Stochastic Processes Homework 3

Due Date: November 21, 2020

Instruction: Please type or write your answers clearly and show your work. You are encouraged to use the Rmarkdown version of this assignment as a template to submit your work. Unless stated otherwise, all programming references in the assignment will be in `R`. For this assignment, problems roughly covers content from the lectures 6 - 8.

1. Simulate a realization of a Poisson process, $N_t, t \leq 0$, in R using the following method:

1. Let initial arrival time $S_0 = 0$
2. Generate i.i.d. exponential random variables $X_1, X_2, ...$
3. Let $S_n = X_1 + ... + X_n$, for $n = 1, 2, 3, ...$
4. For each $k = 0, 1, 2, ...$, let $N_t = k$ for $S_k \leq t < S_{k+1}$

Provide your code and plot the simulation with $\lambda = 0.5$ on the interval $[0, 50]$.

2. Starting at 9 a.m., patients arrive at a doctor's office according to a Poisson process. On average, three patients arrive every hour.

a) Find the probability that at least two patients arrive by 9:30 a.m.
b) Find the probability that 10 patients arrive by noon and eight of them come to the office before 11 a.m.

3. See the definition for the spatial and nonhomogeneous Poisson processes. Define a nonhomogeneous, spatial Poisson process in R^2. Consider such a process $(N_A, A \subset R^2)$ with intensity function $\lambda(x, y) = e^{-(x^2+y^2)}, -\infty < x, y < \infty$.

Let C denote the unit circle, that is, the circle of radius 1 centered at the origin. Find $P(N_C = 0)$. (Hint: the rate would require you to integrate $\lambda(x, y)$ over a unit circle, which can be evaluated using polar coordinate transformation)

4. Starting at 9 a.m., customers arrive at a store according to a nonhomogeneous Poisson process with intensity function $\lambda(t) = t^2, t > 0$, where the time unit is hours. Find the probability mass function of the number of customers who enter the store by noon.

5 Consider the following implementation of simulating spatial Poisson processes.

```
# spatialPoisson.R
# Spatial Poisson process

lambda <- 100
squarearea <- 1
trials <- 10000
simlist <- numeric(trials)
for (i in 1:trials) {
  # simulate number of points in the area
    N <- rpois(1, lambda*squarearea)
    # assign uniformly distributed coordinates to each point
    xpoints <- runif(N,0,1)

    ypoints <- runif(N,0,1)
    # count number of points within centered at (0.7,0.7) with a radius of 0.2
```

```r
    ct <- sum(((xpoints-0.7)^2 + (ypoints-0.7)^2) <= 0.2^2)
    # collect number of points in circle in a vector
    simlist[i] <- ct
}
# mean and variance from the simulation
#mean(simlist)
#var(simlist)
# Compare to theoretical mean
#lambda*pi*(0.2)^2
```

a) Modify it to simulate a spatial Poisson process in R^3, with $\lambda = 10$ and on the box of volume 8 with vertices at the 8 points $(\pm 1, \pm 1, \pm 1)$.
b) Estimate the mean and variance of the number of points in the sphere centered at the origin of radius 1.
c) Compare the simulated count with the theoretical value