Markov Chain — Core Concepts (Step by Step)

What is a Markov Chain? (Quick reminder)

A **Markov chain** is a stochastic process

$$X_0, X_1, X_2, \dots$$

where the **future depends only on the present**, not the past:

$$P(X_{n+1} = j \mid X_n = i, X_{n-1}, \dots) = P(X_{n+1} = j \mid X_n = i)$$

This is called the **Markov property**.

1️⃣ What is the Transition Matrix? [2p]

Definition

The **transition matrix** P contains the probabilities of moving from one state to another in **one step**.

If the state space is $S = \{1,2,\dots,k\}$, then

$$P = (p_{ij}), \quad \text{where } p_{ij} = P(X_{n+1}=j \mid X_n=i)$$

Properties

- $p_{ij} \ge 0$
- Each row sums to 1:

$$\sum_{j} p_{ij} = 1$$

Example

Suppose we have **3 states**:

- State 1
- State 2
- State 3

Transition probabilities:

- From state 1 → 2 with prob 1
- From state 2 → 1 or 3 (each 0.5)
- From state 3 → 3 with prob 1

Then

P=(0100.500.5001)P = \begin{pmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix}P=00.5010000.51

Intuition

Each **row** answers:

> "If I am in state *i* now, where can I be next?"

2️⃣ Is the Markov Chain Irreducible? [2p]

Definition

A Markov chain is **irreducible** if **every state can be reached from every other state** (possibly in multiple steps).

Mathematically:

∀ i,j ∃ n≥1 such that Pn(i,j)>0\forall i,j \quad \exists n \ge 1 \text{ such that } P^n(i,j) > 0∀ i,j ∃ n≥1 such that Pn(i,j)>0

Example 1: Irreducible Chain

$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- From 1 → 2
- From 2 → 1

✔ Every state communicates with every other → **irreducible**

Example 2: Not Irreducible

$P = \begin{pmatrix} 1 & 0 \\ 0.4 & 0.6 \end{pmatrix}$

- From state 1 → stuck forever
- State 2 can reach 1, but 1 cannot reach 2

❌ **Not irreducible**

Intuition

> Can I eventually go **everywhere** from **anywhere**?

3️⃣ Is the Markov Chain Aperiodic? What is the Period? [3p]

Definition of Period

The **period** of state i is:

$$d(i) = \gcd \{ n \ge 1 : P^n(i,i) > 0 \}$$

- If $d(i) = 1$, state is **aperiodic**
- If $d(i) > 1$, state is **periodic**

Example 1: Periodic Chain

P=(0110)P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}P=(0110)

- From state 1 → back to 1 only in **even steps**
- Possible return times: 2, 4, 6, …

d(1)=gcd(2,4,6,…)=2d(1) = \gcd(2,4,6,\dots) = 2d(1)=gcd(2,4,6,…)=2

❌ **Periodic with period 2**

Example 2: Aperiodic Chain

P=(0.50.50.30.7)P = \begin{pmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{pmatrix}P=(0.50.30.50.7)

- You can stay in the same state in **1 step**
- So return time includes 1

d(i)=1d(i) = 1d(i)=1

✔ **Aperiodic**

Important Theorem (Exam Favorite)

If a Markov chain is **irreducible**, then **all states have the same period**.

Intuition

> Are you forced to return in a **fixed cycle**, or can you come back at **any time**?

4️⃣ Does the Markov Chain Have a Stationary Distribution? [3p]

Definition

A **stationary distribution** π satisfies:

$$\pi = \pi P \quad \text{and} \quad \sum_i \pi_i = 1$$

This means:

> If the chain starts with distribution π, it **stays the same forever**.

Existence Condition (Key Result)

A **finite**, **irreducible** Markov chain **always has** a stationary distribution.

Example

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

Let $\pi = (\pi_1, \pi_2)$.

Solve:

$$\begin{cases} \pi_1 = 0.7\pi_1 + 0.4\pi_2 \\ \pi_2 = 0.3\pi_1 + 0.6\pi_2 \\ \pi_1 + \pi_2 = 1 \end{cases}$$

From first equation:

$$0.3\pi_1 = 0.4\pi_2 \Rightarrow \pi_1 = \frac{4}{3}\pi_2$$

Normalization:

$$\frac{4}{3}\pi_2 + \pi_2 = 1 \Rightarrow \pi_2 = \frac{3}{7}, \quad \pi_1 = \frac{4}{7}$$

✔ Stationary distribution:

$$\pi = \left(\frac{4}{7}, \frac{3}{7}\right)$$

Intuition

> Long-run proportion of time spent in each state

---

5 Is the Markov Chain Reversible? [3p]

Definition

A Markov chain is **reversible** if there exists a stationary distribution π such that:

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \forall i,j$$

This is called **detailed balance**.

Physical Meaning

> Flow from $i \to j$ equals flow from $j \to i$

Example: Reversible Chain

Using the previous example:

$$\pi = \left(\frac{4}{7}, \frac{3}{7}\right)$$

Check:

$$\pi_1 p_{12} = \frac{4}{7} \times 0.3 = \frac{12}{70}$$
$$\pi_2 p_{21} = \frac{3}{7} \times 0.4 = \frac{12}{70}$$

✔ Detailed balance holds → **reversible**

Example: Non-Reversible Chain

If

$$\pi_i p_{ij} \neq \pi_j p_{ji}$$

for **any pair**, chain is **not reversible**.

Intuition

> Running the movie backward looks statistically the same 🎥

🔑 Summary Table (Exam Gold)

Concept	Key Idea
Transition Matrix	One-step transition probabilities
Irreducible	All states communicate
Aperiodic	No fixed return cycle
Stationary Distribution	Long-run stable distribution
Reversible	Detailed balance holds

If you want, next I can:

- Solve a **full exam-style problem**
- Explain **convergence to stationarity**
- Connect this with **MCMC / Metropolis–Hastings**
- Give **intuition using graphs**

Just tell me 👍