CSE 221: Algorithms

Graph Algorithms

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References

1. T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms, Second Edition*. The MIT Press, September 2001.
2. Jon Kleinberg and Éva Tardos, *Algorithm Design*. Pearson Education, 2006.
3. M. Goodrich and R. Tamassia, *Algorithm Design*. John-Wiley and Sons. 2002.

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Introduction to graph algorithms

- All about weighted graphs.

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- Minimum-cost Spanning Tree algorithms.

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- Excellent applications of Greedy and Dynamic Programming strategies.

Contents

Spanning trees

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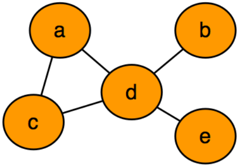

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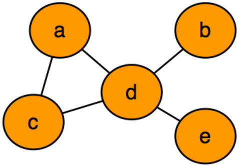

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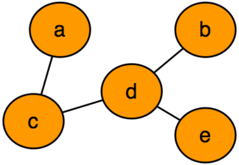

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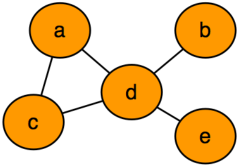

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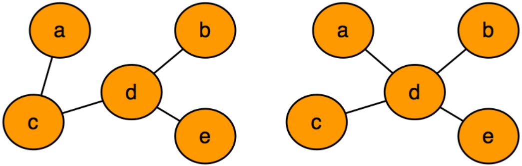

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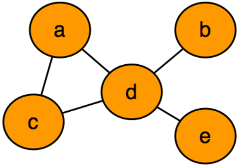

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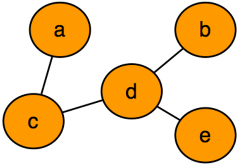

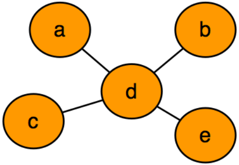

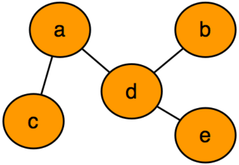

Spanning trees of weighted graphs

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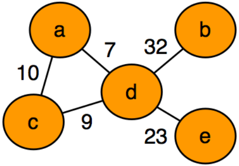

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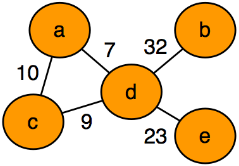
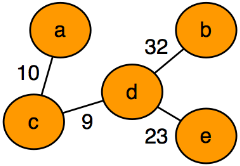
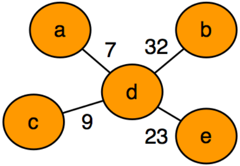


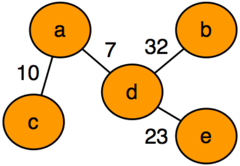
The spanning trees (with associated total weights) are:



$\sum_{e \in T} w(e) = 74$



$\sum_{e \in T} w(e) = 71$



$\sum_{e \in T} w(e) = 72$

Minimum-cost Spanning Tree (MST)

Definition

The minimum-cost spanning tree of a graph A spanning tree T of a undirected graph $G = (V, E)$ is a minimum-cost spanning tree of G if the total weight $w(T) = \sum_{(u,v) \in T} w(u, v)$ is minimized.

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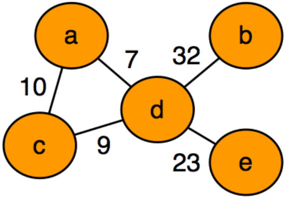

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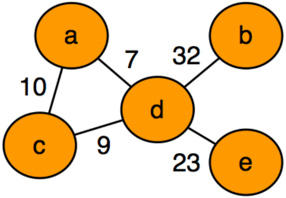
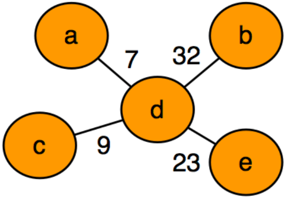


MST

$\Longrightarrow$



Minimum-cost Spanning Tree (continued)

Uniqueness of MST

The minimum-cost spanning tree may not be unique!

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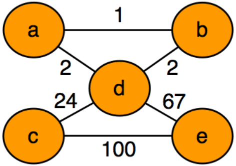

Graph G

Minimum-cost Spanning Tree (continued)

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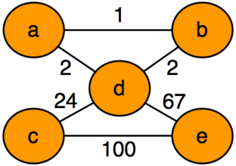

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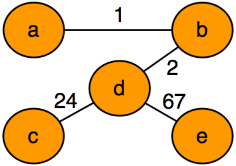

$\sum w_T = 94$

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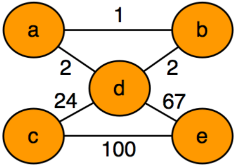

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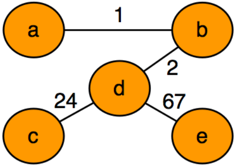

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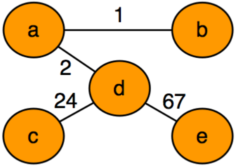

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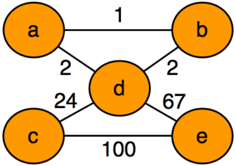

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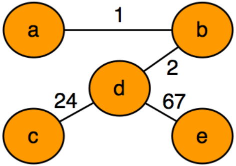

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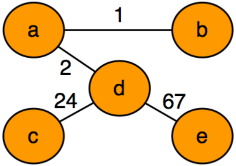

$\sum w_T = 94$

Key observation

However, if the weights are all distinct (i.e., $w(u_i, v_j) \neq w(u_k, v_l)$ unless $i = k$ and $j = l$), then it is indeed unique.

Computing an MST

- We grow the tree one edge at a time, starting with a graph $G' = (V, \emptyset)$.
- At each step, add a new safe edge, ensuring that it does not create a cycle (why?).
- If adding an edge guarantees that the tree after each step is a subset of some MST, then the final result will be an MST.

Computing an MST

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Key question

How do we pick the next safe edge?

Computing an MST

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- At each step, add a new safe edge, ensuring that it does not create a cycle (why?).
- If adding an edge guarantees that the tree after each step is a subset of some MST, then the final result will be an MST.

Key question

How do we pick the next safe edge?
Which algorithm design strategy does this question remind you of?

Prim's algorithm to compute an MST

```
MST-PRIM(G, w, r)
for each u ∈ V[G]
        do key[u] ← ∞
           π[u] ← NIL
key[r] ← 0
Q ← V[G]
while Q ≠ ∅
        do u ← EXTRACT-MIN(Q)
           for each v ∈ Adj[u]
               do if v ∈ Q and w(u, v) < key[v]
                     then π[v] ← u
                          key[v] ← w(u, v)
```

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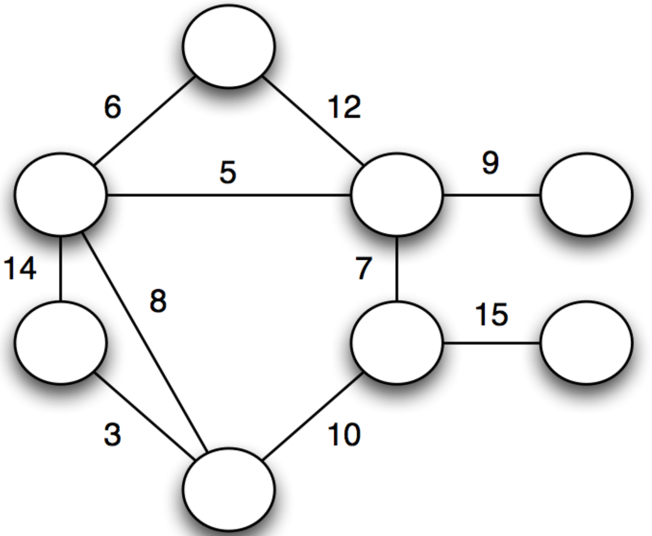
$key[r] \leftarrow 0$
$Q \leftarrow V[G]$
while $Q \neq \emptyset$
  do $u \leftarrow \text{EXTRACT-MIN}(Q)$
    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
        then $\pi[v] \leftarrow u$
          $key[v] \leftarrow w(u, v)$

Running time

$O(V \lg V + E \lg V) = O(E \lg V)$

Prim's algorithm in action



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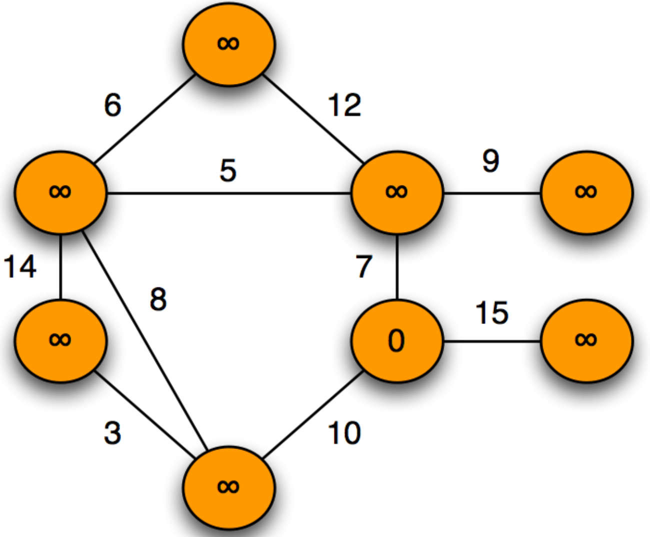

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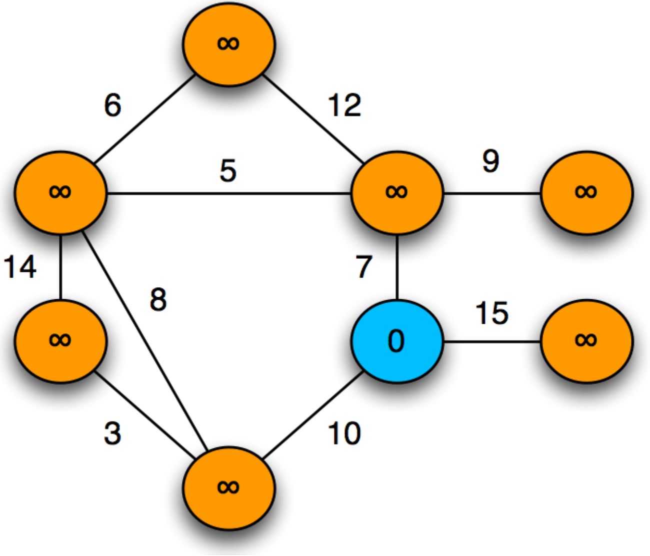

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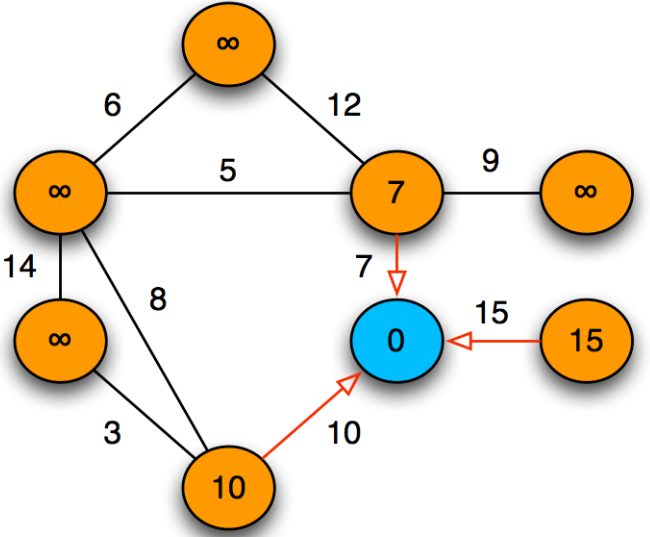

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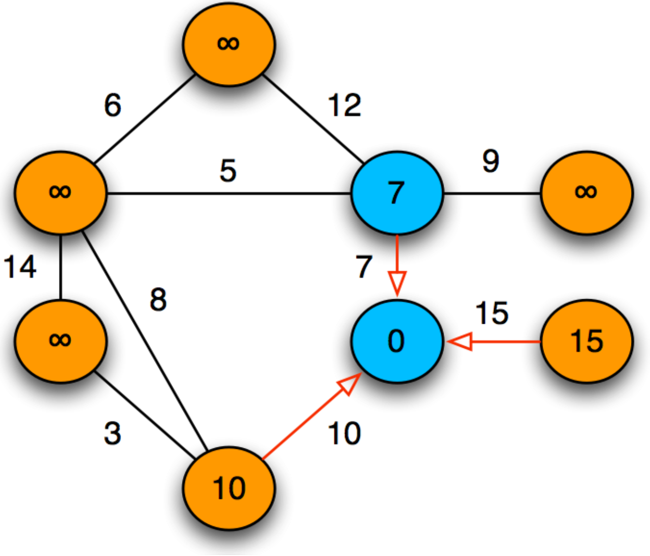

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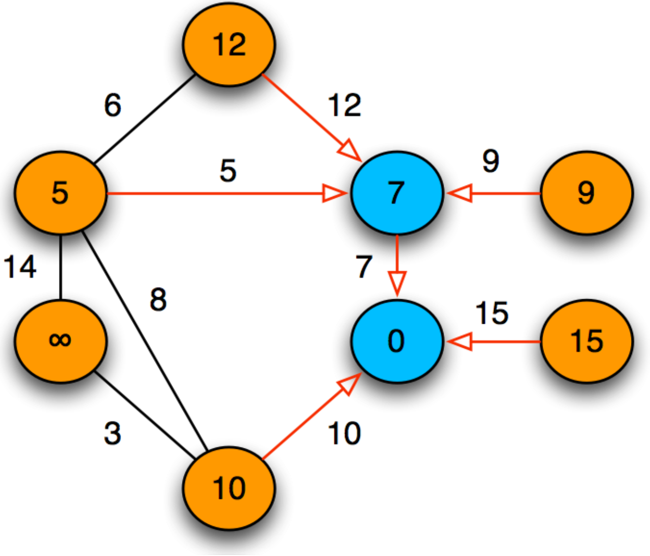

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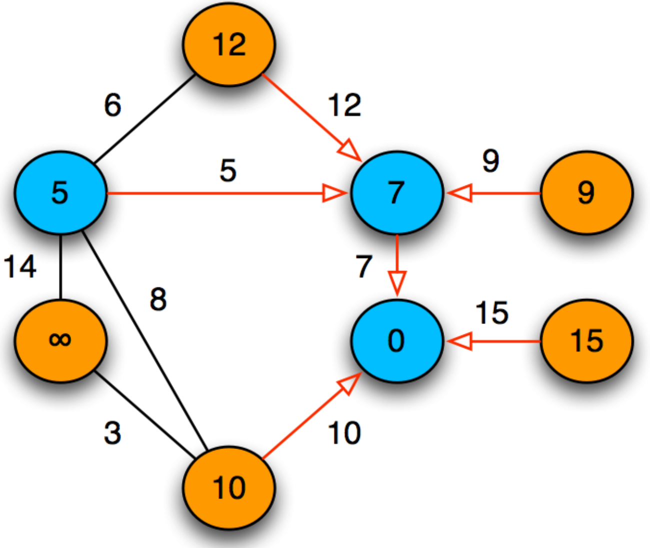

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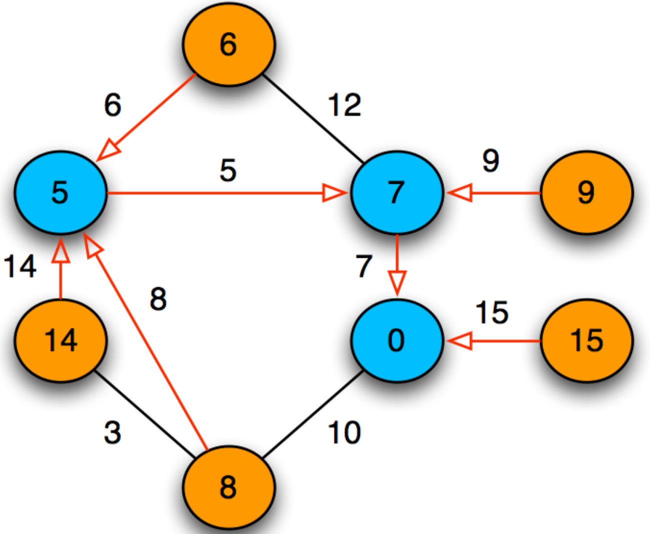

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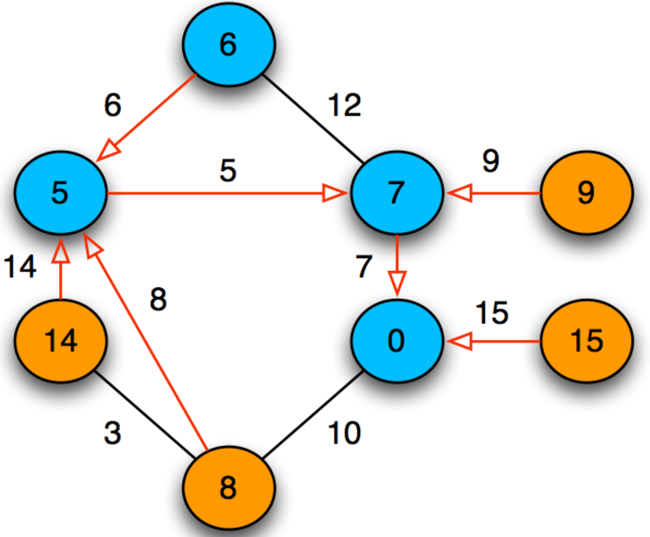

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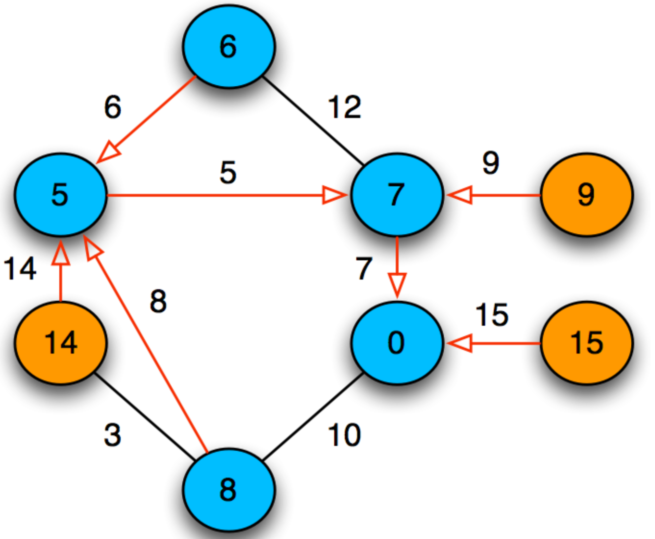

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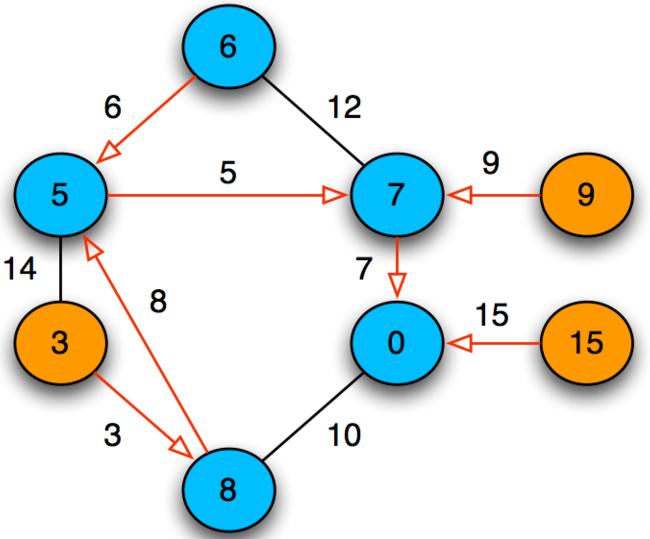

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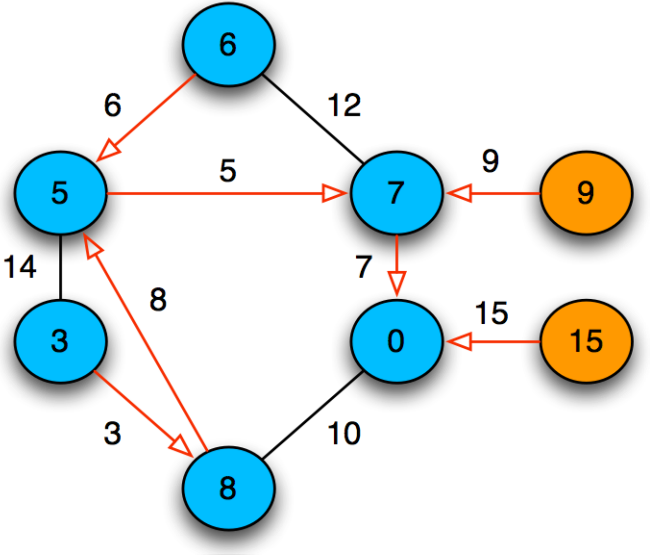

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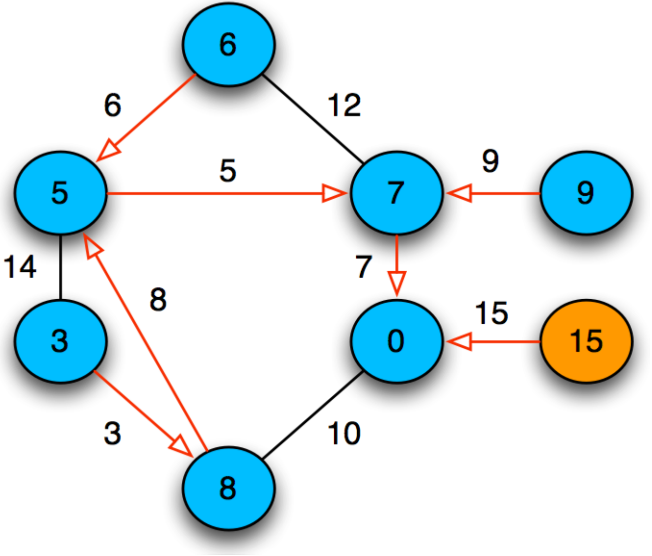

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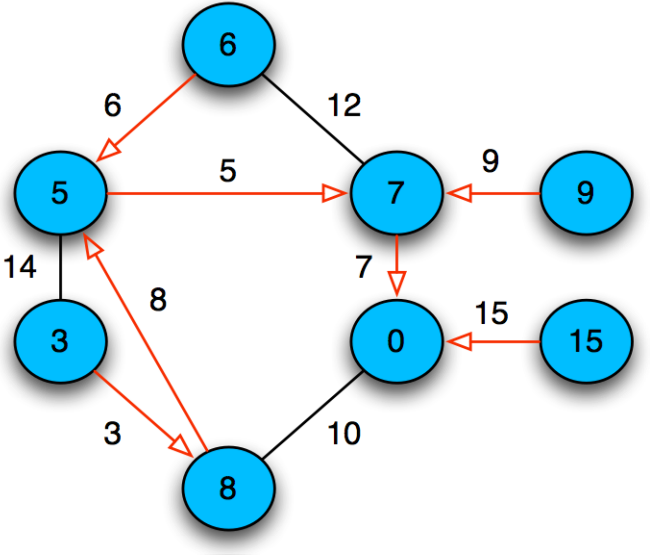

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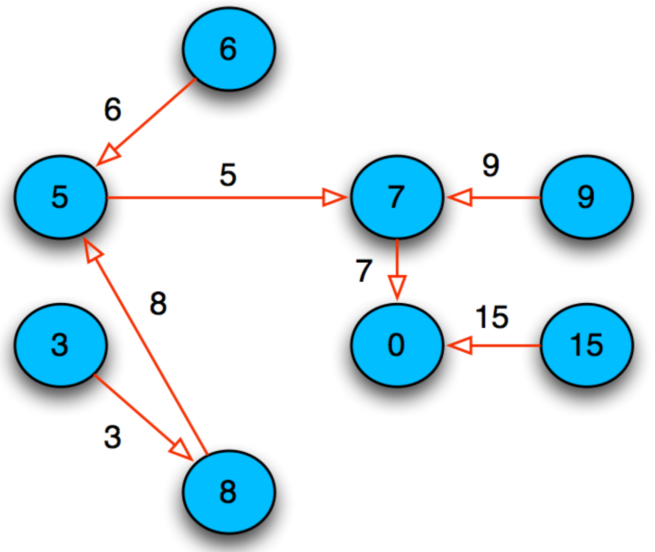

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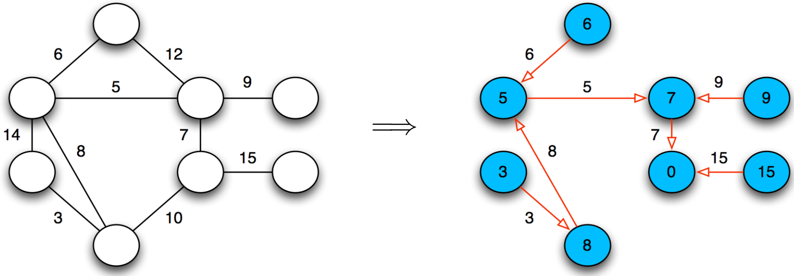


Kruskal's algorithm to compute an MST

```
MST-KRUSKAL(G, w)
A ← ∅
for each vertex v ∈ V[G]
        do MAKE-SET(v)
sort the edges of E into non-decreasing order by weight w
for each edge (u, v) ∈ E, taken in non-decreasing order by weight
        do if FIND-SET(u) ≠ FIND-SET(v)
              then A ← A ∪ {(u, v)}
                   UNION(u, v)
return A
```


Kruskal's algorithm to compute an MST

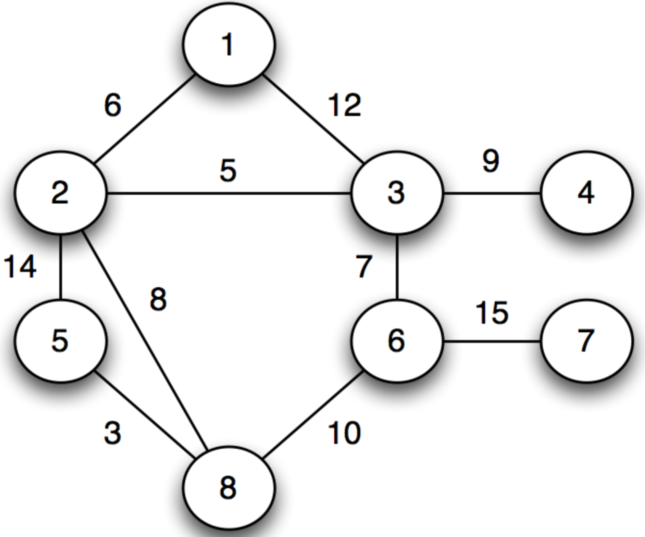
MST-KRUSKAL(G, w)

$A \leftarrow \emptyset$
for each vertex $v \in V[G]$
    do MAKE-SET(v)
sort the edges of E into non-decreasing order by weight w
for each edge $(u, v) \in E$, taken in non-decreasing order by weight
    do if FIND-SET$(u) \neq$ FIND-SET(v)
        then $A \leftarrow A \cup \{(u, v)\}$
            UNION(u, v)
return A

Running time

$O(E \lg E)$

Kruskal's algorithm in action



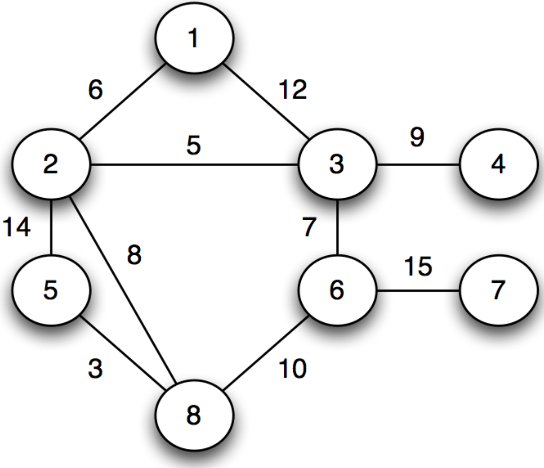
Kruskal's algorithm in action

$w(u,v)$	(u,v)
3	(5,8)
5	(2,3)
6	(1,2)
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)

$|V| = 8$

$|E| = 10$

$|T| = 0$



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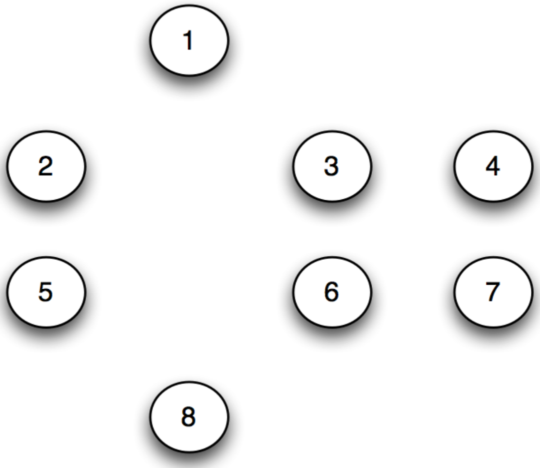

$$|V| = 8$$
$$|E| = 10$$
$$|T| = 0$$

Vertex sets:

$\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
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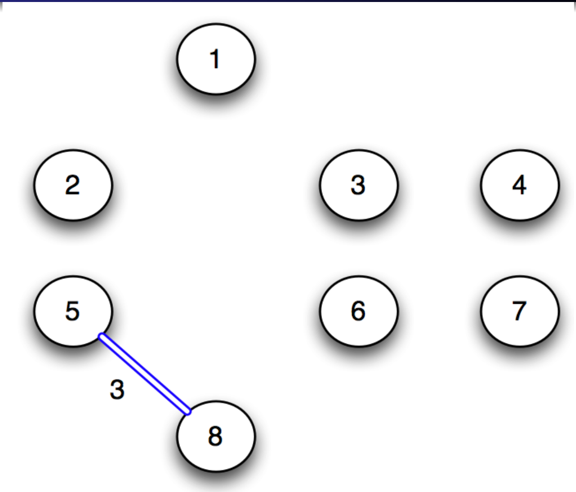

$|V| = 8$

$|E| = 10$

$|T| = \boxed{1}$

Vertex sets:

$\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\} \implies \{1\},\{2\},\{3\},\{4\},\boxed{\{5, 8\}},\{6\},\{7\}$

Kruskal's algorithm in action

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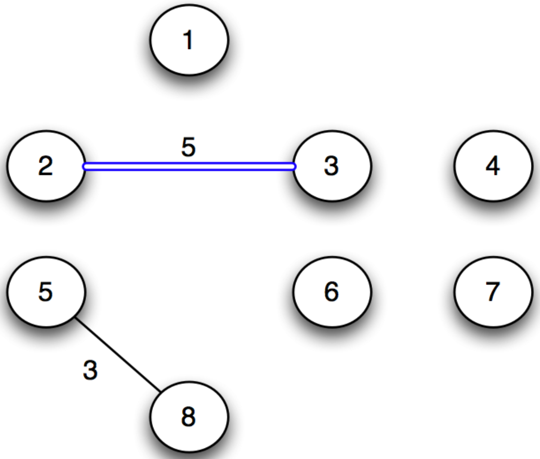

$|V| = 8$

$|E| = 10$

$|T| = \boxed{2}$

Vertex sets:

$\{1\},\{2\},\{3\},\{4\},\{5, 8\},\{6\},\{7\} \Longrightarrow \{1\},\boxed{\{2, 3\}},\{4\},\{5, 8\},\{6\},\{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
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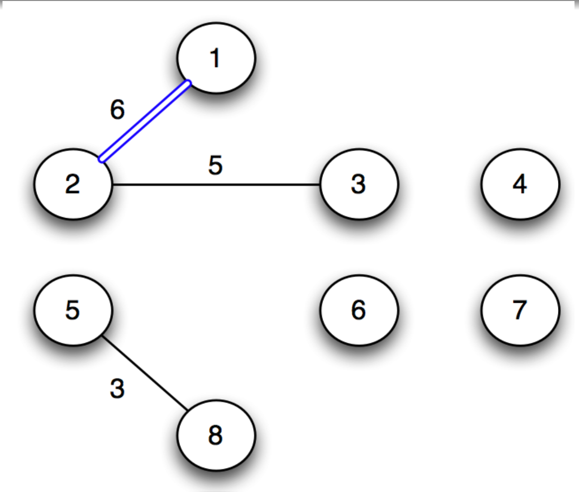
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

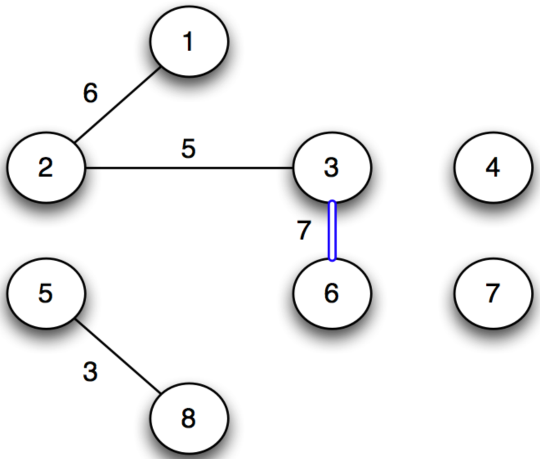
$|T| = \boxed{3}$

Vertex sets:

$\{1\},\{2,3\},\{4\},\{5,8\},\{6\},\{7\} \Longrightarrow \boxed{\{1,2,3\}},\{4\},\{5,8\},\{6\},\{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

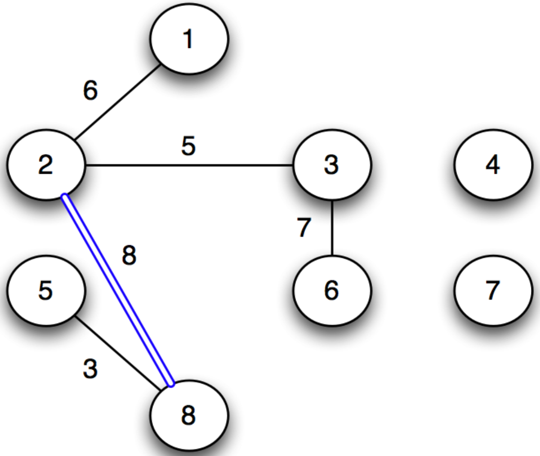
$|T| = \boxed{4}$

Vertex sets:

$\{1, 2, 3\},\{4\},\{5, 8\},\{6\},\{7\} \Longrightarrow \boxed{\{1, 2, 3, 6\}},\{4\},\{5, 8\},\{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

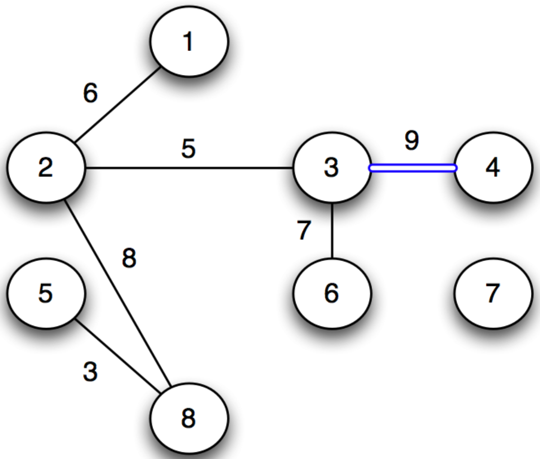
$|T| = \boxed{5}$

Vertex sets:

$\{1, 2, 3, 6\}, \{4\}, \{5, 8\}, \{7\} \Longrightarrow \boxed{\{1, 2, 3, 5, 6, 8\}}, \{4\}, \{7\}$

Kruskal's algorithm in action

$w(u, v)$	(u, v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$$|V| = 8$$

$$|E| = 10$$

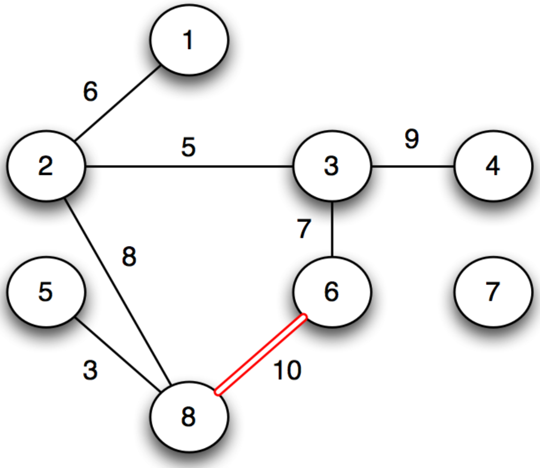
$$|T| = \boxed{6}$$

Vertex sets:

$\{1, 2, 3, 5, 6, 8\}, \{4\}, \{7\} \Longrightarrow \boxed{\{1, 2, 3, 4, 5, 6, 8\}}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

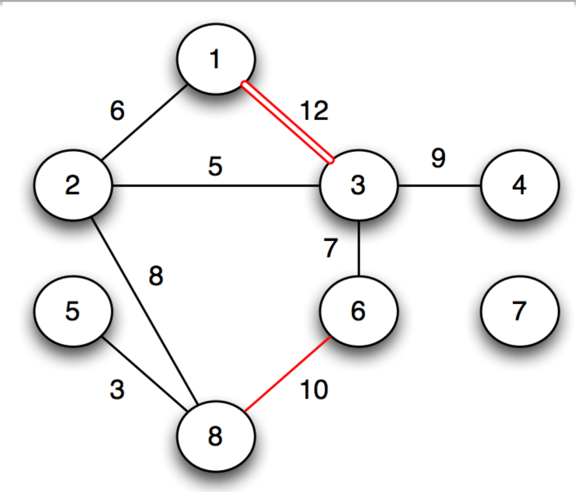
$|T| = \boxed{6}$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 8\}, \{7\} \Longrightarrow \{1, 2, 3, 4, 5, 6, 8\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
12	(1,3)
14	(2,5)
15	(6,7)



$$|V| = 8$$

$$|E| = 10$$

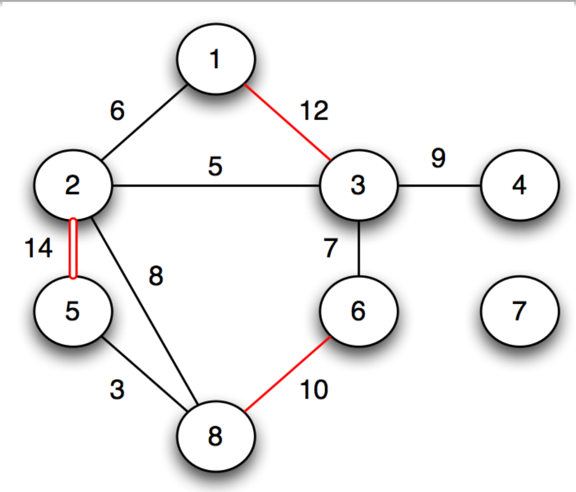
$$|T| = \boxed{6}$$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 8\}, \{7\} \Longrightarrow \{1, 2, 3, 4, 5, 6, 8\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
× 12	(1,3)
14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

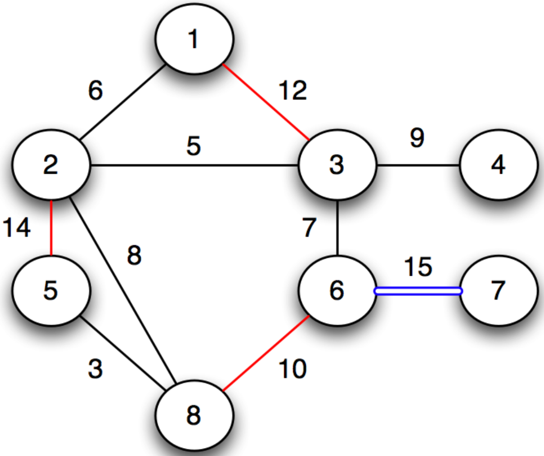
$|T| = \boxed{6}$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 8\}, \{7\} \Longrightarrow \{1, 2, 3, 4, 5, 6, 8\}, \{7\}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
× 12	(1,3)
× 14	(2,5)
15	(6,7)



$|V| = 8$

$|E| = 10$

$|T| = \boxed{7}$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 8\}, \{7\} \implies \boxed{\{1, 2, 3, 4, 5, 6, 7, 8\}}$

Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
× 12	(1,3)
× 14	(2,5)
✓ 15	(6,7)

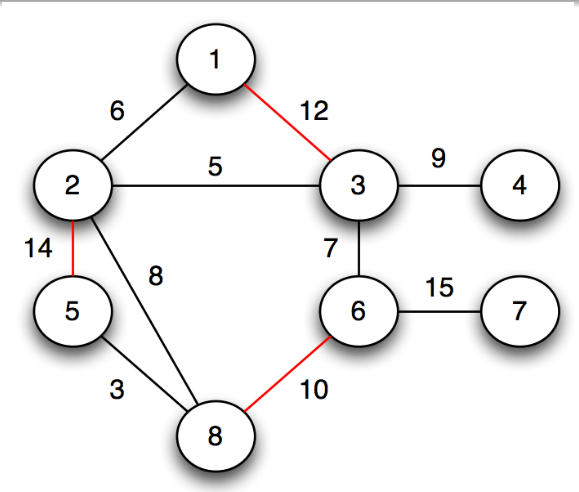
$$|V| = 8$$

$$|E| = 10$$

$$|T| = \boxed{7}$$

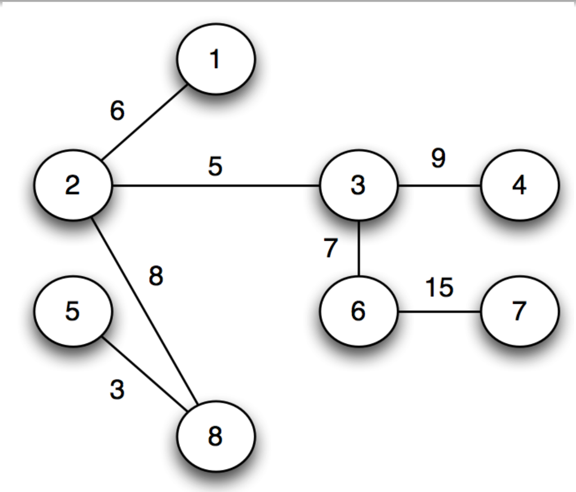
Vertex sets:

$\{1, 2, 3, 4, 5, 6, 7, 8\}$



Kruskal's algorithm in action

$w(u,v)$	(u,v)
✓ 3	(5,8)
✓ 5	(2,3)
✓ 6	(1,2)
✓ 7	(3,6)
✓ 8	(2,8)
✓ 9	(3,4)
× 10	(6,8)
× 12	(1,3)
× 14	(2,5)
✓ 15	(6,7)



$|V| = 8$

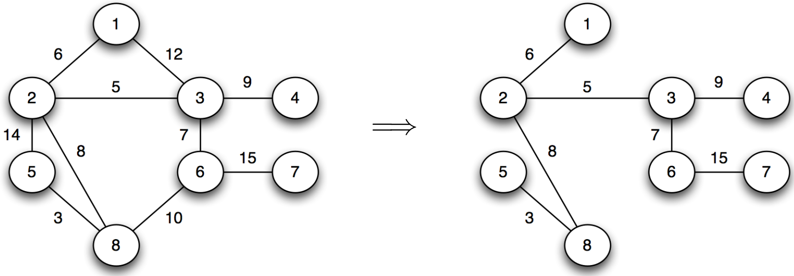
$|E| = 10$

$|T| = \boxed{7}$

Vertex sets:

$\{1, 2, 3, 4, 5, 6, 7, 8\}$

Kruskal's algorithm in action



Contents

Dijkstra's algorithm for SSSP

$\textsc{Dijkstra}(G, s)$

for each $v \in V[G]$
    do $d[v] \leftarrow \infty$
        $\pi[v] \leftarrow \mathrm{NIL}$
$d[s] \leftarrow 0$
$S \leftarrow \emptyset$
$Q \leftarrow V[G]$
while $Q \neq \emptyset$
    do $u \leftarrow \textsc{extract-min}(Q)$
        $S \leftarrow S \cup \{u\}$
        for each vertex $v \in Adj[u]$
            do if $d[v] > d[u] + w(u, v)$
                then $d[v] \leftarrow d[u] + w(u, v)$
                        $\pi[v] \leftarrow u$

Dijkstra's algorithm for SSSP

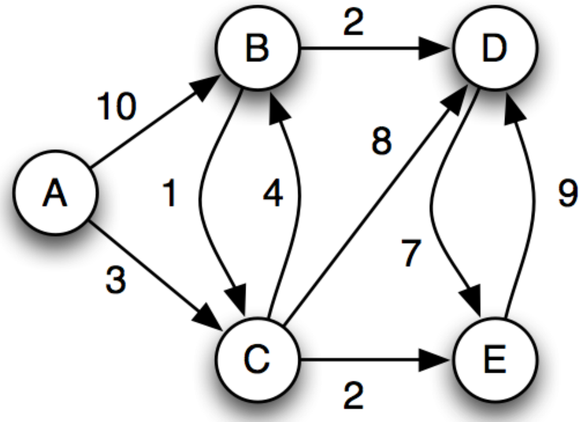
DIJKSTRA(G, s)

```
for each v ∈ V[G]
      do d[v] ← ∞
         π[v] ← NIL
d[s] ← 0
S ← ∅
Q ← V[G]
while Q ≠ ∅
      do u ← EXTRACT-MIN(Q)
         S ← S ∪ {u}
         for each vertex v ∈ Adj[u]
              do if d[v] > d[u] + w(u, v)
                    then d[v] ← d[u] + w(u, v)
                         π[v] ← u
```

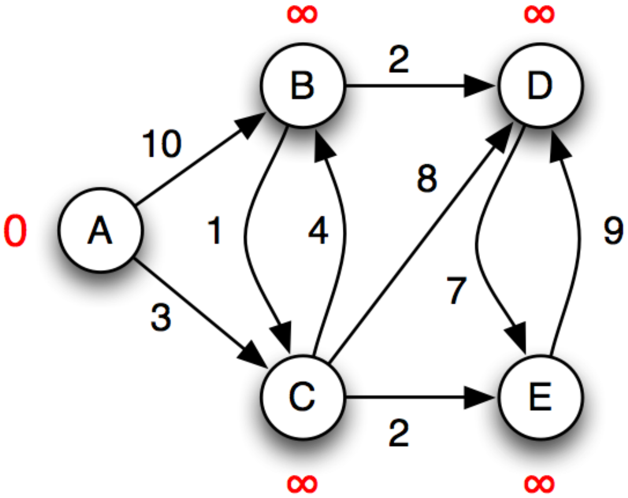
Running time

$O((V + E) \lg V)$

Dijkstra's SSSP algorithm in action



Dijkstra's SSSP algorithm in action



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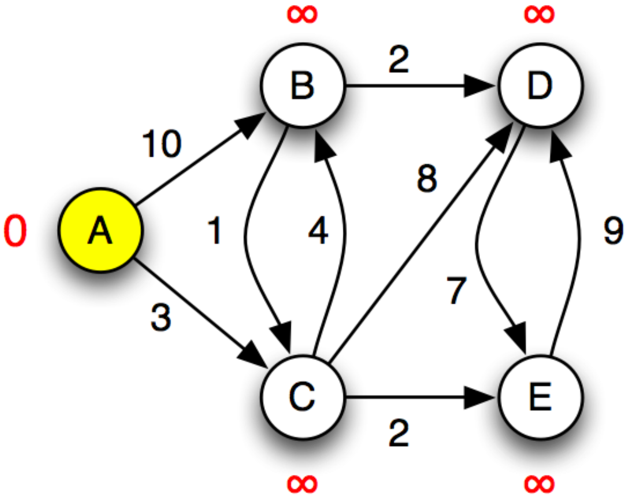

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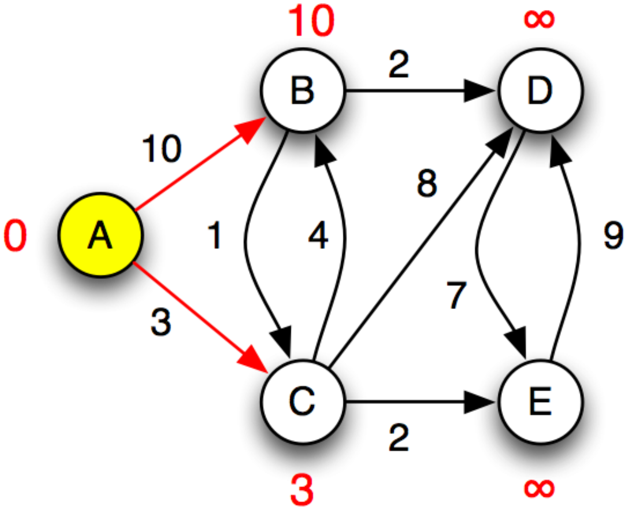

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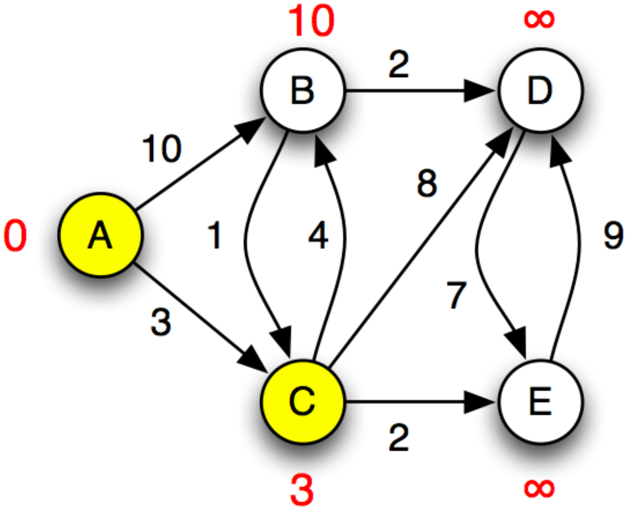

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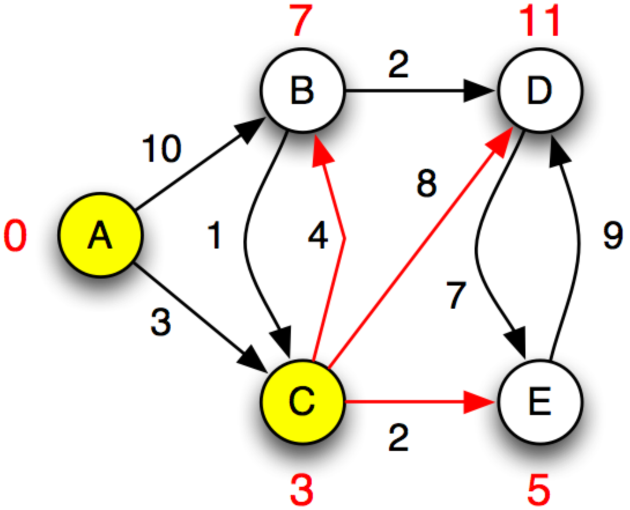

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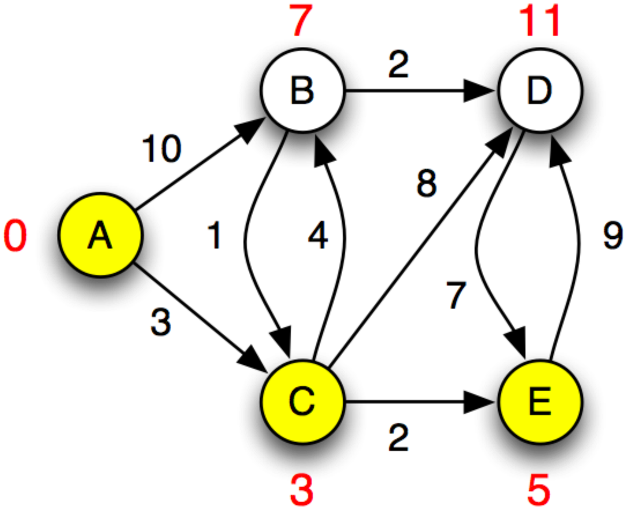

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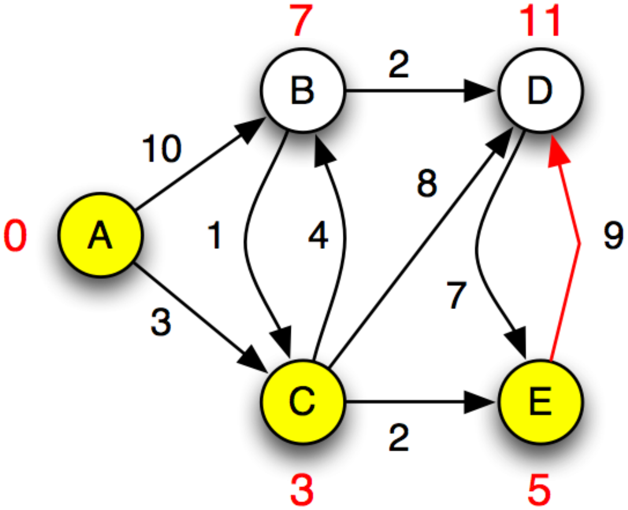

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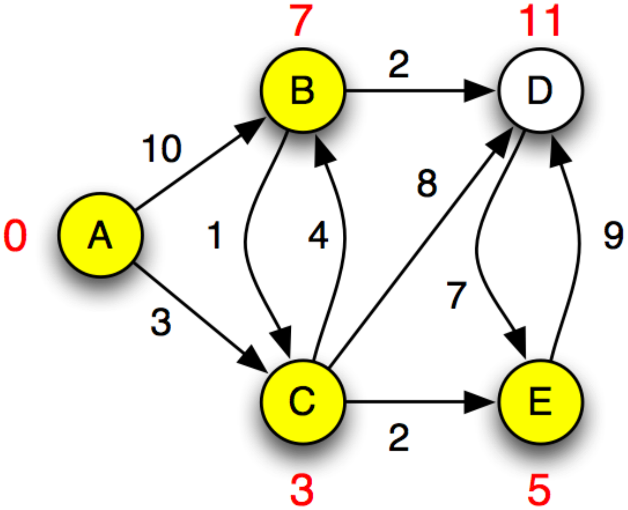

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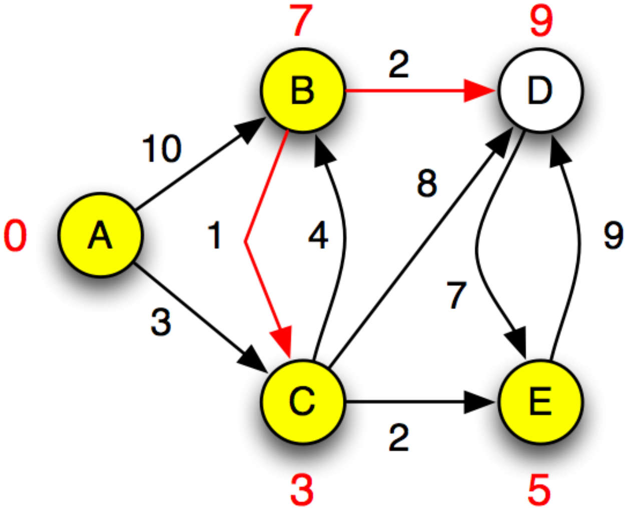

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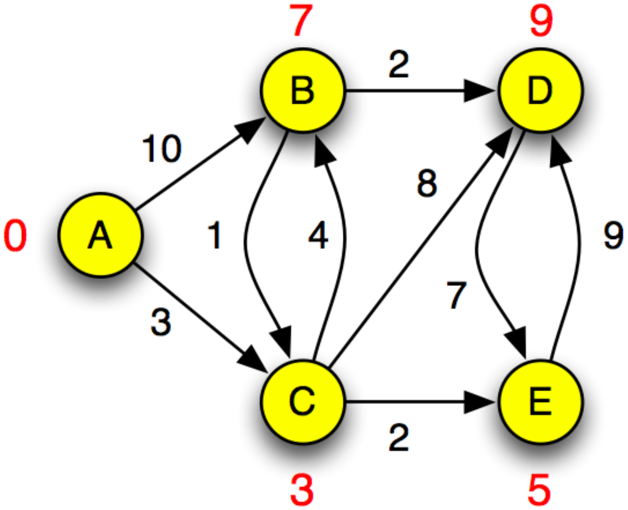

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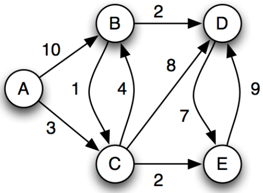

$\Longrightarrow$