Homework 2 MATH 250

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Question 5:

Consider the system of equations

$$\begin{cases} x + ay - z = 0 \\ ax + y + z = 0 \\ x + y + az = a^2 \end{cases} \tag{1}$$

Solution:

We'll take all the coefficients of each variable and put them into a matrix and solve for the identity matrix:

$$= \begin{bmatrix} 1 & a & -1 & | & 0 \\ a & 1 & 1 & | & 0 \\ 1 & 1 & a & | & a^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a & -1 & | & 0 \\ 0 & 1-a^2 & 1+a & | & 0 \\ 0 & 1-a & 1+a & | & a^2 \end{bmatrix} \qquad (R_2 - aR_1 \to R_2 \ \& \ R_3 - R_1 \to R_3)$$

$$= \begin{bmatrix} 1 & a & -1 & | & 0 \\ 0 & 1-a^2 & 1+a & | & 0 \\ 0 & a(1-a) & 0 & | & a^2 \end{bmatrix} \qquad (R_3 - R_2 \to R_3)$$

$$= \begin{bmatrix} 1 & a & -1 & | & 0 \\ 0 & (1-a)(1+a) & 1+a & | & 0 \\ 0 & a(1-a) & 0 & | & a^2 \end{bmatrix} \qquad (\text{foil } (1-a^2))$$

$$= \begin{bmatrix} 1 & a & -1 & | & 0 \\ 0 & 1+a & 1+a & | & 0 \\ 0 & a & 0 & | & \frac{a^2}{1-a} \end{bmatrix} \qquad (\tfrac{1}{1-a}R_2 \ \& \ \tfrac{1}{1-a}R_3)$$

$$= \begin{bmatrix} 1 & a & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & 0 & | & \frac{a}{1-a} \end{bmatrix} \qquad (R_2\tfrac{1}{1+a} \ \& \ R_3\tfrac{1}{a})$$

$$= \begin{bmatrix} 1 & a & -1 & | & 0 \\ 0 & 0 & 1 & | & \frac{a}{a-1} \\ 0 & 1 & 0 & | & \frac{a}{1-a} \end{bmatrix} \qquad (R_2 - R_3 \to R_2)$$

$$= \begin{bmatrix} 1 & a & -1 & | & 0 \\ 0 & 1 & 0 & | & \frac{a}{1-a} \\ 0 & 0 & 1 & | & \frac{a}{a-1} \end{bmatrix} \qquad (\text{swap } R_2 \ \& \ R_3)$$

$$= \begin{bmatrix} 1 & a & 0 & | & \frac{a}{a-1} \\ 0 & 1 & 0 & | & \frac{a}{1-a} \\ 0 & 0 & 1 & | & \frac{a}{a-1} \end{bmatrix} \qquad (R_1 + R_3 \to R_1)$$

$$= \begin{bmatrix} 1 & 0 & 0 & | & \frac{a}{a-1} - \frac{a^2}{1-a} \\ 0 & 1 & 0 & | & \frac{a}{1-a} \\ 0 & 0 & 1 & | & \frac{a}{a-1} \end{bmatrix} \qquad (R_1 - aR_2 \to R_1)$$

Now that our matrix is finally in the identity matrix, we know what x, y, and z are since column one represents x, column 2 represents y, and column 3 represents z.

So we have the following:

$$\begin{aligned} x &= \frac{a}{a-1} - \frac{a^2}{1-a} \\ y &= \frac{a}{1-a} \\ z &= \frac{a}{a-1} \end{aligned}$$

Now we can answer all 3 parts of this question.

Part (a):

Find the values of a for which the previous system of equations has no solutions.

Solution:

if $a = 1$, then we will not have a solution since we will have 0 in our denominator for all 3 variables, therefore making this undefined.

Part (b):

Solve the system for those values of a for which there are infinitely many solutions.

Solution:

if $a = -1$, we will have infinite solutions because on our second step when solving for the identity matrix, if we plug $a = -1$, then the second row would be filled with 0, which tells us there are infinite solutions.

Part (c):

Find the values of a for which there is a unique solution.

Solution:

if $a \neq 1$ and $a \neq -1$, then there will be a unique solution.

Question 7:

Consider the system of equations

$$\begin{cases} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (a^2 - 5)z = a \end{cases} \tag{2}$$

Solution:

We'll take all the coefficients of each variable and put them into a matrix and solve for the identity matrix:

$$= \begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 1 & 2 & 1 & | & 3 \\ 1 & 1 & a^2 - 5 & | & a \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & a^2 - 4 & | & a - 2 \end{bmatrix} \qquad (R_2 - R_1 \to R_2 \ \& \ R_3 - R_1 \to R_3)$$

Since the bottom left of our matrix is all 0, we can start solving for x, y, and z.

We'll begin by solving for z:

$$\begin{aligned} (a-2)(a=2)z &= a - 2 && \text{(difference of squares)} \\ (a+2)z &= 1 && \text{(divide both sides by } a - 2\text{)} \\ z &= \frac{1}{a+2} && \text{(divide both sides by } a + 2\text{)} \end{aligned}$$

We can now solve for y now that we know z:

$$
\begin{aligned}
y + 2z &= 1 \\
y + 2\left(\frac{1}{a+2}\right) &= 1 && \text{(substitute } z\text{)} \\
y + \frac{2}{a+2} &= 1 && \text{(simplify)} \\
y &= 1 - \frac{2}{a+2} && \text{(simplify)} \\
y &= \frac{a}{a+2}
\end{aligned}
$$

Finally, we can solve for x now that we know y and z:

$$
\begin{aligned}
x + y - z &= 2 \\
x + \frac{a}{a+2} + \frac{1}{a+2} &= 2 && \text{(substitute } x \text{ and } y\text{)} \\
x + \frac{a+1}{a+2} &= 2 && \text{(simplify)} \\
x &= \frac{2(a+2) - (a+1)}{a+2} && \text{(simplify)} \\
x &= \frac{2a + 4 - a - 1}{a+2} && \text{(simplify)} \\
x &= \frac{a+3}{a+2} && \text{(simplify)}
\end{aligned}
$$

Now that we know x, y, and z, we can answer all 3 parts of this question.

Part (a):

Find the values of a for which the system has no solution.

Solution:

if $a = -2$, all three variables will be undefined since we will have 0 in the denominator, which gives us no solution.

Part (b):

Solve the system for the value of a which yields infinitely many solutions.

Solution:

if $a = -2$, we will have no solution because on our final step for solving the identity matrix, $a = -2$ will make the third row all 0, which means we will have infinite solutions.

Part (c):

Find the values of a for which there is a unique solution.

Solution:

if $a \neq 2$ and $a \neq -2$ for unique solutions.

Question 15:

The parabola $y = a + bx + cx^2$ goes through the points $(1, 4), (2, 8)$ and $(3, 14)$. Find and solve a matrix equations for the unknowns (a, b, c)

Solution:

Using our given we can come up with a system of equations:

$$\begin{cases} a + b + c = 4 \\ a + 2b + 4c = 8 \\ a + 3b + 9c = 14 \end{cases} \tag{3}$$

Now we'll put our coefficients into a matrix and solve for the identity matrix:

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 1 & 2 & 4 & | & 8 \\ 1 & 3 & 9 & | & 14 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 1 & 3 & | & 4 \\ 0 & 2 & 8 & | & 10 \end{bmatrix} && (R_2 - R_1 \to R_2 \ \& \ R_3 - R_1 \to R_3) \\
&= \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 1 & 3 & | & 4 \\ 0 & 1 & 4 & | & 5 \end{bmatrix} && (\tfrac{1}{R} R_3) \\
&= \begin{bmatrix} 1 & 1 & 1 & | & 4 \\ 0 & 1 & 3 & | & 4 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} && (R_3 - R_2 \to R_3)
\end{aligned}
$$

Since the bottom left 3 are all 0, we know $c = 1$ and can now solve for a and b.

We'll first solve for b:

$$
\begin{aligned}
b + 3 &= 4 \\
b &= 1 && \text{(simplify)}
\end{aligned}
$$

Now we can solve for a since we know b and c:

$$
\begin{aligned}
a + b + c &= 4 \\
a + 1 + 1 &= 4 && \text{(substitute } b \text{ and } c\text{)} \\
a + 2 &= 4 && \text{(simplify)} \\
a &= 2
\end{aligned}
$$

So our final answers are $a = 2$, $b = 1$, and $c = 1$.

Question 56:

The transpose of a matrix, called A^T, is the matrix you obtain by swapping the rows with the columns of a matrix. For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, and if $A = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$.

Part (a):

If A is an m x n matrix, what is the size of A^T?

Solution:

m x m or in other words (2 x 2).

Part (b):

What is $(A^T)^T$?

Solution:

$$(A^T)^T = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$$

Part (c):

A square matrix is called **symmetric** if $A^T = A$. Find all 2 x 2 symmetric matrices.

Solution:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

From these two matrices we can say that $a = a$, $c = b$, $b = c$. $d = d$, therefore:

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Part (d):

A square matrix is called **skew-symmetric** if $A^T = -A$. Find all 2 x 2 skew-symmetric matrices.

Solution:

$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

From these two matrices, we can say that $a = -a \implies a = 0, c = -b, b = -c, d = -d \implies d = 0$, therefore:

$$A = \begin{bmatrix} 0 & -1 \\ a & 0 \end{bmatrix}$$