Experiment 5: Harmonic Oscillator
Part I. Spring Oscillator

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Determining the relationship between the frequencies in free and damped oscillations

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The damped oscillations of a spring can be predicted based on its free oscillations through multiple characteristics: resonant frequency, damping time, and Q-factor. In this experiment, the resonant frequency of a spring's free oscillation is predicted—using its determined spring constant—and then measured using a force sensor and hanging weight with magnets. The motion of the spring is then damped using an aluminum tube. The damped frequency of the oscillations and the ratio of its successive extrema were used to find the period and damping constant; these are used to calculate the Q-factor. These components are used with the measured resonant frequency to see if they predict the damped frequency. While the predicted and measured resonant frequencies were inconsistent, the measured damped frequency of $(0.6904 \pm 0.0008)s^{-1}$ is indistinguishable from the predicted damping frequency of $(0.69 + 0.08)\ s^{-1}$ due to the large uncertainty of the latter. This confirms the relationship between a spring's resonant frequency and damped frequency through the damped oscillation's damping time and Q-factor.

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INTRODUCTION

When a spring oscillates "freely", it ideally loses no energy and thus maintains its amplitude and restoring force. When a spring's motion is damped however, the mechanical energy and thus amplitude of the oscillations decrease over time. The only two factors that do not change over time for either type of oscillation are its frequency and period. This is because these measurements only depend on the spring constant, the mass on the spring, and if present, the constant damping force. It thus becomes important to understand how to determine the frequency and period of free and damped oscillations, and to relate these frequencies by characterizing the system's damping loss.

In this experiment, the spring has a hanging weight with magnets and freely oscillates while attached to a force sensor. Its resonant frequency is predicted using the mass and spring constant. This same frequency is then measured experimentally using the voltage vs. time curve produced by the force sensor to confirm its definition. The motion is then damped by having the magnetic weight oscillate through a metal tube, which leads to energy dissipation. The experimental damped frequency as well as the constant ratio between the amplitudes of two successive peaks—since the damping force is proportional to velocity—can be used to find the damping time τ and Q. The measurements of τ and Q can be used to "predict" the damped frequency based on the measured resonant frequency. If the predicted damped frequency is consistent with the measured damped frequency, the idea that the damping time and Q can quantify a spring's damping loss will be confirmed.

METHODS

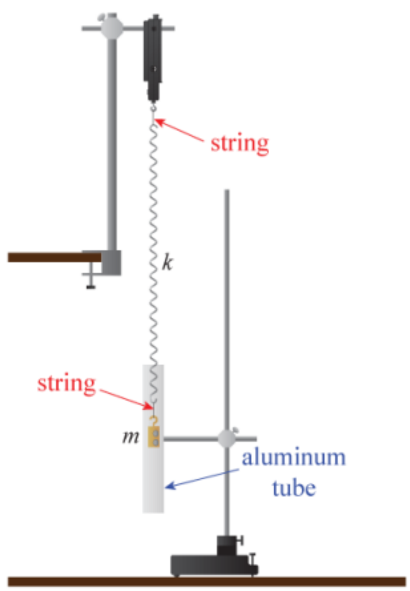

Figure 1. Experiment Setup. Above is the set up for the damped oscillation. The determination of the spring constant and the free undamped oscillation have the same setup except the aluminum tube is not used. The figure is reproduced (with permission) from Fig. 5.1 by Campbell, W. C. et al. Physics 4AL: Mechanics Lab Manual (ver. May 12, 2017). (Univ. California Los Angeles, Los Angeles, California)

The first step is to determine the spring constant k of involved in the system. While the force sensor is not used in this part, it makes sense to place the spring under the same conditions as it would be in the main experiment. To do this, set up the force sensor using clamps and rods in the configuration shown in Figure 1. The spring is hung from the force sensor hook via a string connected both—this is to decouple the spring during the main experiment. The mass of five different weights is recorded. Each of these weights are then hung on the spring to evaluate the effect of an applied force on the displacement of the spring. Instead of using two measurements to determine the displacement, the distance from the floor to the end of the string is measured. While this value is not equal to the displacement, both measurements are affected linearly by the force; since the magnitude of the linear change is equivalent it becomes convenient to simply record the distance from the floor.

The next part is to record data for the free oscillations. The DAQ should now be set up to record the output from the force sensor. While User Defined sensor is chosen for the hardware setup, force or voltage can be displayed versus time for the table and graph. This is because the extrema of data for both force and voltage is at the same time, which is the only measurement used in the quantitative analysis. The sample rate should be anywhere between 20 Hz and 50 Hz to optimize clarity of the extrema without too many points.

After the mass of the weight with metals is measured, it is attached to the end of the spring with a string. The string on both sides of the spring prevents it from rotating during oscillation. The undamped oscillation is started by releasing the weight-spring system after pulling it above its equilibrium point. The output vs. time data for the oscillations is recorded for 20-60 seconds for ample oscillations to analyze.

The same procedure of releasing and recording is done again for the damped oscillation except an aluminum tube is added to the set up as shown in Figure 1. This should be placed and the weight should be released from a displacement such that the weight oscillates mostly (if not completely) in the tube.

As an extra point of analysis, during the free oscillation, add the Fast Fourier Transform (FFT) graph to the display, which shows voltage versus frequency. Estimate and record the value and width of the peak (resonance).

ANALYSIS

Part I: Determination of Spring Constant

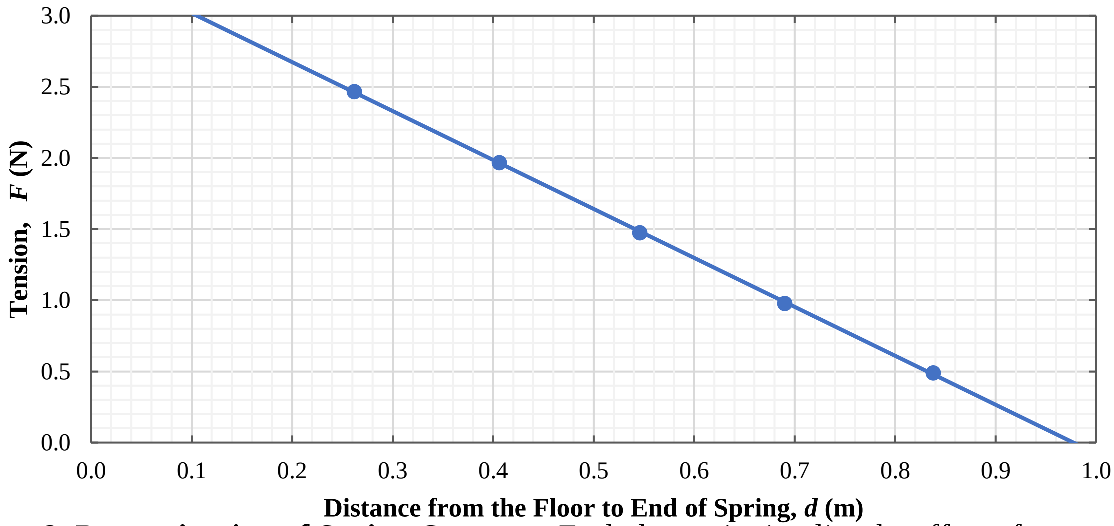

Figure 2. Determination of Spring Constant. Each data point implies the effect of a certain applied on the spring's displacement. The applied tension F, provided by the hanging weight of mass m, was calculated by $F = mg$, where $g = 9.8\, m/s^2$. The displacement x of the spring, while not equivalent, is implied by the distance d from the floor displayed. (Note: The last two weights fell to the ground from original height used, prompting us to adjust to a greater height for the spring. The distances were still recorded and adjusted based on the height differences to conform to the initial conditions.) The magnitude of the slope, $k,$ is the same for both measurements, except the slope is negative for distance since greater the vertical displacement, the closer the distance to the floor. The line fitted to the plot has equation $F(d) = (-3.440 \pm 0.025)\frac{N}{m}d + 3.36\, N$. The spring constant is determined to be $(3.440 \pm 0.025)N/m$.

To calculate the theoretical value of the resonant frequency, which is going to be compared to the experiment values found, the spring constant must be determined. As seen in Figure 2, this was determined to be $k = (3.440 \pm 0.025)\mathrm{N/m}$. The mass of the weight with metals is also measured to be $(0.1738\ \pm 0.0005)kg$. With the given formula, $f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, we thus use these measurements to predict resonant frequency $f_{o,p}$ and its uncertainty:

$$f_{o,p} = \frac{1}{2\pi}\sqrt{\frac{(3.440 \pm 0.025)\mathrm{N/m}}{(0.1738\ \pm 0.0005)kg}}$$

$$f_{o,p_{best}} = \frac{1}{2\pi}\sqrt{\frac{(3.440)\frac{kg * \frac{m}{s^2}}{m}}{(0.1738)kg}} = \frac{1}{2\pi}\sqrt{\frac{3.440}{0.1738}\ s^{-2}}\ =\ 0.708052\, s^{-1}$$

To estimate the uncertainty, equations ii.23, ii.24, and ii.22 from the lab manual are respectively used for division $g = \frac{k}{m}$, square root $u = g^{\frac{1}{2}}$, and multiplication by a constant $f_{o,p} = \frac{1}{2\pi}u$

$$\delta g = \left|\frac{k_{best}}{m_{best}}\right| \sqrt{\left(\frac{\delta k}{k_{best}}\right)^2 + \left(\frac{\delta m}{m_{best}}\right)^2} = \left|\frac{3.440}{0.1738}\right| \sqrt{\left(\frac{0.025}{3.440}\right)^2 + \left(\frac{0.0005}{0.1738}\right)^2} = 0.1547 s^{-2}$$

$$\delta u = \sqrt{\frac{k_{best}}{m_{best}}} * \frac{1}{2} * \frac{\delta g}{g_{best}} = 0.01739 s^{-1}$$

$$\delta f_{o,p} = \frac{1}{2\pi} \delta u = 0.0028 s^{-1}$$

The predicted value for the resonant frequency is therefore $f_{o,p} = (0.7081 \pm 0.0028)s^{-1}$**.** The experimental value of the resonant frequency is then determined for the free oscillations of the weight-spring system.

Part II: Free Oscillation

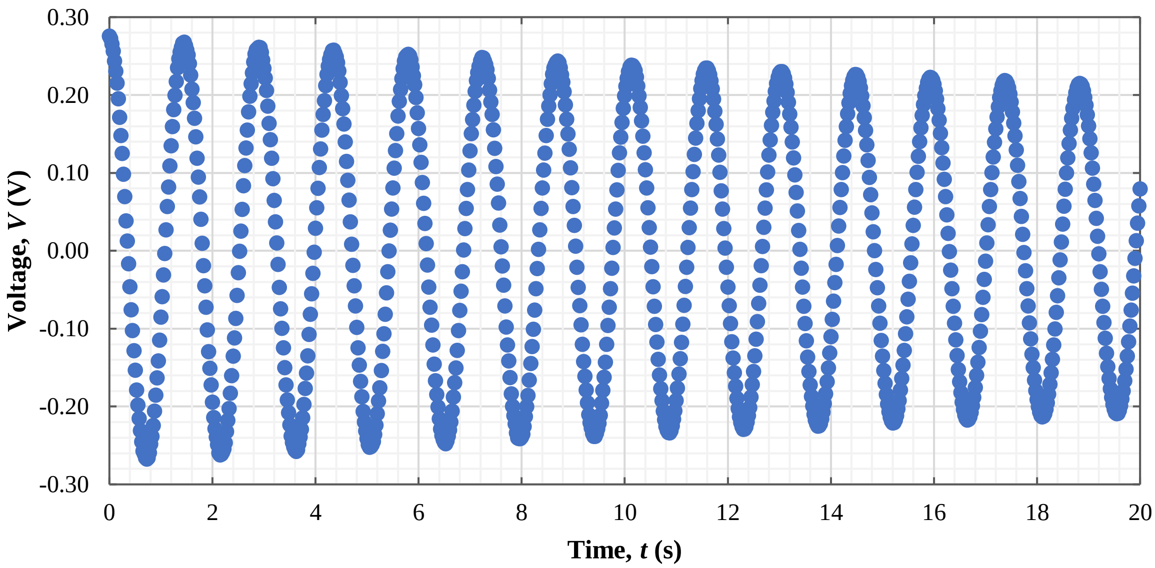

Figure 3. Free Oscillation Frequency Measurement. The voltage measurement generated by the force sensor due to the oscillations was recorded versus time. While data was collected for a longer period (~45 seconds), only 20 seconds of this data is analyzed for the calculation of the resonant frequency. The first peak of the period of data selected is aligned at $t = 0\ s$. The very subtle downward slope of the peaks represents the natural loss of some energy due to air resistance and friction, values which depend on the environmental conditions and material of the real spring involved. This little damping in the system is assumed to be negligible for the purposes of the experiment.

To measure the resonant frequency $f_{o,m}$, the time of each peak point shown in Figure 1 was estimated in Excel by hovering over a point in the peak. The estimated time region was then used to locate the exact time point at which the sensor recorded the most voltage for each peak. The average frequency between the 1st peak and each successive nth peak was calculated by dividing n-1 by the time elapsed between the points (i.e. $f_{o,avg} = \frac{n-1}{t_{nth\ peak}}$ since $t_i = 0s$ at the 1st peak). The best value was determined to be the average frequency between the 1st and 14th (last) and the uncertainty was estimated with the systematic uncertainty $\delta f_{o,m} = \frac{1}{\sqrt{13}}$ *STDEV.S(All calculated frequencies). Therefore $f_{o,m}$ was measured to be

$$f_{o,m} = \frac{13}{18.825\ s} = (0.69057 \pm 0.00022)s^{-1}$$

The experimental resonant frequency of $f_{o,m} = (0.69057 \pm 0.00022)s^{-1}$ and the theoretical resonant frequency $f_{o,p} = (0.7081 \pm\ 0.0028)s^{-1}$ are used to find the theoretical and experimental period for the free oscillation. The values for the period are calculated using $T = 1/f_o$ and $\delta T = |T_{best}| - 1| * \delta f_o$. This yields the corresponding values $T_m = (1.4481 \pm 0.0003)s$ and $T_p = (1.412 \pm 0.004)\ s$. As the bounds of these values do not overlap, the measurements for the period of the free oscillation do not agree. Since $f_{o,m} < f_{o,p}$ with a difference $f_{o,m} - f_{o,p} = (-0.0175 \pm 0.0028)\ s^{-1}$, the disagreement can be attributed partially to the natural damping of the spring system discussed in Figure 3. The theoretical resonant frequency was based on an ideal spring system, where the spring is massless and has no natural damping losses. Since there was a subtle loss of energy, the average frequency decreased over time; inversely, the period increased. Any unintended deformation of spring prior to or during the run would also affect the real spring constant, affecting the frequency and period as well.

Part III: Damped Oscillations

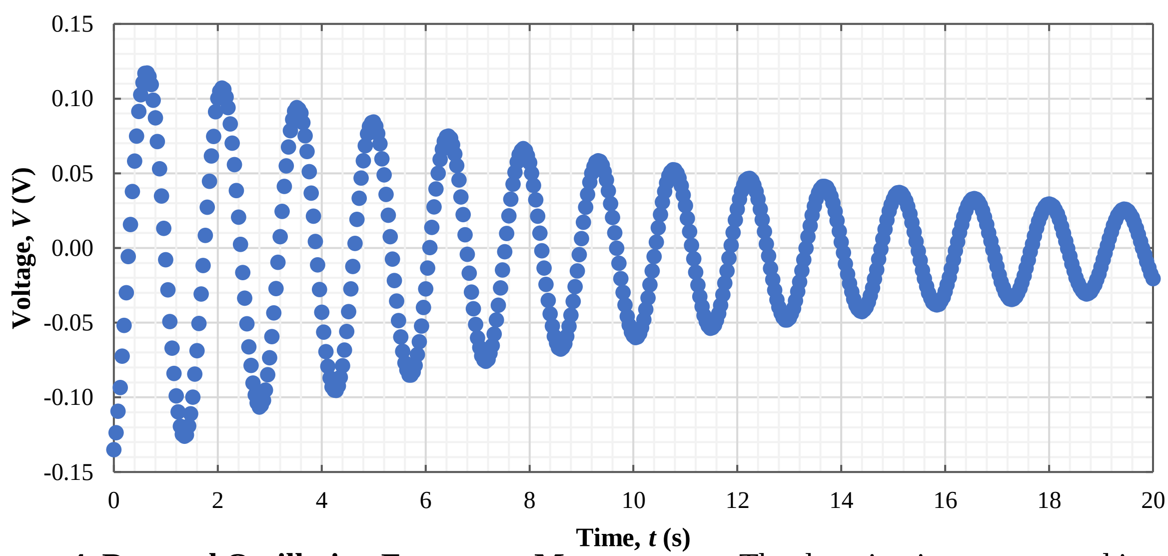

Figure 4. Damped Oscillation Frequency Measurement. The damping is apparent and is caused by the production of eddy current heating due to the motion of the weight's magnets inside the aluminum tube. The energy dissipation leads to significant decay in the oscillations and thus less force applied on the sensor over time, outputting less voltage. Approximately 25 seconds of the oscillations were recorded; only 20 seconds of this data was used to compare the free and damped oscillations with the same time domain. The graph was shifted vertically to be centered around 0 V. This is necessary to later get the ratio of the amplitudes to measure the decay time τ.

The same process described for the free oscillations is used to find the damped frequency $f_{damped,m}$ for the damped oscillations. The best value for the frequency is defined as the average frequency between the 1st and 14th (last) peak. The uncertainty is estimated with the systematic uncertainty, which is calculated by measuring the average frequency at each peak—starting from the first peak at $t = 0.61\ s$. The uncertainty is then calculated with the Excel formula ($\delta f_{damped,m} = \frac{1}{\sqrt{13}} * STDEV.S(All\ 13\ frequencies)$. The frequency for this data is therefore

$$f_{damped,m} = \frac{13}{19.44s - 0.61s} = (0.6904 \pm 0.0008)s^{-1}$$

The period T of the damped oscillations is calculated to be $T = (1.4484 \pm 0.0011)s$. We now want to calculate the damping time τ, which is defined as the time is takes to decrease the amplitude of the oscillations by a factor of $\frac{1}{e}$. This means that the ratio of the peaks must be analyzed in some way to find the damping time.

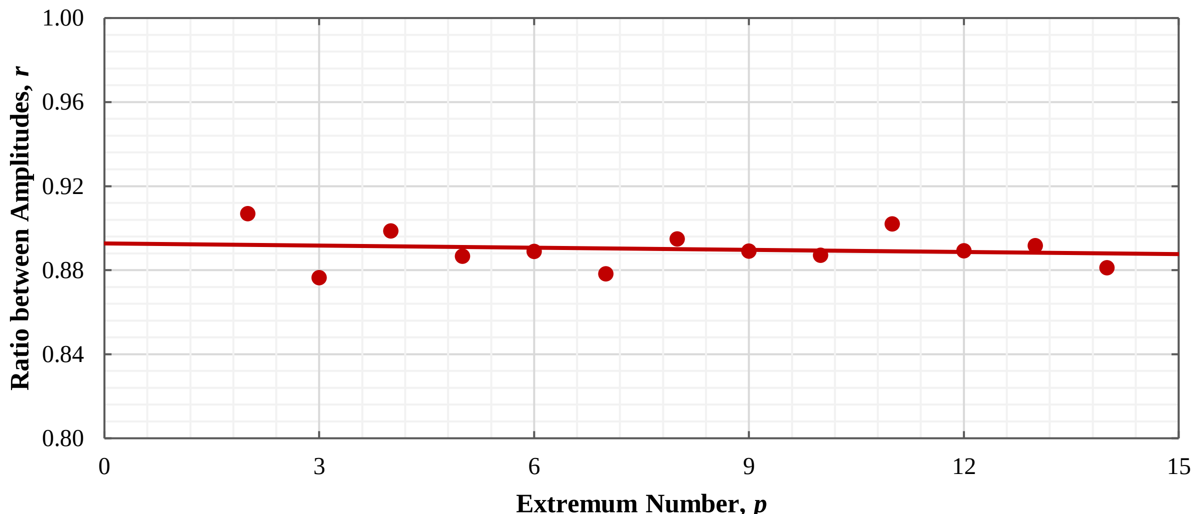

Figure 5. Ratio Between Successive Amplitudes. The amplitude (in voltage) for each peak was found using the peak times determined in the earlier step. The ratio between every two peaks was found by dividing the second peak's amplitude by the first peaks amplitude. This ratio is labelled with the extremum number of the second peak in each pair. The ratio is expected to be constant; the fitted red line with equation $r(p) = (-0.0003 \pm 0.0007)p + (0.893 \pm 0.006)$ agrees with this as the slope is consistent with 0. The ratio based on this figure is $r = 0.893 \pm 0.006$. The constant value of the ratio between each peak indicates that the effect of damping is linear. This agrees with the idea that damping is proportional to velocity

To determine the exact relationship between the amplitude ratio and damping time, the equation of a damped oscillation must be used. The equation that defines a damped oscillation is $x(t) = Ae^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}t} \times e^{-\frac{bt}{2m}}$. Since $\tau = \frac{2m}{b}$, we can rewrite $e^{-\frac{bt}{2m}}$ as $e^{-\frac{t}{\tau}}$ and thus the whole equation as $x(t) = Ae^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}t} \times e^{-\frac{t}{\tau}}$. Therefore, the ratio between the amplitudes of two consecutive peaks, which occur a period of T seconds apart, can be found. The expression for each peak is first defined (with x being replaced with V as the measurement of amplitude):

$$V(t) = Ae^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}t} \times e^{-\frac{t}{\tau}}$$

$$V(t+T) = Ae^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}(t+T)} \times e^{-\frac{(t+T)}{\tau}}$$

The exponential decay of the amplitude is only represented by $e^{-\frac{t}{\tau}}$ and $e^{-\frac{t+T}{\tau}}$. The first term for each expression is the amplitude if the spring had no damping; therefore, since the values are a period apart, they must be equivalent. The knowledge that $Ae^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}(t+T)} = Ae^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}t}$ can be used to calculate and simplify the ratio:

$$\frac{V(t+T)}{V(t)} = \frac{Ae^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}(t+T)} \times e^{-\frac{t+T}{\tau}}}{Ae^{i\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}t} \times e^{-\frac{t}{\tau}}} = \frac{e^{-\frac{(t+T)}{\tau}}}{e^{-\frac{t}{\tau}}} = e^{\frac{-t-T+t}{\tau}} = e^{-\frac{T}{\tau}}$$

Taking the natural log of the leftmost and rightmost expressions and some algebra yields the relationship.

$$\tau = -\frac{T}{ln\left(\frac{V(t+T)}{V(t)}\right)}$$

The period T was previously calculated to be $T = (1.4484 \pm 0.0011)s$. Therefore, each ratio $r = \frac{V(t+T)}{V(t)}$ plotted in Figure 5 can be converted to a measurement of τ. The best value of τ is the mean of these conversions. The uncertainty is determined through the systematic uncertainty, which is STDEV.S(All the conversions)/SQRT(13). The damping time is therefore $\tau = (12.80 \pm 0.14)s$.

The final characteristic that will be measured for this spring system is the Q- factor. This measures the quality of the oscillating system's preservation of its mechanical energy during a single cycle. Q is defined in that it satisfies the flowing relationship

$$f_{damped} \equiv f_o\sqrt{1-\frac{1}{4Q^2}}$$

It is important find a way to define Q based on the other characteristics of an oscillation, especially since f_{damped} is predicted using Qinstead of the other way around. Fortunately, this equation can be compared to another definition where

$$f_{damped} = \frac{1}{2\pi}\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$

In fact, this can be rearranged to look quite similar to the definition involving Q:

$$f_{damped} = \frac{1}{2\pi}\sqrt{\frac{k}{m}\left(1-\frac{b^2}{4km}\right)} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\sqrt{1-\frac{b^2}{4km}} = f_o\sqrt{1-\frac{1}{4Q^2}}$$

Since we know that $f_o = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$, this can be cancelled out in the equality along with some other operations to isolate Q.

$$\sqrt{1-\frac{b^2}{4km}}=\sqrt{1-\frac{1}{4Q^2}}$$
$$\frac{b^2}{4km}=\frac{1}{4Q^2}=\frac{1}{\left(\frac{4km}{b^2}\right)}$$
$$Q^2=\frac{km}{b^2}\rightarrow\ Q=\frac{\sqrt{km}}{b}$$
While b is a constant, it is an unknown, unmeasured value in our experiment. However, what is known is the definition of the damping time, where $\tau=\frac{2m}{b}$. This can be rearranged to define Q with terms that can be measured in this experiment:
$$b=\frac{2m}{\tau}$$
$$Q=\frac{\sqrt{km}}{b}\rightarrow Q=\frac{\tau\sqrt{km}}{2m}=\frac{\tau}{2}\sqrt{\frac{k}{m}}$$
The familiarity of the term $\sqrt{\frac{k}{m}}$ can lead to yet another definition of Q involving the resonant frequency f_o:
$$f_o=\frac{1}{2\pi}\sqrt{\frac{k}{m}}\rightarrow\ \sqrt{\frac{k}{m}}=\ 2\pi f_o$$
$$Q=\frac{\tau}{2}\sqrt{\frac{k}{m}}\rightarrow\ Q=\frac{\tau}{2}2\pi f_o=\tau f_o\pi$$
This definition of Q, the measured damping time $\tau=(12.80\pm0.14)s$, and the measured resonant frequency $\mathrm{f}_{o,m}=(0.69057\pm0.00022)s^{-1}$ allow us to predict the damped frequency $f_{damped,p}$. This can then be compared to the measured damped frequency $f_{damped,m}$.
$$Q=\ \tau f_{o.m}\pi=\ (12.80\pm0.14)s\times(0.69057\pm0.00022)s^{-1}\times\pi$$
$$\delta Q=\pi\times(12.80\times0.69057)\sqrt{\left(\frac{0.14}{12.80}\right)^2+\left(\frac{0.00022}{.69057}\right)^2}$$
$$Q=27.8\pm\ 0.3$$
Now the damped frequency can be predicted,
$$f_{damped,p}=f_o\sqrt{1-\frac{1}{4Q^2}}=\ (0.69057\pm0.00022)s^{-1}\sqrt{1-\frac{1}{4(27.8\pm\ 0.3)^2}}$$
$$f_{damped,pbest}=0.69046\ s^{-1}$$
The uncertainty was propagated as the following:
$$\delta\left(1-\frac{1}{4Q^2}\right)=\frac{1}{4}*(\delta Q^{-2})*|-2|=\ 0.5*(\delta Q^{-2})$$

$$\delta\left(\sqrt{\left(1-\frac{1}{4Q^2}\right)}\right)=\frac{1}{2}*\sqrt{\left(1-\frac{1}{4Q^2}\right)}*\frac{\delta\left(1-\frac{1}{4Q^2}\right)}{\left(1-\frac{1}{4Q^2}\right)}=\frac{0.5*(\delta Q^{-2})}{2\sqrt{\left(1-\frac{1}{4Q^2}\right)}}$$

$$\delta f_{damped,p}=f_{damped,best}\sqrt{\left(\frac{\delta f_o}{f_{o,best}}\right)^2+\left(\frac{\delta\left(\sqrt{\left(1-\frac{1}{4Q^2}\right)}\right)}{\sqrt{\left(1-\frac{1}{4Q^2}\right)}}\right)^2}$$

$$\delta f_{damped,p}=0.69057\sqrt{1-\frac{1}{4(27.8^2)}}\sqrt{\left(\frac{0.00022}{0.69057}\right)^2+\left(\frac{0.5*0.3^{-2}}{2\left(1-\frac{1}{4(27.8^2)}\right)}\right)^2}=0.08\ s^{-1}$$

Therefore, the predicted damped frequency is $f_{damped,p}$=$(0.69+0.08)\ s^{-1}$. The measured damped frequency $f_{damped,m}=(0.6904\pm 0.0008)s^{-1}$ agrees very closely with this predicted value. Due to the large uncertainty of the predicted frequency, the measured frequency is completely within the bounds of the predicted frequency. This essentially indistinguishable.

Part IV: Exploration with Pasco FFT Feature

Another method of determining the Q-factor of an oscillation is to use the Fast Fourier Transform (FFT) feature in Capstone. This displays the frequency response of a time-domain signal. The convenience of this is that if $Q$is quite large—which is the case for free oscillations—Q can be calculated as the ratio of resonance frequency to resonance width.

Resonance frequency f_o can be estimated by taking the value of the peak of the resonance curve produced by the FFT feature, which graphs a frequency vs. voltage curve for our experiment. The resonance width Δf_o is the width of the peak, bounded by frequencies that produce $\frac{1}{\sqrt{2}}V_{max}$.

Unfortunately, the resonance curve was not saved, but the f_o and Δf_o were estimated and recorded.

$$f_o=0.692\pm 0.003 \qquad V_{max}=0.28$$

$$f_l=0.677\pm 0.003, f_h=0.705\pm 0.003, \quad V_{1/\sqrt{2}}=0.20\rightarrow \Delta f_o=0.028\pm 0.006$$

$$Q_{FFT}=\frac{f_o}{\Delta f_o}=\frac{0.692\pm 0.003}{0.028\pm 0.006}=22.3\pm 0.3$$

Q_{FFT} is lower than Q measured in through the time constant. The disagreement is $(22.3\pm 0.3)-27.8\pm\ 0.3=\ -5.5\pm 0.6$. This difference is quite large. Reasons for this could be change in the spring conditions during the damped run, or mistake in estimating the resonance width.

CONCLUSIONS

This experiment was conducted to understand how to characterize free and damped oscillations of a string. First the predicted value for the resonant frequency, determined by the spring constant and mass, was compared to the measured value of the frequency obtained by the force sensor during the free oscillations. The measured resonant frequency of $f_{o,m} = (0.69057 \pm 0.00022)s^{-1}$ and the predicted resonant frequency $f_{o,p} = (0.7081 \pm\ 0.0028)s^{-1}$ do not agree as the bounds of the values do not overlap. This inconsistency can be attributed to the natural damping of a real spring as well as any unintended deformations of the spring before or during the run. This can mostly be improved with the quality of the spring and simply being careful.

The focus is then directed to characterizing damped motion. The measure damped frequency was $(0.6904 \pm 0.0008)s^{-1}$. This was used to find the period, which was used with the constant ratio between successive peak amplitudes to determine the damping constant $\tau = (12.80 \pm 0.14)s$. This gave enough information to find that the damped oscillation's quality-factor $Q = 27.8 \pm\ 0.3$. Q was used with the measured resonant frequency to predict that the damping frequency $(0.69 + 0.08)\ s^{-1}$. Since the measured damped frequency is completely contained within the prediction, the relationship between $f_o, f_{damped}, \tau,$ and Q of an oscillation is confirmed.

A general source of error in this experiment is the downward drifting of the force sensor. This was manually adjusted for the measured data, but this leads to greater uncertainty in that the data recorded may have been higher than what it expected. This could have been improved by "warming up" the sensor by letting the spring oscillate for a few more minutes; this would lead to more precise data. The drift can be quantified beforehand by apply the same force on the sensor several times and see how much it drifts over time. The equation of this linear drift can then be applied to the oscillation data.