Math Olympiad Problem Solving
Stanford University EPGY Summer Institutes 2008
Problem Set: Algebraic Expressions

1. (1987 AIME) Find $3x^2y^2$ if x, y are integers such that

$$y^2 + 3x^2y^2 = 30x^2 + 517.$$

2. Find all positive integer solutions (x, y) of the equation $x^2 - y^2 = 20$.

3. Find all positive integer solutions (x, y) of the equation $xy + 5x + 3y = 200$.

4. (1991 AIME) Find $x^2 + y^2$ if x and y are positive integers and

$$xy + x + y = 71 \text{ and } x^2y + xy^2 = 880.$$

5. Find all integer solutions (n, m) of the equation

$$n^4 + 2n^3 + 2n^2 + 2n + 1 = m^2.$$

6. If $x^2 + y^2 + z^2 = 49$ and $x + y + z = x^3 + y^3 + z^3 = 7$, find xyz.

7. (1987 AHSME #11) Let c be a constant. The simultaneous equations

$$\begin{aligned} x - y &= 2 \\ cx + y &= 3 \end{aligned}$$

have a solution (x, y) inside Quadrant I if and only if

(A) $c = -1$ (C) $c < 3/2$ (E) $-1 < c < 3/2$
(B) $c > -1$ (D) $0 < c < 3/2$

8. (1987 AHSME #15) If (x, y) is a solution to the system

$$xy = 6 \text{ and } x^2y + xy^2 + x + y = 63,$$

find $x^2 + y^2$.

(A) 13   (B) 1173/32   (C) 55   (D) 69   (E) 81

9. (1986 AHSME #30) The number of real solutions (x, y, z, w) of the system of equations

$$
\begin{aligned}
2y &= x + \frac{17}{x} \\
2z &= y + \frac{17}{y} \\
2w &= z + \frac{17}{z} \\
2x &= w + \frac{17}{w}
\end{aligned}
$$

is

(A) 1   (B) 2   (C) 4   (D) 8   (E) 16

10. Solve for x:
$$
2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}.
$$

11. Solve
$$
(x - 7)(x - 3)(x + 5)(x + 1) = 1680.
$$

12. Solve
$$
x^4 + x^3 - 4x^2 + x + 1 = 0.
$$

13. Solve the equation
$$
2^{\sin^2 x} + 5 \cdot 2^{\cos^2 x} = 7.
$$

14. How many real solutions are there to
$$
\sin x = \log_e x?
$$

15. Solve the equation
$$
|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2.
$$

16. Find the real solutions of
$$
\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1.
$$

17. Solve the equation
$$
6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0.
$$

18. Solve the equation
$$
x(2x + 1)(x - 2)(2x - 3) = 63.
$$

19. Find (by hand, without using a calculator) the value of

$$\sqrt{30 \cdot 31 \cdot 32 \cdot 33 + 1}.$$

20. Solve

$$\frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} + \frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}} = 98.$$

21. Find a real solution to

$$(x^2 - 9x - 1)^{10} + 99x^{10} = 10x^9(x^2 - 1).$$

   Hint: Write this equation as

$$(x^2 - 9x - 1)^{10} - 10x^9(x^2 - 9x - 1) + 9x^{10} = 0.$$

22. Solve the equation

$$\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{\vdots}{1 + \cfrac{1}{x}}}}} = x.$$

   where the fraction is repeated n times.

23. Solve for x

$$\sqrt{x + \sqrt{x + 11}} + \sqrt{x + \sqrt{x - 11}} = 4.$$

24. Let a,b, c be real constants, $abc \neq 0$. Solve

$$x^2 - (y - z)^2 = a^2,$$
$$y^2 - (z - x)^2 = b^2,$$
$$z^2 - (x - y)^2 = c^2.$$

25. Solve

$$x^3 + 3x^2y + y^3 = 8,$$
$$2x^3 - 2x^2y + xy^2 = 1.$$

26. Solve the system

$$x + 2 + y + 3 + \sqrt{(x + 2)(y + 3)} = 39,$$
$$(x + 2)^2 + (y + 3)^2 + (x + 2)(y + 3) = 741.$$

27. Solve the system
$$x^4 + y^4 = 82,$$
$$x - y = 2.$$

28. Solve the system
$$x^2 + x + y = 8,$$
$$y^2 + 2xy + z = 168,$$
$$z^2 + 2yz + 2xz = 12480.$$