Simplex Solver

April 30, 2024

Problem

Given the following linear system and objective function, find the optimal solution.

$$\max x_1 + x_2 + x_3$$
$$\begin{cases} x_1 + x_2 + x_3 \leq 3 \\ x_3 \leq 1 \\ x_1 + x_3 \leq 2 \\ 2x_1 + 5x_3 \leq 8 \\ --7x_1 + 8x_2 \leq 0 \\ x_1 + 2x_2 - x_3 \leq 1 \end{cases}$$

Solution

Add slack variables to turn all inequalities to equalities.

$$\begin{cases} x_1 + x_2 + x_3 + s_1 = 3 \\ x_3 + s_2 = 1 \\ x_1 + x_3 + s_3 = 2 \\ 2x_1 + 5x_3 + s_4 = 8 \\ --7x_1 + 8x_2 + s_5 = 0 \\ x_1 + 2x_2 - x_3 + s_6 = 1 \end{cases}$$

Create the initial tableau of the new linear system.

$$\begin{array}{c} \left[\begin{array}{ccccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 2 & 0 & 5 & 0 & 0 & 0 & 1 & 0 & 0 & 8 \\ -7 & 8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ \\ \end{array}$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_1 and the departing variable is s_6.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$
\left[\begin{array}{ccccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b \\
\hline
0 & -1 & 2 & 1 & 0 & 0 & 0 & 0 & -1 & 2 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & -2 & 2 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\
0 & -4 & 7 & 0 & 0 & 0 & 1 & 0 & -2 & 6 \\
0 & 22 & -7 & 0 & 0 & 0 & 0 & 1 & 7 & 7 \\
1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right]
\begin{array}{c}
\\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ x_1 \\ \\
\end{array}
$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_3 and the departing variable is s_3.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$
\left[\begin{array}{ccccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b \\
\hline
0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & -1/2 & 0 & 0 & 1/2 & 1/2 \\
0 & -1 & 1 & 0 & 0 & 1/2 & 0 & 0 & -1/2 & 1/2 \\
0 & 3 & 0 & 0 & 0 & -7/2 & 1 & 0 & 3/2 & 5/2 \\
0 & 15 & 0 & 0 & 0 & 7/2 & 0 & 1 & 7/2 & 21/2 \\
1 & 1 & 0 & 0 & 0 & 1/2 & 0 & 0 & 1/2 & 3/2 \\
\hline
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2
\end{array}\right]
\begin{array}{c}
\\ s_1 \\ s_2 \\ x_3 \\ s_4 \\ s_5 \\ x_1 \\ \\
\end{array}
$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is x_2 and the departing variable is s_2.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$
\left[\begin{array}{ccccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b \\
\hline
0 & 0 & 0 & 1 & -1 & -1/2 & 0 & 0 & -1/2 & 1/2 \\
0 & 1 & 0 & 0 & 1 & -1/2 & 0 & 0 & 1/2 & 1/2 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -3 & -2 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -15 & 11 & 0 & 1 & -4 & 3 \\
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1/2 & 0 & 0 & 1/2 & 5/2
\end{array}\right]
\begin{array}{c}
\\ s_1 \\ x_2 \\ x_3 \\ s_4 \\ s_5 \\ x_1 \\ \\
\end{array}
$$

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$s_1 = \frac{1}{2}, s_2 = 0, s_3 = 0, s_4 = 1, s_5 = 3, s_6 = 0, x_1 = 1, x_2 = \frac{1}{2}, x_3 = 1, z = \frac{5}{2}$$