CSE 221: Algorithms

Quicksort

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References

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2. Erik Demaine and Charles Leiserson, *6.046J Introduction to Algorithms*. MIT OpenCourseWare, Fall 2005. Available from: `ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-046JFall-2005/CourseHome/index.htm`

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Why do we want to study Quicksort?

One of the most widely used, and extensively studied, sorting algorithms.

Divide and conquer

Quicksort an *n*-element array:

1. *Divide* Partition the array into subarrays around a *pivot* x such that the elements in lower subarray $\leq x \leq$ elements in the upper subarray.

$\leqslant x$	x	$\geqslant x$

2. *Conquer* Recursively sort the two subarrays.
3. *Combine* Trivial – just concatenate the lower subarray, *pivot*, and the upper subarray.

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Key

Linear-time partitioning algorithm.

Quicksort in action

1	2	3	4	5	6	7	8
6	10	13	5	8	3	2	11

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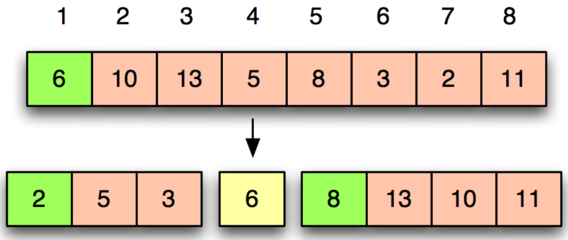

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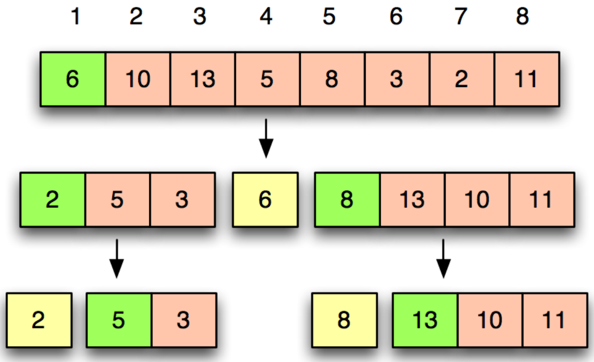

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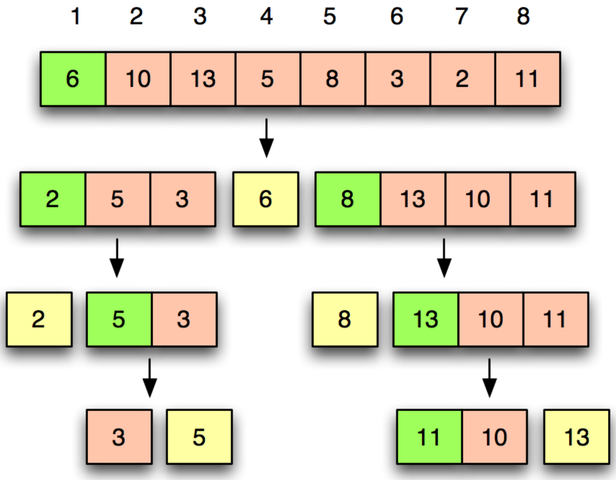

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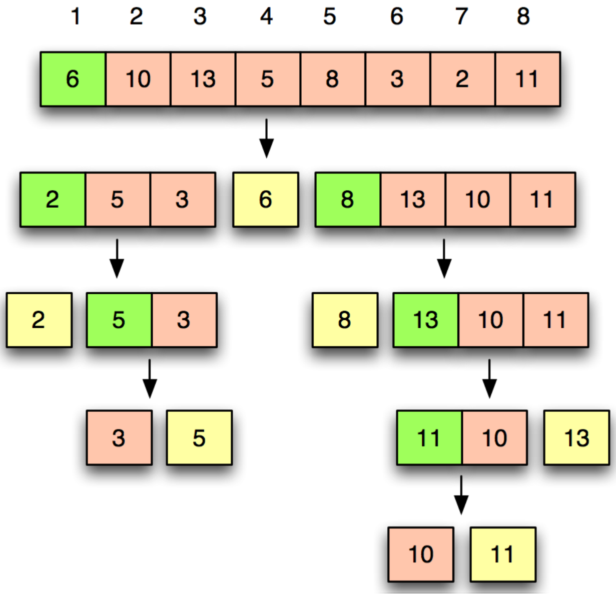

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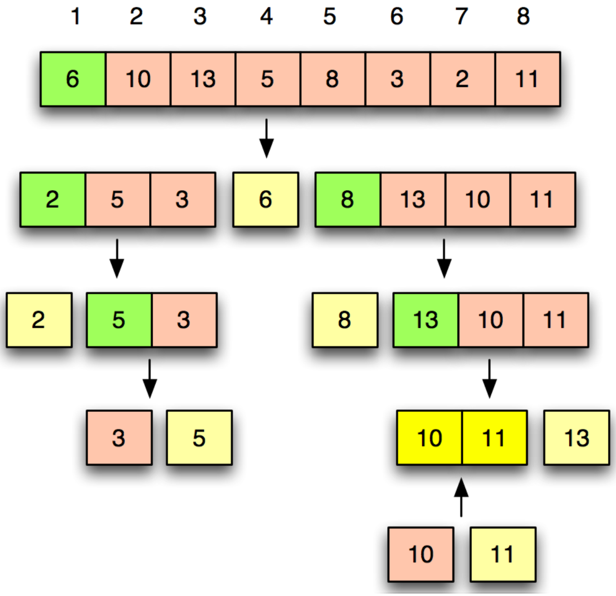

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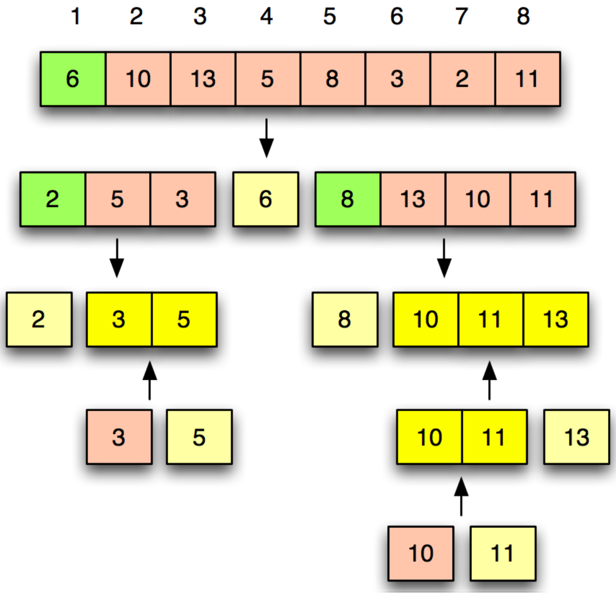

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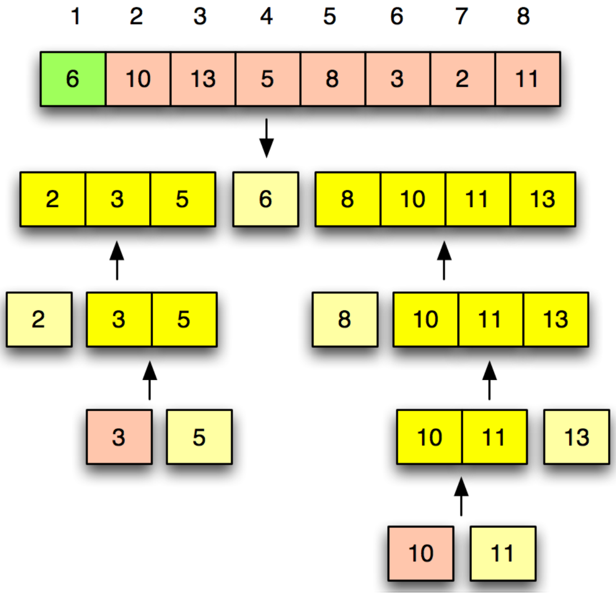

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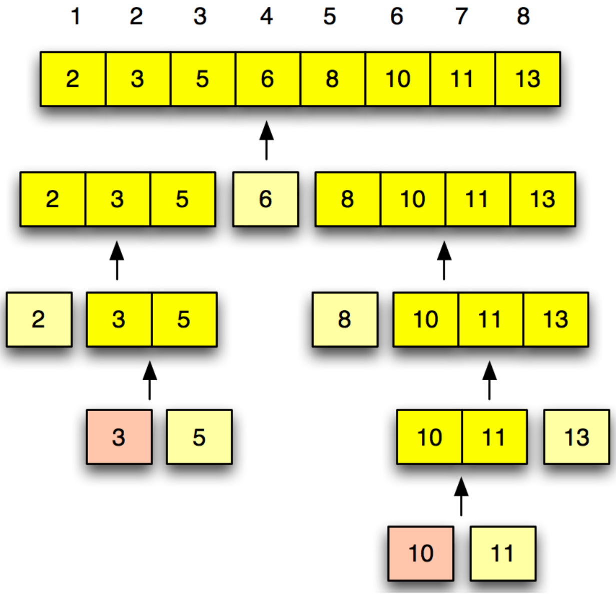

Contents

Partitioning algorithm

Algorithm

```
PARTITION(A, p, q) ▷ A[p . . q]
x ← A[p]                ▷ pivot = A[p]
i ← p
for j ← p + 1 to q
        do if A[j] ≤ x
                then i ← i + 1
                     exchange A[i] ↔ A[j]
exchange A[p] ↔ A[i]
return i
```

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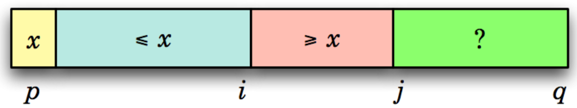
Invariant



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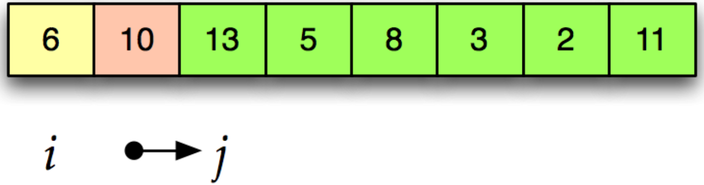

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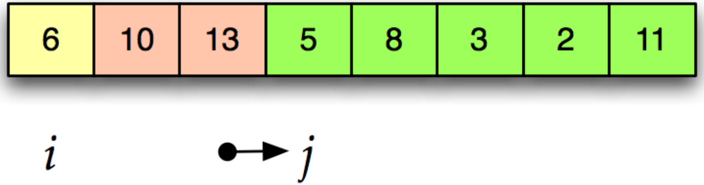

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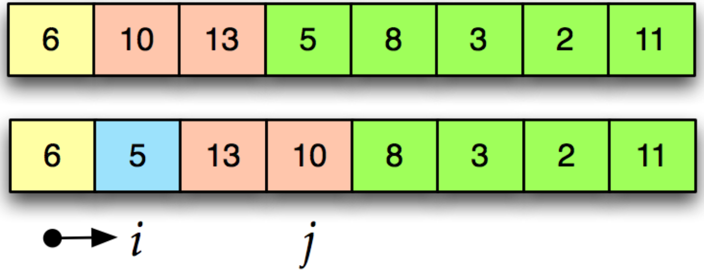

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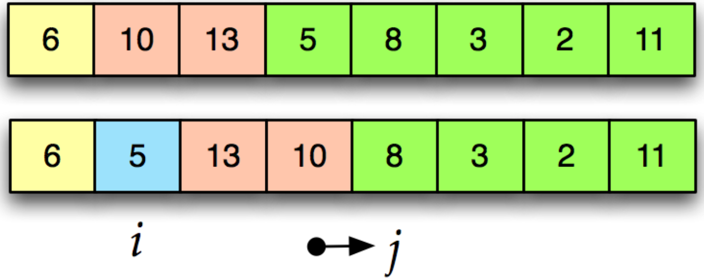

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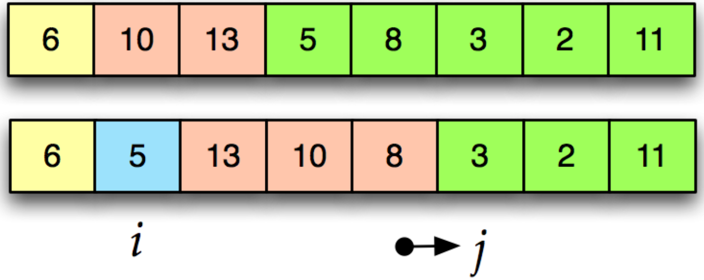

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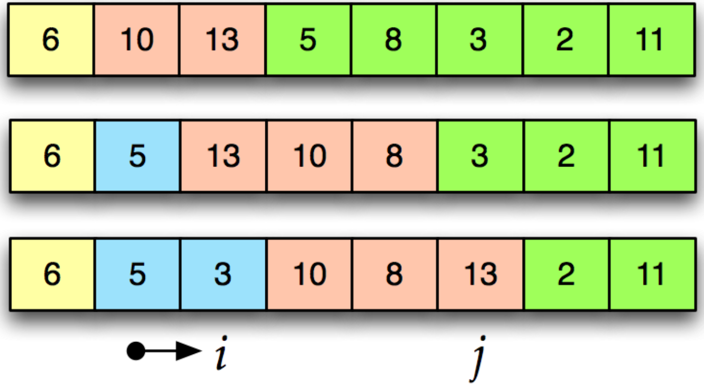

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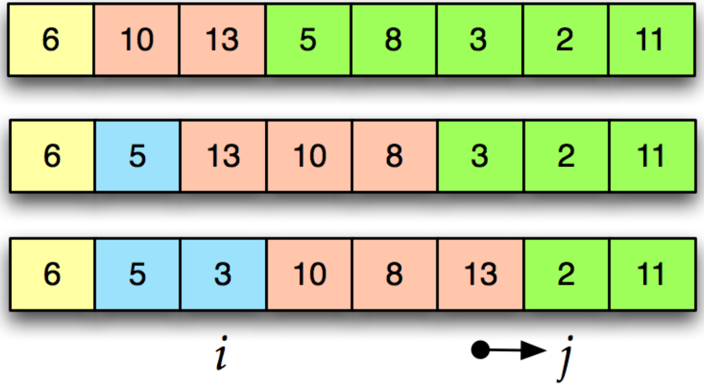

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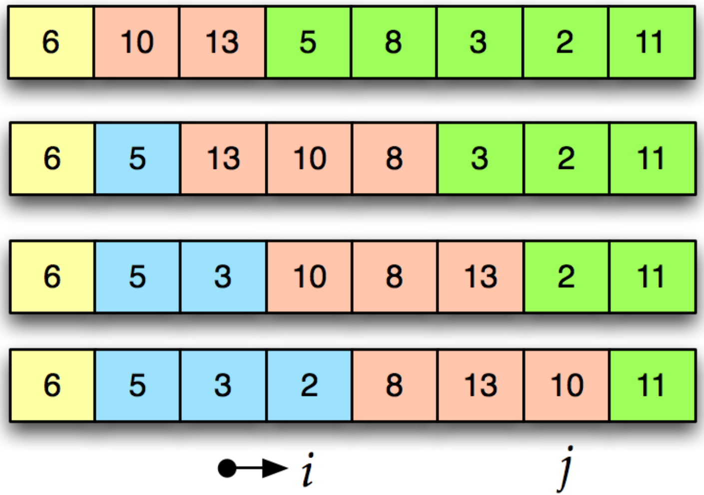

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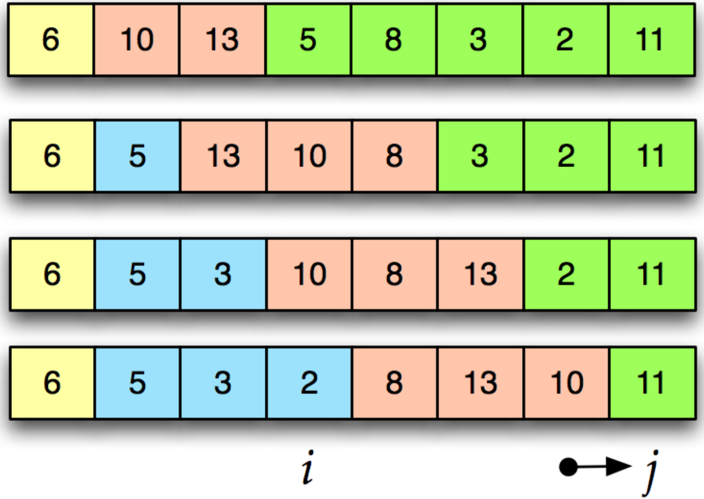

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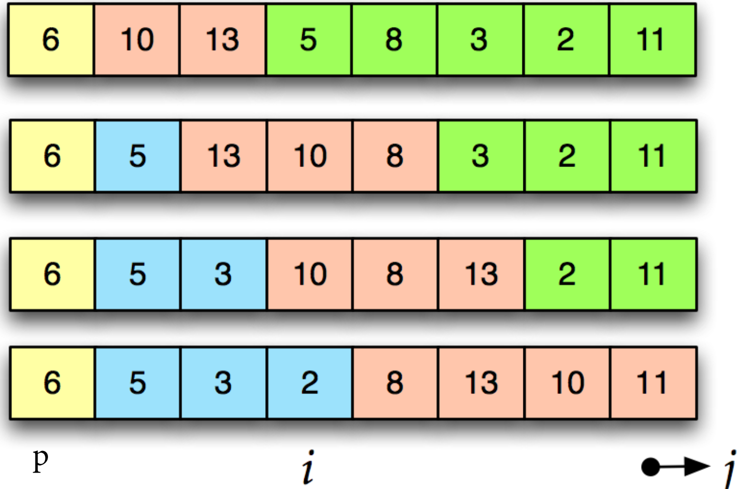

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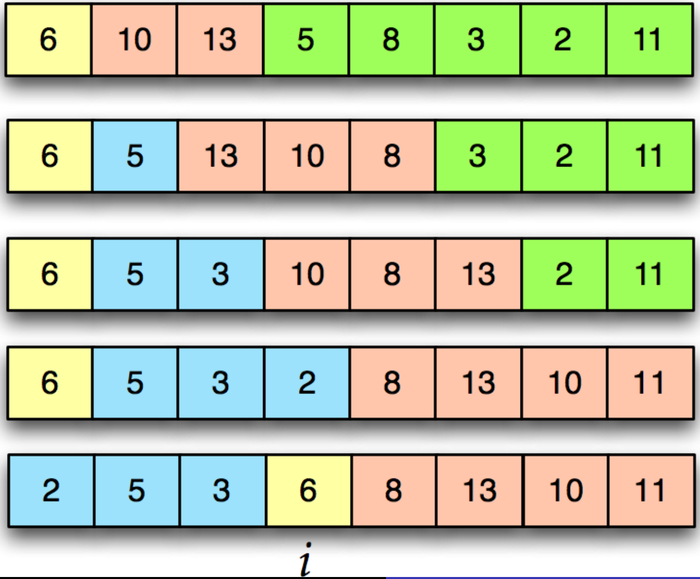

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Quicksort algorithm

Algorithm

$\textsc{Quicksort}(A, p, r) \triangleright A[p \,..\, r]$

```
if p < r
   then q ← PARTITION(A, p, r)
        QUICKSORT(A, p, q − 1)
        QUICKSORT(A, q + 1, r)
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Initial call

$\textsc{Quicksort}(A, 1, n)$

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- Worst-case happens when pivot is always the minimum or maximum element.

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Worst-case analysis

(Note: the worst-case running time for partitioning is $\Theta(n)$.)

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \boxed{\Theta(n^2)} \end{aligned}$$

Worst-case recursion tree

$$T(n) = T(0) + T(n-1) + cn$$

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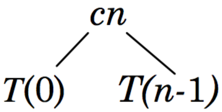

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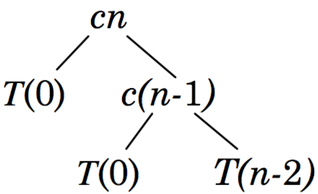

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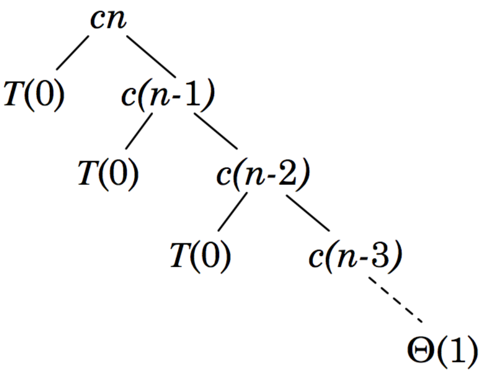

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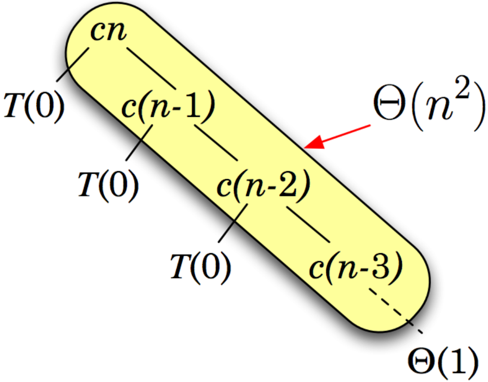

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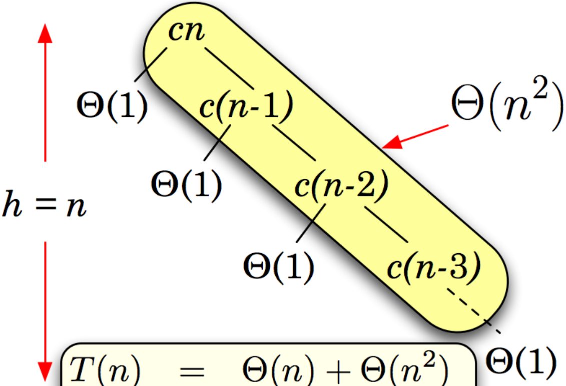

$$\begin{aligned} T(n) &= \Theta(n) + \Theta(n^2) \\ &= \Theta(n^2) \end{aligned}$$

Best- and almost-worst case performances

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How to ensure that we don't *usually* hit the worst-case?

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RANDOMIZED-PARTITION(A, p, r) $\triangleright$ $A[p \mathinner{.\,.} r]$

```
1   i ← RANDOM(p, r)                    ▷ i = [p . . r]
2   exchange A[p] ↔ A[i]
3   return PARTITION(A, p, r)
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$\textsc{Randomized-Partition}(A, p, r) \triangleright A[p \mathinner{..} r]$

$i \leftarrow \textsc{Random}(p, r) \qquad \triangleright i = [p \mathinner{..} r]$

exchange $A[p] \leftrightarrow A[i]$

return $\textsc{Partition}(A, p, r)$

$\textsc{Randomized-Quicksort}(A, p, r)$

if $p < r$

  then $q \leftarrow \textsc{Randomized-Partition}(A, p, r)$

    $\textsc{Randomized-Quicksort}(A, p, q-1)$

    $\textsc{Randomized-Quicksort}(A, q+1, r)$

Conclusion

- One of the most widely used sorting algorithm.
- While it runs in $O(n^2)$ time in the worst-case, it runs in $O(n \lg n)$ time on the average.
- Runs almost twice as fast as merge-sort.
- Can be *tuned* substantially.
- Almost all program language runtime library provide some variant of Quicksort (java.util.Arrays.sort() in Java, qsort() in C, std::sort() in C++, etc).

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- What are the worst, best and average case performances?
- Is it in-place?
- Is it stable?