CS 205: Introduction to Discrete Structures I Homework 2

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Problem 1. [10 points]

Question:

Prove that the sum of the first n odd positive integers is n^2. In other words, show that $1 + 3 + 5 + ... + (2n + 1) = (n + 1)^2$ for all $n \in \mathbb{Z}$.

Solution:

Base Case:

When $n = 1$:

$$\begin{aligned}
1 + (2(1) + 1) &= (1 + 1)^2 && \text{(substitute } n = 1\text{)} \\
1 + (2 + 1) &= (2)^2 && \text{(simplify)} \\
1 + 3 &= 4 && \text{(simplify)} \\
4 &= 4 && \text{(this expression holds true for when } n = 1\text{)}
\end{aligned}$$

Inductive Step:

We will assume $n = k$ is true.

$p_k = 1 + 3 + 5 + ... + (2k + 1) = (k + 1)^2$ for all $k \in \mathbb{Z}$

We need to prove this holds true for $k + 1$:

$$
\begin{aligned}
1+3+5+....+(2k+1)+(2(k+1)+1) &= (k+1+1)^2 \\
(k+1)^2+(2(k+1)+1) &= (k+2)^2 && \text{(replace } 1+3+5... \text{ with } (k+1)^2\text{)} \\
k^2+2k+1+2k+2+1 &= k^2+4k+4 && \text{(distribute and foil)} \\
k^2+4k+4 &= k^2+4k+4 && \text{(combine like terms)}
\end{aligned}
$$

Since both sides of the equation are equal, $k+1$ is true and hence the sum of the first n odd positive integers is n^2. ■

Problem 2. [10 points]

Question:

Suppose that a and b are real numbers with $0 < b < a$. Prove that for all positive integers n, the following holds: $a^n - b^n \leq na^{n-1}(a-b)$.

Solution:

Base Case:

When $n = 1$:

$$
\begin{aligned}
a^1 - b^1 &\leq 1a^{1-1}(a-b) && \text{(substitute n = 1)} \\
a^1 - b^1 &\leq 1a^0(a-b) && \text{(simplify)} \\
a-b &\leq a-b && \text{(this expression holds true for when n = 1)}
\end{aligned}
$$

Inductive Step:

We will assume $n = k$ is true.

$p_k = a^k - b^k \leq ka^{k-1}(a-b)$

We need to prove this holds true for $k+1$:

$$
a^{k+1} - b^{k+1} \leq (k+1)a^k(a-b) \qquad \text{(substitute k+1)}
$$

*Use identity $(a-b)a^k = a^{k+1} - b^{k+1} + b^k(a-b)$

$$
\begin{aligned}
a^{k+1} - b^{k+1} &= (a-b)a^k - b^k(a-b) && \text{(identity used)}\\
a^{k+1} - b^{k+1} &= (a-b)k(a^{k-1}) - b^k(a-b)\\
a^{k+1} - b^{k+1} &= ka^k(a-b) - b^k(a-b) && \text{(simplify)}\\
a^{k+1} - b^{k+1} + (a-b)b^k &= k(a^k)(a-b) + (a-b)b^k && \text{(add } (a-b)b^k \text{ to both sides)}\\
a^{k+1} - b^{k+1} + (a-b)b^k &= (k(a^k) + b^k)(a-b) && \text{(factor } (a-b) \text{ on right-hand side.)}\\
a^{k+1} - \frac{b^{k+1}}{a-b} + b^k &\leq k(a^k) + b^k && \text{(divide both sides by } (a-b))\\
a^{k+1} - \frac{b^{k+1}}{a-b} + b^k &\leq k(a^k) + a^k && \text{(since } 0 < b < a\text{, simplify right-hand side)}\\
a^{k+1} - \frac{b^{k+1}}{a-b} + b^k &\leq (k+1)a^k && \text{(factor } a^k \text{ on right-hand side)}\\
a^{k+1} - b^{k+1} &\leq (k+1)a^k(a-b) && \text{(multiply both sides by } (a-b))
\end{aligned}
$$

The inequality has been proven for all positive n integers. ■

Problem 3. [15 points]

Part (a):

Find a formula for the following sum:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n. You may want to find the sum for $n = 1, 2, 3, ...$ and then conjecture.

Solution:

Let s_n be the sum of n terms for this sequence as we try to find the pattern:

$$
\begin{aligned}
s_1 &= \frac{1}{1 \cdot 2} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{1+1}\\
s_2 &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = 1 - \frac{1}{3} = 1 - \frac{1}{2+1}\\
s_3 &= s_2 + \frac{1}{3 \cdot 4} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4} = 1 - \frac{1}{4} = 1 - \frac{1}{3+1}
\end{aligned}
$$

We can start to see a pattern and based off that pattern we can say that our conjecture is $1 - \frac{1}{n+1}$.

Part (b):

Prove the formula you conjectured in first part by mathematical induction.

Solution:

Base Case: When $n = 1$:

$$\frac{1}{1(1+1)} = \frac{1}{2} \qquad \text{(substitute } n = 1 \text{ into } \tfrac{1}{n(n+1)}\text{)}$$

$$1 - \frac{1}{1+1} = \frac{1}{2} \qquad \text{(substitute } n = 1 \text{ into } 1 - \tfrac{1}{n+1} \text{ (conjecture))}$$

$\frac{1}{2} = \frac{1}{2}$ so our base case is true for when $n = 1$.

Inductive Step:

We will assume n $= k + 1$.

$$\begin{aligned}
&= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+1+1)} && \text{(substitute } k+1\text{)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{(common denominator and simplify)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} && \text{(common denominator)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} && \text{(foil numerator)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} && \text{(factor numerator)} \\
&= \frac{k+1}{k+2} && \text{(cancel common factor)} \\
&= \frac{(k+2)-1}{k+2} && \text{(rewrite numerator)} \\
&= 1 - \frac{1}{k+2} && \text{(separate common denominator)} \\
&= 1 - \frac{1}{(k+1)+1} && \text{(rewrite denominator)}
\end{aligned}$$

$1 - \frac{1}{(k+1)+1}$ holds true for the series, thus proved by induction. ∎

Problem 4. [15 points]

Part (a):

Find the smallest positive integer n_0 such that for all $n \geq n_0$, we can form n cents of postage by only using 3-cent and 10-cent stamps.

Solution:

If we compute the Frobenius number, which is $a \cdot b - a - b$, where $a = 3$ and $b = 10$, we will get the following:

$$3 \cdot 10 - 3 - 10 = 17$$

This tells us that every amount of postage of 18 cents or more can be formed using 3-cent or 10-cent stamps since it is impossible to form a 17-cent stamp. So $n_0 = 18$ and thus $n \geq 18$.

Part (b):

Prove your answer to the first part by using strong induction.

Solution:

<u>Basis:</u>

$P(18)$ is true using 6, 3-cent stamps $(6 \cdot 3 = 18)$.
$P(19)$ is true using 1, 10-cent stamp and 3, 3-cent stamps $(10 + 3 \cdot 3 = 19)$.
$P(20)$ is true using 2, 10-cent stamps $(10 + 10 = 20)$.

<u>Inductive Hypothesis:</u>

Assume $P(j)$ is true for all $18 \leq j \leq k$ and $k \in \mathbb{Z}$ with $k \geq 20$. We can make postage for n-cents using 3-cent and 10-cent stamps. To make postage for $P(k+1)$ stamps, we use our induction hypothesis and find postage for $P(k-2)$-cent is true using only 3-cent and 10-cent stamps. As $k \geq 20$, $k - 2 \geq 18$, we can add one 3-cent stamp for $k - 2 + 3 = (k+1)$-cents, hence proves $P(k+1)$ is true. We now know that $P(n)$ is true for all $n \geq 20$. ∎

Problem 5. [15 points]

Question:

Suppose we have the following sequence defined as

- $a_1 = 1$
- $a_2 = 3$
- $a_k = 2a_{k-2} + a_{k-1}$ for all integers ≥ 3

Part (a):

Find the next 4 terms of the sequence a_3, a_4, a_5, a_6.

Solution:

Find a_3:

$$\begin{aligned}
a_3 &= 2a_{3-2} + a_{3-1} && \text{(substitute } k = 3\text{)} \\
&= 2a_1 + a_2 && \text{(simplify)} \\
&= 2(1) + 3 && \text{(substitute } a_1 \text{ and } a_2\text{)} \\
a_3 &= 5
\end{aligned}$$

Find a_4:

$$\begin{aligned}
a_4 &= 2a_{4-2} + a_{4-1} && \text{(substitute } k = 4\text{)} \\
&= 2a_2 + a_3 && \text{(simplify)} \\
&= 2(3) + 5 && \text{(substitute } a_2 \text{ and } a_3\text{)} \\
&= 6 + 5 && \text{(simplify)} \\
a_4 &= 11
\end{aligned}$$

Find a_5:

$$\begin{aligned}
a_5 &= 2a_{5-2} + a_{5-1} && \text{(substitute } k = 5\text{)} \\
&= 2a_3 + a_4 && \text{(simplify)} \\
&= 2(5) + 11 && \text{(substitute } a_3 \text{ and } a_4\text{)} \\
&= 10 + 11 \\
a_5 &= 21
\end{aligned}$$

Find a_6:

$$\begin{aligned}
a_6 &= 2a_{6-2} + a_{6-1} && \text{(substitute } k = 6\text{)} \\
&= 2a_4 + a_5 && \text{(simplify)} \\
&= 2(11) + 21 && \text{(substitute } a_4 \text{ and } a_5\text{)} \\
&= 22 + 21 && \text{(simplify)} \\
a_6 &= 43
\end{aligned}$$

Part (b):

State a property about the sequence a_k that might be true $\forall\ k \geq 1$.

Solution:

One property about the sequence a_k that might be true for all $k \geq 1$ and can be proven by strong induction is that $a_k \geq 2^{k-1}$.

Part (c):

Prove your conjecture (or property) using strong induction.

Solution:

Basis:

$$\begin{aligned}
a_1 &= 1 \geq 2^0 = 1 && \text{(substitute } k = 1 \text{ and this is true)} \\
a_2 &= 3 \geq 2^{2-1} = 2 && \text{(substitute } k = 2 \text{ and this is true)} \\
a_3 &= 5 \geq 4 && \text{(substitute } k = 3 \text{ and this is true)} \\
a_4 &= 21 \geq 8 && \text{(substitute } k = 4 \text{ and this is true)}
\end{aligned}$$

Induction Hypothesis:

We have $a_n \geq 2^{n-1}$ and $1 \leq n \leq k$.

Inductive Step:

We need to prove that $a_{k+1} \geq 2^k$:

$$a_{k+1} = 2a_{k-1} + a_k \qquad \text{(using the recurrence relation)}$$

By our induction hypothesis, we have $a_{k-1} \geq 2^{k-2}$ and $a_k \geq 2^{k-1}$ so we have:

$$\begin{aligned} a_{k+1} &\geq 2(2^{k-2}) + 2^{k-1} \\ &= 2^{k-1} + 2^{k-1} \\ &= 2^k \end{aligned}$$

Therefore by mathematical induction, the inequality $a_{k+1} \geq 2^k$ holds true and $a_k \geq 2^{k-1}$ holds for all $k \geq 1$. ■

Problem 6. [10 points]

Question:

Prove using the <u>well-ordering principle</u> that there is no solution over the positive integers to the equation:

$$4a^3 + 2b^3 = c^3.$$

Solution:

We will assume for the sake of contradiction that there exists a positive integer solution to the equation $4a^3 + 2b^3 = c^3$. We need to show that this leads to a contradiction, and that there are no positive integer solutions to the equation.

Consider the set $S = \{c \mid \text{there exist positive integers } a, b, \text{ such that } 4a^3 + 2b^3 = c^3\}$. By our assumption, S is non-empty since there exists at least one positive integer solution to the equation. By the well-ordering principle, S has a least element, which we will call n.

Let a, b be positive integers such that:

$$\begin{aligned} 4a^3 + 2b^3 &= n^3 && \text{(replace } c^3 \text{ with } n^3\text{)} \\ 2(2a^3 + b^3) &= n^3 && \text{(factor out a 2 on left-hand side)} \end{aligned}$$

Since $(2a^3 + b^3)$ is an integer, we have n^3 is even $\implies$ n is even. We can write $n = 2m, m \in \mathbb{Z}$, and m is positive.

We can rewrite the equation as the following:

$$\begin{aligned}
4a^3 + 2b^3 &= (2m)^3 && \text{(substitute } n = 2m\text{)} \\
4a^3 + 2b^3 &= 8m^3 && \text{(simplify right-hand side)} \\
2a^3 + b^3 &= 4m^3 && \text{(divide by 2 on both sides)} \\
b^3 &= 4m^3 - 2a^3 && \text{(solve for } b^3\text{)} \\
b^3 &= 2(2m^3 - a^3) && \text{(factor 2 on right-hand side)}
\end{aligned}$$

Since $(2m^3 - a^3)$ is an integer and we have b^3 is even $\implies$ b is even. We can write $b = 2k$, $k \in \mathbb{Z}$ and positive. We can rewrite the equation as the following:

$$\begin{aligned}
(2k)^3 &= 4m^3 - 2a^3 && \text{(substitute } b = 2k\text{)} \\
8k^3 &= 4m^3 - 2a^3 && \text{(simplify left-hand side)} \\
4k^3 &= 2m^3 - a^3 && \text{(divide by 2 on both sides)} \\
a^3 &= 2m^3 - 4k^3 && \text{(solve for } a^3\text{)} \\
a^3 &= 2(m^3 - 2k^3) && \text{(factor 2 on right-hand side)}
\end{aligned}$$

Since $(m^3 - 2k^3)$ is an integer, we have a^3 is even $\implies$ a is even. We can write $a = 2x, x \in \mathbb{Z}$ and positive.

We can rewrite the equation as the following:

$$\begin{aligned}
(2x)^3 &= 2m^3 - 4k^3 && \text{(substitute } a = 2x\text{)} \\
8x^3 &= 2m^3 - 4k^3 && \text{(simplify left-hand side)} \\
4x^3 &= m^3 - 2k^3 && \text{(divide by 2 on both sides)}
\end{aligned}$$

Finally, we can rearrange the equation one last time to:

$$2k^3 + 4x^3 = m^3$$

Since m, k, x are all positive integers, we have that m belongs to S. But since $m < n$, it contradicts the minimality of S. Hence proved. ■