ASSIGNMENT 1

KAJ NYSTRÖM

ABSTRACT. Assignment 1 consists of four problems. Follow the instructions below to ensure your work is clear, complete, and well-organized.

General Guidelines.

- Submit your assignment as a single PDF document. If handwritten, scan your work clearly and legibly.
- Use clear headings to separate solutions for each problem.
- Show all derivations and intermediate steps in your calculations. Partial credit will be awarded for correct reasoning and partial solutions.
- Interpret your results and discuss their implications where applicable. Focus on providing a mathematical and conceptual understanding of your findings.
- When comparing or evaluating results, clearly articulate the reasoning behind your conclusions.
- Use correct mathematical notation and proper grammar throughout your work.

Formatting Requirements.

- Use a structured layout for each problem:
  a. **Problem Statement:** Restate the question in your own words (optional but recommended for clarity).
  b. **Solution:** Provide detailed derivations and calculations, clearly indicating each step.
  c. **Interpretation:** Explain the meaning of your results and any assumptions made.
- Highlight your final answers by underlining, boxing, or using boldface text.
- If you make use of external tools (e.g., software for calculations or plots), include the relevant code or methodology as an appendix.

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1. Estimation methods

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables drawn from the (truncated) *Pareto distribution* with parameters $\alpha > 0$ and $\beta > 0$. The probability density function is given by

$$f(x, \alpha, \beta) = \begin{cases} \frac{\alpha\beta^\alpha}{x^{\alpha+1}}, & x \geq \beta, \\ 0, & x < \beta. \end{cases}$$

In this problem you are to estimate the parameters α and β using both the **Method of Moments (MOM)** and the **Maximum Likelihood Estimation (MLE)**.

Problem A: Method of moments.

(1) Derive the first and second moments, $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$, of the Pareto distribution in terms of α and β, assuming $\alpha > 2$.
(2) Express α and β as functions of the moments, $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.
(3) Using the sample moments $\hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^n X_i$ and $\hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^n X_i^2$, derive the MOM estimators for α and β.

Problem B: Maximum likelihood estimation.

(1) Write down the likelihood function $L(\alpha, \beta)$ based on the given probability density function for the sample
$$\{X_1, X_2, \ldots, X_n\}.$$
(2) Derive the log-likelihood function $\ell(\alpha, \beta)$.
(3) Find the MLE for β assuming $\beta \leq \min(X_1, \ldots, X_n)$.
(4) Substitute the MLE of β into $\ell(\alpha, \beta)$ and find the MLE for α. (Hint: You may need to solve an equation involving the logarithm of the sample values.)

Problem C: Comparison.

(1) Compare the MOM and MLE estimators for α and β. Under what conditions would these estimators be close?
(2) Simulate a small dataset of $n = 10$ observations from a Pareto distribution with known parameters $\alpha = 3$ and $\beta = 1$. Compute the MOM and MLE estimates for this dataset.

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2. Principles of estimation

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables drawn from the exponential distribution with parameter $\lambda > 0$. The probability density function is given by

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

You are tasked with estimating the parameter λ using the following two estimators

$$T_1 = \frac{1}{\bar{X}_n}, \quad T_2 = \frac{n-1}{n} \cdot \frac{1}{\bar{X}_n},$$

where $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ is the sample mean.

Problem A: Properties of the estimators.

(1) Show that both T_1 and T_2 are consistent estimators of λ.
(2) Compute the bias of T_1 and T_2 as estimators of λ. Is either estimator unbiased?

Problem B: Mean squared error (MSE).

(1) Derive the variance of T_1 and T_2.
(2) Compute the MSE for T_1 and T_2 as estimators of λ.

Problem C: Comparison of estimators.

(1) Compare T_1 and T_2 based on their MSEs. For which values of n and λ is T_2 a better estimator than T_1?
(2) Simulate $n = 20$ observations from an exponential distribution with $\lambda = 2$. Compute the sample mean $\bar{X}_n$ and the estimates T_1 and T_2. Which estimator performs better based on MSE for this dataset?

Problem D: Sufficient statistic.

(1) Show that the sample mean $\bar{X}_n$ is a sufficient statistic for λ using the Fisher-Neyman factorization theorem.
(2) Explain why T_2, as a function of $\bar{X}_n$, is still a sufficient statistic for λ.

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3. Sampling distributions and confidence intervals

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a $N(\mu, \sigma^2)$ distribution. In this assignment, you will explore the construction and properties of confidence intervals (CIs) for the mean μ and their dependence on known or unknown variance.

Problem A: Known variance σ^2.

(1) Derive the $(1-\alpha)$ confidence interval for μ under the assumption that σ^2 is known. Use the pivotal quantity approach discussed in the lectures.
(2) Explain how the width of the confidence interval changes with: - (a) The sample size n, - (b) The confidence level $1-\alpha$.
(3) Simulate $n = 25$ samples from $N(5, 4)$. Construct the 95% confidence interval for μ using $\sigma^2 = 4$. Provide the observed interval and explain its interpretation.

Problem B: Unknown variance σ^2.

(1) Derive the $(1-\alpha)$ confidence interval for μ when σ^2 is unknown. Use the t-distribution and the natural estimator for σ^2, $\hat{s}_n^2$, as discussed in the lectures.
(2) Explain why the t-distribution is used instead of the standard normal distribution in this case. What happens to the t-distribution as $n \to \infty$?
(3) Simulate $n = 20$ samples from $N(10, 9)$. Compute the sample variance $\hat{s}_n^2$ and construct the 90% confidence interval for μ. Compare the length of this interval to the length of the interval obtained in Part 1 when using the same sample size and confidence level.

Problem C: Large-sample approximation (CLT).

(1) Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. from an unknown distribution F with mean μ and variance σ^2. Show how the Central Limit Theorem (CLT) can be used to construct an approximate $(1-\alpha)$ CI for μ when n is large and σ^2 is known.
(2) Simulate $n = 100$ observations from an exponential distribution with rate parameter $\lambda = 2$ (mean $\mu = \frac{1}{\lambda}$). Compute an approximate 95% CI for μ using the CLT approach. How does the observed interval compare to the true value of μ?

Problem D: Comparing confidence intervals.

(1) Suppose you are estimating the proportion θ of successes in a Bernoulli trial. Derive an approximate $(1-\alpha)$ CI for θ using the CLT when the sample size n is large.
(2) Simulate $n = 50$ Bernoulli trials with success probability $p = 0.6$. Construct an approximate 95% CI for θ. Compare the length of this interval to the length of the CI for μ derived in Part 3 for the exponential distribution. Which is shorter, and why?

Problem E: Interpretation and discussion.

(1) Explain the correct interpretation of a confidence interval for a parameter θ. Why is it incorrect to say, "θ lies in the interval with probability $1-\alpha$"?
(2) Based on your simulations and derived confidence intervals, discuss the trade-offs between confidence level, sample size, and interval length. Provide specific examples from the simulated data to support your points.

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4. Information theory and the Cramér-Rao bound

Problem A: Fisher information for exponential distribution.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. samples from an exponential distribution with parameter $\theta > 0$:

$$f(x, \theta) = \theta e^{-\theta x}, \quad x \geq 0.$$

(1) Derive the score function $\dot{\ell}(x, \theta)$ and the Fisher information $I(\theta)$ for a single observation.
(2) Show that the Fisher information for n observations is $I_n(\theta) = nI(\theta)$.
(3) Verify that the sample mean $\bar{X}_n$ is an unbiased estimator of $g(\theta) = 1/\theta$.
(4) Using the Cramér-Rao inequality, derive the lower bound for the variance of any unbiased estimator of $g(\theta) = 1/\theta$. Check whether $\bar{X}_n$ achieves this bound.

Problem B: Efficiency of an estimator in the context of the uniform distribution.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. samples from a uniform distribution $U(0, \theta)$, where $\theta > 0$.

(1) Show that the maximum likelihood estimator (MLE) for θ is $\hat{\theta}_n = \max\{X_1, X_2, \ldots, X_n\}$.
(2) Compute the Fisher information $I(\theta)$ for this model.
(3) Using the Cramér-Rao inequality, determine the lower bound for the variance of any unbiased estimator of θ^2.
(4) Construct an unbiased estimator of θ^2 and verify whether it achieves the Cramér-Rao lower bound.

Problem C: Large-sample confidence interval in the context of the Gamma distribution.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. samples from a Gamma distribution $\text{Gamma}(\alpha, \beta)$, where the shape α is known, and the scale $\beta > 0$ is unknown:

$$f(x, \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x \geq 0.$$

(1) Show that the MLE for β is $\hat{\beta}_n = \frac{1}{n\alpha}\sum_{i=1}^n X_i$.
(2) Derive the Fisher information $I_n(\beta)$ for β.
(3) Using the asymptotic normality of the MLE, construct an approximate $(1-\alpha)$-level confidence interval for β.

Problem D: Asymptotic efficiency of MLE.

For Problem A, consider the MLE $\hat{\theta}_n = 1/\bar{X}_n$ as an estimator for θ.

(1) Prove that $\hat{\theta}_n$ is consistent for θ.
(2) Using the Delta method, show that $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \sigma^2(\theta))$, where $\sigma^2(\theta)$ matches the Cramér-Rao lower bound.
(3) Interpret the result in terms of the asymptotic efficiency of the MLE.

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