# CSE 221

# Design and Analysis of Algorithms

Lecture 1:

Logistics, introduction, and multiplication!

*The original slides are from Stanford University*

# How was your break?

# The big questions

- Who are we?
  - Professor, TAs, students?
- Why are we here?
  - Why learn about algorithms?
- What is going on?
  - What is this course about?
  - Logistics?
- Can we multiply integers?
  - And can we do it quickly?

# Who are we?

- **Instructor:**
  - **Rayhan Rashed**
- **Course Coordinator:**
  - **Shaily Roy**

# Where are you?

# Why are we here?

- I’m here because I’m super excited about algorithms!

**Yay Algorithms!**

You are better equipped to answer this question than I am, but I'll give it a go anyway...

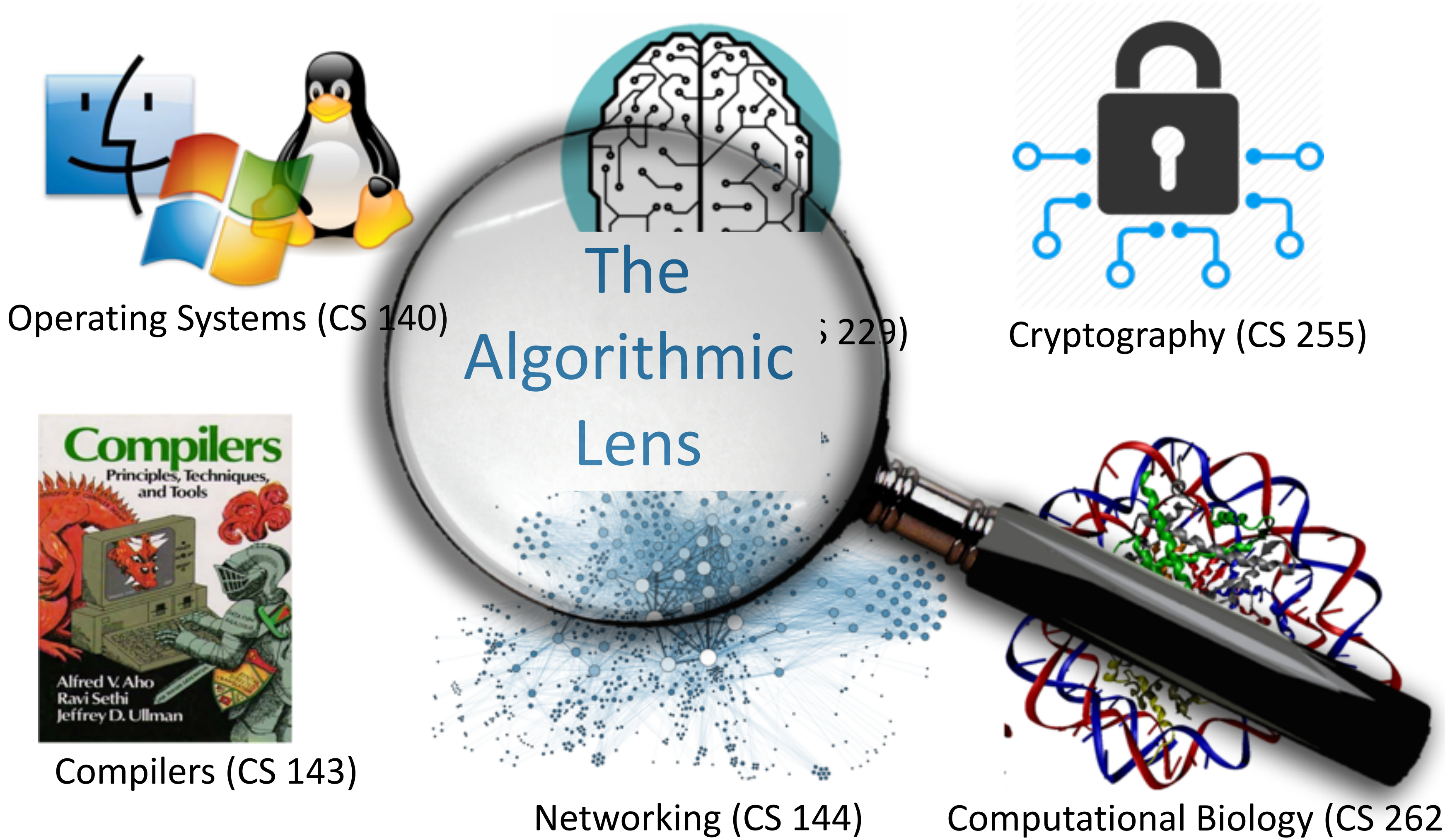
# Why are you here?

- Algorithms are fundamental.
- Algorithms are useful.
- Algorithms are fun!
- CS161 is a required course.

# Why is CSE221 required?

- Algorithms are fundamental.
- Algorithms are useful.
- Algorithms are fun!

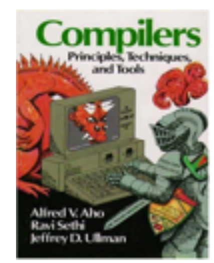
# Algorithms are fundamental



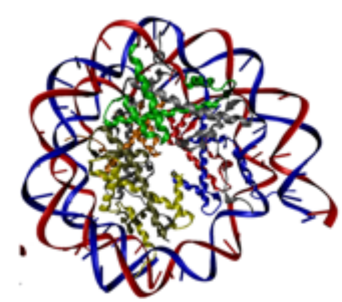
# Algorithms are useful

- All those things without the course numbers.
- As inputs get bigger and bigger, having good algorithms becomes more and more important!









# Algorithms are fun!

- Algorithm design is both an art and a science.
- Many surprises!
- Many exciting research questions!

# What’s going on?

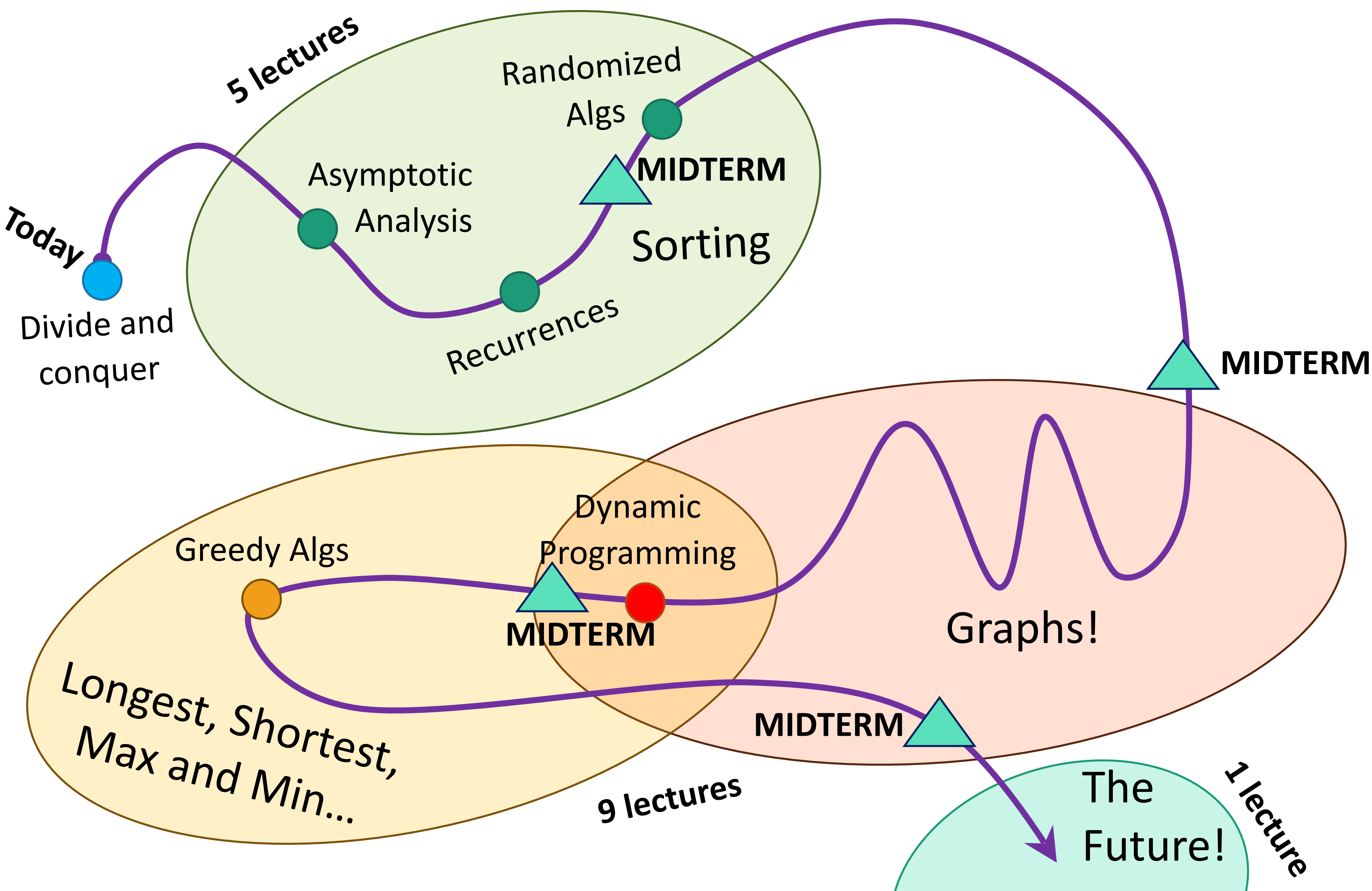
- Course goals/overview
- Logistics

# Course goals

- The design and analysis of algorithms
  - These go hand-in-hand
- In this course you will:
  - Learn to think analytically about algorithms
  - Flesh out an "algorithmic toolkit"
  - Learn to communicate clearly about algorithms

More detailed schedule on the website!

# Roadmap



# Our guiding questions:

Does it work?

Is it fast?

Can I do better?

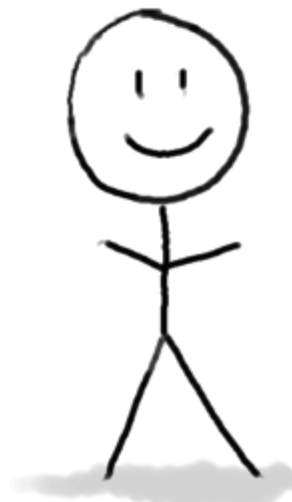
# Our internal monologue...

What exactly do we mean by better? And what about that corner case? Shouldn't we be zero-

Plucky the
Pedantic Penguin

Detail-oriented
Precise
Rigorous

Does it work?
Is it fast?
Can I do better?



Dude, this is just like that other time. If you do the thing and the stuff like you did then, it'll totally work real fast!

Lucky the
Lackadaisical Lemur

Big-picture
Intuitive
Hand-wavey

Both sides are necessary!

# Aside: the bigger picture

- Does it work?
- Is it fast?
- Can I do better?

- Should it work?
- Should it be fast?

- We want to reduce crime.
- It would be more “efficient” to put cameras in everyone’s homes/cars/etc.

- We want advertisements to reach to the people to whom they are most relevant.
- It would be more “efficient” to make everyone’s private data public.

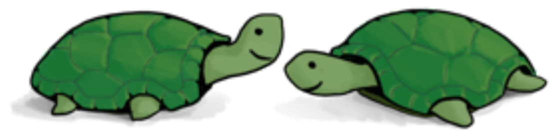
- We want to design algorithms, that work well, on average, in the population.
- It would be more “efficient” to focus on the majority population.

# Course elements and resources

- Course website:
    - BuX
- Lectures
- Discord
- Assignments
- Quiz
- Exams
- Presentation

# How to get the most out of lectures

- **During lecture:**
  - Participate live (if you can), ask questions.
  - Engage with in-class questions.
- **Before lecture:**
  - Do ***pre-lecture exercises*** on the website.
- **After lecture:**
  - Go through the exercises on the slides.



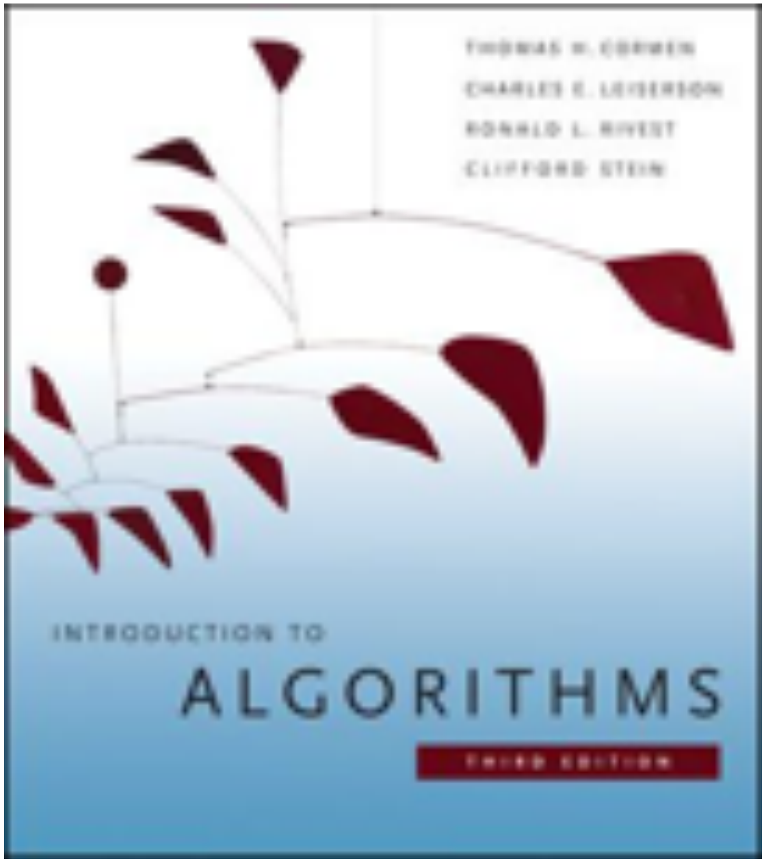
Think-~~Pair~~-Share Terrapins
(in-class questions)

Siggi the Studious Stork
(recommended exercises)

Ollie the Over-achieving Ostrich
(challenge questions)

- ***Do the reading***
  - either before or after lecture, whatever works best for you.
  - **do not wait to “catch up” the week before the exam.**

# Optional References




**"CLRS": Introduction to Algorithms** by Cormen, Leiserson, Rivest, and Stein. Available FOR FREE ONLINE through the Stanford library.

# Homework!

- Weekly assignments, posted Wednesday by 12:30pm, due the next Wednesday 11:59pm.
- First HW posted this Wednesday!

# How to get the most out of homework

- HW has two parts: exercises and problems.
- Do the exercises on your own.
- Try the problems on your own **before** discussing it with classmates.

- If you get help from a CA at office hours:
  - **Try the problem first**.
  - Ask: **"I was trying this approach and I got stuck here."**
  - After you've figured it out, **write up your solution from scratch**, without the notes you took during office hours.

# Exams

- There will be 4 **midterms.**
  (2 hour exams to be taken in a 48 hour window)
    - **Midterm 1:** Thu Jan 28 – Fri Jan 29
    - **Midterm 2:** Thu Feb 11 – Fri Feb 12
    - **Midterm 3:** Mon Mar 1 – Tue Mar 2
    - **Midterm 4:** Mon Mar 15 – Tue Mar 16
- We will drop the lowest score of first 3 midterms; last midterm cannot be dropped.
- Weighting: **Homeworks** (55%), **Midterms** (45%)
- If you have a conflict with the midterm times, email

  cs161-win2021-staff@lists.stanford.edu ASAP!!!!!

# Talk to us!

- Stay connected at Discord:
  - See course website (Resources) for link: "sign up for Ed”
  - Course announcements will be posted there
  - Discuss material with STs and your classmates
- Office hours (on Nooks):
  - See course website for schedule

# Talk to each other!

- Answer your peers’ questions on Discord!
- There is a bonus for helping out!

# Course elements and resources

- Course website:
  - BuX
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- Quiz
- Exams
- Presentation

# Collaboration

- We encourage collaboration on homeworks (but strongly recommend you do exercises on your own)
- Valid and invalid modes of collaboration detailed on the course website.
  - Briefly, you can exchange ideas with classmates, but must write up solutions on your own.
- You must cite all collaborators, as well as all sources used (outside of course materials).

# Bug bounty! 

- We hope all PSETs and slides will be bug-free.
- However, we sometmes maek typos.
- **If you find a typo** (that affects understanding*) on slides, IPython notebooks, Section material or PSETs:
  - Let us know! (Post on Ed or tell a CA).
  - The first person to catch a bug gets a bonus point.

Bug Bounty Hunter

*So, typos lke thees onse don't count, although please point those out too. Typos like 2 + 2 = 5 do count, as does pointing out that we omitted some crucial information.

# Feedback!

- We will have an anonymous feedback form on the course website (bottom of the main page).
- Please help us improve the course!

# How are you approaching CSE221?

# Everyone can succeed in this class!

1. **Work hard**
2. **Work smart**
3. **Ask for help**

# The big questions

- Who are we?
  - Professor, TA’s, students?
- Why are we here?
  - Why learn about algorithms?
- What is going on?
  - What is this course about?
  - Logistics?
- Can we multiply integers?
  - And can we do it quickly?

# Course goals

- Think analytically about algorithms
- Flesh out an “algorithmic toolkit”
- Learn to communicate clearly about algorithms

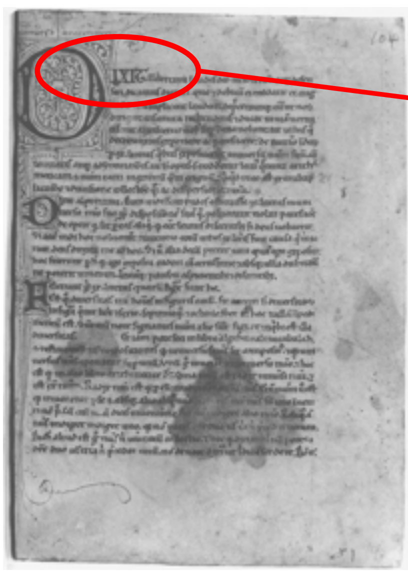
# Today’s goals

- Karatsuba Integer Multiplication
- Algorithmic Technique:
  - Divide and conquer
- Algorithmic Analysis tool:
  - Intro to asymptotic analysis

# Let’s start at the beginning

# Etymology of “Algorithm”

- Al-Khwarizmi was a 9th-century scholar, born in present-day Uzbekistan, who studied and worked in Baghdad during the Abbassid Caliphate.
- Among many other contributions in mathematics, astronomy, and geography, he wrote a book about how to multiply with Arabic numerals.
- His ideas came to Europe in the 12th century.



*Dixit algorizmi*
(so says Al-Khwarizmi)

- Originally, “Algorisme” [old French] referred to just the Arabic number system, but eventually it came to mean “Algorithm” as we know today.

# This was kind of a big deal

$$\mathrm{XLIV} \times \mathrm{XCVII} = ?$$

$$\begin{array}{r} 44 \\ \times\ 97 \\ \hline \end{array}$$

# Integer Multiplication

$$\begin{array}{r} 44 \\ \times\ 97 \\ \hline \end{array}$$

# Integer Multiplication

$$
\begin{array}{r}
12345678959314 13 \\
\times \quad 4563823520395533 \\
\hline
\end{array}
$$

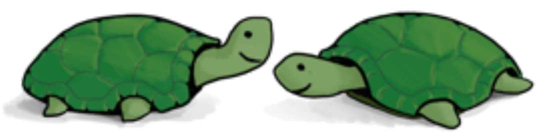
# Integer Multiplication

$$
\begin{array}{rr}
 & \overbrace{1233925720752752384623764283568364918374523856298}^{n} \\
\times & 4562323582342395285623467235019130750135350013753 \\
\hline
\end{array}
$$

???

**How fast is the grade-school multiplication algorithm?**

(How many one-digit operations?)



Think-pair-share Terrapins

About $n^2$ one-digit operations

Plucky the Pedantic Penguin

At most $n^2$ multiplications,

and then at most $n^2$ additions (for carries)

and then I have to add n different 2n-digit numbers…

# Big-Oh Notation

- We say that Grade-School Multiplication

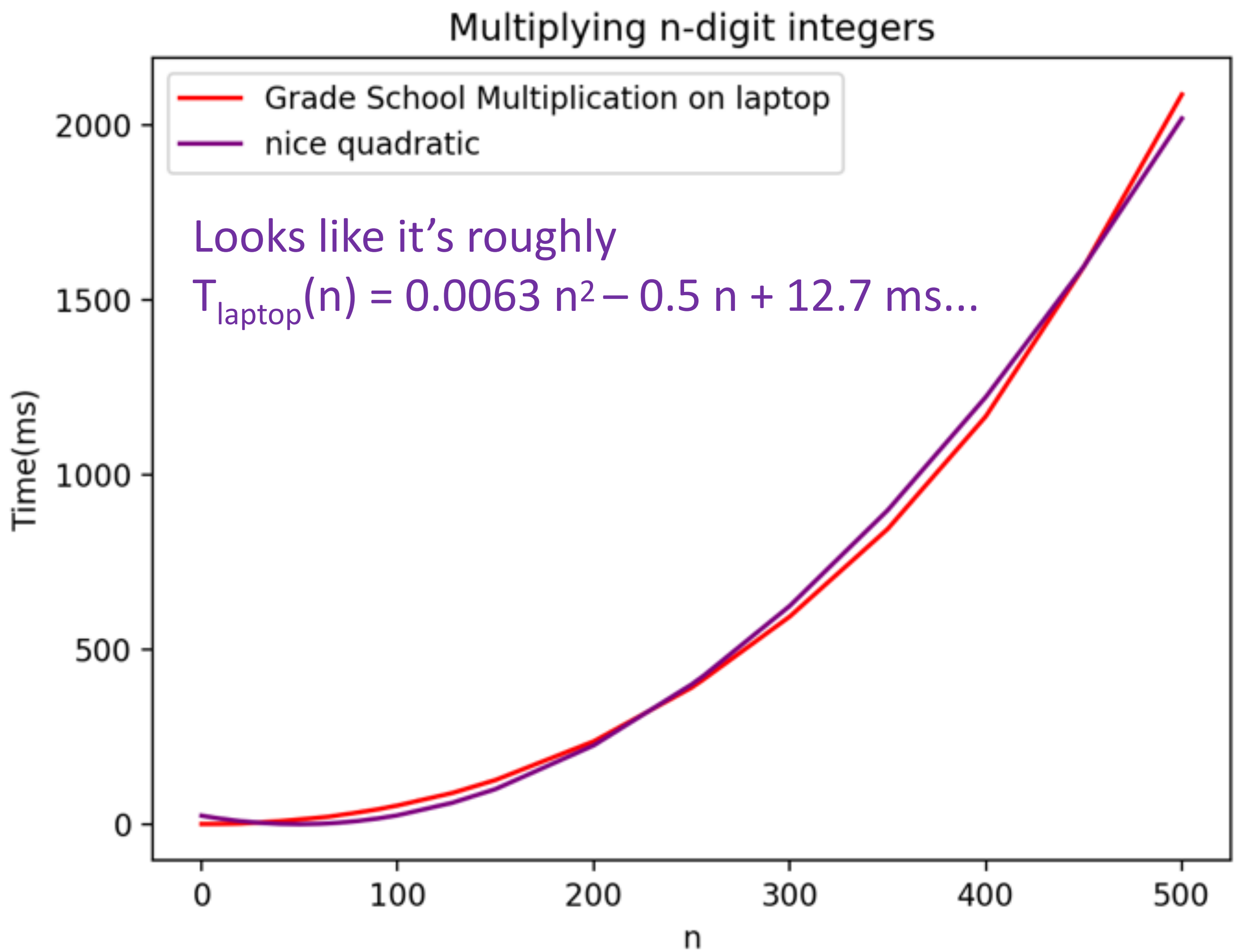
  “runs in time $O(n^2)$”

- Formal definition coming Wednesday!
- Informally, big-Oh notation tells us how the running time scales with the size of the input.

# Implemented in Python, on my laptop

highly non-optimized

The runtime "scales like" $n^2$



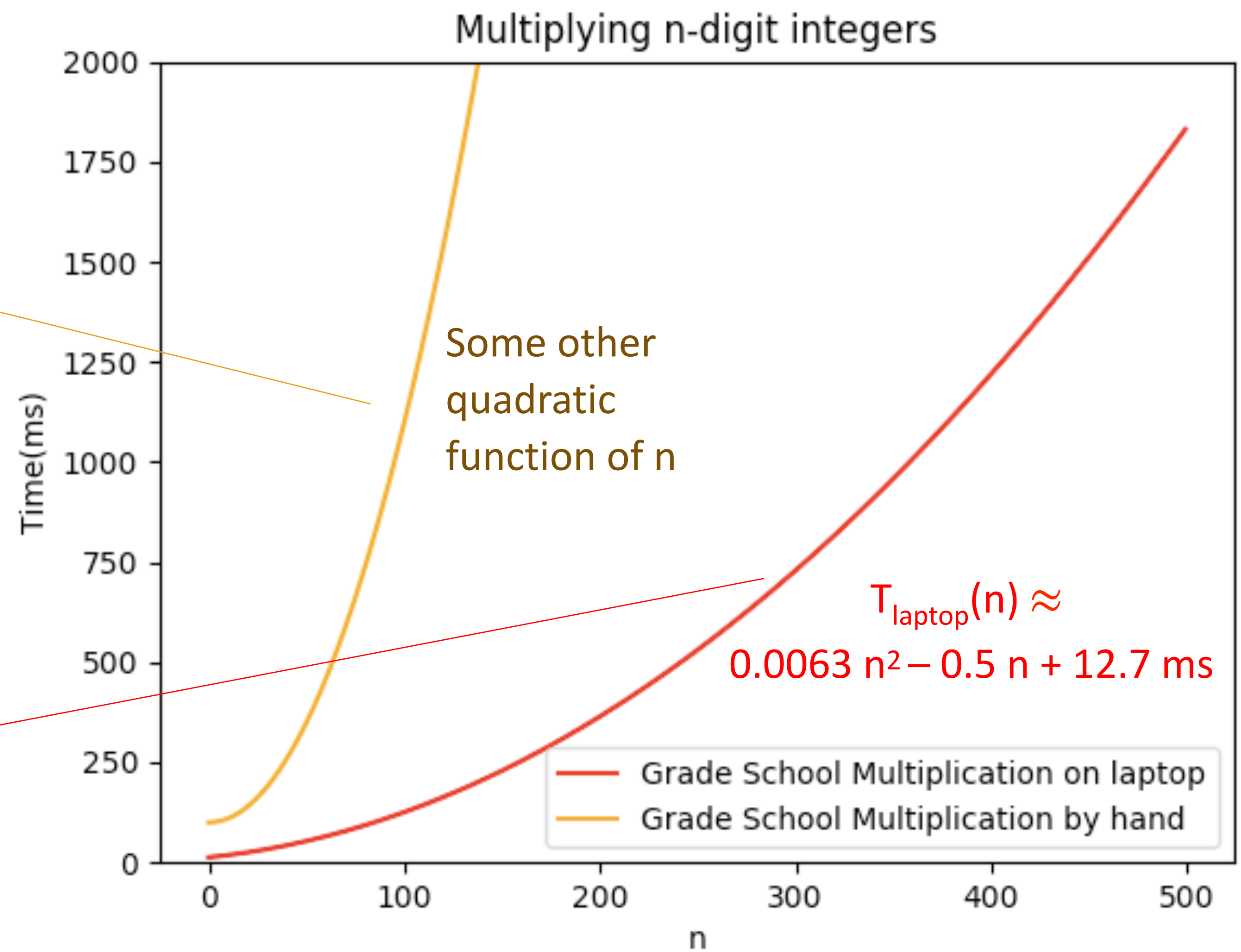

Looks like it's roughly

$T_{laptop}(n) = 0.0063\, n^2 - 0.5\, n + 12.7$ ms...

python™

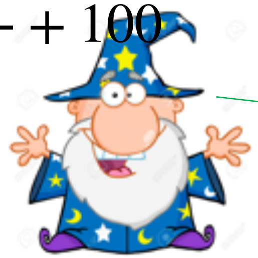
# Implemented by hand

The runtime still “scales like” $n^2$



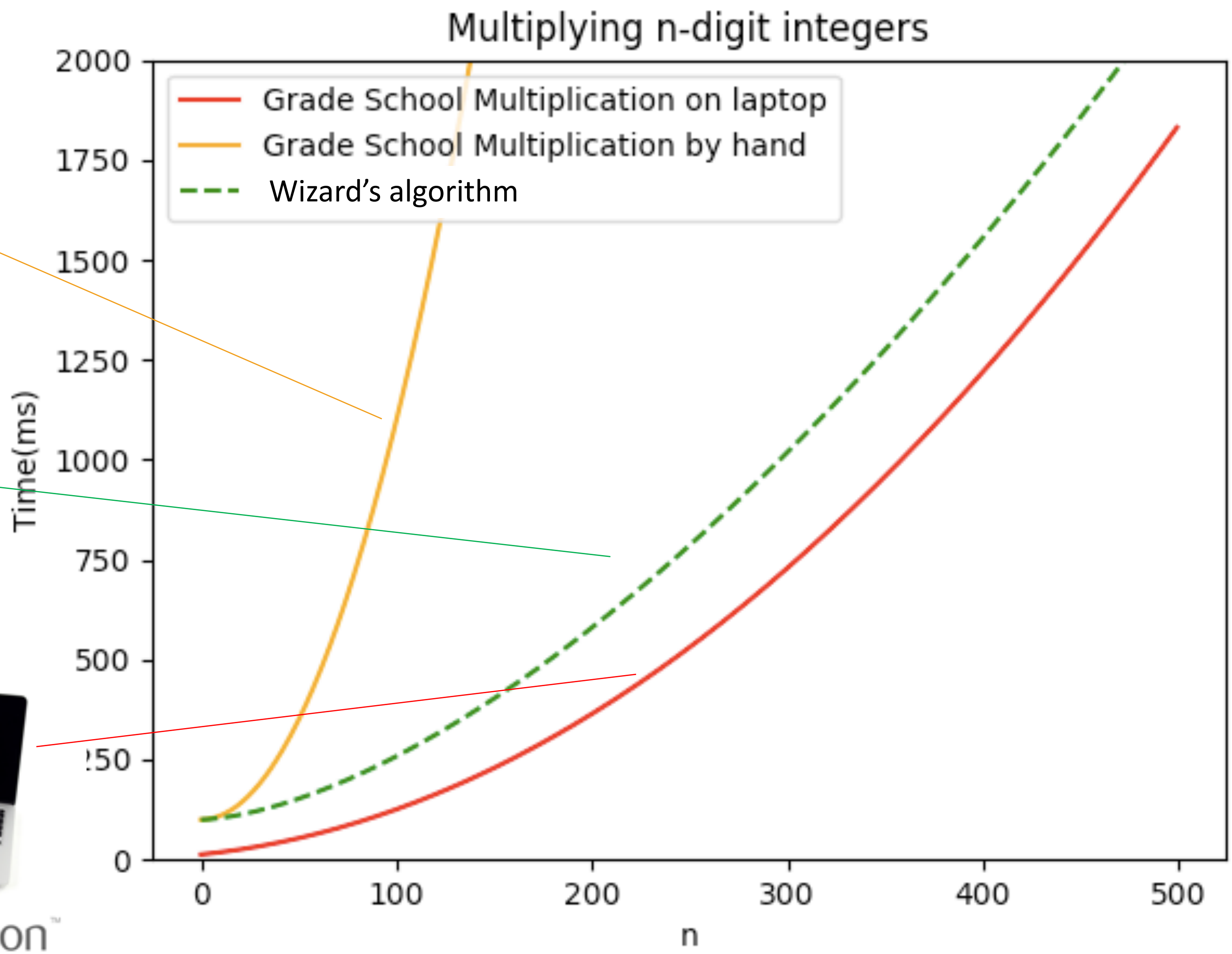
# Why is big-Oh notation meaningful?

$$\approx \frac{n^{1.6}}{10} + 100$$



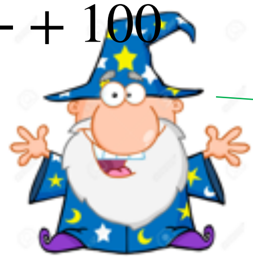
$$\approx .0063n^2$$

python

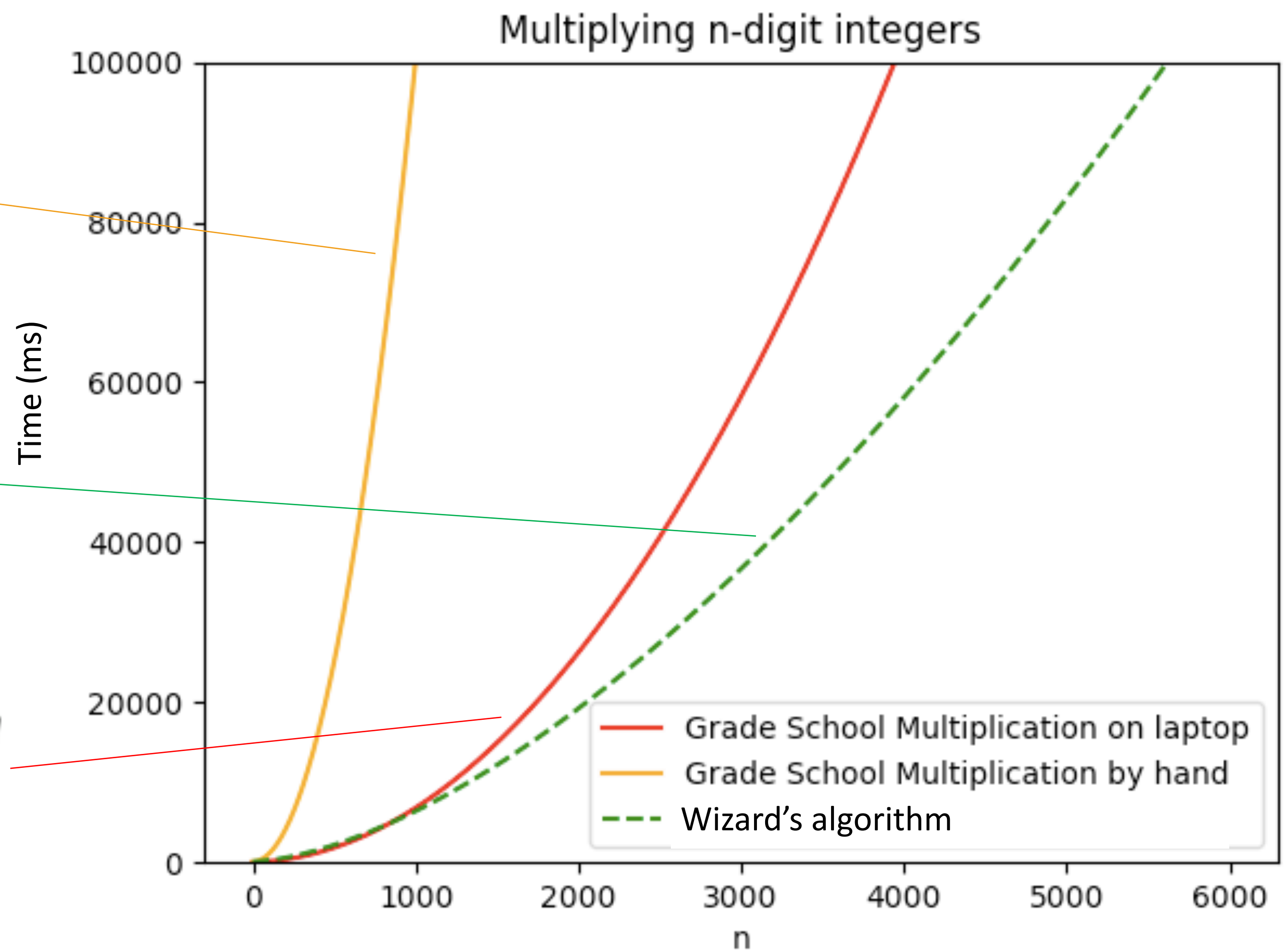


# Let n get bigger…

$$\approx \frac{n^{1.6}}{10} + 100$$



$$\approx .0063n^2$$



# Take-away

- An algorithm that runs in time $O(n^{1.6})$ is “better” than an an algorithm that runs in time $O(n^2)$.

- So the question is…

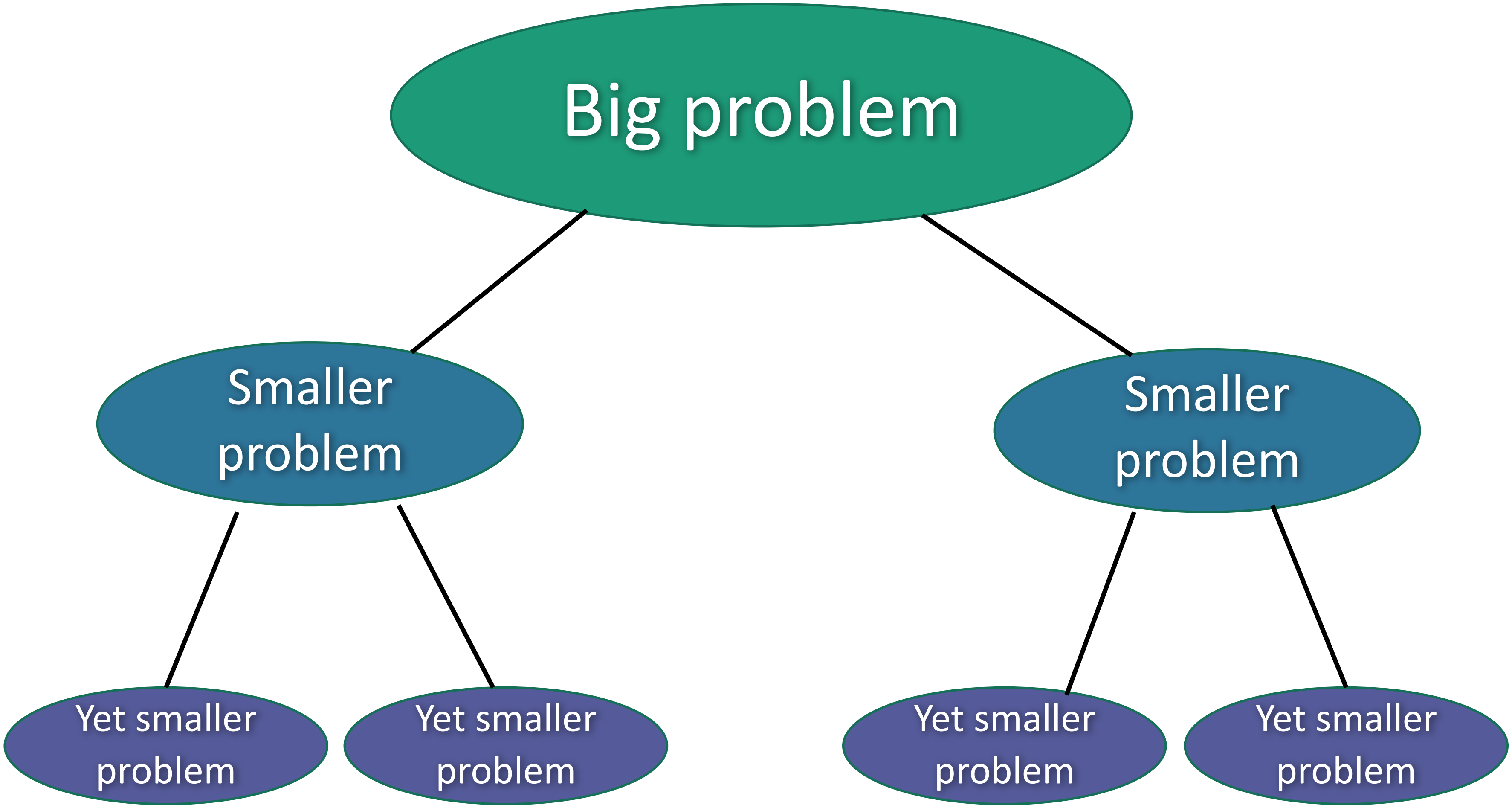
# Can we do better?

Can we multiply n-digit integers faster than $O\left(n^2\right)$?

# Let’s dig in to our algorithmic toolkit…

# Divide and conquer

Break problem up into smaller (easier) sub-problems



# Divide and conquer for multiplication

Break up an integer:

$$1234 = 12 \times 100 + 34$$

$$
\begin{aligned}
1234 \times 5678 \\
&= (12 \times 100 + 34)(56 \times 100 + 78) \\
&= (\underbrace{12 \times 56}_{1})10000 + (\underbrace{34 \times 56}_{2} + \underbrace{12 \times 78}_{3})100 + (\underbrace{34 \times 78}_{4})
\end{aligned}
$$

One 4-digit multiply → Four 2-digit multiplies

# More generally

Suppose n is even

Break up an n-digit integer:

$$[x_1 x_2 \cdots x_n] = [x_1 x_2 \cdots x_{n/2}] \times 10^{n/2} + [x_{n/2+1} x_{n/2+2} \cdots x_n]$$

$$\begin{aligned} x \times y &= (a \times 10^{n/2} + b)(c \times 10^{n/2} + d) \\ &= \underbrace{(a \times c)}_{1} 10^n + (\underbrace{a \times d}_{2} + \underbrace{c \times b}_{3}) 10^{n/2} + \underbrace{(b \times d)}_{4} \end{aligned}$$

One n-digit multiply $\longrightarrow$ Four (n/2)-digit multiplies

# Divide and conquer algorithm

not very precisely…

(Assume n is a power of 2…)

x,y are n-digit numbers

**Multiply**$(x, y)$:

- **If** n=1:
  - **Return** xy

Base case: I've memorized my 1-digit multiplication tables…

- Write $x = a\ 10^{\frac{n}{2}} + b$
- Write $y = c\ 10^{\frac{n}{2}} + d$

a, b, c, d are n/2-digit numbers

- Recursively compute $ac,\ ad, bc, bd$:
  - ac = **Multiply**(a, c), etc..
- Add them up to get $xy$:
  - xy = ac $10^n$ + (ad + bc) $10^{n/2}$ + bd

Make this pseudocode more detailed! How should we handle odd n? How should we implement "multiplication by $10^n$"?

See the Lecture 1 Python notebook for actual code!

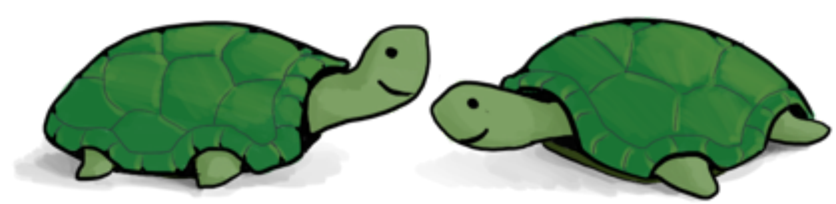
Siggi the Studious Stork

# Think-Pair-Share

- We saw that this 4-digit multiplication problem broke up into four 2-digit multiplication problems

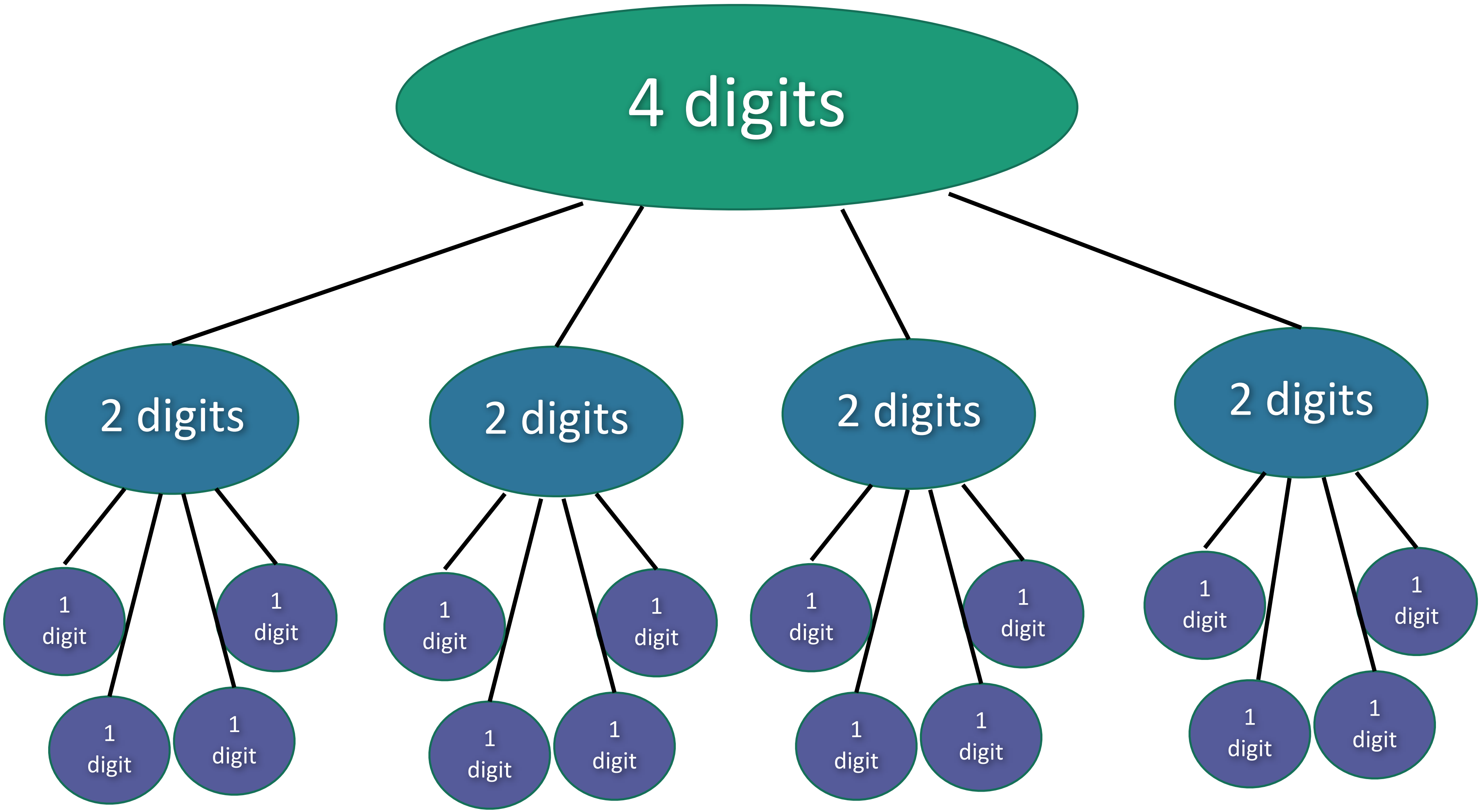
$$1234 \times 5678$$

- If you recurse on those 2-digit multiplication problems, how many 1-digit multiplications do you end up with total?



# Recursion Tree

16 one-digit multiplies!



# What is the running time?

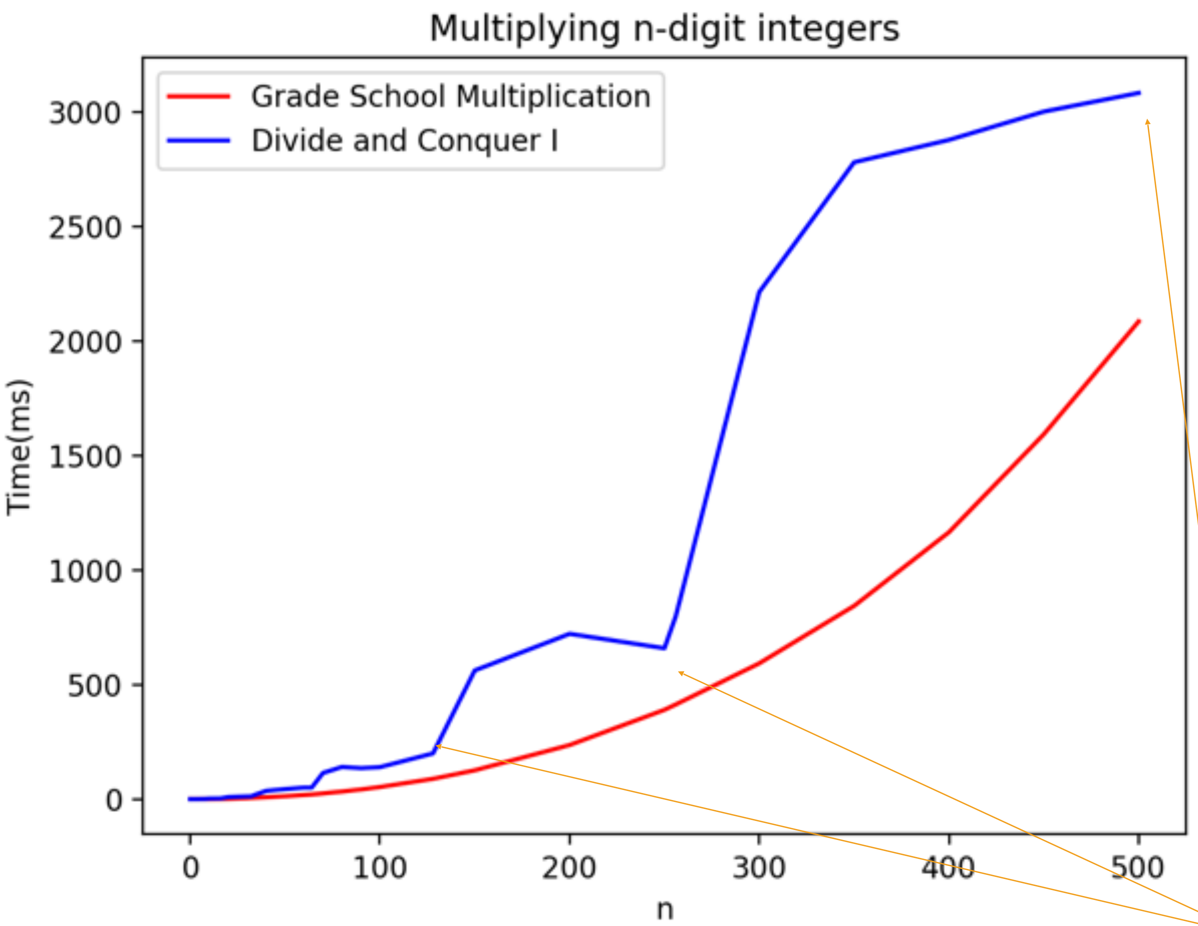
- Better or worse than the grade school algorithm?

- How do we answer this question?
  1. Try it.
  2. Try to understand it analytically.

# 1. Try it.

Check out the Lecture 1 IPython Notebook

Conjectures about running time?




Doesn't look too good but hard to tell…

Maybe one implementation is slicker than the other?

Maybe if we were to run it to n=10000, things would look different.

Something funny is happening at powers of 2…

# 2. Try to understand the running time analytically

- Proof by meta-reasoning:

~~It must be faster than the grade school algorithm, because we are learning it in an algorithms class.~~

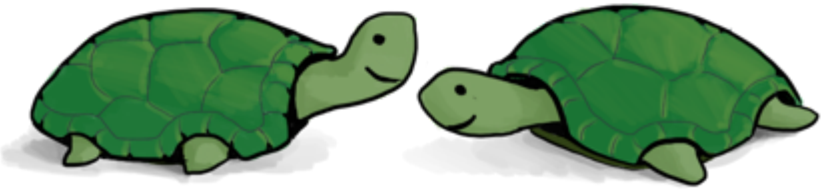
**Not sound logic!**

Plucky the Pedantic Penguin

# 2. Try to understand the running time analytically

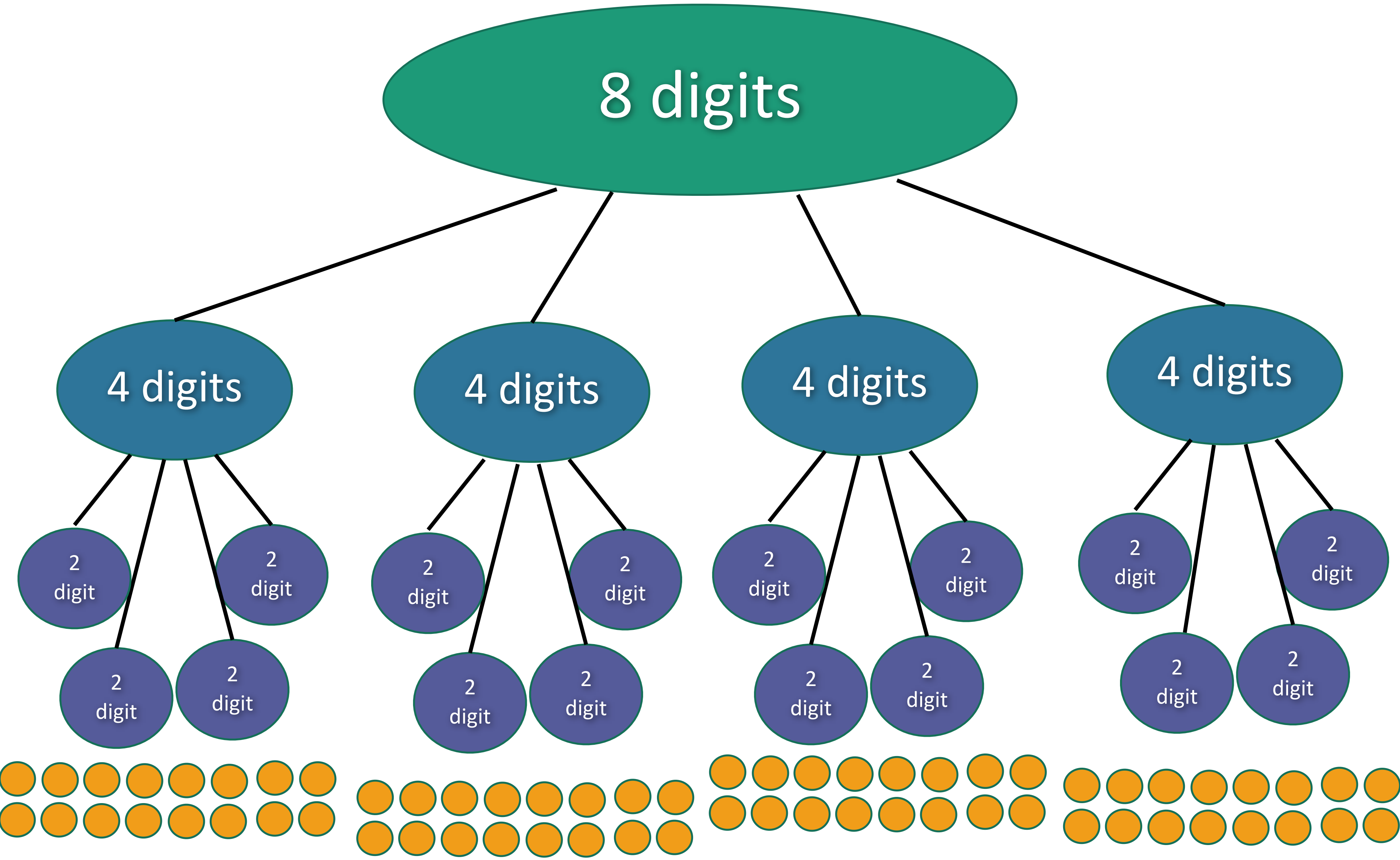
Think-Pair-Share:

- We saw that multiplying 4-digit numbers resulted in 16 one-digit multiplications.
- How about multiplying 8-digit numbers?
- What do you think about n-digit numbers?



# Recursion Tree

64 one-digit multiplies!

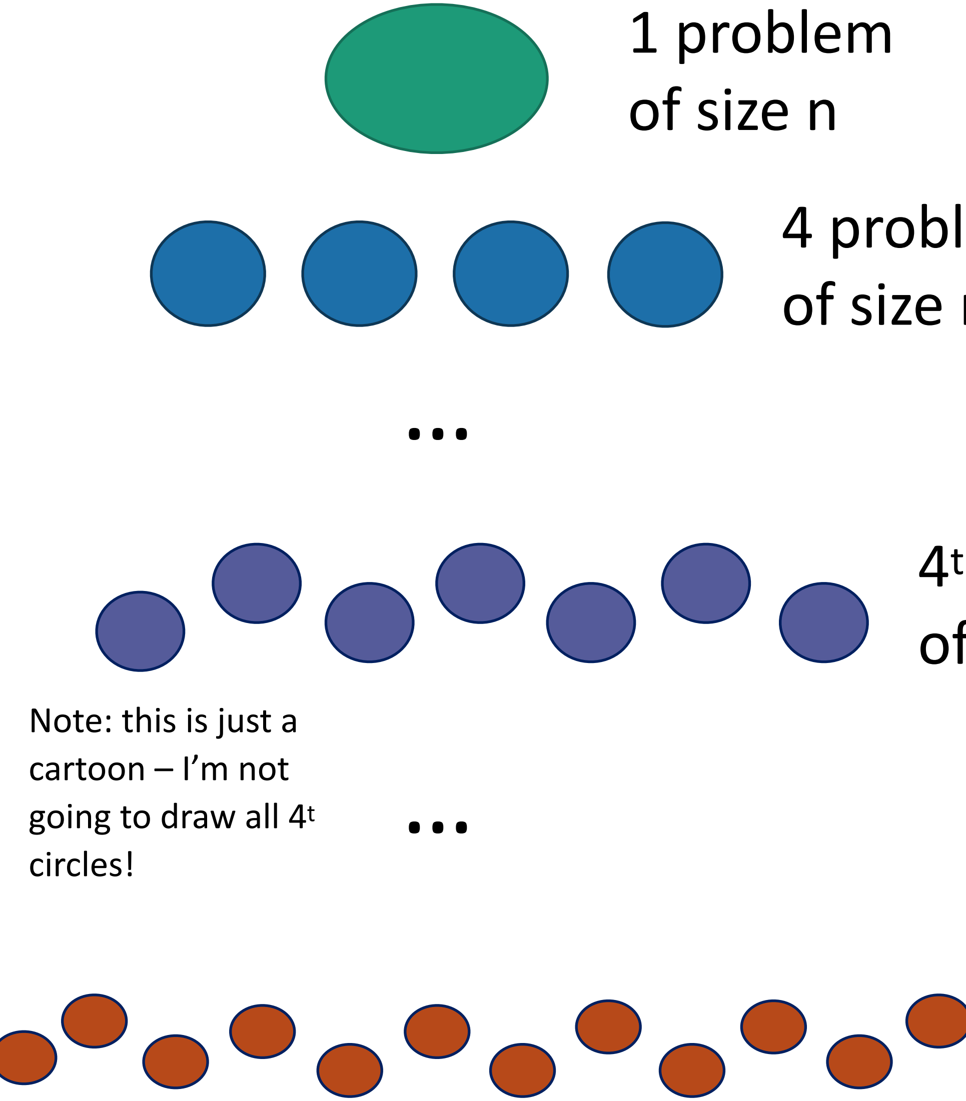


# 2. Try to understand the running time analytically

Claim:

The running time of this algorithm is AT LEAST $n^2$ operations.

# There are $n^2$ 1-digit problems



1 problem
of size n

4 problems
of size n/2

…

$4^t$ problems
of size $n/2^t$

Note: this is just a cartoon – I'm not going to draw all $4^t$ circles!

…

$\underline{n^2}$ problems
of size 1

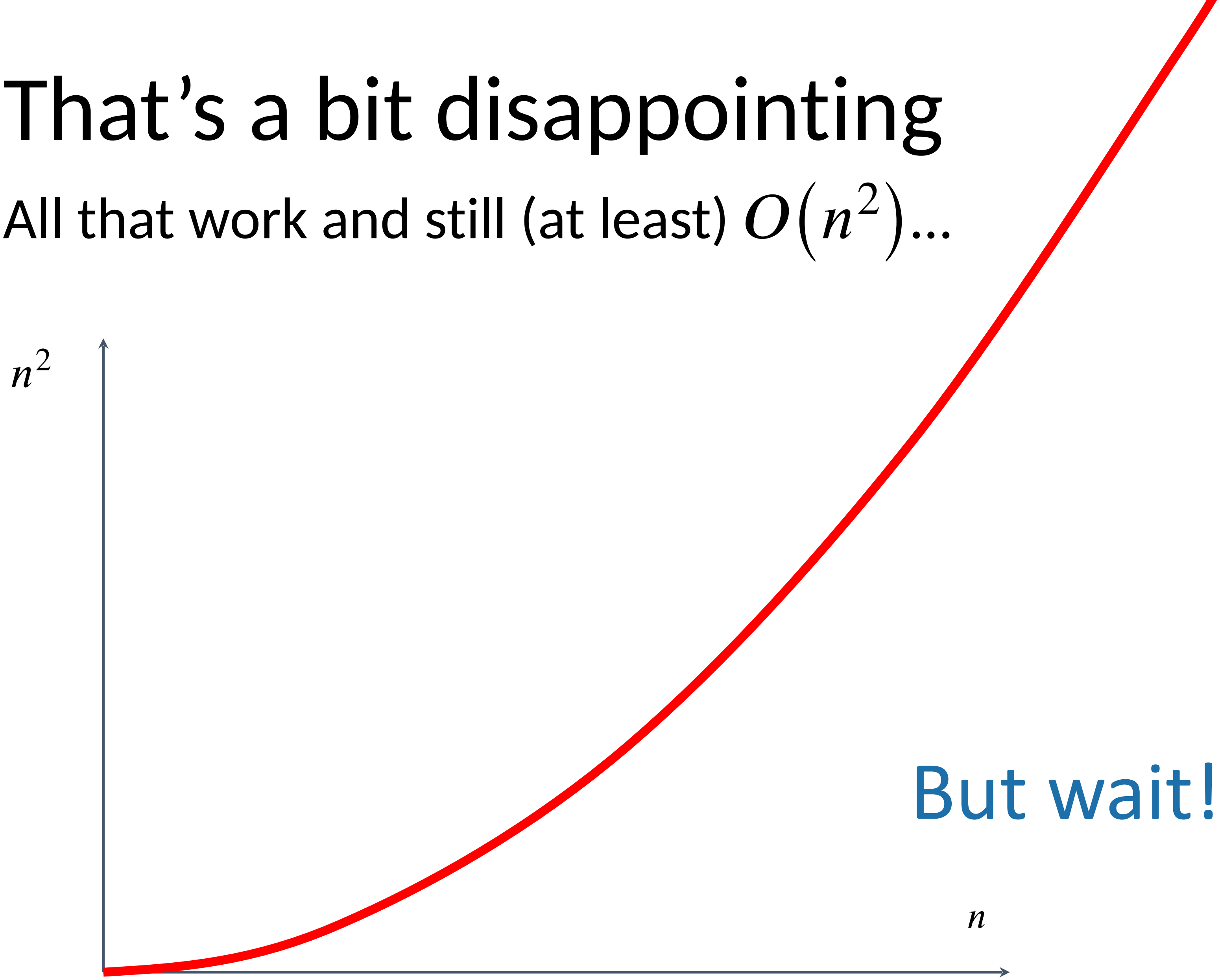
- If you cut n in half $\log 2(n)$ times, you get down to 1.
- So at level
  $t = \log 2(n)$
  we get…

$$4^{\log_2 n} = n^{\log_2 4} = n2$$

problems of size 1.

# That's a bit disappointing

All that work and still (at least) $O\left(n^2\right)$...




But wait!!

# Divide and conquer can actually make progress

- Karatsuba figured out how to do this better!

$$xy = (a \cdot 10^{n/2} + b)(c \cdot 10^{n/2} + d)$$
$$= ac \cdot 10^n + (ad + bc)10^{n/2} + bd$$

Need these three things

- If only we could recurse on three things instead of four…

# Karatsuba integer multiplication

- Recursively compute these THREE things:
  - ac
  - bd
  - (a+b)(c+d)

Subtract these off

get this

(a+b)(c+d) = ac + bd + bc + ad

- Assemble the product:

$$
\begin{aligned}
xy &= (a \cdot 10^{n/2} + b)(c \cdot 10^{n/2} + d) \\
&= ac \cdot 10^{n} + (ad + bc)10^{n/2} + bd
\end{aligned}
$$

# How would this work?

(Still not super precise, see IPython notebook for detailed code. Also, still assume n is a power of 2.)

x,y are n-digit numbers

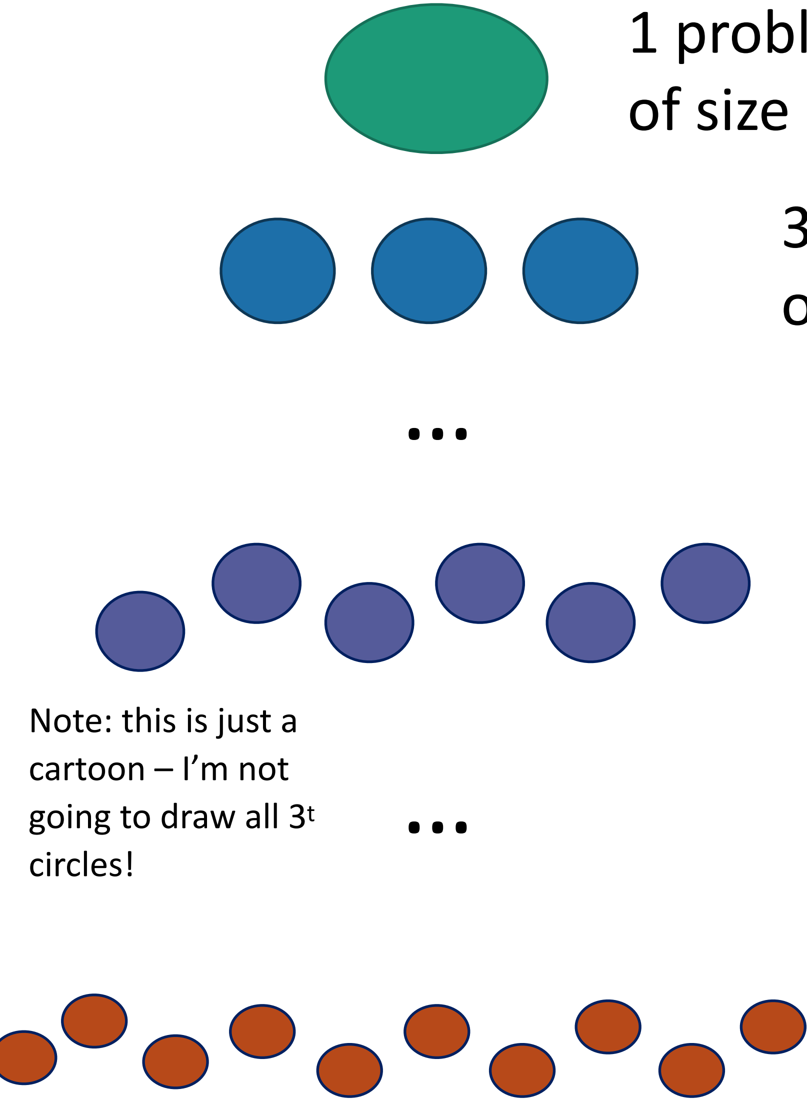
**Multiply**$(x, y)$:

- **If** n=1:
  - **Return** xy
- Write $x = a\ 10^{\frac{n}{2}} + b$ and $y = c\ 10^{\frac{n}{2}} + d$

a, b, c, d are n/2-digit numbers

- ac = **Multiply**(a, c)
- bd = **Multiply**(b, d)
- z = **Multiply**(a+b, c+d)
- xy = ac $10^n$ + (z – ac - bd) $10^{n/2}$ + bd
- Return xy

# What's the running time?



1 problem
of size n

3 problems
of size n/2

...

$3^t$ problems
of size $n/2^t$

Note: this is just a cartoon – I'm not going to draw all $3^t$ circles!

...

$\underline{\;n^{1.6}\;}$ problems
of size 1

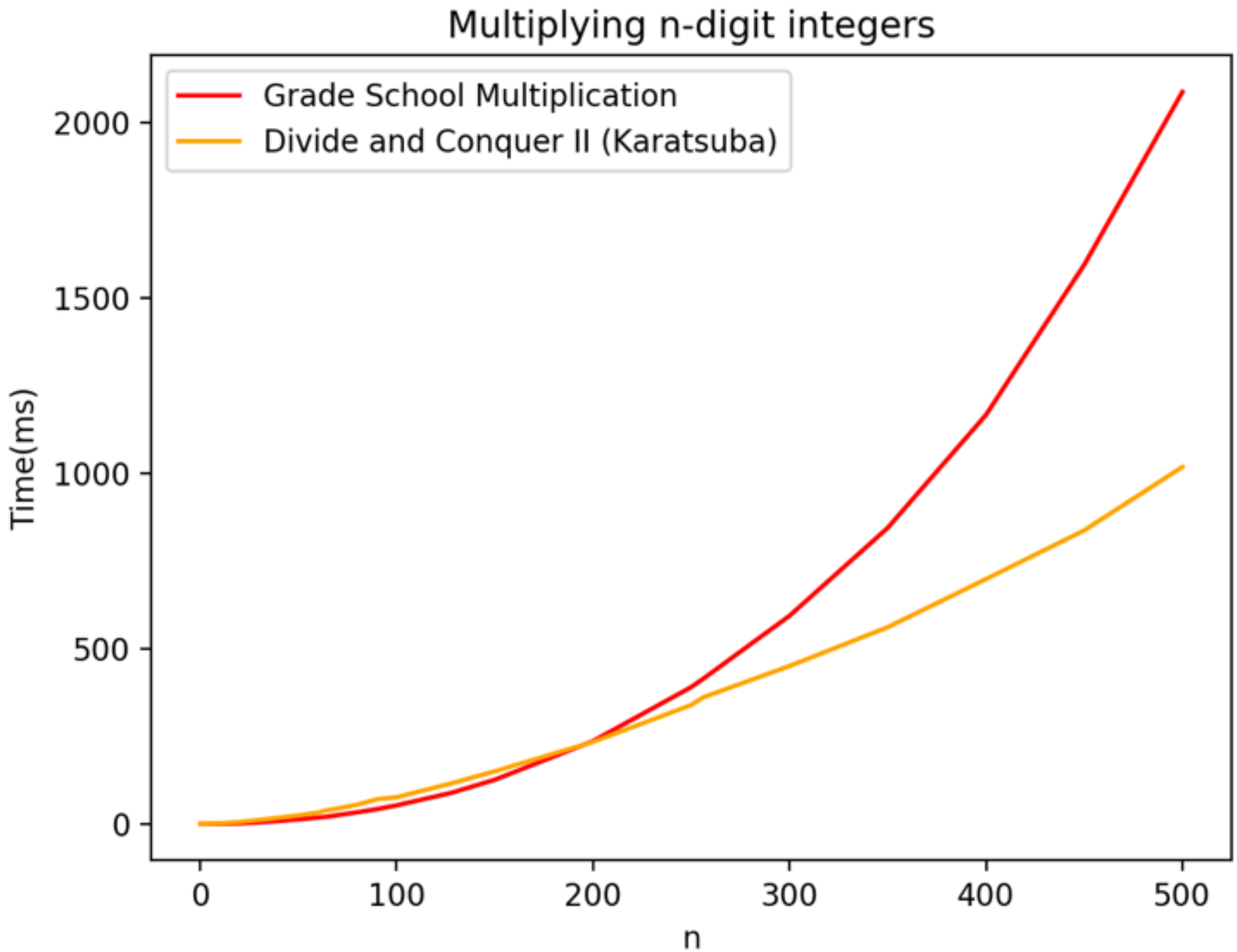
- If you cut n in half $\log 2(n)$ times, you get down to 1.
- So at level
  $t = \log 2(n)$
  we get...

  $3^{\log_2 n} = n^{\log_2 3} \approx n^{1.6}$
  problems of size 1.

We aren't accounting for the work at the higher levels!
But we'll see later that this turns out to be okay.

# This is much better!

# We can even see it in real life!



# Can we do better?

- **Toom-Cook** (1963): instead of breaking into three n/2-sized problems, break into five n/3-sized problems.
  - Runs in time $O\left(n^{1.465}\right)$

Try to figure out how to break up an n-sized problem into five n/3-sized problems! **(Hint: start with nine n/3-sized problems).**

Ollie the Over-achieving Ostrich

Given that you can break an n-sized problem into five n/3-sized problems, where does the 1.465 come from?

Siggi the Studious Stork

- **Schönhage–Strassen** (1971):
  - Runs in time $O(n\log(n)\log\log(n))$
- **Furer** (2007)
  - Runs in time $n\log(n) \cdot 2^{O(\log^*(n))}$
- **Harvey and van der Hoeven** (2019)
  - Runs in time $O(n\log(n))$

[This is just for fun, you don't need to know these algorithms!]

# Course goals

- Think analytically about algorithms
- Flesh out an “algorithmic toolkit”
- Learn to communicate clearly about algorithms

# Today’s goals

- Karatsuba Integer Multiplication
- Algorithmic Technique:
  - Divide and conquer
- Algorithmic Analysis tool:
  - Intro to asymptotic analysis

How was the pace today?

# The big questions

- Who are we?
  - Professor, TA's, students?
- Why are we here?
  - Why learn about algorithms?
- What is going on?
  - What is this course about?
  - Logistics?
- Can we multiply integers?
  - And can we do it quickly?
- Wrap-up

# Wrap up

- Algorithms are fundamental, useful and fun!
- In this course, we will develop both algorithmic intuition and algorithmic technical chops
- Karatsuba Integer Multiplication:
  - You can do better than grade school multiplication!
  - Example of divide-and-conquer in action
  - Informal demonstration of asymptotic analysis

# Next time

- Sorting!
- Asymptotics and (formal) Big-Oh notation
- Divide and Conquer some more

# BEFORE Next time

- ***Pre-lecture exercise!*** On the course website!