CS 205: Introduction to Discrete Structures I Homework 2

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Problem 1. [12 points]

Question:

Provide answers to the following questions.

Part (a)

Given $R = \{1, 2, 3\}$ and $S = \{3, 4\}$ find $R \times S$ and $S \times R$. Is the Cartesian product relation commutative? That is, is it true that $R \times S = S \times R$ for every two sets R and S? If the answer is YES, then give a proof. If the answer is NO, then give a counterexample.

Solution:

$R \times S = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$

$S \times R = \{(3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$

A Cartesian product relation is commutative when a certain element x is present in the first set but not in the second set. Then the Cartesian product of the first set and second set will have a pair that has x.

We have sets R and S and for every two sets $R \times S \neq S \times R$ is not always commutative.

We can set our element $x = 1$ which will show up in $R \times S$ but not $S \times R$.

If we look at $(1, 4)$ for $R \times S$ and $(3, 2)$ for $S \times R$, it is not commutative because the number 1 does not show up for $S \times R$, and instead it is $(3, 2)$.

Therefore, $R \times S \neq S \times R$ and the Cartesian product relation is not true for every two sets R and S.

Part (b)

Given the predicate $P(x) : 0 < x^2 < 100$ where x is an integer, find the truth set for the predicate. Express the answer in set notation.

Solution:

We can begin plugging in the following values to find the truth set for the predicate:

$$\begin{aligned}
P(1) &= 0 < 1 < 100 && \text{(input 1 into } x^2\text{)} \\
P(2) &= 0 < 4 < 100 && \text{(input 2 into } x^2\text{)} \\
P(3) &= 0 < 9 < 100 && \text{(input 3 into } x^2\text{)} \\
P(4) &= 0 < 16 < 100 && \text{(input 4 into } x^2\text{)} \\
P(5) &= 0 < 25 < 100 && \text{(input 5 into } x^2\text{)} \\
P(6) &= 0 < 36 < 100 && \text{(input 6 into } x^2\text{)} \\
P(7) &= 0 < 49 < 100 && \text{(input 7 into } x^2\text{)} \\
P(8) &= 0 < 64 < 100 && \text{(input 8 into } x^2\text{)} \\
P(9) &= 0 < 81 < 100 && \text{(input 9 into } x^2\text{)}
\end{aligned}$$

We can also input negative numbers $1 - 9$ and get the same answers.

So the truth set for the predicate is the following:
{(-9),(-8),(-7),(-6),(-5),(-4),(-3),(-2),(0),(-1),(1,),(2),(3),(4),(5),(6),(7),(8),(9)}

$$\{x \in Z | (-10 < x < 10)\}$$

Part (c)

Prove or disprove: If powerset(A) = powerset(B), then $A = B$. To disprove you only need to give a counterexample. To prove it, you need to show detailed work.

Solution:

Let's assume powerset(A) = powerset(B).

Since $A \in$ powerset(A) $\implies$ $A \in$ powerset(B).

Conclusion 1: This tells us that $A \subset B$.

Since $B \in$ powerset(B) $\implies$ $B \in$ powerset(A).

Conclusion 2: This tell us that $B \subset A$

Therefore based off both conclusions, $A = B$.

Problem 2. [10 points]

Question:

The second associative law states that if A, B, and C are sets, then $A \cap (B \cap C) = (A \cap B) \cap C)$. We prove this statement by proving the below two assertions.

Part (a)

Prove that $A \cap (B \cap C) \subseteq (A \cap B) \cap C$. Hint: To prove this, start with an element $x \in A \cap (B \cap C)$ and show that $x \in (A \cap B) \cap C$.

Solution:

$$x \in A \cap (B \cap C)$$

Since x is part of $B \cap C$, then x belongs to B and C. All the information above gives us the following about the left hand side:

$$\begin{aligned}
x &\in A && (x \text{ belongs to } A \text{ because of intersection}) \\
x &\in B && (\text{since } x \in B \cap C) \\
x &\in C && (\text{since } x \in B \cap C)
\end{aligned}$$

If we look at the right hand side, we have the following too:

$$\begin{aligned} x &\in A && \text{(since } x \in A \cap B) \\ x &\in B && \text{(since } x \in A \cap B) \end{aligned}$$

When we compare the left hand side to the right hand side, x is a element for both sides. Because $x \in (A \cap B)$ and $x \in C$, this tells us that $x \in A \cap (B \cap C) \implies x \in (A \cap B) \cap C$.

Since this is true for any element x in $A \cap (B \cap C)$ therefore, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

Part (b)

Prove that $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

Solution:

$$x \in (A \cap B) \cap C$$

Since x is an element for A, B, C, this gives us the following information for the left hand side:

$$\begin{aligned} x &\in A \\ x &\in B \\ x &\in C \end{aligned}$$

Now, if we look at the right hand side, x is also an element for $A \cap (B \cap C)$:

$$\begin{aligned} x &\in A \\ x &\in B \\ x &\in C \end{aligned}$$

When we compare the left hand side to the right hand side, x has the same conditions that are the same.

Therefore, every element in $(A \cap B) \cap C$ is in $A \cap (B \cap C) \implies (A \cap B) \cap C \subseteq A \cap (B \cap C)$.

Problem 3. [10 points]

Question:

For every real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Show that if x is a real number and m is an integer, then $\lceil (x+m) \rceil = \lceil x \rceil + m$.

Solution:

We will start by looking at the left hand side. $\lceil (x+m) \rceil$ tells us that it is the smallest integer $\geq (x+m)$.

$$p \in Z, p \geq (x+m)$$

We will now look at the right side. $\lceil x \rceil$ tells us that it is the smallest integer $\geq x$.

$$q \in Z, q \geq x$$

$\lceil x \rceil + m$ is an integer that adds m to the smallest integer $\geq x$. So, $\lceil x \rceil + m \geq x + m$.

Now, we can combine our statements above:

$$\begin{aligned} p &\geq (x+m) \\ \lceil x \rceil + m &\geq x + m \end{aligned}$$

Because p is the smallest integer $\geq (x+m) \implies p \geq \lceil x \rceil$. This can be expressed as $p \geq \lceil x \rceil + m$.

However, we must make sure that p is the smallest integer that is $\geq \lceil x \rceil + m$.

We can assume the following:

$$\begin{aligned} r &\in Z \\ r &\geq \lceil x \rceil + m \\ r &< p \end{aligned}$$

Because r is an integer, it can be written as $r = \lceil x \rceil + s, s \in Z$. This tell us the following:

$$\begin{aligned} r - m &= \lceil x \rceil + s - m \\ r - m &\geq x + s \\ r &> p - m \end{aligned}$$

$r > p - m$ contradicts our assumption that $r < p$. It is impossible for $r \geq \lceil x \rceil + m$ and $r < p$. This tells us that p is the smallest integer that is $\geq \lceil x \rceil + m$.

Conclusions:

$$\lceil (x+m) \rceil = p$$
$$\lceil x \rceil + m = \lceil x \rceil + (p - (x+m)) = p$$

Because $p \geq (x+m)$ and $\lceil x \rceil + (p - (x+m)) = p$, we can finally conclude that $\lceil (x+m) \rceil = \lceil x \rceil + m$.

Problem 4. [8 points]

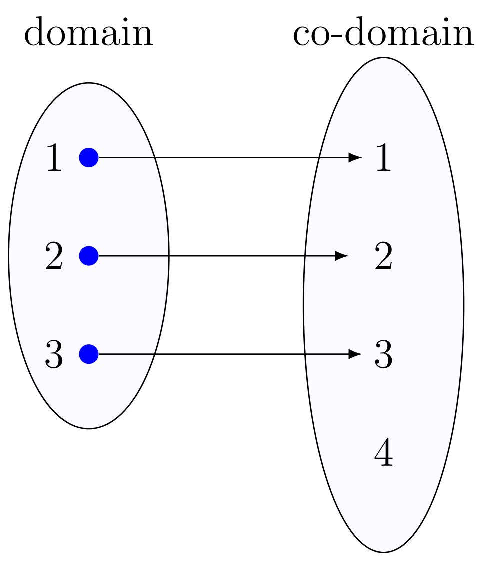
Question:

Give examples of functions $f\colon N \to N$ for the following cases.

Part (a)

f is one-to-one but not onto.

Solution:

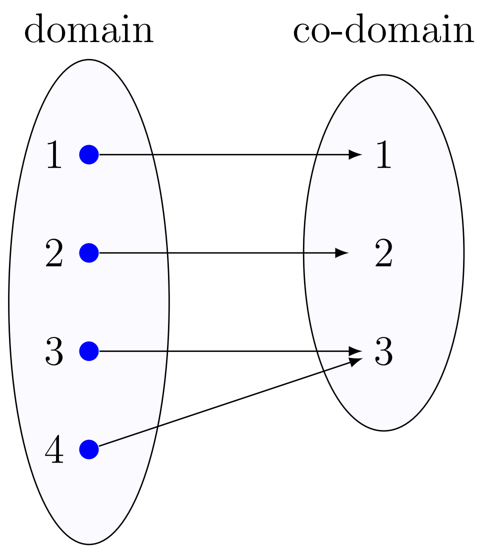


f is one-to-one because every domain value has a unique co-domain, but not onto because every co-domain value is not used.

Part (b)

f is onto but not one-to-one.

Solution:

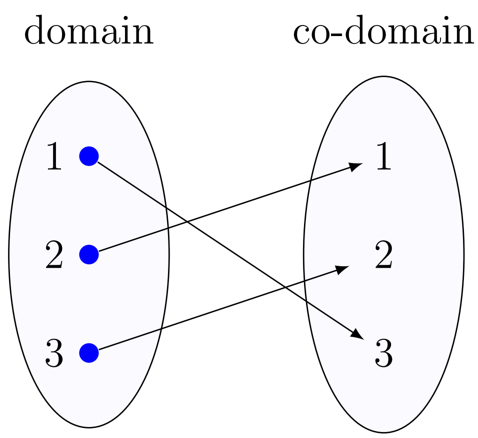


f is onto because every value in the co-domain is an output for the domain, but not one-to-one because there are values of the domain that share the same co-domain output.

Part (c)

f is both onto and one-to-one (but different from the identity function).

Solution:

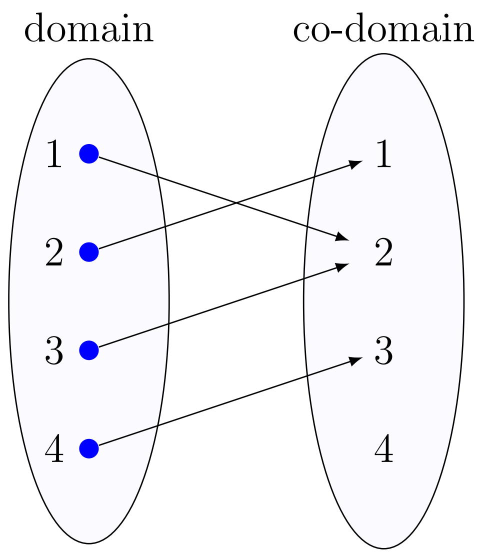


f is onto because every co-domain value is an output for the domain and one-to-one because every domain has a unique co-domain value output.

Part (d)

f is neither one-to-one nor onto.

Solution:



f is not one-to-one because there are values in the domain that share the same co-domain output and is not onto because not every value in the co-domain is being used as an output.

Problem 5. [10 points]

Question:

Consider the function g: $R \to R$ defined as follows:

$$g(x) = -\frac{2}{3}x - 5$$

Does g have an inverse? If yes, what is it (and prove that it is the inverse)? If not, why not?

Solution:

We can begin by solving for x in terms of y to find the inverse.

$$y = -\frac{2}{3}x - 5 \qquad \text{(set } g(x) \text{ to } y\text{)}$$

$$y + 5 = -\frac{2}{3}x \qquad \text{(add 5 to both sides)}$$

$$x = -\frac{3}{2}(y + 5) \qquad \text{(multiply both sides by } -\tfrac{3}{2}\text{)}$$

$$x = -\frac{3}{2}y - \frac{15}{2} \qquad \text{(distribute } -\tfrac{3}{2}\text{)}$$

$$y = -\frac{3}{2}x - \frac{15}{2} \qquad \text{(swap } x \text{ and } y\text{)}$$

$$g^{-1}(x) = -\frac{3}{2}x - \frac{15}{2} \qquad \text{(replace } y \text{ with } g^{-1}(x)\text{)}$$

As shown, $g(x)$ does have an inverse and is one-to-one because if you were to graph $g(x)$, it passes the horizontal line test. There is also nothing that can make $g(x)$ undefined because x is in the numerator and every transformation performed above to calculate $g^{-1}(x)$ was one-to-one. Since the composition of one-to-one functions on R is one-to-one then so too is the resultant inverse function.

Problem 6. [10 points]

Question:

If f and g are one-to-one, then prove that $f \circ g$ is also one-to-one.

Solution:

Since f and g are one-to-one functions, we can say x_1 and x_2 are distinct elements in $f \circ g$, we can say the following:

$$\exists y_1 \wedge y_2, y_1 = g(x_1), y_2 = g(x_2)$$

$y_1 = g(x_1)$ and $y_2 = g(x_2)$ are in the domain of f such that $y_1 \neq y_2$. $f(y_1) \neq f(y_2)$ because f is a one-to-one function. Since $y_1 = g(x_1)$ and $y_2 = g(x_2)$, we can write the following:

$$(f \circ g)(x_1) = f(g(x_1)) = f(y_1)$$
$$(f \circ g)(x_2) = f(g(x_2)) = f(y_2)$$

Because $f(y_1) \neq f(y_2) \implies (f \circ g)(x_1) \neq (f \circ g)(x_2)$, therefore $f \circ g$ is one-to-one.

Problem 7. [15 points]

Question:

If A is a countably infinite set and B is a finite set, then prove that $A \cup B$ is a countably infinite set.

Solution:

Since set A is countably infinite, then there must exist a bijection between it and the natural numbers.

$$\exists f : N \to A$$

Now let a_n be the following:

$$a_n = f(n)$$

We can then write all the elements of A as follows: $A = \{a_0, a_1, a_2, ...a_n\}$

Note that this covers all elements of A because it is bijective with N. Now consider $C = B - A$. C is just B with the elements of A removed. As such, $A \cup B = A \cup C$. C has size n.

$C = \{c_0, c_1, c_2, ...c_n\} = n$

To show that $A \cup C$ is countably infinite, we can write the following:

$$A \cup C = \{c_0, c_1, c_2, ...c_n, a_0, a_1, a_2, ...a_n\}$$
$$N = \{0, 1, 2, ..., n, n+1, n+2, n+3, ...\}$$

$A \cup C$ is countably infinite because there is a bijection, g, between $A \cup C$ and the natural numbers, so $A \cup C$ is countably infinite.

Since $A \cup C = A \cup B$, this means that $A \cup B$ is **countably infinite.**