

# CS 205: Introduction to Discrete Structures I Homework 0

Sami Kamal

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## Problem 1. [16 points]

Let  $p$  be the proposition “Hiking in the New Jersey forest terrain is allowed”, and let  $q$  be the proposition “Bears have been spotted in New Jersey forests”. Express each of these compound propositions as an English sentence.

a)  $\neg q$

**Solution:** No bears have been spotted in New Jersey forests.

(b)  $p \wedge q$

**Solution:** Hiking in the New Jersey forest terrain is allowed **and** bears have been spotted in New Jersey forests.

(c)  $\neg p \vee q$

**Solution:** No hiking in the New Jersey forest terrain is allowed **or** bears have been spotted in New Jersey forests.

(d)  $p \implies \neg q$

**Solution:** If hiking in the New Jersey forest terrain is allowed, **then** bears have **not** been spotted in New Jersey forests.

(e)  $\neg q \implies p$

**Solution:** No bears have been spotted in New Jersey forests, **so** hiking in the New Jersey forest terrain is allowed.

(f)  $\neg p \implies \neg q$

**Solution:** If **no** hiking in the New Jersey forest terrain is allowed, then **no** bears have been spotted in New Jersey forests.

$$(g) p \iff \neg q$$

**Solution:** Hiking in the New Jersey forest terrain is allowed **if and only if** bears have **not** been spotted in New Jersey forests.

$$(h) \neg p \wedge (p \vee \neg q)$$

**Solution:** **No** hiking in the New Jersey forest terrain is allowed **and** Hiking in the New Jersey forest terrain is allowed **or** bears have **not** been spotted in New Jersey forests.

## Problem 2. [12 points]

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie.

On Day 1, you meet two inhabitants: Zoey and Mel. Zoey tells you that Mel is a knave. Mel says, “Neither Zoey nor I are knaves.”

On Day 2, you meet two inhabitants: Peggy and Zippy. Peggy tells you that “of Zippy and I, exactly one is a knight”. Zippy tells you that only a knave would say that Peggy is a knave.

On Day 3, you meet two inhabitants: Sue and Zippy. Sue says that Zippy is a knave. Zippy says, “I and Sue are knights.”

On Day 4, you meet two inhabitants: Sally and Zippy. Sally claims, “I and Zippy are not the same.” Zippy says, “Of I and Sally, exactly one is a knight.”

On Day 5, you meet two inhabitants: Homer and Bozo. Homer tells you, “At least one of the following is true: that I am a knight or that Bozo is a knight.” Bozo claims, “Homer could say that I am a knave.”

On Day 6, you meet two inhabitants: Marge and Zoey. Marge says, “Zoey and I are both knights or both knaves.” Zoey claims, “Marge and I are the same.”

Can you determine who are the knights and who are the knaves? Show the complete work for your deductions.

**Solution:**

Day 1:

$\neg m$  = Zoey says "Mel is a knave."

$(z \wedge m)$  = Mel says "Neither Zoey nor I are knaves."

$z$	$m$	$\neg m$	$z \wedge m$
F	T	$m = \mathbf{T}$	$(z \wedge m) = \mathbf{F}$
F	F	$m = \mathbf{F}$	$\neg(z \wedge m) = \mathbf{T}$
T	T	$\neg m = \mathbf{F}$	$(z \wedge m) = \mathbf{T}$
T	F	$\neg m = \mathbf{T}$	$\neg(z \wedge m) = \mathbf{T}$

From the truth table, Zoey is a **knight** and Mel is a **knave**.

Day 2:

$p \oplus z$  = Peggy says "of Zippy and I, exactly one is a knight."

$p$  = Zippy says "Only a knave would say that Peggy is a knave."

$p$	$z$	$p \oplus z$	$p$
F	F	$\neg(p \oplus z) = \mathbf{T}$	$\neg p = \mathbf{T}$
F	T	$\neg(p \oplus z) = \mathbf{F}$	$p = \mathbf{F}$
T	F	$p \oplus z = \mathbf{T}$	$\neg p = \mathbf{F}$
T	T	$p \oplus z = \mathbf{F}$	$p = \mathbf{T}$

From the truth table, Peggy and Zippy are **knaves**.

Day 3:

$\neg z$  = Sue says "Zippy is a knave."

$(s \wedge z)$  = Zippy says "I and Sue are knights."

$s$	$z$	$\neg z$	$s \wedge z$
F	T	$z = \mathbf{T}$	$(s \wedge z) = \mathbf{F}$
F	F	$z = \mathbf{F}$	$\neg(s \wedge z) = \mathbf{T}$
T	T	$\neg z = \mathbf{F}$	$(s \wedge z) = \mathbf{T}$
T	F	$\neg z = \mathbf{T}$	$\neg(s \wedge z) = \mathbf{T}$

From the truth table, Sue is a **knight** and Zippy is a **knave**.

Day 4:

$s \oplus z$  = Sally says "I and Zippy are not the same."

$s \oplus z$  = Zippy says "Of I and Sally, exactly one is a knight."

$s$	$z$	$s \oplus z$	$s \oplus z$
F	T	$\neg(s \oplus z) = \mathbf{F}$	$s \oplus z = \mathbf{T}$
F	F	$\neg(s \oplus z) = \mathbf{T}$	$\neg(s \oplus z) = \mathbf{T}$
T	T	$s \oplus z = \mathbf{F}$	$s \oplus z = \mathbf{F}$
T	F	$s \oplus z = \mathbf{T}$	$\neg(s \oplus z) = \mathbf{F}$

From the truth table, Sally and Zippy are both **knaves**.

Day 5:

$h \vee b$  = Homer says "At least one of the following is true: that I am a knight or that Bozo is a knight."

$(\neg h \wedge b) \vee (h \wedge \neg b)$  = Bozo says "Homer could say that I am a knave."

$h$	$b$	$h \vee b$	$(\neg h \wedge b) \vee (h \wedge \neg b)$
F	T	$\neg(h \vee b) = \mathbf{F}$	$(\neg h \wedge b) \vee (h \wedge \neg b) = \mathbf{F}$
F	F	$\neg(h \vee b) = \mathbf{T}$	$\neg(\neg h \wedge b) \wedge \neg(h \wedge \neg b) = \mathbf{T}$
T	T	$h \vee b = \mathbf{T}$	$(\neg h \wedge b) \vee (h \wedge \neg b) = \mathbf{F}$
T	F	$h \vee b = \mathbf{T}$	$\neg(\neg h \wedge b) \wedge \neg(h \wedge \neg b) = \mathbf{F}$

From the truth table. Homer and Bozo are both **knaves**.

Day 6:

$(m \wedge z) \vee (\neg m \wedge \neg z)$  = Marge says, "Zoey and I are both knights or both knaves."

$m \equiv z$  = Zoey says "Marge and I are the same."

$m$	$z$	$(m \wedge z) \vee (\neg m \wedge \neg z)$	$m \equiv z$
F	T	$(\neg m \vee \neg z) \wedge (m \vee z) = \mathbf{T}$	$m \equiv z = \mathbf{F}$
F	F	$(\neg m \vee \neg z) \wedge (m \vee z) = \mathbf{F}$	$\neg(m \equiv z) = \mathbf{F}$
T	T	$(m \wedge z) \vee (\neg m \wedge \neg z) = \mathbf{T}$	$m \equiv z = \mathbf{T}$
T	F	$(m \wedge z) \vee (\neg m \wedge \neg z) = \mathbf{F}$	$\neg(m \equiv z) = \mathbf{T}$

From the truth table, Marge and Zoey are **knights**.

### Problem 3. [9 points]

#### Question:

Prove or disprove these universally quantified statements. If disproving, you must provide a counterexample, where the domain for all variables consists of all real numbers.

#### Part (a)

Prove or disprove:

$$\forall x, \exists y, \left( x = \frac{1}{y} \right)$$

**Solution:** This statement is **false** if  $x = 0$  then there is no  $y$  such that  $x = \frac{1}{y}$ . Or in other words, 0 has no inverse.

#### Part (b)

Prove or disprove:

$$\forall x, \exists y, (y^2 - x < 100)$$

**Solution:** This statement is **false**. Consider  $x = -200$

$$\begin{aligned} y^2 - x &< 100 \\ y^2 - (-200) &< 100 && (x = -200) \\ y^2 + 200 &< 100 \\ y^2 &< -100 \end{aligned}$$

And so we have  $y^2 < -100$ , but this is not possible because  $y$  is a real number and any real number squared is non-negative.

#### Part (c)

Prove or disprove:

$$\forall x, \exists y, (x^2 \neq y^3)$$

**Solution:** This statement is **false**. Consider  $x = 0$  and  $y = 0$ ,  $x = y$  so this statement is false.

#### Problem 4. [10 points]

##### Question:

Prove that if  $n$  and  $m$  are perfect squares, then  $(n \cdot m) + 2$  is not a perfect square.

##### Solution:

###### Lemma 1:

$$n = l^2, \text{ where } l \in \mathbb{Z}$$

$$n + 2 = z^2 \text{ where } z \in \mathbb{Z}$$

$$\text{Now } z^2 - l^2 = 2$$

$$(z + l)(z - l) = z^2 - l^2 = (n + 2) - n = 2$$

It is impossible for  $(z + l)$  or  $(z - l) = 1$  and for the other to  $= 2$ .

###### Contradiction Proof:

Let's assume  $(n \cdot m) + 2$  is a perfect square.

$$(n \cdot m) + 2 = k^2, \text{ where } k \in \mathbb{Z}$$

Because  $n$  and  $m$  are perfect squares, we can rewrite these as the following:

$$n = l^2, \in \mathbb{Z}$$

$$m = z^2, \in \mathbb{Z}$$

$l^2$  and  $z^2$  can now be substituted into  $(n \cdot m) + 2 = k^2$ :

$$(l^2 \cdot z^2) + 2 = k^2$$

$$(lz)^2 + 2 = k^2$$

Note that  $l$  and  $z$  are integers and so their product is an integer. And so by lemma 1, it is impossible for  $(lz)^2 + 2 = k^2$  to be a perfect square.

This shows that the assumption of  $(n \cdot m) + 2$  is a perfect square is **false**. So if  $n$  and  $m$  are perfect squares, we can conclude that  $(n \cdot m) + 2$  is **not** a perfect square.

## Problem 5

### Question:

Show that these 3 statements about a real number  $x$  are equivalent:

- a.  $x$  is rational.
- b.  $\frac{x}{2}$  is rational.
- c.  $3x - 1$  is rational.

### Solution:

Since  $x$  is rational we have the following:

$$\exists n, m \in \mathbb{Z} \wedge m \neq 0, \quad x = \frac{n}{m}$$

And so:

$$x = \frac{n}{m} \quad (\text{assume a})$$

$$\frac{x}{2} = \frac{n}{2m} \quad (\text{divide both sides by 2})$$

And since  $n$  and  $2m$  are integers (integers are closed under multiplication) we have that,  $\frac{x}{2}$  is an integer. And so we have  $a \rightarrow b$ .

Next we will show that  $b \rightarrow c$ .

$$\frac{x}{2} = \frac{n}{m} \quad (\text{assume b, } n, m \in \mathbb{Z} \wedge m \neq 0)$$

$$x = \frac{2n}{m} \quad (\text{multiply both sides by 2})$$

$$3x - 1 = \frac{6n}{m} - 1 \quad (\text{multiply both sides by 3, subtract 1})$$

$$3x - 1 = \frac{3n - m}{m} \quad (\text{common denominator})$$

Note that because  $n$  and  $m$  are integers (assumed from b), we have that  $3n - m$  is an integer (integers are closed under multiplication and addition). And since both

the numerator and denominator are integers, we have that  $3x - 1$  is rational, giving us  $b \rightarrow c$ .

Finally we will show that  $c \rightarrow a$ :

$$\begin{aligned}
 3x - 1 &= \frac{n}{m} && (\text{assume } c, \quad n, m \in \mathbb{Z} \wedge m \neq 0) \\
 x &= \frac{\frac{n}{m} + 1}{3} \\
 &= \frac{n}{3m} + \frac{1}{3} \\
 &= \frac{n + m}{3m} && (\text{common denominator})
 \end{aligned}$$

Note that because  $n$  and  $m$  are integers (assumed from because we have that  $n + m$  and  $3m$  are integers (integers are closed under multiplication and addition)). Since both the numerator and denominator are integers, we have that  $x$  is rational, giving us  $c \rightarrow a$ .

Putting this all together we have a chain of implications:

$$a \rightarrow b \rightarrow c \rightarrow a$$

This of course implies our result:

$$a \equiv b \equiv c$$

## Problem 6 [18 points]

### Question:

Prove the following two statements. You need to give a proof starting with the definition. You will not get any credit for providing examples.

### Part (a)

Prove that the sum two rational numbers is rational.

### Solution:

Since  $x$  and  $y$  are rational we have the following:

$$p, q, r, s \in \mathbb{Z} \wedge q \neq 0, s \neq 0$$



$$x = \frac{p}{q}, y = \frac{r}{s}$$

We can express the sum of  $x$  and  $y$  as rational numbers:

$$\begin{aligned} x + y &= \frac{p}{q} + \frac{r}{s} \\ &= \frac{ps + rq}{qs} \end{aligned} \quad \text{(common denominator)}$$

Because  $ps$  and  $rq$  are integers,  $q \neq 0$  and  $s \neq 0$ ,  $x + y$  can be shown as a ratio of two integers, therefore  $x + y$  is a rational number.

### Part (b)

Prove that the sum of an irrational number and a rational number must be irrational.

#### Solution:

Let's assume that  $x$  is rational,  $y$  is irrational, and  $x + y$  is rational expression. Because  $x + y$  is a rational, we can express the following:

$$p, q, r, s \in \mathbb{Z} \wedge q \neq 0, s \neq 0$$

$$x = \frac{p}{q}, x + y = \frac{r}{s}$$

We can now substitute  $x$  into  $x + y = \frac{r}{s}$ :

$$\begin{aligned} \frac{p}{q} + y &= \frac{r}{s} && \text{(substitute } x = \frac{p}{q} \text{)} \\ y &= \frac{r}{s} + \left(-\frac{p}{q}\right) && \text{(subtract both sides by } \frac{p}{q} \text{)} \end{aligned}$$

The expression  $y = \frac{r}{s} + \left(-\frac{p}{q}\right)$  is rational since it is a ratio of two numbers being summed, however our assumption of  $y$  being irrational is contradicted, so the sum of a irrational number and a rational number must be irrational.