1 Introduction

2 Literature Review

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This chapter presents a systematic, two-stage methodology for sizing and validating a rotary-actuated Stewart platform, starting with a one-degree-of-freedom analogue and ending with an optimized six-degree-of-freedom motion simulator.

In the first stage, we isolate the most demanding motion from the dataset, vertical translation along the z-axis, using a single-axis slider-crank model
We derive its forward and inverse kinematics, then formulate the inverse dynamics to find the optimal off-the-shelf servomotor that can meet the dynamic performance requirements.

This analysis serves two purposes: firstly, it establishes whether a Stuart-platform motion simulator can deliver the required dynamic performance and secondly helps us to determine the suitable actuators.

In the second stage, we develop the complete Stewart-platform model. We derive its inverse kinematics to determine the geometric parameters
required to span the entire 6-DOF workspace. Building on that, we formulate the inverse dynamics and optimize the platform for maximum dynamic performance.

3.1 Operational Envelope:

From the processed torso-motion data-set, we identified the key performance targets that the platform must satisfy. Table below summarizes
the required translation, acceleration, and rotational limits in each axis along with fundamental associated frequencies.

Feature	Axis	Value	Fundamental Frequency (Hz)
Translation	Z	± 4 cm	3
Rotation	Z (transverse)	$\pm 15.5°$	1.5
Rotation	Y (sagittal)	$\pm 2.5°$	3
Rotation	X (frontal)	$\pm 3.0°$	1.5
Linear Acceleration	Z	$-1g$ to $+6g$	3

Table 1: Torso Motion Dataset

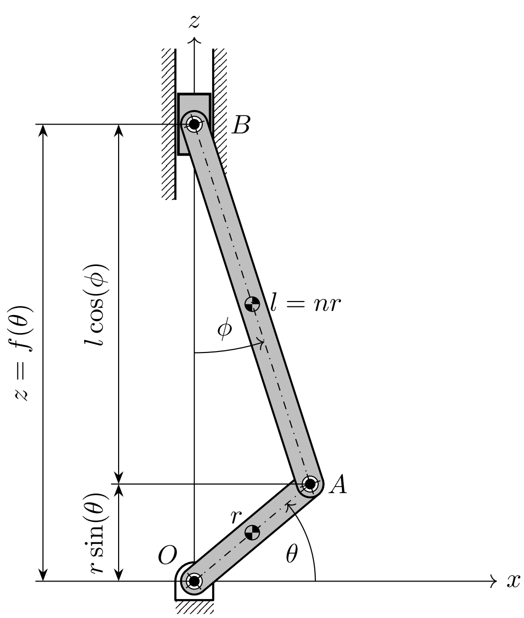

Figure 1: Slider-crank mechanism

From the table above, we use the translation and rotation features to define the kinematic workspace requirements and the linear acceleration feature for determining the dynamic requirements.

3.2 1-DOF Analysis(Pick a good name):

A slider-crank mechanism is a four-link system composed of three revolute joints and one prismatic joint. The kinematics and dynamics of a Stewart platform operating strictly along the vertical z-axis can be approximated using 6 identical slider-crank mechanisms, each contributing equally to the motion and load distribution(proof in appendix).

3.2.1 Forward Kinematics:

We consider a vertical slider–crank mechanism in which a crank of length $r$drives a slider B via a connecting rod of length l, The vertical displacement of the slider, z, can be written as:

$$z = r\sin(\theta) + l\cos(\phi) \tag{1}$$

Here, θ is the angle between the crank arm and the horizontal x-axis, and ϕ is the angle between the connecting rod and the vertical z-axis.

Applying the sine rule to $\triangle OAB$, we obtain :

$$\frac{r}{\sin(\phi)} = \frac{l}{\sin\left(\frac{\pi}{2} - \theta\right)} \tag{2}$$

We define a ratio $n = l/r$ for convenience of calculation. substituting $l = nr$ and solving for ϕ and we get-

$$\phi = \sin^{-1}\left(\frac{\cos\theta}{n}\right) \tag{3}$$

So the displacement equation becomes -

$$z = f(\theta) = r\sin(\theta) + nr\cos(\sin^{-1}(\frac{\cos\theta}{n})) \tag{4}$$

Equation(4) gives the forward kinematics , the vertical position z of the slider as a function of crank angle θ.

3.2.2 Inverse Kinematics:

The inverse kinematics problem involves determining the crank angle θ required to achieve a given vertical slider position z. In $\triangle OAB$ applying cosine rule, we obtain :

$$\angle AOB = \frac{\pi}{2} - \theta = \cos^{-1}(\frac{h^2 + r^2 - l^2}{2hr}) \tag{5}$$

substituting $l = nr$ and solving for θ and we obtain -

$$\theta = \sin^{-1}\left(\frac{h^2 + r^2(1 - n^2)}{2hr}\right) \tag{6}$$

This expression allows us to compute the crank angle θ given the slider displacement h.

3.2.3 Inverse Dynamics:

The dynamics of the system are derived using Lagrangian mechanics. We consider the slider has a mass m and the inertia of motor driving the crank is J in the crank frame. At this stage of the analysis we assume the crank arm and the connecting rod has no mass. In the *theta* as the generalized co-ordinate system. The total kinetic energy of the system can be written as:

$$\begin{aligned} T &= T_{\text{motor}} + T_{\text{slider}} \\ &= \tfrac{1}{2}J\,\dot{\theta}^2 + \tfrac{1}{2}m\,\dot{z}^2 \\ &= \tfrac{1}{2}J\,\dot{\theta}^2 + \tfrac{1}{2}m\left(\tfrac{dz}{d\theta}\right)^2\dot{\theta}^2 \\ &= \tfrac{1}{2}\left(J + m\left(\tfrac{dz}{d\theta}\right)^2\right)\dot{\theta}^2 \end{aligned} \tag{7}$$

The potential energy of the system :

$$V = mgz = mgr(\sin(\theta) + n\cos(\sin^{-1}(\frac{\cos\theta}{n})) \tag{8}$$

Hence the Lagrangian of the system is :

$$\mathcal{L} = \tfrac{1}{2}\left(J + m\left(\tfrac{dz}{d\theta}\right)^2\right)\dot{\theta}^2 - mgz(\theta) \tag{9}$$

For the generalized co-ordinate system q, the Euler-Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = Q \tag{10}$$

Where, Q represents the generalized force. For our system, $q = \theta$, and thus required torque becomes $Q = \tau$ .Substituting we get:

$$\begin{aligned}
\tau &= \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\Big) - \frac{\partial \mathcal{L}}{\partial \theta} \\
&= \frac{d}{dt}\Big[(J + m\,(\frac{dz}{d\theta})^2)\,\dot{\theta}\Big] - \Big[\tfrac{1}{2}\big(2m\,\frac{dz}{d\theta}\,\frac{d^2z}{d\theta^2}\big)\,\dot{\theta}^2 - mg\,\frac{dz}{d\theta}\Big] \\
&= (J + m\,(\frac{dz}{d\theta})^2)\,\ddot{\theta} + 2m\,\frac{dz}{d\theta}\,\frac{d^2z}{d\theta^2}\,\dot{\theta}^2 - \big(m\,\frac{dz}{d\theta}\,\frac{d^2z}{d\theta^2}\,\dot{\theta}^2 - mg\,\frac{dz}{d\theta}\big) \\
&= (J + m\,(\frac{dz}{d\theta})^2)\,\ddot{\theta} + m\,\frac{dz}{d\theta}\,\frac{d^2z}{d\theta^2}\,\dot{\theta}^2 + mg\,\frac{dz}{d\theta}\,.
\end{aligned} \tag{11}$$

Here,

$$\begin{aligned}
\frac{dz}{d\theta} &= r\cos\theta + nr\Big[-\sin\!\big(\sin^{-1}(\tfrac{\cos\theta}{n})\big)\,\frac{d}{d\theta}\sin^{-1}\!\big(\tfrac{\cos\theta}{n}\big)\Big] = r\cos\theta + \frac{r\,\cos\theta\,\sin\theta}{\sqrt{n^2 - \cos^2\theta}}, \\
\frac{d^2z}{d\theta^2} &= -r\sin\theta + r\Big(\frac{\cos 2\theta}{\sqrt{n^2 - \cos^2\theta}} - \frac{\cos^2\theta\,\sin^2\theta}{(n^2 - \cos^2\theta)^{3/2}}\Big).
\end{aligned} \tag{12}$$

3.2.4 Optimal Configuration:

In order replicate the torso motion as accurately as possible, we must satisfy the position workspace requirements as well reproduce the g-forces experienced by a runner.To satisfy the kinematics worksapce

The maximum g-force is experienced, vertically, when the runner is at the lowest point in the trajectory. To generate maximum acceleration, at the lowest point in the trajectory, the configuration dependent load inertia of the system must match the motor inertia in the load frame. From Equation 11 we find the the total effective inertia of the system:

$$J_{total} = J + m\,(\frac{dz}{d\theta})^2 = J + m(r\cos\theta + \frac{r\,\cos\theta\,\sin\theta}{\sqrt{n^2 - \cos^2\theta}})^2 \tag{13}$$

Therefore, the condition for inertia matching is

$$y \tag{14}$$