ECE368: Probabilistic Reasoning
Lab 3: Hidden Markov Model

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You should hand in: 1) A scanned .pdf version of this sheet with your answers (file size should be under 2 MB); 2) one Python file inference.py that contains your code. The files should be uploaded to Quercus.

1. (a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(\mathbf{z}_i|(\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{N-1}, \hat{y}_{N-1}))$ for $i = 0, 1, \dots, N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on the messages. (1 **pt**)

<u>Forward Message</u>

Initialization: $\alpha(z_0) = P(z_0) * P((\hat{x}_0, \hat{y}_0) | z_0)$

Recursion: $\alpha(z_i) = P((\hat{x}_i, \hat{y}_i) | z_i) * \sum_{z_{i-1}} \left(\alpha(z_{i-1}) * P(z_i | z_{i-1})\right)$

<u>Backward Message</u>

Initialization: $\beta(z_{N-1}) = 1$

Recursion: $\beta(z_i) = \sum_{z_{i+1}} \left(\beta(z_{i+1}) * P(z_{i+1}|z_i) * P((\hat{x}_{i-1}, \hat{y}_{i+1}) | z_{i+1})\right)$

<u>Marginal Distribution:</u>

$P(z_i | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{N-1}, \hat{y}_{N-1})) = \alpha(z_i) * \beta(z_i)$

And be sure to normalize each of them across z_i.

(b) After you run the forward-backward algorithm on the data in test.txt, write down the obtained marginal distribution of the state at $i = 99$ (the last time step), i.e., $p(\mathbf{z}_{99}|(\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (2 **pt**)

$$p(z_{99} | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{99}, \hat{y}_{99})) = \begin{cases} 0.810263 & \text{if } z_{99} = (11, 0, \text{'stay'}) \\ 0.179608 & \text{if } z_{99} = (11, 0, \text{'right'}) \\ 0.010128 & \text{if } z_{99} = (10, 1, \text{'down'}) \end{cases}$$

2. Modify your forward-backward algorithm so that it can handle missing observations. After you run the modified forward-backward algorithm on the data in test_missing.txt, write down the obtained marginal distribution of the state at $i = 30$, i.e., $p(\mathbf{z}_{30}|(\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{99}, \hat{y}_{99}))$. Only include states with non-zero probability in your answer. (1 **pt**)

$$p(z_{30} | (\hat{x}_0, \hat{y}_0), \dots, (\hat{x}_{30}, \hat{y}_{30})) = \begin{cases} 0.913043 & \text{if } z_{30} = (6, 7, \text{'right'}) \\ 0.043478 & \text{if } z_{30} = (5, 7, \text{'stay'}) \\ 0.043478 & \text{if } z_{30} = (5, 7, \text{'right'}) \end{cases}$$

3. (a) Write down the formulas of the Viterbi algorithm using $\mathbf{z}_i$ and $(\hat{x}_i, \hat{y}_i), i = 0, 1, \ldots, N-1$. Your answer should contain the initialization of the messages and the recursion of the messages in the Viterbi algorithm. (1 **pt**)

Initialization

$$\omega_0(z_0) = \ln(P(z_0)) + \ln(P((\hat{x}_0, \hat{y}_0) | z_0)) = \ln(P(z_0) \times P((\hat{x}_0, \hat{y}_0) | z_0))$$

Recursion

$$\omega_i(z_i) = \ln(P((\hat{x}_i, \hat{y}_i) | z_i)) + \max_{z_{i-1}} \{\ln(P(z_i | z_{i-1})) + \omega_{i-1}(z_{i-1})\}$$

(b) After you run the Viterbi algorithm on the data in test_missing.txt, write down the last 10 hidden states of the most likely sequence (i.e., $i = 90, 91, 92, \ldots, 99$) based on the MAP estimate. (3 **pt**)

The last 10 hidden states:

(11, 5, 'down')	i = 90	(10, 7, 'left')	i = 95
(11, 6, 'down')	i = 91	(9, 7, 'left')	i = 96
(11, 7, 'down')	i = 92	(8, 7, 'left')	i = 97
(11, 7, 'stay')	i = 93	(7, 7, 'left')	i = 98
(11, 7, 'stay')	i = 94	(6, 7, 'left')	i = 99

4. Compute and compare the error probabilities of $\{\tilde{\mathbf{z}}_i\}$ and $\{\check{\mathbf{z}}_i\}$ using the data in test_missing.txt. The error probability of $\{\tilde{\mathbf{z}}_i\}$ is $\boxed{0.03}$. The error probability of $\{\check{\mathbf{z}}_i\}$ is $\boxed{0.02}$. (1 **pt**)

5. Is sequence $\{\check{\mathbf{z}}_i\}$ a valid sequence? If not, please find a small segment $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$ that violates the transition model for some time step i. You answer should specify the value of i as well as the corresponding states $\check{\mathbf{z}}_i, \check{\mathbf{z}}_{i+1}$. (1 **pt**)

No, $\{\check{z}_i\}$ is not a valid sequence because it violates the transition model for $i = 64$.

In which case: $\check{z}_{64}$ = (3, 7, 'stay')

$\check{z}_{65}$ = (2, 7, 'stay')