Notes on Euclidean Geometry

Kiran Kedlaya
based on notes for the Math Olympiad Program (MOP)
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Please send all comments and corrections to the author at `kedlaya@math.mit.edu`. Thank you!

Contents

Introduction

This book is a compilation and distillation of my notes, as participant and later as instructor, from the Math Olympiad Program (MOP), the annual summer program to prepare U.S. high school students for the International Mathematical Olympiad (IMO). As such, it has an overt and a covert mission. The overt mission is to assist students in preparing for the USA Mathematical Olympiad (USAMO) and the IMO, as American students have historically fared poorly on problems in Euclidean geometry. The covert, and perhaps more important, mission is to introduce said students (and anyone else who happens to read this) to a lineage of mathematics stretching from ancient times to the present; hence the inclusion of advanced topics in inversive and projective geometry which may segue into the study of complex analysis, algebraic geometry, or the like.

The model for this book has been the slender classic *Geometry Revisited* by H.S.M. Coxeter and S. Greitzer, with which many American IMO participants, myself included, have supplemented their education in Euclidean geometry. We have gone further by including some topics neglected there (the circle of Apollonius, directed angles, concurrent perpendiculars) and providing numerous problems. Think of this book as *"Geometry Revisited" Revisited*, if you will.

Some words about terminology are in order at this point. For the purposes of this book, a *theorem* is an important result which I either prove or tell you where a proof can be found (usually the former, but a few of the proofs would take me too far afield). A *corollary* is a result which is important in its own right, but is easily deduced from a nearby theorem. A *fact* is a result which is important and/or useful, but whose proof is simple enough to be left as an *exercise*. On the other hand, while an exercise is usually a routine application of standard techniques, a *problem* requires some additional insight.

I have attributed my source for each problem to the best of my knowledge. USAMO, IMO and Putnam problems are listed by year and number; problems from other national contests (as well as the Kürschák contest of Hungary, the Balkan Mathematical Olympiad and others) are listed by country and year. "Arbelos" refers to Samuel Greitzer's student publication from 1982–1987, and "Monthly" to the *American Mathematical Monthly*. Problems listed as "Original" are my own problems which have not before appeared in print.

The reader stuck on a problem can consult the hints section in the back, but we strongly advise not doing so before making a considerable effort to solve the problem unaided. Note that many of the problems presented have solutions other than those we describe, so it is probably worth reading the hints even for problems which you solve on your own, in case you've found another approach.

No list of acknowledgments could possibly include everyone who has made helpful suggestions, but at least I should mention those from whom I learned the subject via MOP: Titu Andreescu, Răzvan Gelca, Anne Hudson, Gregg Patruno, and Dan Ullman. Thanks also to the participants of the 1997, 1998 and 1999 MOPs for suffering through preliminary versions

of this book.

Chapter 1

Tricks of the trade

We begin with a chapter that highlights a small core of basic techniques that prove useful in a large number of problems. The point is to show how much one can accomplish even with very little advanced knowledge.

1.1 Slicing and dicing

One of the most elegant ways of establishing a geometric result is to dissect the figure into pieces, then rearrange the pieces so that the result becomes obvious. The quintessential example of this technique is the ancient Indian proof of the Pythagorean theorem.

Theorem 1.1 (Pythagoras). *If ABC is a right triangle with hypotenuse BC, then $AB^2 + BC^2 = AC^2$.*

Proof. Behold! DIAGRAM. □

Other useful dissections include a proof of the fact that the area of a triangle is half its base times its height DIAGRAM, a proof that the median to the hypotenuse of a right triangle divides the triangle into two isosceles triangles DIAGRAM, and in three dimensions, an embedding of a tetrahedron in a box (rectangular parallelepiped) DIAGRAM.

Problems for Section 1.1

1. (MOP 1997) Let Q be a quadrilateral whose side lengths are a, b, c, d, in that order. Show that the area of Q does not exceed $(ac + bd)/2$.

2. Let ABC be a triangle and M_A, M_B, M_C the midpoints of the sides BC, CA, AB, respectively. Show that the triangle with side lengths AM_A, BM_B, CM_C has area $3/4$ that of the triangle ABC.

3. In triangle ABC, points D, E, F are marked on sides BC, CA, AB, respectively, so that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2.$$

   Show that the triangle formed by the lines AD, BE, CF has area $1/7$ that of the triangle ABC.

4. In the hexagon $ABCDEF$, opposite sides are equal and parallel. Prove that triangles ACE and BDF have the same area.

5. On the Scholastic Aptitude Test (an American college entrance exam), students were once asked to determine the number of faces of the polyhedron obtained by gluing a regular tetrahedron to a square pyramid along one of the triangular faces. The answer expected by the test authors was 7, since the two polyhedra have 9 faces, 2 of which are removed by gluing, but a student taking the exam pointed out that this is incorrect. What is the correct answer?

6. A regular tetrahedron and a regular octahedron have edges of the same length. What is the ratio between their volumes?

1.2 Angle chasing

A number of problems in Euclidean geometry can be solved by careful bookkeeping of angles, which allows one to detect similar triangles, cyclic quadrilaterals, and the like. We offer as an example of such "angle chasing" a theorem attributed to Miquel. DIAGRAM.

Theorem 1.2. *Let ABC be a triangle and let P, Q, R be any points on the sides BC, CA, AB, respectively. Then the circumcircles of ARQ, BPR, CQP pass through a common point.*

Proof. Let T be the intersection of the circumcircles of ARQ and BPR. By collinearity of points,

$$\angle TQA = \pi - \angle CQT, \quad \angle TRB = \pi - \angle ART, \quad \angle TPC = \pi - \angle BPT.$$

In a cyclic quadrilateral, opposite angles are supplementary. Therefore

$$\angle TQA = \pi - \angle ART, \angle TRB = \pi - \angle BPT.$$

We conclude $\angle TPC = \pi - \angle CQT$. Now conversely, a quadrilateral whose opposite angles are supplementary is cyclic. Therefore T also lies on the circumcircle of CQP, as desired. $\square$

A defect of the technique is that the relevant theorems depend on the configuration of the points involved, particularly on whether certain points fall between certain other points. For example, one might ask whether the above theorem still holds if P, Q, R are allowed to lie on the extensions of the sides of ABC. It does hold, but the above proof breaks down because some of the angles claimed to be equal become supplementary, and vice versa. DIAGRAM.

Euclid did not worry much about configuration issues, but later observers realized that carelessness in these matters could be disastrous. For example, consider the following pseudotheorem from [7].

Pseudotheorem. *All triangles are isosceles.*

Pseudoproof. Let ABC be a triangle, and let O be the intersection of the internal angle bisector of A with the perpendicular bisector of BC. DIAGRAM (showing O INSIDE!)

Let D, Q, R be the feet of perpendiculars from O to BC, CA, AB, respectively. By symmetry across OD, $OB = OC$, while by symmetry across AO, $AQ = AR$ and $OQ = OR$. Now the right triangles ORB and OQC have equal legs $OR = OQ$ and equal hypotenuses $OB = OC$, so they are congruent, giving $RB = QC$. Finally, we conclude

$$AB = AR + RB = AQ + QC = AC,$$

and hence the triangle ABC is isosceles. □

Problems for Section 1.2

1. Where is the error in the proof of the Pseudotheorem?

2. (USAMO 1994/3) A convex hexagon $ABCDEF$ is inscribed in a circle such that $AB = CD = EF$ and diagonals AD, BE, CF are concurrent. Let P be the intersection of AD and CE. Prove that $CP/PE = (AC/CE)^2$.

3. (IMO 1990/1) Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line of E to the circle through D, E, M intersects the lines BC and AC at F and G, respectively. If $AM/AB = t$, find EG/EF in terms of t.

1.3 Sign conventions

Euclid's angles, lengths and areas were always nonnegative quantities, but often it is more convenient to allow negative quantities in a systematic fashion, so that equations like

$$AB + BC = AC$$

hold without conditions on the relative positions of the objects involved. In this section, we describe standard sign conventions, their advantages (in avoiding configuration dependencies and errors as in the Pseudoproof) and limitations.

When measuring segments on or parallel to a given line, one may speak of *directed lengths* by fixing a choice of the positive direction, and regarding the segment AB as having positive length when the ray from A through B points in the positive direction, and negative otherwise. Of course this depends on the choice of the positive direction, but often lengths come in ratios such as AB/BC, or in pairwise products such as $AB \cdot BC$; in those cases, the choice does not matter and the directed ratio or product is well-defined.

Directed areas are less complicated than directed lengths because all areas can be directed at once. Conventionally, the area of a triangle or other polygon whose vertices are denoted in counterclockwise order is taken to be positive. This convention is generally compatible with directed lengths, though in any case you should pause for a moment to decide whether your intended manipulations make sense. For example, if A, B, C are collinear, the ratio of directed areas $[PAB]/[PBC]$ equals the ratio of directed lengths AB/BC.

Directed angles are more complicated than directed lengths or areas, and the bulk of this section will be devoted to explaining their uses and drawbacks. The *directed angle* $\angle(\ell_1, \ell_2)$ between lines ℓ_1 and ℓ_2, modulo π, is the angle of rotation required to bring ℓ_1 parallel to ℓ_2. The directed angle $\angle ABC$ made by the points A, B, C is then defined as the angle between the lines AB and BC. Pay close attention to the fact that angles are regarded modulo π and not 2π; this difference will help in many cases but has its own limitations.

One can now verify the following "axioms of directed angle arithmetic," all of which are independent of configuration.

Fact 1.3. *The following relations hold among directed angles:*

1. $\angle ABC = -\angle CBA$. *(Definition)*
2. $\angle APB + \angle BPC = \angle APC$.
3. $\angle ABC = \angle ABD$ *if and only if* B, C, D *are collinear. In particular,* $\angle ABC = 0$ *if and only if* A, B, C *are collinear.*
4. $\angle ABD = \angle ACD$ *if and only if* A, B, C, D *are concyclic.*
5. $\angle ABC = \angle ACD$ *if and only if* CD *is tangent to the circle passing through* A, B, C.
6. $\angle ABC + \angle BCA + \angle CAB = 0$. *(Angles of a triangle)*
7. $2\angle ABC = \angle AOC$ *if* A, B, C *lie on a circle centered at* O.
8. $\angle ABC$ *equals* $\frac{1}{2}$ *of the measure of the arc* $\overset{\frown}{AC}$ *of the circumcircle of* ABC.

For example, if A, B, C, D lie on a circle in that order, then we have $\angle ABD = \angle ACD$ as undirected angles. On the other hand, if theye have $\angle ABD = \pi - \angle CDA$, so in terms of directed angles

$$\angle ABD = \pi - \angle CDA = -\angle CDA = \angle ADC.$$

It should be noted that this coincidence is a principal reason why one works modulo π and not 2π! (The other principal reason is of course so that collinear points always make an angle of 0.)

The last two axioms ought to raise some eyebrows, because division by 2 is a dangerous thing when working modulo π. To be precise, the equation $2\angle A = 2\angle B$ of directed angles does not imply that $\angle A = \angle B$, for the possibility also exists that $\angle A = \angle B + \pi/2$. (Those familiar with elementary number theory will recognize an analogous situation: one cannot divide by 2 in the congruence $2a \equiv 2b \pmod{c}$ when c is even.) This explains why we do not write $\angle ABC = \frac{1}{2}\angle AOC$: the latter expression is not well-defined.

On the other hand, directed arcs can be unambiguously measured mod 2π, so dividing an arc by 2 gives an angle mod π. If all of this seems too much to worry about, do not lose hope; the conventions are easily learned with a little practice.

We start with an easy example as a warmup. The result is one we will invoke in the proof of Pascal's theorem (Theorem 4.7). DIAGRAM

Theorem 1.4. *Let A and B be the intersections of circles ω_1 and ω_2. Let CD be any chord on ω_1, and let E and F be the second intersections of the lines CA and BD, respectively, with ω_2. Then EF is parallel to CD. (If $E = F$, we take EF to mean the tangent to ω_2 at E.)*

Proof. We chase angles as follows:

$$\begin{aligned}
\angle CDF &= \angle CDB && \text{(collinearity of } B, D, F) \\
&= \angle CAB && \text{(cyclic quadrilateral } ABCD) \\
&= \angle EAB && \text{(collinearity of } A, C, E) \\
&= \angle EFB && \text{(cyclic quadrilateral } ABEF).
\end{aligned}$$

Hence the lines CD and EF make the same angle with BF, and so are parallel. □

Don't forget that directed angles can be expressed in terms of lines as well as in terms of points. This alternate perspective simplifies some proofs, as in the following example. (For a situation where this diagram occurs, see Problem 4.4.3.)

Theorem 1.5. *Let ABC be a triangle. Suppose that the lines ℓ_1 and ℓ_2 are perpendicular, and meet each side (or its extension) in a pair of points symmetric across the midpoint of the side. Then the intersection of ℓ_1 and ℓ_2 is concyclic with the midpoints of the three sides. DIAGRAM.*

Proof. Let M_A, M_B, M_C be the midpoints of the sides BC, CA, AB, respectively, and let $P = \ell_1 \cap \ell_2$. Since the lines ℓ_1, ℓ_2, BC form a right triangle and M_B is the midpoint of the hypotenuse, the triangle formed by the points P, M_B and $\ell_2 \cap BC$ is isosceles with DIAGRAM

$$\angle(M_B P, \ell_2) = \angle(\ell_2, BC).$$

By a similar argument,

$$\angle(\ell_2, M_A P) = \angle(CA, \ell_2),$$

and adding these gives

$$\angle M_B P M_A = \angle BCA = \angle M_B M_C M_A$$

since the sides of the triangle $M_A M_B M_C$ are parallel to those of ABC. We conclude that M_A, M_B, M_C, P are concyclic, as desired. $\square$

Problems for Section 1.3

1. Let $A_0B_0C_0$ be a triangle and P a point. Define a new triangle whose vertices A_1, B_1, C_1 as the feet of the perpendiculars from P to B_0C_0, C_0A_0, A_0B_0, respectively. Similarly, define the triangles $A_2B_2C_2$ and $A_3B_3C_3$. Show that $A_3B_3C_3$ is similar to $A_0B_0C_0$.

2. Two circles intersect at points A and B. An arbitrary line through B intersects the first circle again at C and the second circle again at D. The tangents to the first circle at C and the second at D intersect at M. Through the intersection of AM and CD, there passes a line parallel to CM and intersecting AC at K. Prove that BK is tangent to the second circle.

3. Let C_1, C_2, C_3, C_4 be four circles in the plane. Suppose that C_1 and C_2 intersect at P_1 and Q_1, C_2 and C_3 intersect at P_2 and Q_2, C_3 and C_4 intersect at P_3 and Q_3, and C_4 and C_1 intersect at P_4 and Q_4. Show that if P_1, P_2, P_3, and P_4 lie on a line or circle, then Q_1, Q_2, Q_3, and Q_4 also lie on a line or circle. (This is tricky; see the proof of Theorem 7.2.)

1.4 Working backward

A common stratagem, when trying to prove that a given point has a desired property, is to construct a phantom point with the desired property, then reason backwards to show that it coincides with the original point. We illustrate this point with an example. DIAGRAM

Theorem 1.6. *Suppose the triangles ABC and $AB'C'$ are directly similar. Then the points A, B, C and $BB' \cap CC'$ lie on a circle.*

Proof. Since we want to show that $BB' \cap CC'$ lies on the circle through A, B, C, and analogously on the circle through A, B', C', we define the point P to be the intersection of these two circles. Then

$$\angle APB = \angle ACB = \angle AC'B' = \angle APB'$$

and so P lies on the line BB', and similarly on the line CC'. □

Problems for Section 1.4

1. (IMO 1994/2) Let ABC be an isosceles triangle with $AB = AC$. Suppose that

   1. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
   2. Q is an arbitrary point on the segment BC different from B and C;
   3. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

   Prove that OQ is perpendicular to EF if and only if $QE = QF$.

2. (Morley's theorem) Let ABC be a triangle, and for each side, draw the intersection of the two angle trisectors closer to that side. (That is, draw the intersection of the trisectors of A and B closer to AB, and so on.) Prove that these three intersections determine an equilateral triangle.

Chapter 2

Concurrence and Collinearity

This chapter is devoted to the study of two fundamental and reciprocal questions: when do three given points lie on a single line, and when do three given lines pass through a single point? The techniques we describe in this chapter will later be augmented by more sophisticated approaches, such as the radical axis theorem and the Pascal-Brianchon theorems.

We say that three points are *collinear* if they lie on a single line, and that three lines are *concurrent* if they pass through a single point, or if all three are parallel. The latter convention addresses situations where a point of concurrence has moved "to infinity"; we will formalize this in our discussion of projective geometry later.

2.1 Concurrent lines: Ceva's theorem

We begin with a simple but useful result, published in 1678 by the Italian engineer Giovanni Ceva (1647-1734). In his honor, a segment joining a vertex of a triangle to a point on the opposite side is called a *cevian*. DIAGRAM.

Theorem 2.1 (Ceva). *Let ABC be a triangle, and let P, Q, R be points on the lines BC, CA, AB, respectively. Then the lines AP, BQ, CR are concurrent if and only if (using directed lengths)*

$$\frac{BP}{PC}\frac{CQ}{QA}\frac{AR}{RB} = 1. \tag{2.1}$$

Proof. First suppose that the lines AP, BQ, CR concur at a point T. Then the ratio of lengths BP/PC is equal, by similar triangles, to the ratio of the distances from B and C to the line AP. On the other hand, that ratio is also equal to the ratio of areas $[ATB]/[CTA]$, by calculating these areas as half of base times height, with AT as the base. DIAGRAM. By this argument, we get

$$\frac{BP}{PC}\frac{CQ}{QA}\frac{AR}{RB} = \frac{[ATB]}{[CTA]}\frac{[BTC]}{[ATB]}\frac{[CTA]}{[BTC]} = 1.$$

Conversely, suppose that (2.1) holds; we will apply the trick of working backward. The lines AP and BQ meet at some point T, and the line CT meets AB at some point, which we call R'. By construction, AP, BQ, CR' are concurrent. However, using Ceva in the other direction (which we just proved), we find that

$$\frac{BP}{PC}\frac{CQ}{QA}\frac{AR'}{R'B} = 1.$$

Combining this equation with (2.1) yields

$$\frac{AR}{RB} = \frac{AR'}{R'B}.$$

Since $AR + RB = AR' + R'B = AB$, adding 1 to both sides gives

$$\frac{AB}{RB} = \frac{AB}{R'B},$$

from which we conclude that $RB = R'B$, and hence $R = R'$. $\square$

In certain cases, Ceva's Theorem is more easily applied in the following trigonometric form.

Fact 2.2 ("Trig Ceva"). *Let ABC be a triangle, and let P, Q, R be any points in the plane distinct from A, B, C, respectively. Then AP, BQ, CR are concurrent if and only if*

$$\frac{\sin\angle CAP}{\sin\angle APB}\frac{\sin\angle ABQ}{\sin\angle QBC}\frac{\sin\angle BCR}{\sin\angle RCA} = 1.$$

One can either deduce this from Ceva's theorem or prove it directly; we leave both approaches as exercises. Be careful when using Trig Ceva with directed angles, as signs matter; the ratio $\sin\angle CAP/\sin\angle APB$ must be defined in terms of angles modulo 2π, but the sign of the ratio itself only depends on the line AP, not on the choice of P on one side or the other of A.

Problems for Section 2.1

1. Suppose the cevians AP, BQ, CR meet at T. Prove that

$$\frac{TP}{AP} + \frac{TQ}{BQ} + \frac{TR}{CR} = 1.$$

2. Let ABC be a triangle and D, E, F points on sides BC, CA, AB, respectively, such that the cevians AD, BE, CF are concurrent. Show that if M, N, P are points on EF, FD, DE, respectively, then the lines AM, BN, CP concur if and only if the lines DM, EN, FP concur. (Many special cases of this question occur in the problem literature.)

3. (Hungary-Israel, 1997) The three squares ACC_1A'', $ABB_1'A'$, $BCDE$ are constructed externally on the sides of a triangle ABC. Let P be the center of $BCDE$. Prove that the lines $A'C, A''B, PA$ are concurrent.

4. (Răzvan Gelca) Let ABC be a triangle and D, E, F the points where the incircle touches the sides BC, CA, AB, respectively. Let M, N, P be points on the segments EF, FD, DE, respectively. Show that the lines AM, BN, CP intersect if and only if the lines DM, EN, FP intersect.

5. (USAMO 1995/3) Given a nonisosceles, nonright triangle ABC inscribed in a circle with center O, and let A_1, B_1, and C_1 be the midpoints of sides BC, CA, and AB, respectively. Point A_2 is located on the ray OA_1 so that ΔOAA_1 is similar to ΔOA_2A. Points B_2 and C_2 on rays OB_1 and OC_1, respectively, are defined similarly. Prove that lines AA_2, BB_2, and CC_2 are concurrent.

6. Given triangle ABC and points X, Y, Z such that $\angle ABZ = \angle XBC$, $\angle BCX = \angle YCA$, $\angle CAY = \angle ZAB$, prove that AX, BY, CZ are concurrent. (Again, many special cases of this problem can be found in the literature.)

7. Let A, B, C, D, E, F, P be seven points on a circle. Show that AD, BE, CF are concurrent if and only if

$$\frac{\sin \angle APB \sin \angle CPD \sin \angle EPF}{\sin \angle BPC \sin \angle DPE \sin \angle FPA} = -1,$$

using directed angles modulo 2π. (The only tricky part is the sign.)

2.2 Collinear points: Menelaos' theorem

When he published his theorem, Ceva also revived interest in an ancient theorem attributed to Menelaos (a mathematician from the first century A.D.[1], not the brother of Agamemnon in Homer's *Iliad*). DIAGRAM.

Theorem 2.3 (Menelaos). *Let ABC be a triangle, and let P, Q, R be points on the lines BC, CA, AB, respectively. Then P, Q, R are collinear if and only if*

$$\frac{BP}{PC}\frac{CQ}{QA}\frac{AR}{RB} = -1.$$

[1]Which stands for "Arbitrary Delineation."

Proof. Assume that P, Q, R are collinear. Let x, y, z be the directed distances from A, B, C, respectively, to the line PQR. DIAGRAM. Then $BP/PC = -y/z$ and so forth, so

$$\frac{BP}{PC}\frac{CQ}{QA}\frac{AR}{RB} = (-1)(-1)(-1)\frac{y}{z}\frac{z}{x}\frac{x}{y} = -1.$$

The converse follows by the same argument as for Ceva's Theorem. □

An important consequence of Menelaos' theorem is the following result of Desargues, which is most easily stated by introducing two pieces of terminology. Two triangles ABC and DEF are said to be *perspective from a point* if the lines AD, BE, CF are concurrent. They are said to be *perspective from a line* if the points $AB \cap DE, BC \cap EF, CD \cap FA$ are collinear.

Theorem 2.4 (Desargues). *Two triangles ABC and DEF are perspective from a point if and only if they are perspective from a line.*

Proof. We only prove that if ABC and DEF are perpective from a point, they are perspective from a line. We leave it as an exercise to deduce the reverse implication from this (stare at the diagram); we will do this again later, using duality.

Suppose that AD, BE, CF concur at O, and let $P = BC \cap EF$, $Q = CA \cap FD$, $R = AB \cap DE$. To show that P, Q, R are collinear, we want to show that

$$\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = -1$$

and invoke Menelaos' theorem. To get hold of the first term, we apply Menelaos to the points R, D, E on the sides of the triangle OAB, giving

$$\frac{AR}{RB}\frac{BD}{DO}\frac{OE}{EA} = -1.$$

Analogously,

$$\frac{BP}{PC}\frac{CE}{EO}\frac{OF}{FB} = \frac{CQ}{QA}\frac{AF}{FO}\frac{OD}{DC} = -1.$$

When we multiply these three expressions together and cancel equal terms, we get precisely the condition of Menelaos' theorem. □

Another important consequence of Menelaos' theorem is the following result of Pappus.

Theorem 2.5 (Pappus). *Let A, C, E be three collinear points and B, D, F three other collinear points. Then the points $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ are collinear.*

The proof is similar, but more complicated; we omit it, save to say that one applies Menelaos repeatedly using the triangle formed by the lines AB, CD, EF. If you can't make the cancellation work, see [2].

Note that Desargues' and Pappus' theorems only involve points and lines, with no mention of distances or angles. This makes them "theorems of projective geometry," and we will see later how projective transformations often yield simple proofs of such theorems.

Problems for Section 2.2

1. Prove the reverse implication of Desargues' theorem.

2. Let A, B, C be three points on a line. Pick a point D in the plane, and a point E on BD. Then draw the line through $AE \cap CD$ and $CE \cap AD$. Show that this line meets the line AC in a point P that depends only on A, B, C. (We will study the relationship among the points A, B, C, P further in the chapter on projective geometry.)

3. (MOP 1990) Let A, B, C be three collinear points and D, E, F three other collinear points. Let $G = BE \cap CF$, $H = AD \cap CF$, $I = AD \cap CE$. If $AI = HD$ and $CH = GF$, prove that $BI = GE$.

4. (Original) Let ABC be a triangle and P a point in its interior, not lying on any of the medians of ABC. Let A_1, B_1, C_1 be the intersections of PA with BC, PB with CA, PC with AB, respectively, and let A_2, B_2, C_2 be the intersections of B_1C_1 with BC, C_1A_1 with CA, A_1B_1 with AB, respectively. Prove that the midpoints of A_1A_2, B_1B_2, C_1C_2 are collinear. (See also Problem 4.5.2.)

2.3 Concurrent perpendiculars

Some of the special points of a triangle are constructed by drawing perpendiculars to the sides of a triangle. For example, the circumcenter can be constructed by drawing the perpendicular bisectors. It is convenient that a result analogous to Ceva's Theorem holds for perpendiculars; the analogy is so strong that we can safely leave the proof to the reader (see Problem 1). DIAGRAM.

Fact 2.6. *Let ABC be a triangle, and let P, Q, R be three points in the plane. Then the lines through P, Q, R perpendicular to BC, CA, AB, respectively, are concurrent if and only if*

$$BP^2 - PC^2 + CQ^2 - QA^2 + AR^2 - RB^2 = 0.$$

A surprising consequence is that the lines through P, Q, R perpendicular to BC, CA, AB, respectively, are concurrent if and only if the lines through A, B, C perpendicular to QR, RP, PQ, respectively, are concurrent!

Problems for Section 2.3

1. Prove that the lines AB and CD are perpendicular if and only if $AC^2 - AD^2 = BC^2 - BD^2$. (Use vectors, coordinates or Pythagoras.) Then prove Fact 2.6.

2. (Germany, 1996) Let ABC be a triangcle, and construct squares $ABB_1A_2, BCC_1B_2, CAA_1C_2$ externally on its sides. Prove that the perpendicular bisectors of the segments A_1A_2, B_1B_2, C_1C_2 are concurrent.

3. Let ABC be a triangle, ℓ a line and L, M, N the feet of the perpendiculars to ℓ from A, B, C, respectively. Prove that the perpendiculars to BC, CA, AB through L, M, N, respectively, are concurrent. Their intersection is called the *orthopole* of the line ℓ and the triangle ABC.

2.4 Additional problems

Problems for Section 2.4

1. (USAMO 1997/2) Let ABC be a triangle, and draw isosceles triangles DBC, AEC, ABF external to ABC (with BC, CA, AB as their respective bases). Prove that the lines through A, B, C perpendicular to EF, FD, DE, respectively, are concurrent. (Several solutions are possible.)

2. (MOP 1997) Let ABC be a triangle, and D, E, F the points where the incircle touches sides BC, CA, AB, respectively. The parallel to AB through E meets DF at Q, and the parallel to AB through D meets EF at T. Prove that the lines CF, DE, QT are concurrent.

3. (Stanley Rabinowitz) The incircle of triangle ABC touches sides BC, CA, AB at D, E, F, respectively. Let P be any point inside triangle ABC, and let X, Y, Z be the points where the segments PA, PB, PC, respectively, meet the incircle. Prove that the lines DX, EY, FZ are concurrent. (The diagram for this problem serves as the logo for Mathpro Press, founded by Rabinowitz.)

Chapter 3

Transformations

In geometry, it is often useful to study transformations of the plane (i.e. functions mapping the plane to itself) preserving certain properties. In fact, Felix Klein DATES went so far as to define "geometry" as the study of properties invariant under a particular set of transformations!

3.1 Rigid motions

A *rigid motion* of the Euclidean plane is a map which preserves distances; that is, if P maps to P' and Q to Q', then we have $PQ = P'Q'$. The rigid motions are the following:

- Translation: each point moves a fixed distance in a fixed direction, so that $PQQ'P'$ is always a parallelogram.
- Rotation with center O and angle θ: each point P maps to the point P' such that $OP = OP'$ and $\angle POP' = \theta$, where the angle is directed and measured modulo 2π (not the usual π!).
- Reflection through the line ℓ: each point P maps to the point P' such that ℓ is the perpendicular bisector of PP'.
- Glide reflection along the line ℓ: reflection through ℓ followed by a translation along ℓ.

Theorem 3.1. *Given two congruent figures, each not contained in a line, one can be mapped onto the other by a unique rigid motion. If the figures are directly congruent, the rigid motion is a translation or a rotation; if oppositely congruent, a reflection or glide reflection.*

The use throughout this chapter of the term "figure" includes the assumption that all points of the figure are distinguishable, to rule out the possibility of nontrivial maps carrying the figure to itself (e.g. the symmetries of a regular polygon, or a circle).

Proof. We first address the uniqueness. If there were two rigid motions carrying the first figure to the second, then composing one with the inverse of the other would yield a nontrivial rigid motion leaving one entire figure in place. By assumption, however, this figure contains three noncollinear points A, B, C, and a point P is uniquely determined by its distances to these three points (see Problem 1), so every point is fixed by the rigid motion, a contradiction. Thus the motion is unique if it exists.

Now we address existence. let A, B, C be three noncollinear points of the first figure, and A', B', C' the corresponding points of the second figure. There exists a translation mapping A to A'; following that with a suitable rotation (since $AB = A'B'$), we can ensure that B also maps to B'. Now we claim C maps either to C' or to its reflection across $A'B'$; in other words, given two points A, B and a point C not on AB, C is determined up to reflection across AB by the distances AC and BC. Of course this holds because this data fixes C to lie on two distinct circles.

Now if P is any point of the first figure, then P is uniquely determined by the distances AP, BP, CP, and so it must map to the corresponding point of the second figure. This completes the proof of existence.

For the final assertion, first assume triangles ABC and $A'B'C'$ are directly congruent. If the perpendicular bisectors of AA' and BB' are parallel, then $ABB'A'$ is a parallelogram, so there is a translation taking A to A' and B to B'. Otherwise, let these perpendicular bisectors meet at O. Draw the circle through B and B' centered at O; there are (at most) two points on this circle whose distance to A' is the length AB. One point is the reflection of B across the perpendicular bisector of AA'; by our assumption, this cannot equal B'. Thus B' is the other point, which is the image of B under the rotation about O taking B to B'. DIAGRAM.

In either case, we have a direct rigid motion taking A to A' and B to B'; this rigid motion must also take C to C', completing the proof in this case.

Next, assume ABC and $A'B'C'$ are oppositely congruent. The lines through which AB reflects to a line parallel to $A'B'$ form two perpendicular families of parallel lines. In each family there is one line passing through the midpoint of AA'; the glide reflection through this line taking A to A' takes B either to B' or to its half-turn about A'. In the latter case, switching to the other family gives a glide reflection taking B to B'. As in the first case, C automatically goes to C', and we are done. $\square$

In particular, the composition of two rotations is either a rotation or translation. In fact, one can say more.

Fact 3.2. *The composition of a rotation of angle θ_1 with a rotation of angle θ_2 is a rotation of angle $\theta_1 + \theta_2$ if this is not a multiple of 2π, and a translation otherwise.*

On the other hand, given two rotations, it is not obvious where the center of their composition is; in particular, it generally depends on the order of the rotations. (In fancy language, the group of rigid motions is not commutative.)

Problems for Section 3.1

1. Prove that a point not lying on AB is uniquely determined up to reflection across AB by its distances to two points A and B, and that any point at all is uniquely determined by its distances to three points A, B, C. (The latter is the "triangulation" principle used once upon a time for navigation at sea, and nowadays in the Global Positioning System.)

2. (MOP 1997) Consider a triangle ABC with $AB = AC$, and points M and N on AB and AC, respectively. The lines BN and CM intersect at P. Prove that MN and BC are parallel if and only if $\angle APM = \angle APN$.

3. (IMO 1986/2) A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a sequence of points $P_1, P_2, P_3, \dots$ such that P_{k+1} is the image of P_k under rotation with center A_{k+1} through angle 120° clockwise (for $k = 0, 1, 2, \dots$). Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

4. (MOP 1996) Let $AB_1C_1, AB_2C_2, AB_3C_3$ be directly congruent equilateral triangles. Prove that the pairwise intersections of the circumcircles of triangles $AB_1C_2, AB_2C_3, AB_3C_1$ form an equilateral triangle congruent to the first three.

3.2 Homothety

Having classified the rigid motions, we move now to the *similarities*, those transformations which preserve lengths up to a scalar factor. That is, $P'Q'/PQ$ does not depend on P or Q. An important class of similarities not yet encountered are the homotheties, which are sufficiently useful for problems that we shall devote an entire section to them.

Given a point P and a real number r, the *homothety* with center P and ratio r maps each point P maps to the point P' on the ray OP such that $OP' = rOP$. We use the usual directed lengths convention, so r may be positive or negative. DIAGRAM.

Homotheties have the property that they map every side of a figure to a parallel segment. Aside from translations (which might be thought of as degenerate homotheties with center "at infinity"), this property characterizes homotheties; the following theorem is often useful as a concurrence criterion.

Theorem 3.3. *Two directly similar but not congruent figures with corresponding sides parallel are homothetic. In particular, the lines AA', where A and A' are corresponding points, all pass through a common point.*

As for rotations, we conclude that the composition of two homotheties is a homothety, though again it is less than obvious where the center is!

Problems for Section 3.2

1. Given a triangle ABC, construct (with straightedge and compass) a square with one vertex on AB, one vertex on AC, and two (adjacent) vertices on BC.

2. (USAMO 1992/4) Chords $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ of a sphere meet at an interior point P but are not contained in a plane. The sphere through A, B, C, P is tangent to the sphere through A', B', C', P. Prove that $AA' = BB' = CC'$.

3. (Putnam 1996/A-2) Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

4. Given three nonintersecting circles, draw the intersection of the external tangents to each pair of the circles. Show that these three points are collinear.

3.3 Spiral similarity

A *similarity* is any transformation that takes any figure to a figure similar to it. In other words, it preserves angles up to sign (and hence collinearity) and multiplies lengths by a constant factor. We distinguish between *direct* and *opposite* similarities, which respectively preserve and reverse the sign of directed angles.

A *spiral similarity* of angle θ and ratio r centered at a point P consists of a homothety of ratio r centered at P followed by a rotation of angle θ centered at P. (The order of these operations do not matter; one easy way to see this is to express both operations in terms of complex numbers.)

One might imagine there are more complicated similarities, e.g. a homothety followed by a rotation about different centers. As it turns out, there are no more similarities left to describe! (This proof uses complex numbers; we will sketch a synthetic proof later.)

Theorem 3.4. *Every direct similarity is either a translation or a spiral similarity.*

Proof. First we show that every direct similarity can be expressed in terms of homothety, translation, and rotation. Let A and B be two points with images C and D. If we perform a homothety about A of ratio CD/AB, then a translation mapping A to C, then a suitable rotation, we get another similarity mapping A and B to C and D. On the other hand, if P is any point not on the line AB and Q and Q' are its images under the original similarity and the new similarity, then the triangles ABP, CDQ and CDQ' are all similar, implying that C, Q, Q' and D, Q, Q' are collinear, which means $Q = Q'$. In other words, the original similarity coincides with the new one.

The basic transformations can be expressed in terms of complex numbers as follows:

Translation by vector v	$z \mapsto z+v$
Homothety of ratio r, center x	$z \mapsto r(z-x)+x$
Rotation by angle θ, center x	$z \mapsto e^{i\theta}(z-x)+x$

The point is that each of these maps has the form $z \mapsto az+b$ for some complex numbers a, b, and hence all direct similarities have this form. (It is easily seen that any such map is indeed a similarity; in fact, it's the composition of a spiral similarity about 0 and a translation.)

Now let $z \mapsto az+b$ be a direct similarity. If $a=1$, clearly we have a translation by b. Otherwise, let $t = b/(1-a)$ be the unique solution of $t = at+b$. Then our map can be written $z \mapsto a(z-t)+t$, which is clearly a spiral similarity about t. □

For practical applications of the relationship between complex numbers and transformations, see Appendix A.

Problems for Section 3.3

1. (USAMO 1978/2) $ABCD$ and $A'B'C'D'$ are square maps of the same region, drawn to different scales and superimposed. Prove that there is only one point O on the small map which lies directly over point O' of the large map such that O and O' represent the same point of the country. Also, give a Euclidean construction (straightedge and compass) for O.

2. (MOP 1998) Let $ABCDEF$ be a cyclic hexagon with $AB = CD = EF$. Prove that the intersections of AC with BD, of CE with DF, and of EA with FB form a triangle similar to BDF.

3. Let C_1, C_2, C_3 be circles such that C_1 and C_2 meet at distinct points A and B, C_2 and C_3 meet at distinct points C and D, and C_3 and C_1 meet at distinct points E and F. Let P_1 be an arbitrary point on C_1, and define points $P_2, \dots, P_7$ as follows:

P_2	is the second intersection of line P_1A with C_2;
P_3	is the second intersection of line P_2C with C_3;
P_4	is the second intersection of line P_3E with C_1;
P_5	is the second intersection of line P_1B with C_2;
P_6	is the second intersection of line P_2D with C_3;
P_7	is the second intersection of line P_3F with C_1.

   Prove that $P_7 = P_1$.

3.4 Affine transformations

The last type of transformation we introduce in this chapter is the most general, at the price of preserving the least structure. However, for sheer strangeness it does not rival either inversion or projective transformations, which we shall introduce much later.

An affine transformation is most easily defined in terms of coordinates, which is the approach we shall take. An *affine transformation* is a map from the plane to itself which, in coordinates, has the form

$$(x, y) \mapsto (ax + by + c, dx + ey + f)$$

for some constants a, b, c, d, e, f. By our calculations involving complex numbers, every similarity is an affine transformations. Some more exotic affine transformations are the stretch $(x, y) \mapsto (x, cy)$ and the shear $(x, y) \mapsto (x + y, y)$.

Fact 3.5. *Any three noncollinear points can be mapped to any three other noncollinear points by a unique affine transformation.*

As an example of the use of the affine transformation, we offer the following theorem.

Theorem 3.6. *Let $ABCDE$ be a convex pentagon and let $F = BC \cap DE$, $G = CD \cap EA$, $H = DE \cap AB$, $I = EA \cap BC$, $J = AB \cap DE$. Suppose that the areas of the triangles AHI, BIJ, CJF, DFG, EGH are all equal. Then the lines AF, BG, CH, DI, EJ are all concurrent. DIAGRAM*

Proof. Everything in the theorem is affine-invariant, so we may place three of the points anywhere we want. Let us assume A, C, D form an isosceles triangle with $AC = AD$ and $\angle CDA = \pi/5$, which is to say that A, C, D are three vertices of a regular pentagon.

Our first observation is that since CJF and DFG have equal areas, so do CJD and JGD, by adding the area of CDH to both sides. By the base-height formula, this means GJ is parallel to CD, and similarly for the other sides. In particular, since ACD was assumed to be isosceles, F lies on its angle bisector at A, and J and C are the reflections of G and D across AF.

Next we want to show that B and E are mirror images across AF, so let E' and H' be the reflections of E and H, respectively. Since the lines FC and FD are mirror images, we know E' lies on BD, and similarly H' lies on AC. Suppose that $E'D < BD$, or equivalently that E is closer than B to the line CD. Then we also have $DH' < CI'$; since $CJ = DG$, we deduce $JH' < JI$. Now it is evident that the triangle $E'H'J$, being contained in BJI, has smaller area; on the other hand, it has the same area as EHG, which by assumption has the same area as BJI, a contradiction. So we cannot have $E'D < BD$, or $E'D > BD$ by a similar argument. We conclude $E'D = BD$, i.e. B and E are mirror images. DIAGRAM.

In particular, this implies that BE is parallel to CD. Since we could just as well have put B, D, E at the vertices of an isosceles triangle, we also may conclude $AC \parallel DE$ and so forth.

Now let ℓ be the line through C parallel to AD; by the above argument, we know B is the intersection of ℓ with DF. On the other hand, B is also the intersection of ℓ with the line through A parallel to CF. If we move F towards A along the angle bisector of ACD at A, the intersection of DF with ℓ moves away from C, but the intersection of the parallel to CF through A with ℓ moves closer to C. Hence these can only coincide for at most one choice of F, and of course they do coincide when $ABCDE$ is a regular pentagon. We conclude that $ABCDE$ is the image of a regular pentagon under an affine transformation, which in particular implies that AF, BG, CH, DI, EJ are concurrent. DIAGRAM $\square$

Problems for Section 3.4

1. Let AX, BY, CZ be three cevians of the triangle ABC satisfying

$$\frac{BX}{XC} = \frac{CY}{YA} = \frac{AZ}{ZB} = k,$$

   where k is a given constant greater than 1. Find the ratio of the area of the triangle formed by the three cevians to the area of ABC. (Compare your answer with Problem 1.1.3, which is the case $k = 2$.)

2. (France, 1996) Let ABC be a triangle. For any line ℓ not parallel to any side of the triangle, let G_ℓ be the centroid of the degenerate triangle whose vertices are the intersection of ℓ with BC, CA, AB (i.e. the vector average of these three points). Determine the union of the G_ℓ over all ℓ.

Chapter 4

Circular reasoning

In this chapter, we study an ostensibly simple, yet intriguing object: the circle. This chapter consists mainly of "classical" results; the modern technique of inversion, which builds on these results and yields surprising new results and proofs of its own, occupies a subsequent chapter.

4.1 Power of a point

The following is truly a theorem of Euclidean geometry: it appears in the *Elements.*

Theorem 4.1. *Given a fixed circle and a fixed point P, draw a line through P intersecting the circle at A and B. Then the product $PA \cdot PB$ depends only on P and the circle, not on the line.*

Proof. Draw another line through P meeting the circle at C and D, labeled as in one of the diagrams. Then

$$\angle PAC = \angle BAC = \angle BDC = -\angle PDB$$

as directed angles, so the triangles PAC and PDB are (oppositely) similar, giving $PA/PD = PC/PB$, or equivalently $PA \cdot PB = PC \cdot PB$. □

The quantity $PA \cdot PB$ is called the *power* of P with respect to the circle. If O is the center of the circle and r its radius, we may choose OP as our line and so express the power as

$$(OP + r)(OP - r) = OP^2 - r^2.$$

Note that the usual sign convention makes the power positive if P lies outside the circle and negative if P lies inside the circle. Also note that for P outside the circle, the limiting case $A = B$ means that PA is tangent to the circle at A.

The power-of-a-point theorem has an occasionally useful converse.

Fact 4.2. *If the lines AB and CD meet at P, and $PA \cdot PB = PC \cdot PD$ (as signed lengths), then A, B, C, D are concyclic.*

Problems for Section 4.1

1. If A, B, C, D are concyclic and $AB \cap CD = E$, prove that $(AC/BC)(AD/BD) = AE/BE$.

2. (*Mathematics Magazine*, Dec. 1992) Let ABC be an acute triangle, let H be the foot of the altitude from A, and let D, E, Q be the feet of the perpendiculars from an arbitrary point P in the triangle onto AB, AC, AH, respectively. Prove that

$$|AB \cdot AD - AC \cdot AE| = BC \cdot PQ.$$

3. Draw tangents OA and OB from a point O to a given circle. Through A is drawn a chord AC parallel to OB; let E be the second intersection of OC with the circle. Prove that the line AE bisects the segment OB.

4. (MOP 1995) Given triangle ABC, let D, E be any points on BC. A circle through A cuts the lines AB, AC, AD, AE at the points P, Q, R, S, respectively. Prove that

$$\frac{AP \cdot AB - AR \cdot AD}{AS \cdot AE - AQ \cdot AC} = \frac{BD}{CE}.$$

5. (IMO 1995/1) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

6. (USAMO 1998/2) Let ω_1 and ω_2 be concentric circles, with ω_2 in the interior of ω_1. From a point A on ω_1 one draws the tangent AB to ω_2 ($B \in \omega_2$). Let C be the second point of intersection of AB and ω_1, and let D be the midpoint of AB. A line passing through A intersects ω_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

4.2 Radical axis

Given two circles, one with center O_1 and radius r_1, the other with center O_2 and radius r_2, what is the set of points with equal power with respect to the two circles? By our explicit formula for the power of a point, this is simply the set of points P such that $PO_1^2 - r_1^2 = PO_2^2 - r_2^2$, or equivalently such that $PO_1^2 - PO_2^2 = r_1^2 - r_2^2$. By Problem 2.3.1?, this set is a straight line perpendicular to O_1O_2; we call this line the *radical axis* of the two circles.

Theorem 4.3 (Radical axis theorem). *Let $\omega_1, \omega_2, \omega_3$ be three circles. Then the radical axes of ω_1 and ω_2, of ω_2 and ω_3, and of ω_3 and ω_1 either all coincide, or are concurrent (or parallel).*

Proof. A point on two of the radical axes has equal power with respect to all three circles. Hence if two of the axes coincide, so does the third, and otherwise if any two of the axes have a common point, this point lies on the third axis as well. □

Corollary 4.4. *The common chords of three mutually intersecting circles lie on concurrent lines.*

If the radical axes coincide, the three circles are said to be *coaxal*; otherwise, the intersection of the three radical axes is called the *radical center* of the circles. (As usual, this intersection could be "at infinity", if the three lines are parallel.) There are three types of coaxal families, depending on whether the circles have zero, one, or two intersections with the common radical axis; these three cases are illustrated in the diagram below. DIAGRAM. A useful criterion for recognizing and applying the coaxal property is the following simple observation and partial converse.

Fact 4.5. *If three circles are coaxal, their centers are collinear. Conversely, if three circles pass through a common point and have collinear centers, they are coaxal.*

Like the power-of-a-point theorem, the radical axis theorem has an occasionally useful converse.

Fact 4.6. *Suppose that $ABCD$ and $CDEF$ are cyclic quadrilaterals, and the lines AB, CD, EF are concurrent. Then $EFAB$ is also cyclic.*

Problems for Section 4.2

1. Use the radical axis theorem to give another solution for Problem 2.4.1.

2. (MOP 1995) Let BB', CC' be altitudes of triangle ABC, and assume $AB \neq AC$. Let M be the midpoint of BC, H the orthocenter of ABC, and D the intersection of BC and $B'C'$. Show that DH is perpendicular to AM.

3. (IMO 1994 proposal) A circle ω is tangent to two parallel lines ℓ_1 and ℓ_2. A second circle ω_1 is tangent to ℓ_1 at A and to ω externally at C. A third circle ω_2 is tangent to ℓ_2 at B, to ω externally at D and to ω_1 externally at E. Let Q be the intersection of AD and BC. Prove that $QC = QD = QE$.

4. (India, 1995) Let ABC be a triangle. A line parallel to BC meets sides AB and AC at D and E, respectively. Let P be a point inside triangle ADE, and let F and G be the intersection of DE with BP and CP, respectively. Show that A lies on the radical axis of the circumcircles of PDG and PFE.

5. (IMO 1985/5) A circle with center O passes through the vertices A and C of triangle ABC, and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangle ABC and KBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

4.3 The Pascal-Brianchon theorems

An amazing theorem about a hexagon inscribed in a circle was published by Blaise Pascal (1623-1662). His original proof, which was favorably described by Leibniz, has unfortunately been lost; we present here an ingenious proof essentially due to Jan van Yzeren (A simple proof of Pascal's hexagon theorem, textitMonthly, December 1993).

Theorem 4.7 (Pascal). *Let $ABCDEF$ be a hexagon inscribed in a circle. Then the intersections of AB and DE, of BC and EF, and of CD and FA are collinear.*

Proof. Let $P = AB \cap DE, Q = BC \cap EF, R = CD \cap FA$. Draw the circumcircle of CFR, and extend the lines BC and DE to meet this circle again at G and H, respectively. By Theorem 1.4, we have $BE||GH$, $DE||HR$ and $AB||GR$.

Now notice that the triangles RGH and PBE have parallel sides, which means they are homothetic. In other words, the lines BG, EH, PR are concurrent, which means $BG \cap EH = Q$ is collinear with P and R, as desired. $\square$

Some time later, C.J. Brianchon (1760-1854) discovered a counterpart to Pascal's theorem for a hexagon circumscribed about a circle. We will give Brianchon's proof of his theorem, which uses the polar map to reduce it to Pascal's theorem, in Section 8.5; a direct but somewhat complicated proof can be found in [2].

Theorem 4.8 (Brianchon). *Let $ABCDEF$ be a hexagon circumscribed about a circle. Then the lines AD, BE, CF are concurrent.*

In practice, one often applies Pascal's and Brianchon's theorem in certain degenerate cases, where some of the vertices are the same. In Pascal's theorem, if we move two adjacent vertices of the hexagon very close together, the line through them approaches a tangent to the circle. (In calculus, this would be called taking a derivative!) Thus for example, in the diagram below DIAGRAM, we may conclude that $AA \cap CD, AC \cap DE, CD \cap FA$ are collinear, where AA denotes the tangent at A.

As for Brianchon's theorem, the analogous argument shows that the "vertex" between two collinear sides belongs at the point of tangency. DIAGRAM.

Problems for Section 4.3

1. What do we get if we apply Brianchon's theorem with three degenerate vertices? (We will encounter this fact again later.)

2. Let $ABCD$ be a circumscribed quadrilateral, whose incircle touches AB, BC, CD, DA at M, N, P, Q, respectively. Prove that the lines AC, BD, MP, NQ are concurrent.

3. (MOP 1995) With the same notation, let lines BQ and BP intersect the inscribed circle at E and F, respectively. Prove that ME, NF and BD are concurrent.

4. (Poland, 1997) Let $ABCDE$ be a convex quadrilateral with $CD = DE$ and $\angle BCD = \angle DEA = \pi/2$. Let F be the point on side AB such that $AF/FB = AE/BC$. Show that

$$\angle FCE = \angle FDE \quad \text{and} \quad \angle FEC = \angle BDC.$$

4.4 Simson line

The following theorem is often called Simson's theorem, but [2] attributes it to William Wallace.

Theorem 4.9. *Let A, B, C be three points on a circle. Then the feet of the perpendiculars from P to the lines AB, BC, CA are collinear if and only if P also lies on the circle.*

Proof. The proof is by (directed) angle-chasing. Let X, Y, Z be the feet of the respective perpendiculars from P to BC, CA, AB; then the quadrilaterals $PXCY, PYAZ, PZBX$ each have two right angles, and are thus cyclic. Therefore

$$\begin{aligned} \angle PXY &= \angle PCY && \text{(cyclic quadrilateral } PXCY\text{)} \\ &= \angle PCA && \text{(collinearity of } A, C, Y\text{)} \end{aligned}$$

and analogously $\angle PXZ = \angle PBA$. Now X, Y, Z are collinear if and only if $\angle PXY = \angle PXZ$, which by the above equations occurs if and only if $\angle PCA = \angle PBA$; in other words, if and only if A, B, C, P are concyclic. □

For P on the circle, the line described in the theorem is called the *Simson line* of P with respect to the triangle ABC. We note in passing that an alternate proof of the collinearity in this case can be given using Menelaos.

Problems for Section 4.4

1. Let A, B, C, P, Q be points on a circle. Show that the (directed) angle between the Simson lines of P and Q with respect to the triangle ABC equals half of the (directed) arc PQ.

2. Let A, B, C, D be four points on a circle. Prove that the intersection of the Simson line of A with respect to BCD with the Simson line of B with respect to ACD lies on the line through C and the orthocenter of ABD.

3. If A, B, C, P, Q are five points on a circle such that PQ is a diameter, show that the Simson lines of P and Q with respect to ABC intersect at a point concyclic with the midpoints of the sides of ABC.

4. Let I be the incenter of triangle ABC, and D, E, F the projections of I onto BC, CA, AB, respectively. The incircle of ABC meets the segments AI, BI, CI at M, N, P, respectively. Show that the Simson lines of any point on the incircle with respect to the triangles DEF and MNP are perpendicular.

4.5 Circle of Apollonius

The ancient geometer Apollonius DATES is most famous for his early work on conic sections. However, his name has come to be attached to another pretty geometrical construction.

Theorem 4.10. *Let A, B be any two points and $k \neq 1$ a positive real number. Then the locus of points P such that $PA/PB = k$ is a circle whose center lies on AB.*

Proof. One can show this synthetically, but the shortest proof involves introducing Cartesian coordinates such that $A = (a, 0)$ and $B = (b, 0)$. The condition $PA/PB = k$ is equivalent to $PA^2 = k^2 PB^2$, which in coordinates can be written

$$(x-a)^2 + y^2 = k^2[(x-b)^2 + y^2].$$

Combining terms and dividing through by $1 - k^2$, we get

$$x^2 + \frac{2k^2b - 2a}{1-k^2}x + y^2 = \frac{k^2b^2 - a^2}{1-k^2},$$

which is easily recognized as the equation of a circle whose center lies on the x-axis. □

This circle is called the *circle of Apollonius* corresponding to the points A, B and the ratio k. (This term usually also includes the degenerate case $k = 1$, where the "circle" becomes the perpendicular bisector of AB.)

Problems for Section 4.5

1. Use circles of Apollonius to give a synthetic proof of the classification of similarities (Theorem 3.4).

2. (Original) Let ABC be a triangle and P a point in its interior, not lying on any of the medians of ABC. Let A_1, B_1, C_1 be the intersections of PA with BC, PB with CA, PC with AB, respectively, and let A_2, B_2, C_2 be the intersections of B_1C_1 with BC, C_1A_1 with CA, A_1B_1 with AB, respectively. Prove that if some two of the circles with diameters A_1A_2, B_1B_2, C_1C_2 intersect, then they are coaxal. (The case where the circles do not meet is trickier, unless you work in the complex projective plane as described in Section 8.7.) Note that Problem 2.2.4 follows from this.

4.6 Additional problems

Problems for Section 4.6

1. (Archimedes' "broken-chord" theorem) Point D is the midpoint of arc AC of a circle; point B is on minor arc CD; and E is the point on AB such that DE is perpendicular to AB. Prove that $AE = BE + BC$.

2. The convex hexagon $ABCDEF$ is such that

$$\angle BCA = \angle DEC = \angle FAE = \angle AFB = \angle CBD = \angle EDF.$$

Prove that $AB = CD = EF$.

3. (Descartes' circle theorem) Let r_1, r_2, r_3, r_4 be the radii of four mutually externally tangent circles. Prove that

$$\sum_{i=1}^{4} \frac{2}{r_1^2} = \left(\sum_{i=1}^{4} \frac{1}{r_i} \right)^2.$$

Chapter 5

Triangle trivia

To a triangle are associated literally hundreds of special points; in this chapter, we study but a few of the more important ones.

5.1 Centroid

The following is one of the few nontrivial facts proved in standard American geometry courses. The *median* from A in the triangle ABC is the segment joining A and the midpoint of BC.

Fact 5.1. *The medians of a triangle are concurrent. Moreover, the point of concurrency trisects each median. DIAGRAM.*

One can easily show this using Ceva and Menelaos, or by performing an affine transformation making the triangle equilateral. There is also a physical interpretation: if equal masses are placed at each of the vertices of a triangle, the center of mass will lie at the centroid. We leave as an exercise the task of finding physical interpretations of the other special points (or see Section A.2).

5.2 Incenter and excenters

If P is a point inside triangle ABC, then the distances from P to the sides AB and AC are

$$PA \sin \angle PAB \quad \text{and} \quad PA \sin \angle PAC$$

and these are equal if and only if $\angle PAB = \angle PAC$, in other words, if P lies on the internal angle bisector of $\angle A$.

From this it follows that the intersection of two internal angle bisectors is equidistant from all three sides, and consequently lies on the third bisector. This intersection is the *incenter* of ABC, and its distance to any side is the *inradius*, usually denoted r. The terminology comes from the fact that the circle of radius r centered at the incenter is tangent to all three sides of ABC, and thus is called the *inscribed circle*, or *incircle*, of ABC.

Do not forget, though, that the angle A in triangle ABC has *two* angle bisectors, one internal and one external. The locus of points equidistant to the two lines AB and AC is the union of both lines, and so one might expect to find other circles tangent to all three sides. Indeed, the internal angle bisector at A concurs with the external bisectors of the other two angles (by the same argument as above); the point of concurrence is the *excenter* opposite A, and the circle centered there tangent to all three sides is the *escribed circle*, or *excircle*, opposite A.

In studying the incircle and excircles, a fundamental tool is the fact that the two tangents to a circle from an external point have the same length. This fact is equally useful is its own right, and we have included some exercises that take advantage of equal tangents. In any case, the key observation we need is that if D, E, F are the points where the incircle touches BC, CA, AB, respectively, then $AE = AF$ and so on, so a little algebra gives

$$AE = \frac{1}{2}(AE + EB + AF + FC - BD - DC).$$

This establishes the first half of the following result; the second half is no harder.

Fact 5.2. *Let $s = (a + b + c)/2$. Then the distance from A to the point where the incircle touches AB is $s - a$, and the distance from A to the point where the excircle opposite C touches AB is $s - b$. DIAGRAM*

The quantity s is often called the *semiperimeter* of the triangle ABC.

It will often be helpful to know in what ratio an angle bisector divides the opposite side. The answer can be used to give another proof of the concurrence of the angle bisectors.

Fact 5.3 (Angle bisector theorem). *If D is the foot of either angle bisector of A in triangle ABC, then (as unsigned lengths)*

$$\frac{DB}{DC} = \frac{AB}{AC}.$$

Another useful construction for studying incenters is the

Fact 5.4. *Let ABC be a triangle inscribed in a circle with center O, and let M be the second intersection of the internal angle bisector of A with the circle. The following facts are true:*

1. *MO is perpendicular to BC (that is, M is the midpoint of arc BC);*

2. *The circle centered at M passing through B and C also passes through the incenter I and the excenter I_A opposite A; that is, $MB = MI = MC = MI_A$.*

3. $OI^2 = R^2 - 2Rr$, *where* R *is the circumradius and* r *the inradius of* ABC.

Problems for Section 5.2

1. Use the angle bisector theorem to give a synthetic proof of Theorem 4.10.

2. (Arbelos) The two common external tangent segments between two nonintersecting circle cut off a segment along one of the common internal tangents. Show that all three segments have the same length.

3. (USAMO 1991/5) Let D be an arbitrary point on side AB of a given triangle ABC, and let E be the interior point where CD intersects the external common tangent to the incircles of triangles ACD and BCD. As D assumes all positions between A and B, show that E traces an arc of a circle.

4. (Iran, 1997) Let ABC be a triangle and P a varying point on the arc BC of the circumcircle of ABC. Prove that the circle through P and the incenters of PAB and PAC passes through a fixed point independent of P.

5. (USAMO 1999/6) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

6. (IMO 1992/4) In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

5.3 Circumcenter and orthocenter

The locus of points equidistant from two points A and B is the perpendicular bisector of the segment AB. Hence given a triangle ABC, the perpendicular bisectors of any two sides meet at a point equidistant from all three vertices, and this point must then lie on the third bisector. In short, the perpendicular bisectors of a triangle concur at a point called the *circumcenter* of ABC. As you may have guessed, this point is the center of the unique circle passing through the three points A, B, C, and this is known as the *circumcircle* of ABC.

A closely related point is the *orthocenter*, defined as the intersection of the altitudes of a triangle. One can apply Fact 2.6 to show that these actually concur, or one can modify the proof of the following theorem to include this concurrence as a consequence.

Theorem 5.5. *Let ABC be a triangle and O, G, H its circumcenter, centroid and orthocenter, respectively. Then O, G, H lie on a line in that order, and $2OG = GH$.*

The line OGH is called the *Euler line* of triangle ABC.

Proof. The homothety with center G and ratio $-1/2$ carries ABC to the medial triangle $A'B'C'$, where A' is the midpoint of BC and so forth. Moreover, the altitude from A' in the medial triangle coincides with the perpendicular bisector of BC (since both are perpendicular to BC and pass through A'). Hence H maps to O under the homothety, and the claim follows. □

Some of the exercises will use the following facts about the orthocenter, which we leave as exercises in angle-chasing. DIAGRAM.

Fact 5.6. *Let H_A, H_B, H_C be the feet of the altitudes from A, B, C, respectively, and let H be the orthocenter. Then the following statements hold:*

1. *The triangles $AH_BH_C, H_ABH_C, H_AH_BC$ are (oppositely) similar to ABC.*
2. *The altitudes bisect the angles of the triangle $H_AH_BH_C$ (so H is its incenter).*
3. *The reflections of H across BC, CA, AB lie on the circumcircle of ABC.*

The triangle formed by the feet of the altitudes is called the *orthic triangle*.

Problems for Section 5.3

1. Let ABC be a triangle. A circle passing through B and C intersects the sides AB and AC again at C' and B', respectively. Prove that BB', CC', HH' are concurrent, where H and H' are the orthocentes of triangles ABC and $A'B'C'$, respectively.
2. (USAMO 1990/5) An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.
3. Let ℓ be a line through the orthocenter H of a triangle ABC. Prove that the reflections of ℓ across AB, BC, and CA all pass through a common point; show also that this point lies on the circumcircle of ABC.
4. (Bulgaria, 1997) Let ABC be a triangle with orthocenter H, and let M and K denote the midpoints of AB and CH. Prove that the angle bisectors of $\angle CAH$ and $\angle CBH$ meet at a point on the line MK.

5. Let ABC be a triangle with orthocenter H. Define the following points:

   let M_A, M_B, M_C be the midpoints of the sides BC, CA, AB;
   let H_A, H_B, H_C be the feet of the altitudes from A, B, C;
   let A', B', C' be the midpoints of the segments HA, HB, HC.

   1. Show that the triangle $A'B'C'$ is the half-turn of the triangle $M_AM_BM_C$ about its circumcenter.
   2. Conclude that all nine points lie on a single circle, called the *nine-point circle* of ABC.
   3. Show that the center of the nine-point circle is the midpoint of OH.

5.4 Gergonne and Nagel points

These points are less famous than some of the others, but they make for a few interesting problems, so let's get straight to work.

Problems for Section 5.4

1. Prove that the cevians joining each vertex of ABC to the point where the incircle touches the opposite sign are concurrent; this point is the *Gergonne point.* Also show that the cevians joining each vertex to the point where the excircle opposite that vertex touches the opposite side are concurrent; this point is the *Nagel point.*

2. In triangle ABC, let G, I, N be the centroid, incenter, and Nagel point, respectively. Show that G, I, N lie on a line in that order, and that $NG = 2 \cdot IG$. (The proof is analogous to that for the Euler line, but somewhat trickier.)

3. Let P, Q, R be the midpoints of sides BC, CA, AB, respectively. Show that the incenter of PQR is the midpoint of IN. (The analogy continues!)

5.5 Isogonal conjugates

Two points P and Q inside triangle ABC are said to be *isogonal conjugates* if $\angle PAB = \angle QAC$ and so on. In other words, Q is the reflection of P across each of the internal angle bisectors of ABC.

Problems for Section 5.5

1. Prove that every point inside triangle ABC has an isogonal conjugate. What happens if we allow points outside the triangle?

2. Prove that the orthocenter and the circumcenter are isogonal conjugates.

3. The isogonal conjugate of the centroid is called the *Lemoine point*. Draw through the Lemoine point a line parallel to each side of the triangle, and consider its intersections with the other two sides. Show that these six points are concyclic.

4. A cevian through the Lemoine point is called a *symmedian*. Show that the tangents to the circumcircle of a triangle at two vertices intersect on the symmedian of the third vertex. (Don't forget circles of Apollonius!)

5. Let D, E, F be the feet of the symmedians of triangle ABC. Prove that the Lemoine point of ABC is the centroid of DEF.

5.6 Brocard points

The exercises in this section establish the existence and several properties of the Brocard points.

Problems for Section 5.6

1. Show that inside any triangle ABC, there exists a point P such that
$$\angle PAB = \angle PBC = \angle PCA.$$

2. Show that the point P of the previous exercise is unique. It is one of the *Brocard points* of ABC; the other is the isogonal conjugate of P, which satisfies similar relations with the vertices in reverse order.

3. Let ω be the angle such that
$$\cot \omega = \cot A + \cot B + \cot C.$$
Show that the *Brocard angle* $\angle PAB$ is equal to ω.

4. Show that the maximum Brocard angle is $30°$, achieved only by an equilateral triangle. (We will have more to say about this in the section on geometric inequalities.)

5. If K is the area of triangle ABC, show that
$$a^2 + b^2 + c^2 \geq 4\sqrt{3}K$$
by expressing the Brocard angle in terms of a, b, c, K.

6. (IMO 1991/5) Prove that inside any triangle ABC, there exists a point P such that one of the angles $\angle PAB, \angle PBC, \angle PCA$ measures at most $30°$.

5.7 Miscellaneous

Here are a few additional problems concerning triangle trivia. Before proceeding to the problems, we state as facts a few standard formulae for the area of a triangle.

Fact 5.7. *Let ABC be a triangle with side lengths $a = BC, b = CA, c = AB$, inradius r, circumradius R, exradius opposite A r_A, semiperimeter s, and area K. Then*

$$\begin{aligned} K &= \frac{1}{2}ab\sin C \quad \textit{(Law of Sines)} \\ &= \frac{abc}{4R} \quad \textit{(by Extended Law of Sines)} \\ &= rs = r_A(s-a) \\ &= \sqrt{s(s-a)(s-b)(s-c)}. \quad \textit{(Heron's formula)} \end{aligned}$$

Problems for Section 5.7

1. Let D be a point on side BC, and let $m = BD, n = CD$ and $t = AD$. Prove *Stewart's formula*:

$$m^2b + n^2c = a(t^2 + mn).$$

   DIAGRAM.

2. Use Stewart's formula to prove the Steiner-Lehmus theorem: a triangle with two equal angle bisectors must be isosceles. (A synthetic proof is possible but not easy to find.)

3. (United Kingdom, 1997) In acute triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral. Also, what happens if ABC is not acute?

4. The point D lies inside the acute triangle ABC. Three of the circumscribed circles of the triangles ABC, BCD, CDA, DAB have equal radii. Prove that the fourth circle has the same radius, and characterize all such sets of four points. (What happens if ABC need not be acute, or D need not lie in its interior?)

5. (Bulgaria, 1997) Let ABC be a triangle and let M, N be the feet of the angle bisectors of B, C, respectively. Let D be the intersection of the ray MN with the circumcircle of ABC. Prove that

$$\frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}.$$

6. Let $ABCDE$ be a cyclic pentagon such that $r_{ABC} = r_{AED}$ and $r_{ABD} = r_{ACE}$, where r_{XYZ} denotes the inradius of triangle XYZ. Prove that $AB = AE$ and $BC = DE$.

7. (MOP 1990) Let AA_1, BB_1, CC_1 be the altitudes in an acute-angled triangle ABC, and let K and M be points on the line segments A_1C_1 and B_1C_1, respectively. Prove that if the angles MAK and CAA_1 are equal, then the angle C_1KM is bisected by AK.

Chapter 6

Quadrilaterals

6.1 General quadrilaterals

There's not a great deal that can be said about an arbitrary quadrilateral—the extra freedom in placing an additional vertex disrupts much of the structure we found in triangles. What little there is to say we offer in the form of a few exercises.

Problems for Section 6.1

1. Prove that the midpoints of the sides of any quadrilateral form a parallelogram (known as the *Varignon parallelogram*).
2. Let $ABCD$ be a convex quadrilateral, and let θ be the angle between the diagonals AC and BD. Prove that
$$K_{ABCD} = \frac{1}{2} AC \cdot BD \sin\theta.$$
3. Derive a formula for the area of a convex quadrilateral in terms of its four sides and a pair of opposite angles.

6.2 Cyclic quadrilaterals

The condition that the four vertices of a quadrilateral lie on a circle gives rise to a wealth of interesting structures, which we investigate in this section. We start with a classical result of Ptolemy.

Theorem 6.1 (Ptolemy). *Let $ABCD$ be a convex cyclic quadrilateral. Then*
$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

Proof. Mark the point P on BD such that $BP = (AB \cdot CD)/AC$, or equivalently $BP/AB = CD/AC$. Since $\angle ABP = \angle ACD$, the triangles ABP and ACD are similar. DIAGRAM.

On the other hand, we now have

$$\angle DPA = \pi - \angle APB = \pi - \angle ADC = \angle CBA.$$

Thus the triangles APD and ABC are also similar, yielding $DP/BC = AD/AC$. Consequently

$$BD = BP + PD = \frac{AB \cdot CD}{AC} + \frac{AD \cdot BC}{AC}$$

and the theorem follows. $\square$

This proof is elegant, but one cannot help wondering, "How could anyone think of that?" (I wonder that myself; the proof appears in an issue of Samuel Greitzer's *Arbelos*, but he gives no attribution.) The reader might enjoy attempting a proof using trigonometry or complex numbers.

Another important result about cyclic quadrilateral is an area formula attributed to the ancient Indian mathematician Brahmagupta DATES.

Fact 6.2 (Brahmagupta). *If a cyclic quadrilateral has sides a, b, c, d and area K, then*

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = (a+b+c+d)/2$ is the semiperimeter.

Heron's formula for the area of a triangle follows from Brahmagupta's formula by regarding a triangle as a cyclic quadrilateral with one side of length 0. (Brahmagupta's formula is a rare case where Western terminology attributes a result to an Eastern mathematician; the ancient Chinese and Indian discoverers of Pascal's triangle, the solution of the Pell equation, and the arctangent series were not so fortunate.)

Problems for Section 6.2

1. (Brahmagupta) Let $ABCD$ be a cyclic quadrilateral with perpendicular diagonals. Then the line through the intersection of the diagonals and the midpoint of any side is perpendicular to the opposite side.

2. Brahmagupta's formula implies that the area of a cyclic quadrilateral depends only on the lengths of the sides and not the order in which they occur. Can you demonstrate this fact by "slicing and dicing"?

3. Use Ptolemy's theorem and the previous problem to give a formula for the lengths of the diagonals of a cyclic quadrilateral in terms of the lengths of the sides.

4. Let $ABCD$ be a cyclic quadrilateral. Prove that the incenters of triangles ABC, BCD, CDA, DAB form a rectangle.

5. With the same notation, prove that the sum of the inradii of ABC and CDA equals the sum of the inradii of BCD and DAB.

6.3 Circumscribed quadrilaterals

The following theorem characterizes circumscribed quadrilaterals; while it can be proved directly using the equal tangents rule, it proves easier to exploit what we already know about incircles and excircles of triangles. DIAGRAM.

Theorem 6.3. *A convex quadrilateral $ABCD$ admits an inscribed circle if and only if $AB + CD = BC + DA$.*

Proof. Let the sides AB and CD meet at P; without loss of generality, assume A lies between P and B. (We skip the limiting case $AB||CD$.) The quadrilateral $ABCD$ has an inscribed circle if and only if the incircle of triangle PBC coincides with the excircle of triangle PDA. Let Q and R be the points of tangency of line PB with the incircle of PBC and the excircle of PDA, respectively; since both circles are tangent to the sides of the angle $\angle CPB$, they coincide if and only if $Q = R$, or equivalently $PQ = PR$. However, by the usual formulae

$$\begin{aligned} PQ &= \frac{1}{2}(PB + PC - BC) = \frac{1}{2}(PD + DC + PA + AB - BC) \\ PR &= \frac{1}{2}(PA + PD + DA) \end{aligned}$$

and these are equal if and only if $AB + CD = BC + DA$. □

Just as with triangles, a convex quadrilateral can have an escribed circle, a circle not inside the quadrilateral but tangent to all four sides (or rather their extensions). DIAGRAM. We trust the reader can now supply the proof of the analogous characterization of quadraterals admitting an escribed circle.

Fact 6.4. *A convex quadrilateral $ABCD$ admits an escribed circle opposite A or C if and only if $AB + BC = CD + DA$.*

For more problems about circumscribed quadrilaterals, flip back to Section 4.3, where we study them using Brianchon's theorem.

Problems for Section 6.3

1. (IMO 1962/5) On the circle K there are given three distinct points A, B, C. Construct (using only straightedge and compass) a fourth point D on K such that a circle can be inscribed in the quadrilateral thus obtained.

2. (Dick Gibbs) Let $ABCD$ be a quadrilateral inscribed in an ellipse, and let $E = AB \cap CD$ and $F = AD \cap BC$. Show that $ACEF$ can be inscribed in a hyperbola with the same foci as the ellipse. (If you're not familiar with ellipses and hyperbolae, peek ahead to Section 8.3.)

3. (USAMO 1998/6) Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1 A_2 \cdots A_n$ for which exactly k of the quadrilaterals $A_i A_{i+1} A_{i+2} A_{i+3}$ have an inscribed circle, where $A_{n+j} = A_j$.

6.4 Complete quadrilaterals

A *complete quadrilateral* is the figure formed by four lines, no two parallel and no three concurrent; the *vertices* of a complete quadrilateral are the six pairwise intersections of the lines. This configuration has been widely studied; we present here as exercises a number of intriguing properties of the diagram.

In the following exercises, let $ABCDEF$ be the complete quadrilateral formed by the lines ABC, AEF, DBF, DEC. DIAGRAM

Problems for Section 6.4

1. Show that the circles with diameters AD, BE, CF are coaxal. Deduce that the midpoints of the segments AD, BE, CF are collinear. (Can you show the latter directly?)

2. Show that the circumcircles of the triangles ABF, ACE, BCD, DEF pass through a common point. (Many solutions are possible.)

3. We are given five lines in the plane, no two parallel and no three concurrent. To every four of the lines, associate the point whose existence was shown in the previous exercise. Prove these five points lie on a circle. (This assertion and the previous one belong to an infinite chain of such statements: see W.K. Clifford, *Collected Papers* (1877), 38-54.)

Chapter 7

Inversive Geometry

One of the most stunning products of the revival of Euclidean geometry in the 19th century is the method of inversion, introduced by L.J. Magnus in 1831. The power of inversion lies in its ability to convert statements about circles into statements about lines, often reducing the difficult to the trivial.

7.1 Inversion

Let O be a point in the plane and r a positive real number. The *inversion* with center O and radius r is a transformation mapping every point $P \neq O$ in the plane to the point P' on the ray $\vec{OP}$ such that $OP \cdot OP' = r^2$. Since specifying a point and a positive real number is the same as specifying a circle (the point and the positive real corresponding to the center and radius, respectively, of the circle), we can also speak of inversion through a circle using the same definition. DIAGRAM

What happens to the point O? Points near O get sent very far away, in all different directions, so there is no good place to put O itself. To rectify this, we define the *inversive plane* as the usual plane with one additional point, called the *point at infinity.* (We will use the label ∞ throughout this chapter for the point at infinity.) We extend inversion to the entire inversion plane by declaring that O and ∞ are inverses of each other.

As an aside, we note a natural interpretation of the inversive plane. Under stereographic projection (used in some maps), the surface of a sphere, minus the North Pole, is mapped to a plane tangent to the sphere at the South Pole as follows: a point on the sphere maps to the point on the plane collinear with the given point and the North Pole. Then the point at infinity corresponds to the North Pole, and the inversive plane corresponds to the whole sphere. In fact, inversion through the South Pole with the appropriate radius corresponds to reflecting the sphere through the plane of the equator! DIAGRAM.

Returning to Euclidean geometry, we now establish some important properties of inversion. We first make an easy but important observation.

Fact 7.1. *If O is the center of an inversion taking P to P' and Q to Q', then the triangles OPQ and $OQ'P'$ are oppositely similar.*

In particular, we have that $\angle OP'Q' = -\angle OQP$, a fact underlying our next proof.

Theorem 7.2. *The image of a (generalized) circle under an inversion is a (generalized) circle.*

Proof. Let A, B, C, D be four concyclic points and A', B', C', D' their images under some inversion about O. We now chase directed angles, using the similar triangles of Fact 7.1:

$$\begin{aligned}
\angle A'B'C' &= \angle A'B'O + \angle OB'C' \\
&= \angle BAO + \angle OCB \\
&= \angle BAD + \angle DAO + \angle OCD + \angle DCB \\
&= \angle DAO + \angle OCD \\
&= \angle A'D'O + \angle OD'C' \\
&= \angle A'D'C'.
\end{aligned}$$

We see that A', B', C', D' are concyclic as well. DIAGRAM. □

Notice the way the angles are broken up and recombined in the above proof. In some cases, inversion can turn a constraint involving two or more angles in different places into a constraint about a single angle, which then is easier to work with. Some examples can be found in the problems.

Inversion also turns out to "reverse the angles between lines". Since lines are sent to circles in general, we will have to define the angle between two circles to make sense of this statement.

Given two circles ω_1 and ω_2, the (directed) angle between them at one of their intersections P is defined as the (directed) angle from the tangent to ω_1 at P to the tangent of ω_2 at P. In particular, two circles are *orthogonal* if the angle between them is a right angle. Note that the angle between the two circles is only well-defined up to sign without a choice of a point of intersection, since the angle at the other intersection is reversed. (Of course, orthogonality does not depend on this choice.) Note also that a line and a circle are orthogonal if and only if the line passes through the center of the circle.

Fact 7.3. *The directed angle between circles (at a chosen intersection) is reversed under inversion.*

Distances don't fare as well under inversion, but one can say something using Fact 7.1.

Fact 7.4 (Inversive distance formula). *If O is the center of an inversion of radius r sending P to P' and Q to Q', then*

$$P'Q' = PQ \cdot \frac{r^2}{OP \cdot OQ}.$$

Problems for Section 7.1

1. Deduce Theorem 7.2 from Problem 1.3.3 (or use the above proof to figure out how to do that exercise).

2. Give another proof of Theorem 7.2 using the converse of the power-of-a-point theorem (Fact 4.2) and Fact 7.4.

3. The angle between two lines through the origin is clearly preserved under inversion. Why doesn't this contradict the fact that inversion reverses angles?

4. (IMO 1996/2) Let P be a point inside triangle ABC such that

   $$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

   Let D, E be the incenters of triangles APB, APC, respectively. Prove that AP, BD, CE meet in a point. (Many other solutions are possible; over 25 were submitted by contestants at the IMO!)

5. (IMO 1998 proposal) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360°$ and

   $$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1.$$

   Prove that

   $$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

6. Prove that the following are equivalent:

   1. The points A and B are inverses through the circle ω.
   2. The line AB and the circle with diameter AB are both orthogonal to ω.
   3. ω is a circle of Apollonius with respect to A and B.

   In particular, conclude that a circle distinct from ω is fixed (as a whole, not pointwise) by inversion through ω if and only if it is orthogonal to ω.

7. Show that a set of circles is coaxal if and only if there is a circle orthogonal to all of them. Deduce that coaxal circles remain that way under inversion. Also, try drawing a family of coaxal circles and some circles orthogonal to them; the picture is very pretty.

8. Prove that any two nonintersecting circles can be inverted into concentric circles.

7.2 The power of inversion

Steiner was able to give nearly trivial proofs of some very hard-looking statements using inversion. In this section, we take a quick look at some of his dazzling results.

We start with a classical result attributed to Pappus. The figure formed by the three semicircles is known as an *arbelos* ("shoemaker's knife" in Greek), and was a favorite of Archimedes. (It was also a favorite of one-time USA IMO coach Samuel Greitzer, who for several years authored a journal for high school students of the same name.)

Theorem 7.5 (Pappus). *Let ω be a semicircle with diameter AB. Let ω_1 and ω_2 be two semicircles externally tangent to each other at C, and internally tangent to ω at A and B, respectively. Let $C_1, C_2, \dots$ be a sequence of circles, each tangent to ω and ω_1, such that C_i is tangent to C_{i+1} and C_1 is tangent to ω_2, as in the diagram. Let r_n be the radius of C_n and d_n the distance from the center of C_n to AB. Then for all n,*

$$d_n = 2nr_n.$$

DIAGRAM

Proof. Perform an inversion with center A, and choose the radius of inversion so that C_n remains fixed. Then ω and ω_1 map to lines perpendicular to AB and tangent to C_n, and $C_{n-1}, \dots, C_1$ to a column of circles between the lines, with ω_2' at the bottom of the column. The relation $d_n = 2nr_n$ is now obvious. □

The following theorem is known as *Steiner's porism.*

Theorem 7.6. *Suppose two nonintersecting circles have the property that one can fit a "ring" of n circles between them, each tangent to the next, as in the diagram. DIAGRAM. Then one can do this starting with any circle tangent to both given circles.*

Proof. By Problem 7.1.8, a suitable inversion takes the given circles to concentric circles, while preserving tangency of circles. The result is now obvious. □

Problems for Section 7.2

1. Suppose that, in the hypotheses of Pappus' theorem, we assume that C_0 is tangent to ω, ω_1 and the line AB (instead of the semicircle ω_2). Show that in this case $d_n = (2n-1)r_n$.

2. (Romania, 1997) Let ω be a circle and AB a line not intersecting ω. Given a point P_0 on ω, define the sequence $P_0, P_1, \ldots$ as follows: P_{n+1} is the second intersection with ω of the line through B and the second intersection of the line AP_n with ω. Prove that for a positive integer k, if $P_0 = P_k$ for some choice of P_0, then $P_0 = P_k$ for any choice of P_0.

7.3 Inversion in practice

So much for the power of inversion; how is it useful for real problems? The remainder of this chapter will be devoted to several examples of how inversion can be used to solve olympiad-style problems. The paradigm will almost always be: invert the given information, invert the conclusion, and proceed to solve the new problem. Beware that in somes, though, it may be necessary to superimpose the original and inverted diagrams (as in the proof of Theorem 7.5), or to compare the original and inverted diagrams (e.g. using Fact 7.4).

A general principle behind this method is that problems with few circles are easier than those with many circles. Hence when inverting, one should find a "busy point," one with many circles and lines going through it, and invert there.

Problems for Section 7.3

1. Make up an inversion problem by reversing the paradigm: start with a result that you know, invert about some point, and see what you get. The tricky part is choosing things well enough so that the resulting problem doesn't have an obvious busy point; such a problem would be too easy!

2. Given circles C_1, C_2, C_3, C_4 such that C_i and C_{i+1} are externally tangent for $i = 1, 2, 3, 4$ (where $C_5 = C_1$). Prove that the four points of tangency are concyclic.

3. (Romania, 1997) Let ABC be a triangle, D a point on side BC and ω the circumcircle of ABC. Show that the circles tangent to ω, AD, BD and to ω, AD, DC are tangent to each other if and only if $\angle BAD = \angle CAD$.

4. (Russia, 1995) Given a semicircle with diameter AB and center O and a line which intersects the semicircle at C and D and line AB at M ($MB < MA$, $MD < MC$). Let K be the second point of intersection of the circumcircles of triangles AOC and DOB. Prove that angle MKO is a right angle.

5. (USAMO 1993/2) Let $ABCD$ be a convex quadrilateral with perpendicular diagonals meeting at O. Prove that the reflections of O across AB, BC, CD, DA are concyclic. (For an added challenge, find a non-inversive proof as well.)

6. (Apollonius' problem) Given three nonintersecting circles, how many circles are tangent to all three? And how can they be constructed with straightedge and compass?

7. (IMO 1994 proposal) The incircle of ABC touches BC, CA, AB at D, E, F, respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and touches CX and XB at Y and Z, respectively. Prove that $EFZY$ is a cyclic quadrilateral.

8. (Israel, 1995) Let PQ be the diameter of semicircle H. Circle O is internally tangent to H and tangent to PQ at C. Let A be a point on H and B a point on PQ such that AB is perpendicular to PQ and is also tangent to O. Prove that AC bisects $\angle PAB$.

9. (Ptolemy's inequality) If $ABCD$ is a convex quadrilateral, then

$$AC \cdot BD \leq AB \cdot CD + BC \cdot DA,$$

with equality if and only if $ABCD$ is cyclic. (See also Theorem A.11.)

10. (IMO 1993/2) Let A, B, C, D be four points in the plane, with C, D on the same side of line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = \pi/2 + \angle ACB$. Find the ratio $(AB \cdot CD)/(AC \cdot BD)$ and prove that the circumcircles of triangles ACD and BCD are orthogonal.

11. (Iran, 1995) Let M, N, P be the points of intersection of the incircle of $\triangle ABC$ with sides BC, CA, AB, respectively. prove that the orthocenter of $\triangle MNP$, the incenter of $\triangle ABC$, and the circumcenter of $\triangle ABC$ are collinear. (The paradigm does not hold here: invert through the incircle, then superimpose the original and inverted diagrams.)

12. (MOP 1997) Let ABC be a triangle and O its circumcenter. The lines AB and AC meet the circumcircle of triangle BOC again at B_1 and C_1, respectively. Let D be the intersection of lines BC and B_1C_1. Show that the circle tangent to AD at A and having its center on B_1C_1 is orthogonal to the circle with diameter OD.

Chapter 8

Projective Geometry

Projective geometry arose during the Renaissance, when painters switched from a flat style of drawing to perspective drawing. The artists were interested in geometric properties of figures that were invariant under "changes of perspective," such as collinearity of a set of points. The formalism of projective geometry makes a discussion of such properties possible, and exposes some remarkable facts, such as the duality of points and lines.

The reader should be warned that the material here is less "contest-oriented" than that of the preceding chapters. While several techniques (projective transformations, cross-ratio, and to a lesser extent, the polar map) can be useful on contest problems, much of the material here is considered "too advanced" for contests. This is a pity, as some of the most beautiful classical geometry appears in the projective setting. We hope even the most pragmatic reader will indulge us a bit as we take a rather brief tour through projective geometry.

8.1 The projective plane

We begin with a lengthy description of the formalism of the projective plane. The impatient reader may wish to read only the next paragraph at first, then skip to the later sections and come back to this section as needed.

The *projective plane* consists of the standard Euclidean plane, together with a set of points called *points at infinity*, one for each collection of parallel lines. We say that a line passes through the point at infinity corresponding to its direction (and no others), and that all of the points at infinity lie on a *line at infinity*. (Note that three parallel lines now indeed have a common point at infinity, which retroactively justifies our calling such lines "concurrent".)

An alternate description of the projective plane turns out to be quite useful, and corresponds more closely to the artists' conception. View the Euclidean plane as some plane in three-dimensional space, and fix a point O not on the plane (corresponding to the eye). Then each point on the plane corresponds to a line through O passing through that point, but not all lines through O correspond to points on the Euclidean plane. In fact, they correspond to the points at infinity. In other words, we can identify the projective plane with the set of lines in space passing through a fixed point.

This description also yields a natural coordinate system for the projective plane, using what are known as *homogeneous coordinates*. Each point in the projective plane can be specified with a triple of numbers $[x : y : z]$, where x, y, z are not all zero. Be careful, though: for any nonzero real number λ, $[x : y : z]$ and $[\lambda x : \lambda y : \lambda z]$ are the same point! (Hence the name "homogeneous coordinates".) The colons are meant to remind you that it is the ratios between the coordinates that are well-defined, not the individual coordinates themselves.

How are homogeneous coordinates related to the usual Cartesian coordinates on the Euclidean plane? If we embed the Euclidean plane in space as the plane $z = 1$, then the point with Cartesian coordinates (x, y) has homogeneous coordinates $[x : y : 1]$, and the points at infinity are the points of the form $[x : y : 0]$ for some x, y not both zero.

8.2 Projective transformations

The original definition of a projective transformation corresponded to the process of projecting an image in the "real world" onto an artist's canvas. Again, fix a point O in three-dimensional space, and now select two planes not passing through O. The mapping that takes each point P on the first plane to the intersection of the line OP with the second plane was defined as a projective transformation. (Do you see why this map makes sense over the whole projective plane?)

One can also give an algebraic description of projective transformations that accommodates degenerate cases slightly more easily. In terms of homogeneous coordinates, a projective transformation takes the form

$$[x : y : z] \mapsto [ax + by + cz : dx + ey + fz : gx + hy + iz],$$

where the variables $a, \ldots, i$ form an invertible 3×3 matrix. From this description it is clear that affine transformations are projective as well, since they occur when $g = h = 0$. Since we have two additional parameters (it looks like three, but by homogeneity one parameter is superfluous), the following analogue of Fact 3.5 is no surprise.

Fact 8.1. *Any four points, no three collinear, can be mapped to any other four such points by a unique projective transformation.*

The most common use of a projective transformation in problem-solving is to map a particular line to the point at infinity. (As with inversion, it pays to look for a "busy" line for this purpose.) If the statement to be proved is well-behaved under projective transformations, this can yield drastic simplifications. The "well-behaved" concepts mainly consist of incidence properties between points and lines (concurrence, collinearity, and the like); as with affine transformations, angles and distances are not preserved, nor are areas or ratios of lengths along segments (unlike affine transformations).

We demonstrate the power of projection by proving Desargues' theorem (Theorem 2.4).

Theorem 8.2. *Suppose the triangles ABC and DEF are such that the points $AB \cap DE$, $BC \cap EF$, $CA \cap FD$ are collinear. Then the lines AD, BE, CF are concurrent.*

Proof. Apply a projective transformation to place the points $AB \cap DE$ and $BC \cap EF$ at infinity. If triangles ABC and DEF are perspective from a line, they now have parallel sides and so are homothetic; thus the lines AD, BE, CF concur at the center of homothety (or at a point at infinity, in case ABC and DEF are congruent). Conversely, if the lines AD, BE, CF concur at P, consider the homothety centered at A carrying A to D. It preserves the line BE and carries the line AB to the parallel line DE through B, so it maps D to E. Similarly, the homothety maps C to F, and so CA and FD are also parallel, implying that ABC and DEF are perspective from a line. □

Beware that angles, circles, and other "metric" objects are not preserved under projection; we will learn more about getting around this difficulty later in the chapter.

Problems for Section 8.2

1. Use a projective transformation to give an alternate proof of Pappus' theorem.

2. (Original) Let $ABCDE$ be the vertices of a pentagon, and let $F = BC \cap DE$, $G = CD \cap EA$, $H = DE \cap AB$, $I = EA \cap BC$, $J = AB \cap CD$. Show that $BD \cap CE$ lies on the line AF if and only if $GH \cap IJ$ does. DIAGRAM.

8.3 A conic section

A *conic section* is classically defined as a cross-section of a right circular cone by a plane not passing through a vertex, where the cone extends infinitely far in *both* directions. The section is a called an *ellipse*, a *parabola*, or a *hyperbola*, depending on whether the angle between the plane and the axis of the cone is greater than, equal to, or less than $\pi/4$. DIAGRAM.

Theorem 8.3. *An ellipse is the locus of points whose sum of distances to two fixed points is constant. Similarly, a hyperbola is the locus of points whose (absolute) difference of distances to two fixed points is constant.*

Proof. This was already known to Apollonius, but the following clever proof is attributed to Dandelin (1794-1847). We will describe only the case of the ellipse, as the hyperbola case is similar.

Inscribe spheres in the cone on either side of the plane of the ellipse, one on the side of the vertex of the cone, tangent to the plane at A, the other tangent to the plane at B. DIAGRAM. For any point on the cone between the two spheres, the sum of the lengths of the tangents to the two spheres is clearly a constant. On the other hand, for any point on the cone also lying in the plane, the segments to A and B are also tangent to the respective spheres, so the sum of their lengths equals this constant. The result follows. $\square$

The two points alluded to in the above theorem are called *foci* (plural of *focus*). The name comes from the fact that if one has an ellipse made of a reflective material and one places a light source at one focus, all of the light rays will be "focused" at the opposite focus (see Problem 2).

In modern times, it was noted that conic sections have a nice description in terms of Cartesian coordinates. If $z^2 = x^2 + y^2$ is the equation of the cone, it is evident that any cross-section is defined by setting some quadratic polynomial in x and y to 0. Hence a conic section can alternatively be defined as the zero locus of a quadratic polynomial; one must impose mild extra conditions to avoid degenerate cases, such as a pair of lines (which geometrically arise from planes through the vertex of the cone). Unless we say otherwise, our conic sections will be required to be nondegenerate.

Here are some standard equations for the conic sections:

Ellipse	$x^2/a^2 + y^2/b^2 = 1$
Parabola	$y = ax^2 + bx + c$
Hyperbola	$x^2/a^2 - y^2/b^2 = 1$

Also, the equation $xy = 1$ defines a *rectangular* hyperbola, one with perpendicular asymptotes. (The *asymptotes* of a hyperbola are its tangent lines at its intersections with the line at infinity.)

Problems for Section 8.3

1. Given an ellipse in the plane, construct its center using only a straightedge.

2. Prove that a tangent to an ellipse makes equal (undirected) angles with the segments from the two foci to the point of tangency. DIAGRAM

3. (Erdős) An infinite set of points in the plane has the property that the distance between any two of the points is an integer. Prove that the points are all collinear.

4. Let P and Q be two points on an ellipse. Prove that there exist ellipses similar to the given one, externally tangent to each other, and internally tangent to the given ellipse at P and Q, respectively, if and only if P and Q are antipodes.

5. Use the previous exercise to prove that the maximum distance between two points on an ellipse is the length of the major axis *without* doing any calculations.

6. (Original) Prove that the convex quadrilateral $ABCD$ contains a point P such that the incircles of triangles PAB and PBC are tangent, as are those of PBC and PCD, of PCD and PDA, and of PDA and PAB, if and only if $ABCD$ has an inscribed circle.

7. Find all points on the conic $x^2 + y^2 = 1$ with *rational* coordinates x, y as follows: pick a point (x, y) with rational coordinates, and project the conic from (x, y) onto a fixed line (e.g. the line at infinity). More generally, given a single rational point on a conic, this procedure allows you to describe all such points.

8.4 Conics in the projective plane

In this section, we discuss conic sections from the point of view of projective geometry. To start, we rephrase the geometric definition of a conic section.

Fact 8.4. *A curve is a conic section if and only if it is the image of a circle under a suitable projective transformation.*

In particular, the theorems of Pascal and Brianchon continue to hold if the circle in the statement of either theorem is replaced with an arbitrary conic. From these one can deduce converse theorems, that a hexagon is inscribed in (resp. circumscribed about) a conic if and only if it satisfies the conclusion of Pascal (resp. Brianchon); thinking of Pappus' theorem, one realizes that the conics in the previous statement must be permitted to be degenerate.

We also note that the classification of conics can be restated in terms of projective geometry.

Fact 8.5. *A conic is an ellipse (or a circle) if and only if it does not meet the line at infinity. A conic is a parabola if and only if it is tangent to the line at infinity. A conic is a hyperbola if and only if it intersects the line at infinity in two distinct points.*

Problems for Section 8.4

1. Prove that a hexagon whose opposite side meets in collinear points is inscribed in a conic (which may degenerate to a pair of lines).

2. Let ABC and BCD be equilateral triangles. An arbitrary line through D meets AB at M and AC at N. Determine the acute angle between the lines BN and CM.

3. (Poncelet-Brianchon theorem) Let A, B, C be three points on a rectangular hyperbola (a hyperbola with perpendicular asymptotes). Prove that the orthocenter of the triangle ABC also lies on the hyperbola. There are other special points of ABC which must lie on this hyperbola; can you find any?

4. (*Monthly*, Oct. 1994) Let $A_1, A_2, A_3, A_4, A_5, A_6$ be a hexagon circumscribed about a conic, and form the intersections $P_i = A_i A_{i+2} \cap A_{i+1} A_{i+3}$ ($i = 1, \ldots, 6$, all indices modulo 6). Show that the P_i are the vertices of a hexagon inscribed in a conic.

5. (Arbelos) Let A, B, C be three noncollinear points. Draw ellipses E_1, E_2, E_3 with foci B and C, C and A, A and B, respectively. Show that:

    1. Each pair of ellipses meet in exactly two points, where a point of tangency counts twice. (In general, two ellipses can meet in as many as four points.)
    2. The three lines determined by these pairs of points are concurrent.

8.5 The polar map and duality

Fix a circle ω with center O. The *polar map* with respect to ω interchanges points and lines in the following manner:

1. If P is a finite point other than O, the *pole* of P is the line p through P' perpendicular to PP', where P' is the inverse of P through ω.

2. If p is a finite line not passing through O, the *polar* of p is the inverse through ω of the foot of the perpendicular from O to p.

3. If P is a point at infinity, the pole of P is the line through O perpendicular to any line through P, and vice versa.

4. If P is O, the pole of P is the line at infinity, and vice versa.

The polar map is also known as *reciprocation*. We keep the notational convention that points are labeled with capital letters and their poles with the corresponding lowercase letters. DIAGRAM.

Fact 8.6. *The polar map satisfies the following properties:*

1. *Every point is the polar of its pole, and every line is the pole of its polar.*

2. *The polar of the line through the points A and B is the intersection of the poles a and b.*

3. *Three points are collinear if and only if their poles are concurrent.*

An obvious consequence of the existence of the polar map is the duality principle.

Fact 8.7 (Duality principle). *A theorem of projective geometry remains true if the roles of points and lines are interchanged.*

For example, the dual of one direction of Desargues' theorem is the other direction.

We can now give Brianchon's original proof of his theorem, using Pascal's theorem and the polar map. There's nothing to it, really: given a hexagon circumscribed about a circle ω, apply the polar map with respect to ω. The result is a hexagon inscribed in ω, and the collinearity of the intersections of opposite sides translates back to the original diagram as the concurrence of the lines through opposite vertices.

Problems for Section 8.5

1. Make up a problem by starting with a result that you know and applying the polar map. Beware that circles not concentric with ω do not behave well under the polar map; see below.

2. State the dual of Pappus' theorem. Can you prove this directly? (A projection may help.)

3. State and prove a dual version of problem 6.3.3. Since circles do not dualize to circles, you will have to come up with a new proof!

4. (China, 1996) Let H be the orthocenter of acute triangle ABC. The tangents from A to the circle with diameter BC touch the circle at P and Q. Prove that P, Q, H are collinear.

5. Let ABC be a triangle with incenter I. Fix a line ℓ tangent to the incircle of ABC (not containing any of the sides). Let A', B', C' be points on ℓ such that

$$\angle AIA' = \angle BIB' = \angle CIC' = \pi/2.$$

   Show that AA', BB', CC' are concurrent.

6. Let A, B, C, D be four points on a circle. Show that the pole of $AC \cap BD$ with respect to this circle passes through $AB \cap CD$ and $AD \cap BC$. Use this fact to give another solution to Problem 4.2.5 (IMO 1985/6).

7. We know what happens to points and lines under the polar map, but what about a curve? If we view the curve as a *locus*, i.e. a set of points, its dual is a set of lines which form an *envelope*, i.e. they are all tangent to some curve. DIAGRAM.

   Show that the dual of a conic, under this definition, is again a conic. However, the dual of a circle need not be a circle.

8. Let ω be a (nondegenerate) conic. Show that there exists a unique map on the projective plane, taking points to lines and vice versa, satisfying the properties in Fact 8.6,

and taking each point on ω to the tangent to ω through that point. This map is known as the *polar map with respect to* ω (and coincides with the first definition if ω is a circle).

9. (IMO 1998/5) Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

8.6 Cross-ratio

From the discussion so far, it may appear that there is no useful notion of "distance" in projective geometry, for projective transformations do not preserve distances, or even ratios of distances along a line (which affine transformations do preserve). There is something to be salvaged here, though; the "ratios of ratios of distances" are preserved.

Given four collinear points A, B, C, D, the *cross-ratio* of these points is defined as the following ratio of (directed) distances:

$$\frac{AC \cdot BD}{AD \cdot BC}.$$

In case one of these points is at infinity, the definition can be extended by declaring that the ratio of two infinite distances is 1. We have left the definition where all of the points lie at infinity as an exercise.

In light of duality, we ought to be able to make this definition for four concurrent lines, and in fact we can: the cross-ratio of four lines a, b, c, d is defined as the cross-ratio of the intersections A, B, C, D of a, b, c, d with some line ℓ not passing through the point of concurrency. The cross-ratio is well-defined by the following observation, which follows from several applications of the Law of Sines.

Fact 8.8. *Let a, b, c, d be four concurrent lines and ℓ a line meeting a, b, c, d at A, B, C, D, respectively. Then*

$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{\sin\angle(a,c)\sin\angle(b,d)}{\sin\angle(b,c)\sin\angle(a,d)}.$$

Fact 8.9. *The cross-ratio is invariant under projective transformations and the polar map.*

In case the cross-ratio is -1, we say C and D are *harmonic conjugates* with respect to A and B (or vice versa). If you did Problem 2.2.2, you witnessed the most interesting property of harmonic conjugates: if P is any point not on the line and Q is any point on PC other than P or C, then $AP \cap BQ$, $AQ \cap BP$ and D are collinear. (Not surprisingly, this property is projection-invariant.)

One nice application of cross-ratios is the following characterization of conics.

Fact 8.10. *Given four points A, B, C, D, the locus of points E such that the cross-ratio of the lines AE, BE, CE, DE constant is a conic.*

Problems for Section 8.6

1. How should the cross-ratio be defined along the line at infinity?
2. Let A, B, C, D be four points on a circle. Show that for E on the circle, the cross-ratio of the lines EA, EB, EC, ED remains constant.
3. Let M be the midpoint of chord XY of a circle, and let AB and CD be chords passing through M. Let $E = AD \cap XY$ and $F = BC \cap XY$. Prove that $EM = MF$.
4. Points A, B, C, and D, in this order, lie on a straight line. A circle k passes through B and C, and AM, AN, DK, and DL are tangent to k at M, N, K, and L. Lines MN and KL intersect BC at P and Q.
    1. Prove that P and Q do not depend on k.
    2. If $AD = a$, $BC = b$, and the segment BC moves along AD, find the minimum length of segment PQ.

8.7 The complex projective plane: a glimpse of algebraic geometry

The homogeneous coordinates we have worked with so far also make sense for complex numbers, though visualizing the result is substantially harder. The set of points they define (i.e. the set of proportionality classes of ordered triples of complex numbers, not all zero) is called the *complex projective plane*. We define lines and conics in this new plane simply as the zero loci of linear and quadratic polynomials, respectively.

One handy feature of the complex projective plane is the following characterization of circles.

Fact 8.11. *A nondegenerate conic is a circle if and only if it passes through the points* $[1 : i : 0]$ *and* $[1 : -i : 0]$.

These two points are called the *circular points at infinity*, or simply the *circular points* for short.

The fact that complex circles always meet the line at infinity in two points, while real circles to not, is a symptom of the key fact that the complex numbers are *algebraically closed*, i.e. every polynomial with complex coefficients has a complex root. (This is the Fundamental Theorem of Algebra, first proved by Gauss.) This means, for example, that we have the following:

Fact 8.12. *In the complex projective plane, two conics meet in exactly four points (counting points of tangency twice).*

In fact, a more general result is true, which we will not prove.

Theorem 8.13 (Bezout). *The zero loci of two polynomials, of degrees m and n, contains exactly mn points if the loci meet transversally everywhere (i.e. at each intersection, each locus has a well-defined tangent line, and the tangent lines are distinct).*

If the loci do not meet transversally, e.g. if they are tangent somewhere, one must correctly assign multiplicities to the intersections to make the count work.

An interesting consequence of Bezout's theorem, which we will prove independently, is due to Chasles (pronounced "shell"). The zero locus of a polynomial of degree 3 is known as a *cubic curve.*

Theorem 8.14 (Chasles). *Let C_1 and C_2 be two cubic curves meeting in exactly nine distinct points. Then any cubic curve passing through eight of the points passes through the ninth point.*

Proof. The set of homogeneous degree 3 polynomials in x, y, z is a 10-dimensional vector space (check by writing a basis of monomials); let Q_1 and Q_2 be polynomials with zero loci C_1 and C_2, respectively, and let $P_1, \dots, P_9$ be the nine intersections of C_1 and C_2. Note that no four of these points lie on a line and no seven lie on a conic, or else each of C_1 and C_2 would have this line or conic as a component, and their intersection would be infinite rather than nine points.

Let d_i be the dimension of the space of degree 3 polynomials vanishing at $D_1, \dots, D_i$ (and put $d_0 = 10$); then for $i \leq 8$, d_i equals either $d_{i-1} - 1$ or d_i, the latter only if every cubic curve passing through $P_1, \dots, P_{i-1}$ also passes through P_i. However, this turns out not to be the case; see the problems. Thus $d_8 = 2$, and we already have two linearly independent polynomials in this space, namely Q_1 and Q_2. (If they were dependent, they would define the same curve, and again the intersection would be infinite.) Thus if C is a cubic curve defined by a polynomial Q that passes through $P_1, \dots, P_8$, then $Q = aQ_1 + bQ_2$ for some $a, b \in \mathbb{C}$, and so Q also vanishes at P_9, as desired. $\square$

These results are just the tip of a rather sizable iceberg. The modern subject of *algebraic geometry* is concerned with the study of zero loci of sets of polynomials in spaces of any dimension. It interacts with almost every other branch of mathematics, including complex analysis, topology, number theory, combinatorics, and mathematical physics. Unfortunately, the subject as practiced today has become technically involved; the novice should start with a book written in the "classical" style, such as Harris [5] or Shafarevich [9], before proceeding to a "modern" text such as Hartshorne [6]. (If it is not already clear from the rhapsodic tone of this section, algebraic geometry, particularly in connection with number theory, ranks among the author's main research interests.)

Problems for Section 8.7

1. Prove that the center of a circle drawn in the plane cannot be constructed with straight-edge alone.

2. Give another proof that there is a unique conic passing through any five points, using the circular points.

3. Make up a problem by taking a projective statement you know and projecting two of the points in the diagram to the circular points. (One of my favorites is the radical axis theorem—which becomes a projective statement if you replace the circles by conics through two fixed points!)

4. Deduce Pascal's theorem from Chasles' theorem applied to a certain degenerate cubic.

5. Prove that given eight or fewer points in the plane, no four on a line and no seven on a conic, one of which is labeled P, there exists a cubic curve passing through all of the points but P.

6. A cubic curve which is nondegenerate, and additionally has no *singular point* (a point where the partial derivatives of the defining homogeneous polynomial all vanish, like the point $[0:0:1]$ on the curve $y^2z = x^3 + x^2z$) is called an *elliptic curve* (the apparent misnomer occurs for historical reasons). Let E be an elliptic curve, and pick a point O on E. Define "addition" of points on E as follows: given points P and Q, let R be the third intersection of the line PQ with E, and let $P+Q$ be the third intersection of the line OR with E. Prove that $(P+Q)+R = P+(Q+R)$ for any three points P, Q, R, i.e. that "addition is associative". (If you know what a group is, show that E forms a group under addition, by showing that there exist inverses and an identity element.) For more on elliptic curves, and their role in number theory, see [10].

7. Give another solution to problem 7.2.2 using a well-chosen projective transformation in the complex projective plane.

8. One can define addition on a curve on a singular cubic in the same fashion, as long as none of the points involved is a singular point of the cubic. Use this fact to give another solution to Problem 7.2.2.

9. Let E be an elliptic curve. Show that there are exactly nine points at which the tangent line at E has a triple, not just a double, intersection with the curve (and so meets the curve nowhere else). These points are called *flexes*. Also show that the line

through any two flexes meets E again at another flex. (Hence the flexes constitute a counterexample to Problem 9.1.8 in the complex projective plane!)

10. ("projective Steiner's porism") Let ω_1 and ω_2 be two conic sections. Given a point P_0 on ω_1, let P_1 be either of the points on ω_1 such that the line P_0P_1 is tangent to ω_2. then for $n \geq 2$, define P_n as the point on ω_1 other than P_{n-2} such that $P_{n-1}P_n$ is tangent to ω_2. Suppose there exists n such that $P_0 = P_n$ for a particular choice of P_0. Show that $P_0 = P_n$ for any choice of P_0.

Chapter 9

Geometric inequalities

The subject of geometric inequalities is so vast that it suffices to fill entire books, two notable examples being the volume by Bottema et al. [1] and its sequel [8]. This chapter should thus be regarded more as a sampler of techniques than a comprehensive treatise.

9.1 Distance inequalities

A number of inequalities involve comparing lengths. Useful tools against such problems include:

- Triangle inequality: in triangle ABC, $AB + BC > BC$.
- Hypotenuse inequality: if $\angle ABC$ is a right angle, then $AC > BC$.
- Ptolemy's inequality (Problem 7.3.9): if $ABCD$ is a convex quadrilateral, then $AB \cdot CD + BC \cdot DA \geq AC \cdot BD$, with equality if and only if $ABCD$ is cyclic.
- Erdős-Mordell inequality: see Section 9.4.

Transformations can also be useful, particularly reflection. For example, to find the point P on a fixed line that minimizes the sum of the distances from P to two fixed points A and B, reflect the segment PB across the line and observe that the optimal position of P is on the line joining A to the reflection of B. DIAGRAM.

A more dramatic example along the same lines is the following solution (by H.A. Schwarz) to *Fagnano's problem*: of the triangles inscribed in a given acute triangle, which one has the least perimeter? Reflecting the triangle as shown implies that the perimeter of an inscribed triangle is at least the distance from A to its eventual image, with equality when the inscribed triangle makes equal angles with each side. As noted earlier, this occurs for the orthic triangle, which is then the desired minimum. DIAGRAM.

Problems for Section 9.1

1. For what point P inside a convex quadrilateral $ABCD$ is $PA + PB + PC + PD$ minimized?

2. (Euclid) Prove that the longest chord whose vertices lie on or inside a given triangle is the longest side. (This is intuitively obvious, but make sure your proof is complete.)

3. (Kürschák, 1954) Suppose a convex quadrilateral $ABCD$ satisfies $AB + BD \leq AC + CD$. Prove that $AB < AC$.

4. (USAMO 1999/2) Let $ABCD$ be a cyclic quadrilateral. Prove that
$$|AB - CD| + |AD - BC| \geq 2|AC - BD|.$$

5. (Titu Andreescu and Răzvan Gelca) Points A and B are separated by two rivers. One bridge is to be built across each river so as to minimize the length of the shortest path from A to B. Where should they be placed? (Each river is an infinite rectangular strip, and each bridge must be a straight segment perpendicular to the sides of the river. You may assume that A and B are separated from the intersection of the rivers by a strip wider than the two rivers combined.)

6. Prove that a quadrilateral inscribed in a parallelogram has perimeter no less than twice the length of the shorter diagonal of the parallelogram. (You may want to first consider the case where the parallelogram is a rectangle.)

7. (IMO 1993/4) For three points P, Q, R in the plane, we define $m(PQR)$ as the minimum length of the three altitudes of $\triangle PQR$. (If the points are collinear, we set $m(PQR) = 0$.)

   Prove that for points A, B, C, X in the plane,
$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

8. (Sylvester's theorem) A finite set of points in the plane has the property that the line through any two of the points passes through a third. Prove that all of the points are collinear. (As noted in exercise 8.7.9, this result is false in the complex projective plane.)

9. (IMO 1973/4) A soldier needs to check on the presence of mines in a region having the shape of an equilateral triangle. The radius of action of his detector is equal to half the altitude of the triangle. The soldier leaves from one vertex of the triangle. What path should he follow in order to travel the least possible distance and still accomplish his mission?

10. Let D be the third vertex of an equilateral triangle constructed externally on BC. For P inside the triangle, show that $PA + PB + PC \geq AD$, and determine when equality holds.

11. Suppose the largest angle of triangle ABC is not greater than $120°$. Deduce from the previous exercise that for P inside the triangle, $PA + PB + PC$ is minimized when $\angle APB = \angle BPC = \angle CPA = 120°$. The point satisfying this condition is known variously as the *Fermat point* or the *Torricelli point*.

12. (IMO 1995/5) Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \pi/3$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 2\pi/3$. Prove that $AG + GB + GH + DH + HE \geq CF$.

9.2 Algebraic techniques

Another class of methods of attack for geometric inequalities involve invoking algebraic inequalities. The most commonly used is the AM-GM inequality: for $x_1, \dots, x_n > 0$,

$$\frac{x_1 + \cdots + x_n}{n} \geq (x_1 \dots x_n)^{1/n}.$$

Often all one needs is the case $n = 2$, which follows from the fact that

$$(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0.$$

A more sophisticated result is the Cauchy-Schwarz inequality:

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \geq (x_1 y_1 + \cdots + x_n y_n)^2,$$

which one proves by noting that the difference between the left side and the right is

$$\sum_{i<j} (x_i y_j - x_j y_i)^2.$$

A trick that often makes an algebraic approach more feasible, when a problem concerns the side lengths a, b, c of a triangle, is to make the substitution

$$x = s - a, \quad y = s - b, \quad z = s - c,$$

where $s = (a + b + c)/2$. A little algebra gives

$$a = y + z, \quad b = z + x, \quad c = x + y.$$

The point is that the necessary and sufficient conditions $a+b>c, b+c>a, c+a>b$ for a,b,c to constitute the side lengths of a triangle translate into the more convenient conditions $x>0, y>0, z>0$.

Don't forget about the possibility of "algebraizing" an inequality using complex numbers; see Section A.3.

Problems for Section 9.2

1. (IMO 1988/5) ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L, respectively. S and T denote the areas of the triangles ABC and AKL, respectively. Show that $S \geq 2T$.

2. Given a point P inside a triangle ABC, let x,y,z be the distances from P to the sides BC, CA, AB. Find the point P which minimizes
$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}.$$

3. If K is the area of a triangle with sides a,b,c, show that
$$ab+bc+ca \geq 4\sqrt{3}K.$$

4. (IMO 1964/2) Suppose a,b,c are the sides of a triangle. Prove that
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc.$$

5. (IMO 1983/6) Let a,b,c be the lengths of the sides of a triangle. Prove that
$$b^2c(b-c)+c^2a(c-a)+a^2b(a-b) \geq 0.$$
(Beware: you may not assume that $a \geq b \geq c$ without loss of generality!)

6. (Balkan, 1996) Let O and G be the circumcenter and centroid of a triangle of circumradius R and inradius r. Show that $OG^2 \leq R^2 - 2Rr$. (This proves Euler's inequality $R \geq 2r$. If you don't know how to compute OG^2, see problem A.2.7.)

7. (Murray Klamkin) Let $n>2$ be a positive integers, and suppose that $a_1, \dots, a_n$ are positive real numbers satisfying the inequality
$$(a_1^2+\cdots+a_n^2)^2 > (n-1)(a_1^4+\cdots+a_n^4).$$
Show that for $1 \leq i < j < k \leq n$, the numbers a_i, a_j, a_k are the lengths of the sides of a triangle.

8. Let ABC be a triangle with inradius r and circumradius R. Prove that

$$\frac{2r}{R} \leq \sqrt{\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2}}.$$

9. (IMO 1995 proposal) Let P be a point inside the convex quadrilateral $ABCD$. Let E, F, G, H be points on sides AB, BC, CD, DA, respectively, such that PE is parallel to BC, PF is parallel to AB, PG is parallel to DA, and PH is parallel to CD. Let K, K_1, K_2 be the areas of $ABCD, AEPH, PFCG$, respectively. Prove that

$$\sqrt{K} \geq \sqrt{K_1} + \sqrt{K_2}.$$

9.3 Trigonometric inequalities and convexity

A third standard avenue of attack involves reducing a geometric inequality to an inequality involving trigonometric functions. Such inequalities can often be treated using Jensen's inequality for convex functions.

A *convex* function is a function $f(x)$ satisfying the rule

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all x, y and all $t \in [0,1]$. Geometrically, this says that the area above the graph of f is a convex set, i.e. that chords of the graph always lie above the graph. Equivalently, tangents to the graph lie below.

Those of you who know calculus can check whether f is convex by checking whether the second derivative of f (if it exists) is always positive. (In some calculus texts, a convex function is called "concave upward", or occasionally is said to "hold water".) Also, if f is continuous, it suffices to check the definition of convexity for $t = 1/2$.

The key fact about convex functions is *Jensen's inequality*, whose proof (by induction on n) is not difficult.

Fact 9.1. *Let $f(x)$ be a convex function, and let $t_1, \dots, t_n$ be nonnegative real numbers adding up to 1. Then for all $x_1, \dots, x_n$,*

$$f(t_1x_1 + \cdots + t_nx_n) \leq t_1f(x_1) + \cdots + t_nf(x_n).$$

For example, the convexity of the function $(-\log x)$ implies the AM-GM inequality.

As a simple example, note that in triangle ABC, we have $\angle A + \angle B + \angle C = \pi$, and the function $f(x) = \sin x$ is concave, so

$$\sin A + \sin B + \sin C \geq 3\sin\pi/3 = 3\sqrt{3}/2.$$

In other words, the minimum perimeter of a triangle inscribed in a fixed circle is achieved by the equilateral triangle.

Also note that convexity can be used in apparently purely geometric circumstances, thanks to the following fact. (Remember, it suffices to verify this for $t = 1/2$, which is easy.)

Fact 9.2. *The distance from a fixed point P is a convex function on the plane. That is, for any points P, Q, R, the distance from P to the point (in vector notation) $tQ + (1-t)R$ is a convex function of t.*

Problems for Section 9.3

1. (Bottema, 2.7) Show that in triangle ABC, $\sin A \sin B \sin C \leq \frac{3}{8}\sqrt{3}$.

2. Recall that the Brocard angle of triangle ABC is defined by the formula
$$\cot \omega = \cot A + \cot B + \cot C.$$
Prove that the Brocard angle never exceeds $\pi/6$. (Be careful: cot is only convex in the range $(0, \pi/2]$.)

3. (Bottema, 2.15) Let α, β, γ be the angles of a triangle. Prove that
$$\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + \sin\frac{\gamma}{2}\sin\frac{\alpha}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2} \leq \frac{3}{4}.$$

4. Prove that of the n-gons inscribed in a circle, the regular n-gon has maximum area.

5. (Bottema, 2.59) Prove that in triangle ABC,
$$1 + \cos A \cos B \cos C \geq \sqrt{3}(\sin A \sin B \sin C).$$

6. Show that for any convex polygon S, the distance from S to a point P (the length of the shortest segment joining P to a point on S) is a convex function of P.

7. (Junior Balkaniad, 1997) In triangle ABC, let D, E, F be the points where the incircle touches the sides. Let r, R, s be the inradius, circumradius, and semiperimeter, respectively, of the triangle. Prove that
$$\frac{2rs}{R} \leq DE + EF + FD \leq s$$
and determine when equality occurs.

8. (MOP 1998) If ABC is a acute triangle with circumcenter O, orthocenter H and circumradius R, show that for any point P on the segment OH,
$$PA + PB + PC \leq 3R.$$

9.4 The Erdős-Mordell inequality

The following inequality is somewhat more sophisticated than the ones we have seen so far, but is nonetheless useful. It was conjectured by the Hungarian mathematician and problemist Pál (Paul) Erdős (1913-1996) in 1935 and first proved by Louis Mordell in the same year.

Theorem 9.3. *For any point P inside the triangle ABC, the sum of the distances from P to A, B, C is at least twice the sum of the distances from P to BC, CA, AB. (Equality occurs only when ABC is equilateral and P is its center.)*

Proof. The unusually stringent equality condition should suggest that perhaps the proof proceeds in two stages, with different equality conditions. This is indeed the case.

Let X, Y, Z be the feet of the respective perpendiculars from P to BC, CA, AB. We will first prove that

$$PA \geq \frac{AB}{BC} PY + \frac{AC}{BC} PZ. \tag{9.1}$$

The only difference between most proofs of this theorem is in the proof of the above inequality. For example, rewrite (9.1) as

$$PA \sin A \geq PY \sin C + PZ \sin B,$$

recognize that $PA \sin A = YZ$ by the Extended Law of Sines, and observe that the right side is the length of the projection of YZ onto the line BC. Equality holds if and only if YZ is parallel to BC.

Putting (9.1) and its analogues together, we get

$$PA + PB + PC \geq PX \left(\frac{CA}{AB} + \frac{AB}{CA} \right) + PY \left(\frac{AB}{BC} + \frac{BC}{AB} \right) + PZ \left(\frac{BC}{CA} + \frac{CA}{BC} \right),$$

with equality if and only if XYZ is homothetic to ABC; this occurs if and only if P is the circumcenter of ABC (Problem 1). Now for the second step: we note that each of the terms in parentheses is at least 2 by the AM-GM inequality. This gives

$$PA + PB + PC \geq 2(PX + PY + PZ),$$

with equality if and only if $AB = BC = CA$. □

Problems for Section 9.4

1. With notation as in the above proof, show that the triangles XYZ and ABC are homothetic if and only if P is the circumcenter of ABC.

2. Give another proof of (9.1) by comparing P with its reflection across the angle bisector of A. (Beware: the reflection may lie outside of the triangle!)

3. Solve problem 5.6.6 using the Erdős-Mordell inequality.

4. (IMO 1996/5) Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

(A certain special case of this result is equivalent to Erdős-Mordell. Modify the proof slightly to accommodate the generalization.)

5. (Nikolai Nikolov) The incircle k of the triangle ABC touches its sides at the points A_1, B_1, C_1. For any point K on k, let d be the sum of the distances from K to the sides of the triangle $A_1B_1C_1$. Prove that $KA + KB + KC > 2d$.

9.5 Additional problems

Problems for Section 9.5

1. Prove that of all quadrilaterals with a prescribed perimeter P, the square has the greatest area. Can you also prove the analogous result for polygons with any number of sides?

2. What is the smallest positive real number r such that a square of side length 1 can be covered by three disks of radius r?

3. Let r be the inradius of triangle ABC. Let r_A be the radius of a circle tangent to the incircle as well as to sides AB and CA. Define r_B and r_C similarly. Prove that

$$r_A + r_B + r_C \geq r.$$

4. Prove that a triangle with angles α, β, γ, circumradius R, and area A satisfies

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \leq \frac{9R^2}{4A}.$$

5. Let a, b, c be the sides of a triangle with inradius r and circumradius R. Show that

$$\left|1 - \frac{2a}{b+c}\right| \leq \sqrt{1 - \frac{2r}{R}}.$$

6. Two concentric circles have radii R and R_1 respectively, where $R_1 > R$. $ABCD$ is inscribed in the smaller circle and $A_1B_1C_1D_1$ in the larger one, with A_1 on the extension of CD, B_1 on that of DA, C_1 on that of AB, and D_1 on that of BC. Prove that the ratio of the areas of $A_1B_1C_1D_1$ and $ABCD$ is at least R_1^2/R^2.

7. With the same notation, prove that the ratio of the perimeters of $A_1B_1C_1D_1$ and $ABCD$ is at least R/r.

Appendix A

Nonsynthetic methods

The idea of using algebra to solve geometric problems, in the guise of rectilinear coordinates, is variously attributed to Descartes or Fermat, though the trigonometric functions were known to the ancients. In any case, this section is devoted to nonsynthetic methods in Euclidean geometry, including trigonometry, vector geometry, complex numbers, and Cartesian coordinates. We have relegated this material to an appendix not to avoid offending the purists (who are probably offended already by the liberal use of nonsynthetic methods in the main text), but to avoid disrupting the logical sequence of the chapters.

A.1 Trigonometry

This is not a course in trigonometry; all we will do here is summarize the important facts and provide a few problems where trigonometry can or must be employed.

Fact A.1 (Law of Sines). *The area of triangle ABC equals $\frac{1}{2}ab\sin C$. In particular,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Fact A.2 (Extended Law of Sines). *If R is the circumradius of triangle ABC, then $BC = 2R\sin A$.*

Fact A.3 (Law of Cosines). *In triangle ABC,*

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Fact A.4 (Addition formulae). *The sine and cosine functions satisfy the following addition rules:*

$$\begin{aligned}
\cos(A+B) &= \cos A\cos B - \sin A\sin B\\
\cos(A-B) &= \cos A\cos B + \sin A\sin B\\
\sin(A+B) &= \sin A\cos B + \cos A\sin B\\
\sin(A-B) &= \sin A\cos B - \cos A\sin B.
\end{aligned}$$

Using the addition formulae, one can convert products of sines and cosines to sums, and vice versa.

Fact A.5 (Sum-to-product formulae).

$$\begin{aligned}
\sin A + \sin B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\\
\sin A - \sin B &= 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\\
\cos A + \cos B &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\\
\cos A - \cos B &= -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.
\end{aligned}$$

In particular, one has the double and half-angle formulas.

Fact A.6 (Double-angle formula).

$$\begin{aligned}
\sin 2A &= 2\sin A\cos A\\
\cos 2A &= 2\cos^2 A - 1 = 1 - 2\sin^2 A\\
\tan 2A &= \frac{2\tan A}{1-\tan^2 A}.
\end{aligned}$$

Fact A.7 (Half-angle formula).

$$\begin{aligned}
\sin\frac{A}{2} &= \pm\sqrt{\frac{1-\cos A}{2}}\\
\cos\frac{A}{2} &= \pm\sqrt{\frac{1+\cos A}{2}}\\
\tan\frac{A}{2} &= \csc A - \cot A.
\end{aligned}$$

The half-angle formula takes a convenient form for triangles.

Fact A.8. *In triangle ABC with sides a, b, c and semiperimeter s,*

$$\begin{aligned}\sin\frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}}\\ \cos\frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}}.\end{aligned}$$

It may be helpful at times to express certain other quantities associated with a triangle in terms of the angles.

Fact A.9. *If triangle ABC has inradius r and circumradius R, then*

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

We leave the construction of other such formulae to the reader.

Problems for Section A.1

1. For any triangle ABC, prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ and that $\cot A/2 + \cot B/2 + \cot C/2 = \cot A/2 \cot B/2 \cot C/2$.

2. Show that if none of the angles of a convex quadrilateral $ABCD$ is a right angle, then
$$\frac{\tan A + \tan B + \tan C + \tan D}{\tan A \tan B \tan C \tan D} = \cot A + \cot B + \cot C + \cot D.$$

3. Find a formula for the area of a triangle in terms of two angles and the side opposite the third angle. (More generally, given any data that uniquely determines a triangle, one can find an area formula in terms of that data. Can you come up with more examples?)

4. Use trigonometry to give another proof of Ptolemy's theorem (Theorem 6.1).

5. (USAMO 1996/5) Triangle ABC has the following property: there is an interior point P such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$ and $\angle PAC = 40^\circ$. Prove that triangle ABC is isosceles. (For an added challenge, find a non-trigonometric solution!)

6. (IMO 1985/1) A circle has center on the side AB of a cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + DC = AB$.

A.2 Vectors

A *vector* in the place can be defined either as an arrow, where addition of arrows proceeds by the "tip-to-tail" rule illustrated below, or as an ordered pair (x, y) recording the difference in the x and y coordinates between the tip and the tail. Vectors in space are defined similarly, of course using three coordinates instead of two. DIAGRAM.

It is important to remember that a vector is not a point, but rather the "difference of two points"; it encodes relative, not absolute, position. In practice, however, one chooses a point as the *origin* and identifies a point with the vector from the origin to that point. (In effect, one puts the tails of all of the arrows in one place.)

The standard operations on vectors include addition and subtraction, multiplication by real numbers (positive, negative or zero), and the *dot product*, defined geometrically as

$$\vec{A} \cdot \vec{B} = \|\vec{A}\| \cdot \|\vec{B}\| \cos \angle AOB,$$

where O is the origin, and in coordinates as

$$(a_x, a_y) \cdot (b_x, b_y) = a_x b_x + a_y b_y.$$

The key fact here is that $\vec{A} \cdot \vec{B} = 0$ if and only if $\vec{A}$ and $\vec{B}$ are perpendicular.

A more exotic operation is the *cross product*, which is defined for a pair of vectors in space as follows:

$$(a_x, a_y, a_z) \times (b_x, b_y, b_z) = (a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x).$$

Geometrically speaking, $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ and has length

$$\|\vec{A} \times \vec{B}\| = \|\vec{A}\| \cdot \|\vec{B}\| \sin \angle AOB.$$

This length equals the area of the parallelogram with vertices $0, \vec{A}, \vec{A}+\vec{B}, \vec{B}$, or twice the area of the triangle with vertices $0, \vec{A}, \vec{B}$. The sign ambiguity can be resolved by the *right-hand rule*: if you point the fingers of your right hand along $\vec{A}$, then swing them toward $\vec{B}$, your thumb points in the direction of $\vec{A} \times \vec{B}$. DIAGRAM.

Fact A.10. *The following identities hold:*

1. *Triple scalar product identity:* $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$. *Moreover, this quantity equals the volume of a parallelepiped with edges* $\vec{A}, \vec{B}, \vec{C}$.

2. *Triple cross product identity:* $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{C} \cdot \vec{A})\vec{B} - (\vec{B} \cdot \vec{A})\vec{C}$.

The vector equations for some of the special points of a triangle are summarized in the following table. The asterisked expressions assume the circumcenter of the triangle has been chosen as the origin; the origin-independent expressions are not nearly so pleasant to work with!

Circumcenter*	0
Centroid	$\frac{1}{3}(\vec{A}+\vec{B}+\vec{C})$
Orthocenter*	$\vec{A}+\vec{B}+\vec{C}$
Incenter	$\frac{1}{a+b+c}(a\vec{A}+b\vec{B}+c\vec{C})$

Problems for Section A.2

1. (Romania, 1997) Let $ABCDEF$ be a convex hexagon, and let $P = AB \cap CD, Q = CD \cap EF, R = EF \cap AB$, $S = BC \cap DE, T = DE \cap FA, U = FA \cap BC$. Prove that
$$\frac{PQ}{CD} = \frac{QR}{EF} = \frac{RP}{AB} \quad \text{if and only if} \quad \frac{ST}{DE} = \frac{TU}{FA} = \frac{US}{BC}.$$

2. Let P, Q, R be the feet of concurrent cevians in triangle ABC. Determine the vector expression for the point of concurrence in terms of the ratios BP/PC, CQ/QA, AR/RB. Use this formula to extend the above table to other special points. In particular, do so for the Nagel point and obtain an alternate solution to problem 5.4.2.

3. (Răzvan Gelca) Let $ABCD$ be a convex quadrilateral and $O = AC \cap BD$. Let M, N be points on AB so that $AM = MN = NB$, and let P, Q be points on CD so that $CP = PQ = QD$. Show that triangles MOP and NOQ have the same area.

4. (MOP 1995) Five points are given on a circle. A perpendicular is drawn through the centroid of the triangle formed by three of them, to the chord connecting the remaining two. Similar perpendiculars are drawn for each of the remaining nine triplets of points. Prove that the ten lines obtained in this way have a common point.

5. (MOP 1996) Let $ABCDE$ be a convex pentagon, and let F, G, H, I, J be the respective midpoints of CD, DE, EA, AB, BC. If AF, BG, CH, DI pass through a common point, show that EJ also passes through this point.

6. Let A, B, C, D be four points on a circle. Use the result of problem 4.4.2 to show that the Simson line of each point with respect to the triangle formed by the other three passes through the midpoint of the segment joining the center of the circle to the centroid of $ABCD$ (in particular, these four lines are concurrent).

7. Compute the distance between the circumcenter and orthocenter of a triangle in terms of the side lengths a, b, c.

8. (Austria-Poland, 1979) Let A, B, C, D be points in space, M the midpoint of AC and N the midpoint of BD. Prove that
$$4MN^2 = AB^2 + BC^2 + CD^2 + DA^2 - AC^2 - BD^2.$$

9. Show that the distance between the incenter and the nine-point center (see Problem 5.3.5) of a triangle is equal to $R/2 - r$, where r and R are inradius and circumradius, respectively. Deduce *Feuerbach's theorem*, that the incircle and nine-point circle are tangent. (Similarly, one can show the nine-point circle is also tangent to each of the excircles.)

A.3 Complex numbers

In some respects, the use of complex numbers in geometry is an extension of vector geometry. The main difference is that complex numbers come equipped with a mechanism for implementing rotation, while vectors do not. They even come equipped with a mechanism for implementing reflection across the horizontal axis, namely complex conjugation. For more about the relationship between complex numbers and transformations, see Section 3.3.

An interesting use of complex numbers is to prove inequalities, as in the following example (compare with Problem 7.3.9).

Theorem A.11 (Ptolemy's inequality). *Let A, B, C, D be four points in the plane. Then*

$$AC \cdot BD \leq AB \cdot CD + BC \cdot DA,$$

with equality if and only if the quadrilateral $ABCD$ is convex and cyclic.

Proof. Regard A, B, C, D as complex numbers; then we have an identity

$$(A - C)(B - D) = (A - B)(C - D) + (B - C)(A - D).$$

However, the magnitude of $(A - C)(B - D)$ is precisely the product of the lengths of the segments AC and BD, and likewise for the other terms. Thus the desired inequality is simply the triangle inequality applied to these three quantities! (The equality condition is left as an exercise.) $\square$

Problems for Section A.3

1. Prove that x, y, z lie at the corners of an equilateral triangle if and only if either $x + \omega y + \omega^2 z = 0$ or $x + \omega z + \omega^2 y = 0$, where $\omega = e^{2\pi i/3}$.

2. Let A, C, E be three points on a circle. A 60° rotation about the center of the circle carries A, C, E to B, D, F, respectively. Prove that the triangle whose vertices are the midpoints of BC, DE, FA is equilateral.

3. Construct equilateral triangles externally (internally) on the sides of an arbitrary triangle ABC. Prove that the centers of these three triangles form another equilateral triangle. This triangle is known as the *inner (outer) Napoleon triangle* of ABC.

4. Let P, Q, R, S be the centers of squares constructed externally on sides AB, BC, CD, DA, respectively, of a convex quadrilateral $ABCD$. Show that the segments PR and QS are perpendicular to each other and equal in length.

5. Let $ABCD$ be a convex quadrilateral. Construct squares $CDKL$ and $ABMN$ externally on sides AB and CD. Show that if the midpoints of AC, BD, KN, ML do not coincide, then they form a square.

6. (IMO 1977/2) Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments AK, BK, BL, CL, CM, DM, DN, AN are the twelve vertices of a regular dodecagon. (Nowadays the IMO tends to avoid geometry problems such as this one, which have no free parameters, but they are relatively common in single-answer contests such as ARML.)

7. Use complex numbers and the circle of Apollonius (Theorem 4.10) to give another proof that circles map to circles under inversion.

8. Given a point P on the circumference of a unit circle and the vertices A_1, $A_2, \ldots, A_n$ of an inscribed regular n-gon, prove that:

    1. $PA_1^2 + PA_2^2 + \cdots + PA_n^2$ is a constant (independent of P).
    2. $PA_1^4 + PA_2^4 + \cdots + PA_n^4$ is a constant (independent of P).

9. (China, 1998) Let P be an arbitrary point in the plane of triangle ABC with sides $BC = a$, $CA = b$, $AB = c$, and with $PA = x$, $PB = y$, $PC = z$. Prove that

$$ayz + bzx + cxy \geq abc,$$

with equality if and only if P is the circumcenter of ABC.

A.4 Cartesian coordinates

It was first noted independently by Descartes and Fermat that any problem in Euclidean geometry could be reduced to an algebraic problem by introducing rectangular coordinates, which we now call *Cartesian coordinates*. One major result of this discovery was increased interest in what we now call *algebraic geometry* (see Section 8.7).

In practice, algebraizing a problem in Euclidean geometry often leads to a complicated mess whose manipulation is often more time-consuming and surely less pleasant than finding an ingenious synthetic solution. On the other hand, in some cases, the Cartesian point of view leads to unexpectedly short proofs; we have included such proofs in the text when appropriate (see, for example, Theorem 4.10).

In any case, if you do choose to apply coordinates to a problem, there are a few facts that it may help to know, and we summarize them in this section.

The following formula for the area of a polygon is called the *surveyor's formula* (or the *shoelace formula*, after the mnemonic device of writing the variables in a $2 \times n$ array and multiplying along the diagonals).

Fact A.12. *Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the vertices of a polygon without self-intersections (but not necessarily convex). Then the signed area of the polygon is given by*

$$\pm[(x_1y_2 - y_1x_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)].$$

Note that in some cases, it is convenient to allow *oblique coordinates*, in which the coordinate axes are chosen as two lines which are not necessarily parallel. (Alternatively, one may perform an affine transformation and then use normal Cartesian coordinates.) DEMONSTRATE.

Problems for Section A.4

1. (IMO 1988/1) Consider two coplanar circles of radii R and r ($R > r$) with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular l to BP at P meets the smaller circle again at A. (If l is tangent to the circle at P then $A = P$.)

   (i) Find the set of values of $BC^2 + CA^2 + AB^2$.

   (ii) Find the locus of the midpoint of BC.

2. Give a coordinate solution for Problem 1.1.4.

3. Give a coordinate proof of Pappus' theorem (Theorem 2.5), or of your favorite theorem in projective geometry.

4. Prove that the locus of points P such that the ratio of the powers of P to two fixed circles ω_1 and ω_2 equals a constant $k \neq 1$ is a circle.

5. (Greece, 1996) In a triangle ABC the points D, E, Z, H, Θ are the midpoints of the segments BC, AD, BD, ED, EZ, respectively. If I is the point of intersection of BE and AC, and K is the point of intersection of $H\Theta$ and AC, prove that

1. $AK = 3CK$;
2. $HK = 3H\Theta$;
3. $BE = 3EI$;
4. the area of ABC is 32 times that of $E\Theta H$.

6. (Sweden, 1996) Through a point in the interior of a triangle with area T, draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let T_1, T_2, T_3 be the areas of the three triangles. Prove that

$$\sqrt{T} = \sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3}.$$

Appendix B

Hints

Here are the author's suggestions on how to proceed on some of the problems. If you find another solution to a problem, so much the better—but it may not be a bad idea to try to find the suggested solution anyway!

1.1.6 The octahedron has 4 times the volume of the tetrahedron. What happens when you glue them together at a face?

1.4.1 Prove one assertion, then work backward to prove the other.

1.4.2 Construct two of the intersections of the trisectons complete the equilateral triangle, then show that its third vertex is the third intersection. If you're still stuck, see [2].

2.2.1 Draw 10 points: the 6 vertices of the triangles, the three intersections of corresponding sides, and the intersection of the lines joining two pairs of corresponding vertices. If you relabel these 10 points appropriately, this diagram will turn into a case of the forward direction of Desargues!

3.1.4 Consider the triangle AB_1C_1 together with the second intersection of the circumcircles of AB_1C_2 and AB_3C_1. Show that this figure is congruent to the two analogous figures formed from the other triangles. Do this by rotating AB_1C_1 onto C_2AB_2 onto B_3C_3A and tracing what happens to the figure. (Or apply Theorem 1.6.)

3.3.3 How does P_2 depend on P_1?

4.2.5 There are several solutions to this problem, but none are easy to find. In any case, before anything else, find an extra cyclic quadrilateral.

4.3.4 Work backwards, defining G as the point for which the conclusion holds. Also consider the circumcircle of CDE.

4.4.2 Find a cyclic hexagon.

4.4.3 Use Theorem 1.5.

4.6.2 Even using directed angles, the result fails for nonconvex hexagons. Figuring out why may help you determine how to use convexity here.

4.5.1 Given segments AB and CD, what conditions must the center P of a spiral similarity carrying AB to CD satisfy?

4.5.2 By Ceva and Menelaos, one can show $BA_1/A_1C = BA_2/A_2C$. This means the circle with diameter A_1A_2 is a circle of Apollonius with respect to B and C.

5.2.3 The center of the circle lies at C.

5.2.4 The fixed point lies on the circumcircle of ABC.

5.2.5 Show that the point F is the excenter of ACD opposite A.

5.2.6 Use homothety.

5.3.5 For (a), write the half-turn as the composition of two other homotheties and locate the fixed point.

5.6.1 What is the locus of points where one of these equalities holds?

5.7.1 Apply the Law of Cosines to the triangles ABD and ACD.

6.2.4 Use Fact 5.4.

6.3.3 Show that no two consecutive quadrilaterals can both have incircles.

7.1.8 Which circles are orthogonal to two concentric circles?

7.3.6 Reduce to the case where two of the circles are tangent, then invert.

7.3.12 Note that $AB \cdot AB_1 = AC \cdot AC_1$. Also look at the intersection of OA and B_1C_1.

8.4.1 Fix five of the points and compare the locus of sixth points making this condition hold with the conic through the five points.

8.4.3 Apply Pascal's theorem to the hyperbola, using the intersections of the asymptotes with the line at infinity as two of the six points.

8.5.6 Draw the circle with diameter OB, and show that its common chord with the circle centered at O is concurrent with KN and AC.

8.7.1 Find a projective transformation taking the circle to itself but not preserving its center.

8.7.5 In fact, there exists a degenerate cubic with this property.

8.7.7 Find a projective transformation taking the circle to a circle and the line to infinity.

8.7.10 Surprise! Reduce to the case of two concentric circles.

9.1.4 Use the similar triangles formed by the sides and diagonals.

9.2.8 Write everything in terms of $\cot A/2$ and the like. Then turn the result into a statement about homogeneous polynomials using the identity

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2},$$

and solve the result.

9.2.9 Use an affine transformation to make $ABCD$ cyclic, and perform a quadrilateral analogue of the $s-a$ substitution.

A.4.4 Imitate the proof of Theorem 4.10.

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