CS 205: Introduction to Discrete Structures I Homework 1

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Problem 1. [16 points]

Let p be the proposition "Hiking in the New Jersey forest terrain is allowed", and let q be the proposition "Bears have been spotted in New Jersey forests". Express each of these compound propositions as an English sentence.

a) $\neg$q

Solution: **No** bears have been spotted in New Jersey forests.

(b) p $\wedge$ q

Solution: Hiking in the New Jersey forest terrain is allowed **and** bears have been spotted in New Jersey forests.

(c) $\neg p \vee q$

Solution: **No** hiking in the New Jersey forest terrain is allowed **or** bears have been spotted in New Jersey forests.

(d) p $\implies$ $\neg$q

Solution: If hiking in the New Jersey forest terrain is allowed, **then** bears have **not** been spotted in New Jersey forests.

(e) $\neg$q $\implies p$

Solution: **No** bears have been spotted in New Jersey forests, **so** hiking in the New Jersey forest terrain is allowed.

(f) $\neg$p $\implies$ $\neg$q

Solution: If **no** hiking in the New Jersey forest terrain is allowed, then **no** bears have been spotted in New Jersey forests.

(g) $p \iff \neg q$

Solution: Hiking in the New Jersey forest terrain is allowed **if and only if** bears have **not** been spotted in New Jersey forests.

(h) $\neg p \land (p \lor \neg q)$

Solution: **No** hiking in the New Jersey forest terrain is allowed **and** Hiking in the New Jersey forest terrain is allowed **or** bears have **not** been spotted in New Jersey forests.

Problem 2. [12 points]

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie.

On Day 1, you meet two inhabitants: Zoey and Mel. Zoey tells you that Mel is a knave. Mel says, "Neither Zoey nor I are knaves."

On Day 2, you meet two inhabitants: Peggy and Zippy. Peggy tells you that "of Zippy and I, exactly one is a knight'. Zippy tells you that only a knave would say that Peggy is a knave.

On Day 3, you meet two inhabitants: Sue and Zippy. Sue says that Zippy is a knave. Zippy says, "I and Sue are knights."

On Day 4, you meet two inhabitants: Sally and Zippy. Sally claims, "I and Zippy are not the same." Zippy says, "Of I and Sally, exactly one is a knight."

On Day 5, you meet two inhabitants: Homer and Bozo. Homer tells you, "At least one of the following is true: that I am a knight or that Bozo is a knight." Bozo claims, "Homer could say that I am a knave."

On Day 6, you meet two inhabitants: Marge and Zoey. Marge says, "Zoey and I are both knights or both knaves." Zoey claims, "Marge and I are the same."

Can you determine who are the knights and who are the knaves? Show the complete work for your deductions.

Solution:

Day 1:

$\neg m$ = Zoey says "Mel is a knave."

$(z \wedge m)$ = Mel says "Neither Zoey nor I are knaves."

z	m	$\neg m$	$z \wedge m$
F	T	$m = \text{T}$	$(z \wedge m) = \text{F}$
F	F	$m = \text{F}$	$\neg(z \wedge m) = \text{T}$
T	T	$\neg m = \text{F}$	$(z \wedge m) = \text{T}$
T	F	$\neg m = \mathbf{T}$	$\neg(z \wedge m) = \mathbf{T}$

From the truth table, Zoey is a **knight** and Mel is a **knave**.

Day 2:

$p \oplus z$ = Peggy says "of Zippy and I, exactly one is a knight."

p = Zippy says "Only a knave would say that Peggy is a knave."

p	z	$p \oplus z$	p
F	F	$\neg(p \oplus z) = \mathbf{T}$	$\neg p = \mathbf{T}$
F	T	$\neg(p \oplus z) = \text{F}$	$p = \text{F}$
T	F	$p \oplus z = \text{T}$	$\neg p = \text{F}$
T	T	$p \oplus z = \text{F}$	$p = \text{T}$

From the truth table, Peggy and Zippy are **knaves**.

Day 3:

$\neg z$ = Sue says "Zippy is a knave."

$(s \wedge z)$ = Zippy says "I and Sue are knights."

s	z	$\neg z$	$s \wedge z$
F	T	$z = \text{T}$	$(s \wedge z) = \text{F}$
F	F	$z = \text{F}$	$\neg(s \wedge z) = \text{T}$
T	T	$\neg z = \text{F}$	$(s \wedge z) = \text{T}$
T	F	$\neg z = \mathbf{T}$	$\neg(s \wedge z) = \mathbf{T}$

From the truth table, Sue is a **knight** and Zippy is a **knave**.

Day 4:

$s \oplus z =$ Sally says "I and Zippy are not the same."

$s \oplus z =$ Zippy says "Of I and Sally, exactly one is a knight."

s	z	$s \oplus z$	$s \oplus z$
F	T	$\neg(s \oplus z) = \mathrm{F}$	$s \oplus z = \mathrm{T}$
F	F	$\neg(s \oplus z) = \mathbf{T}$	$\neg(s \oplus z) = \mathbf{T}$
T	T	$s \oplus z = \mathrm{F}$	$s \oplus z = \mathrm{F}$
T	F	$s \oplus z = \mathrm{T}$	$\neg(s \oplus z) = \mathrm{F}$

From the truth table, Sally and Zippy are both **knaves**.

Day 5:

$h \vee b =$ Homer says "At least one of the following is true: that I am a knight or that Bozo is a knight."

$(\neg h \wedge b) \vee (h \wedge \neg b) =$ Bozo says "Homer could say that I am a knave."

h	b	$h \vee b$	$(\neg h \wedge b) \vee (h \wedge \neg b)$
F	T	$\neg(h \vee b) = \mathrm{F}$	$(\neg h \wedge b) \vee (h \wedge \neg b) = \mathrm{F}$
F	F	$\neg(h \vee b) = \mathbf{T}$	$\neg(\neg h \wedge b) \wedge \neg(h \wedge \neg b) = \mathbf{T}$
T	T	$h \vee b = \mathrm{T}$	$(\neg h \wedge b) \vee (h \wedge \neg b) = \mathrm{F}$
T	F	$h \vee b = \mathrm{T}$	$\neg(\neg h \wedge b) \wedge \neg(h \wedge \neg b) = \mathrm{F}$

From the truth table. Homer and Bozo are both **knaves**.

Day 6:

$(m \wedge z) \vee (\neg m \wedge \neg z) =$ Marge says, "Zoey and I are both knights or both knaves."

$m \equiv z =$ Zoey says "Marge and I are the same."

m	z	$(m \wedge z) \vee (\neg m \wedge \neg z)$	$m \equiv z$
F	T	$(\neg m \vee \neg z) \wedge (m \vee z) = \mathrm{T}$	$m \equiv z = \mathrm{F}$
F	F	$(\neg m \vee \neg z) \wedge (m \vee z) = \mathrm{F}$	$\neg(m \equiv z) = \mathrm{F}$
T	T	$(m \wedge z) \vee (\neg m \wedge \neg z) = \mathbf{T}$	$m \equiv z = \mathbf{T}$
T	F	$(m \wedge z) \vee (\neg m \wedge \neg z) = \mathrm{F}$	$\neg(m \equiv z) = \mathrm{T}$

From the truth table, Marge and Zoey are **knights**.

Problem 3. [9 points]

Question:

Prove or disprove these universally quantified statements. If disproving, you must provide a counterexample, where the domain for all variables consists of all real numbers.

Part (a)

Prove or disprove:

$$\forall x, \exists y, \left(x = \frac{1}{y}\right)$$

Solution: This statement is **false** if $x = 0$ then there is no y such that $x = \frac{1}{y}$. Or in other words, 0 has no inverse.

Part (b)

Prove or disprove:

$$\forall x, \exists y, (y^2 - x < 100)$$

Solution: This statement is **false**. Consider $x = -200$

$$\begin{aligned} y^2 - x &< 100 \\ y^2 - (-200) &< 100 && \text{(x = -200)} \\ y^2 + 200 &< 100 \\ y^2 &< -100 \end{aligned}$$

And so we have $y^2 < -100$, but this is not possible because y is a real number and any real number squared is non-negative.

Part (c)

Prove or disprove:

$$\forall x, \exists y, (x^2 \neq y^3)$$

Solution: This statement is **false**. Consider $x = 0$ and $y = 0$, $x = y$ so this statement is false.

Problem 4. [10 points]

Question:

Prove that if n and m are perfect squares, then $(n \cdot m) + 2$ is not a perfect square.

Solution:

Lemma 1:

$n = l^2$, where $l \in Z$

$n + 2 = z^2$ where $z \in Z$

Now $z^2 - l^2 = 2$

$(z + l)(z - l) = z^2 - l^2 = (n + 2) - n = 2$

It is impossible for $(z + l)$ or $(z - l) = 1$ and for the other to $= 2$.

Contradiction Proof:

Let's assume $(n \cdot m) + 2$ is a perfect square.

$(n \cdot m) + 2 = k^2$, where $k \in Z$

Because n and m are perfect squares, we can rewrite these as the following:

$n = l^2, \in Z$

$m = z^2, \in Z$

l^2 and z^2 can now be substituted into $(n \cdot m) + 2 = k^2$:

$$(l^2 \cdot z^2) + 2 = k^2$$
$$(lz)^2 + 2 = k^2$$

Note that l and z are integers and so their product is an integer. And so by lemma 1, it is impossible for $(lz)^2 + 2 = k^2$ to be a perfect square.

This shows that the assumption of $(n \cdot m) + 2$ is a perfect square is **false**. So if n and m are perfect squares, we can conclude that $(n \cdot m) + 2$ is **not** a perfect square.

Problem 5

Question:

Show that these 3 statements about a real number x are equivalent:

a. x is rational.

b. $\frac{x}{2}$ is rational.

c. $3x$ - 1 is rational.

Solution:

Since x is rational we have the following:

$$\exists n, m \in Z \land m \neq 0, \quad x = \frac{n}{m}$$

And so:

$$\begin{aligned} x &= \frac{n}{m} && \text{(assume a)} \\ \frac{x}{2} &= \frac{n}{2m} && \text{(divide both sides by 2)} \end{aligned}$$

And since n and $2m$ are integers (integers are closed under multiplication) we have that, $\frac{x}{2}$ is an integer. And so we have $a \rightarrow b$.

Next we will show that $b \rightarrow c$.

$$\begin{aligned} \frac{x}{2} &= \frac{n}{m} && \text{(assume b,} \quad n, m \in Z \land m \neq 0) \\ x &= \frac{2n}{m} && \text{(multiply both sides by 2)} \\ 3x - 1 &= \frac{6n}{m} - 1 && \text{(multiply both sides by 3, subtract 1)} \\ 3x - 1 &= \frac{3n - m}{m} && \text{(common denominator)} \end{aligned}$$

Note that because n and m are integers (assumed from b), we have that $3n - m$ is an integer (integers are closed under multiplication and addition). And since both the numerator and denominator are integers, we have that $3x - 1$ is rational, giving us $b \to c$.

Finally we will show that $c \to a$:

$$\begin{aligned} 3x - 1 &= \frac{n}{m} && \text{(assume c,} \quad n, m \in Z \land m \neq 0) \\ x &= \frac{\frac{n}{m} + 1}{3} \\ &= \frac{n}{3m} + \frac{1}{3} \\ &= \frac{n + m}{3m} && \text{(common denominator)} \end{aligned}$$

Note that because n and m are integers (assumed from because we have that $n+m$ and $3m$ are integers (integers are closed under multiplication and addition). Since both the numerator and denominator are integers, we have that x is rational, giving us $c \to a$.

Putting this all together we have a chain of implications:

$$a \to b \to c \to a$$

This of course implies our result:

$$a \equiv b \equiv c$$

Problem 6 [18 points]

Question:

Prove the following two statements. You need to give a proof starting with the definition. You will not get any credit for providing examples.

Part (a)

Prove that the sum two rational numbers is rational.

Solution:

Since x and y are rational we have the following:

$$p, q, r, s \in Z \land q \neq 0, s \neq 0$$

$$x = \frac{p}{q}, y = \frac{r}{s}$$

We can express the sum of x and y as rational numbers:

$$\begin{aligned} x + y &= \frac{p}{q} + \frac{r}{s} \\ &= \frac{ps + rq}{qs} && \text{(common denominator)} \end{aligned}$$

Because ps and rq are integers, $q \neq 0$ and $s \neq 0$, $x + y$ can be shown as a ratio of two integers, therefore $x + y$ is a rational number.

Part (b)

Prove that the sum of an irrational number and a rational number must be irrational.

Solution:

Let's assume that x is rational, y is irrational, and $x + y$ is rational expression. Because $x + y$ is a rational, we can express the following:

$$p, q, r, s \in Z \land q \neq 0, s \neq 0$$

$$x = \frac{p}{q}, x + y = \frac{r}{s}$$

We can now substitute x into $x + y = \frac{r}{s}$:

$$\begin{aligned} \frac{p}{q} + y &= \frac{r}{s} && \text{(substitute } x = \tfrac{p}{q}\text{)} \\ y &= \frac{r}{s} + (-\frac{p}{q}) && \text{(subtract both sides by } \tfrac{p}{q}\text{)} \end{aligned}$$

The expression $y = \frac{r}{s} + (-\frac{p}{q})$ is rational since it is a ratio of two numbers being summed, however our assumption of y being irrational is contradicted, so the sum of a irrational number and a rational number must be irrational.