}^{\frac{p}{2}} X\left(k\frac{2\pi}{p}\right) e^{i\left(k\frac{2\pi}{p}\right)n} \times \frac{2\pi}{p}.
\end{aligned}$$

This last equation is simply a Riemann sum, so taking $p \to \infty$, we obtain

$$x(n) = \lim_{p\to\infty} \frac{1}{2\pi} \sum_{k=-\frac{p}{2}}^{\frac{p}{2}} X\left(k\frac{2\pi}{p}\right) e^{i\left(k\frac{2\pi}{p}\right)n} \times \frac{2\pi}{p} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{i\omega n} d\omega. \tag{3}$$

The DTFT can be described as a generalized Fourier series and can be applied to almost any kind of signal!

2 Discrete Time Fourier Transforms

So far we have,

$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n} \\
x(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{i\omega n} d\omega.
\end{aligned}$$

Here is a list of common Fourier Transform pairs:

$$\begin{aligned}
x(n) &\overset{\mathcal{FT}}{\longleftrightarrow} X(\omega) \\
x(-n) &\overset{\mathcal{FT}}{\longleftrightarrow} X(-\omega) \\
\delta(n) &\overset{\mathcal{FT}}{\longleftrightarrow} 1 \\
\delta(n-N) &\overset{\mathcal{FT}}{\longleftrightarrow} e^{-i\omega N} \\
x(n-N) &\overset{\mathcal{FT}}{\longleftrightarrow} X(\omega) e^{-i\omega N} \\
x(n) e^{i\omega_0 n} &\overset{\mathcal{FT}}{\longleftrightarrow} X(\omega - \omega_0) \\
\alpha x_1(n) + \beta x_2(n) &\overset{\mathcal{FT}}{\longleftrightarrow} \alpha X_1(\omega) + \beta X_2(\omega) \\
x_1(n) x_2(n) &\overset{\mathcal{FT}}{\longleftrightarrow} \frac{1}{2\pi} X_1(\omega) * X_2(\omega) \\
x_1(n) * x_2(n) &\overset{\mathcal{FT}}{\longleftrightarrow} X_1(\omega) X_2(\omega) \\
x(n) = a^n u(n) &\overset{\mathcal{FT}}{\longleftrightarrow} X(\omega) = \frac{1}{1 - ae^{i\omega}} \\
n x(n) &\overset{\mathcal{FT}}{\longleftrightarrow} X(\omega) = i \frac{d}{d\omega} X(\omega)
\end{aligned}$$