Lecture 11 Quadratic Programs and Linear Program Examples

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Last Time: Linear Programs

Not obvious in general
Standard Form:

$$\min c^\top x \text{ s.t. } Ax \le b$$

Each row of A defines a hyperplane and b determines how far it is from the origin to create an affine set and the feasible set is a polytope. Optimal value in linear programs if attained is a vertex.

Relaxation

Task Assignment Problem

Consider a problem where we have tasks $1 \ldots m$ and workers $1 \ldots n$ and $n \ge m$. Cost of worker j doing task i is m_{ij}.
$M = (m_{ij}) \leftarrow \mathbb{R}^{m \times n}$
Problem: Assign workers to tasks and minimise cost. Let

$$x_{ij} = \begin{cases} 1 \text{ if worker j assigned to task i} \\ 0 \text{ otherwise} \end{cases}$$

x is assignment variable and $X = (x_{ij}) \in \{0,1\}^{m \times n}$

$$\min_{X \in \{0,1\}^{m \times n}} \operatorname{Tr}(X^\top M) = \sum_{i,j} x_{ij} m_{ij}$$

Set up constraints where workers can only work on one task and each task must be done.

$$\sum_j x_{ij} = 1 \;\; i = 1 \ldots m \text{ (each task gets assigned)}$$
$$\sum_i x_{ij} \le 1 \;\; j = 1 \ldots m \text{ (each worker has} \le 1 \text{ task)}$$

Not a linear program as X is restricted to 0 and 1
Rewrite it so (notice the restriction of X)

$$\min_{X \in \mathbb{R}^{m \times n}} \text{ and } x_{ij} \in \{0,1\}\ \forall i,j$$

But still not a linear program due to integral (integer not calculus) constraint. Eliminate the constraint by **relaxing** it

$$\begin{aligned} \min_{x \in \mathbb{R}^{m \times n}} \operatorname{Tr}(X^\top M) \\ \text{s.t. } X\mathbb{1} = \mathbb{1} \\ X^\top \mathbb{1} \leq \mathbb{1} \\ 0^{m \times n} \leq X \leq \mathbb{1}^{m \times n} \end{aligned}$$

The cost will therefore be less than the original problem but surprisingly many times relaxation still gives optimal solution to original problem.

Later: Can show 0 duality gap to show optimal value is the same as original

Examples of LP

Diet Problem

Let's say we have foods s.t.

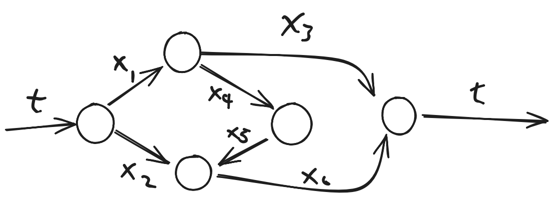
Foods	Calories/g	Vitamin 1/g	Vitamin 2/g	Vitamin 3/g
1	C_1	V_{11}	V_{12}	V_{13}
2	C_2	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	C_m	V_{m1}	V_{m2}	V_{m3}

Let's eat g_i grams of food i per day. Suppose I need b_k units of vitamin k each day. What is min-calorie diet.
Let the last 3 columns be matrix $V \in \mathbb{R}^{m \times 3}$ and the calories and grams be c and

$g = (g_1 \dots g_m)^\top$. *Exam Question Level*

$$\begin{aligned} &\min_{g \in \mathbb{R}^m} c^\top g \\ &\text{s.t. } V^\top g \geq b \\ &\qquad\quad g \geq 0 \\ = &\begin{bmatrix} -V^\top \\ -I \end{bmatrix} g \leq \begin{bmatrix} -b \\ 0 \end{bmatrix} \end{aligned}$$

Network Flow



Link i can support flow $\leq u_i$

What is max flow network can support? We can see a few equations like $t = x_1 + x_2$. We can write

$$\begin{aligned} &\max_{x,t} t \\ &\text{s.t. } Ax = te \\ &0 \leq x \leq u \end{aligned}$$

Where A is the adjacency matrix and $e = (-1, 0, \dots, 0, +1)^\top$ The first constraint shows flow in = flow out. The second constraint entries there is one directional flow and the upper bound.

In standard form:

$$
\begin{aligned}
\max_{x,t} (0,1)^\top \begin{pmatrix} x \\ t \end{pmatrix} \\
\text{s.t. } Ax &\le te \\
-Ax &\le -te \\
x &\le u \\
-x &\le 0 \\
= \begin{bmatrix} A & -e \\ -A & e \\ I & 0 \\ -I & 0 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix} &\le \begin{bmatrix} 0 \\ 0 \\ u \\ 0 \end{bmatrix}
\end{aligned}
$$

Quadratic Programs (QP)

Optimisation problem with (convex) quadratic objective, linear inequality constraints.
QP Standard form:

$$
\begin{aligned}
\min f_0(x) \\
\text{s.t. } Ax \le b
\end{aligned}
$$

Where $f_0(x) = \frac{1}{2}x^\top Hx + c^\top x + d$
d is not necessary as a constant.
Convexity means that H is p.s.d.

Examples of QP

1. Least Squares

$$
\min_x \|Ax - b\|_2^2 = f_0(x) = x^\top A^\top Ax - 2b^\top Ax + \|b\|_2^2
$$

We then see that $H = 2A^\top A$ and $c = -2A^\top b$.

2. Linearly Constrained LS

$$
\begin{aligned}
\min_x \|Ax - b\|_2^2 \\
\text{s.t. } Cx = d
\end{aligned}
$$

Where it's the same as a QP where the constraints are

$$
\begin{bmatrix} C \\ -C \end{bmatrix} x \le \begin{bmatrix} d \\ -d \end{bmatrix}
$$

3. LASSO (ℓ_1 regularised LS)

Reminder: $\|x\|_1 = \sum |x_i|$

$$\min_x \|Ax - b\|_2^2 + \lambda \|x\|_1$$
$$= \min_{x,t} \|Ax - b\|_2^2 + \lambda \sum t_i$$
$$\text{s.t. } x_i \leq t_i$$
$$-x_i \leq t_i \quad i = 1 \dots m$$

Think writing $t_i \geq 0$ is redundant

We can rewrite as quadratic form as

$$(x,t)^\top \begin{pmatrix} A^\top A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + (-2A^\top b, \lambda \mathbb{1})^\top \begin{pmatrix} x \\ t \end{pmatrix} + \|b\|_2^2$$

ℓ_1 regularisation encourages sparsity in the solution

$$\min \|Ax - b\|_2^2$$
$$\text{s.t. } \operatorname{card}(x) \leq k$$

Where $\operatorname{card}(x) =$ nonzero numbers in x vector. This constraint is called a **sparsity constraint**. (x should have ≤ k nonzero values)

λ is a "Lagrange multiplier" meaning that above problem is equivalent to

$$\min_x \|Ax - b\|_2^2 + \lambda \operatorname{card}(x)$$

for suitable λ.

Lagrange multipliers turn constrained problems into unconstrained problems by penalising how by how much the constraint is violated.

Holder's Inequality

$\|x\|_1 \leq \operatorname{card}(x) \|x\|_\infty$

Relax constrained problem:

$$\min \|Ax - b\|_2^2 \text{ s.t. } \|x\|_1 \leq k \|x\|_\infty$$

Which is therefore the lasso problem.