2

$$f(x_1,x_2) = x_1^3 - 12x_1x_2 + 8x_2^3$$

$$\nabla f(x_1,x_2) = \begin{pmatrix} 3x_1^2 - 12x_2 \\ 24x_2^2 - 12x_1 \end{pmatrix} = \begin{pmatrix} 3x_1^2 - 12x_2 \\ -12x_1 + 24x_2^2 \end{pmatrix}$$

$\nabla f(x_1,x_2) = 0$ when $3x_1^2 - 12x_2 = 0$ and $-12x_1 + 24x_2^2 = 0$

from equ1: $x_2 = \frac{3x_1^2}{12} = \frac{x_1^2}{4}$

replace in equ2: $-12x_1 + 24\left(\frac{x_1^2}{2}\right)^2 = 0 \Rightarrow -12x_1 + 6x_1^4 = 0 \Rightarrow -2x_1 + x_1^4 = 0$

$x_1 = 0$, $x_1 = \sqrt[3]{2} = 1.26$, $x_1 = -\sqrt[3]{2}\frac{1}{2} + \sqrt[3]{2}\frac{\sqrt{3}}{2}i$ rejected , $x_1 = -\sqrt[3]{2}\frac{1}{2} - \sqrt[3]{2}\frac{\sqrt{3}}{2}i$ rejected

$\downarrow$ $\qquad\qquad$ $\downarrow$

$x_2 = 0$ $\qquad$ $x_2 = 0.3969$

$$H f(x_1,x_2) = \begin{pmatrix} 6x_1 & -12 \\ -12 & 48x_2 \end{pmatrix}$$

$\Rightarrow$ eigenvalues

$$\det\begin{pmatrix} 6x_1 - \lambda & -12 \\ -12 & 48x_2 - \lambda \end{pmatrix} = 0 \Rightarrow (6x_1 - \lambda)(48x_2 - \lambda) - (12)^2$$

$$(6x_1)(48x_2) - 6x_1\lambda - 48x_2\lambda + \lambda^2 - (12)^2$$

$$\lambda^2 - (6x_1 + 48x_2)\lambda + (6x_1)(48x_2) - (12)^2$$

$*\ (\lambda - r_1)(\lambda - r_2) = \lambda^2 - (r_1 + r_2) + r_1 r_2 = 0$ - let $r_1 = 6x_1$, $r_2 = 48x_2$

if $(6x_1)(48x_2) - (12)^2 > 0$ and $6x_1 + 48x_2 > 0 \rightarrow \lambda_1 > 0\ \lambda_2 > 0 \rightarrow f(x_1,x_2)$ local min

if $(6x_1)(48x_2) - (12)^2 > 0$ and $6x_1 + 48x_2 < 0 \rightarrow \lambda_1 < 0\ \lambda_2 < 0 \rightarrow f(x_1,x_2)$ local max

if $(6x_1)(48x_2) - (12)^2 < 0 \rightarrow \lambda_1 < 0\ \lambda_2 > 0$ or $\lambda_1 > 0\ \lambda_2 < 0 \rightarrow f(x_1,x_2)$ saddle point

$$f(x_1, x_2, x_3) = (2x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - 1)^2$$

$$\frac{\partial f}{\partial x_1} = 4(2x_1 - x_2) = 8x_1 - 4x_2$$

$$\frac{\partial f}{\partial x_2} = -4x_1 + 4x_2 - 2x_3$$

$$\frac{\partial f}{\partial x_3} = -2x_2 + 4x_3 - 2$$

$$H(x) = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

$$\Delta_1 = 8 > 0$$

$$\Delta_2 = \begin{vmatrix} 8 & -4 \\ -4 & 4 \end{vmatrix} = 8(4) - (-4)(-4) > 0$$

$$\Delta_3 = 8\begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix} - (-4)\begin{vmatrix} -4 & -2 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} -4 & 4 \\ 0 & -2 \end{vmatrix}$$

$$= 8[(4)(4) - (-2)(-2)] + 4[(-4)(4)] + 0 > 0$$

since $\Delta_1 > 0$ $\Delta_2 > 0$ $\Delta_3 > 0$ $\Rightarrow f''(x) > 0$ we have a global min

$f'(x) = 0$ to find critical point

$$8x_1 - 4x_2 = 0$$
$$-4x_1 + 4x_2 - 2x_3 = 0 \quad \Rightarrow \quad x_1 = \frac{1}{2}, \quad x_2 = 1, \quad x_3 = 1$$
$$-2x_2 + 4x_3 = 2$$

global min at $\left(\frac{1}{2}, 1, 1\right)$