Lecture 24 Interior Point Method

Note by Samion Suwito on 4/17/25

Last Time

$$\begin{aligned} &\min_x f_0(x) \\ &\text{s.t. } f_i(x) \le 0 \quad i = 1 \dots m \end{aligned} \tag{P}$$

KKT Conditions

1. x primal feasible, $\lambda \geq 0$
2. $\nabla f_0(x) + \sum_i \lambda_i \nabla f_i(x) = 0$
3. $\lambda_i f_i(x) = 0 \quad i = 1 \dots m$
   If strong duality holds, KKT conditions necessarily for optimality
   If f_i's convex, KKT conditions are sufficient for optimality
   $\implies$ If Slater's condition holds, KKT conditions necessary & sufficient.

Remark: Analog of "derivative = 0" for constrained optimality = KKT conditions. Duality is intrinsic.

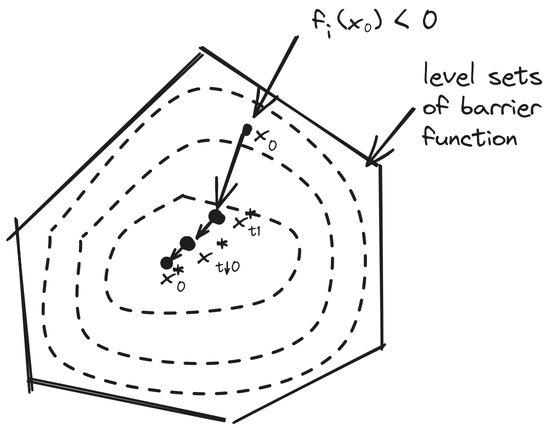
Interior Point Method

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} f_0(x) \\ &\text{s.t. } f_i(x) \le 0 \quad i = 1 \dots m \end{aligned} \tag{P}$$

Assume today f_i's are convex and differentiable.
Many optimisation algorithms have a 2-Phase approach:

- "Phase 1": Find a feasible point.
- "Phase 2": Do the optimisation (usually in an iterative manner)

Let's assume for now that we have a strictly feasible point x_0.
Question: How to execute Phase 2?



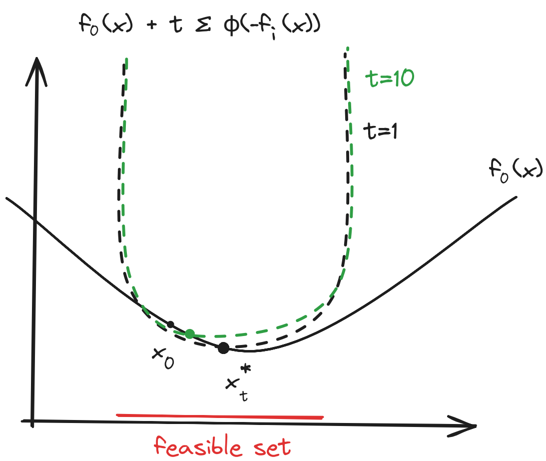
Form auxiliary problem, for $t > 0$:

$$\min_x f_0(x) + t \sum_i \phi(f_i(x))$$

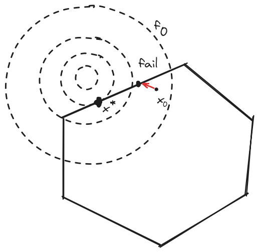
$$\phi(s) = \begin{cases} \log\left(\frac{1}{s}\right) & s > 0 \\ +\infty & s \leq 0 \end{cases}$$

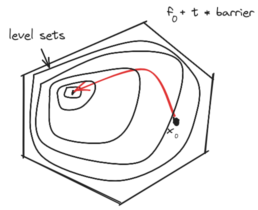
Where $\phi(s)$ is the log barrier function from last time
We plot the level sets of the log barrier function on the picture.

This would be how it would look like with functions notice how as t decreases it gets closer to the original$x^\star$:



This is why we do it instead of gradient descent. It is hard to glide along the boundary especially in high dimensional spaces so by using the log boundary we ensure the descent is strictly in the set.





Given strictly feasible point x_0 determine optimal solution $x_t^\star$ to uncosntrained problem:

$$\min_x f_0(x) + t\sum \phi(-f_i(x))$$

Using e.g. gradient descent.

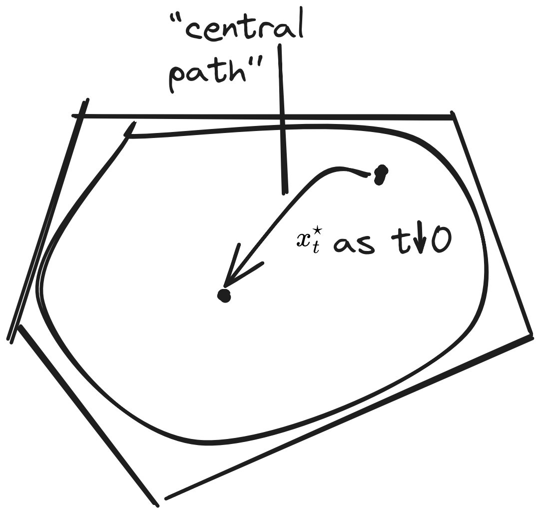
Question: How good is $x_t^\star$ compared to $x^\star$, the optimal solution to (P)
Optimal Solution $x_t^\star$ to unconstrained convex optimisation problem characterised by

$$\nabla_x \left(f_0(x) + t\sum_{i=1}^m \phi(-f_i(x)) \right)\Big|_{x_t^\star} = 0$$
$$\nabla f_0(x_t^\star) - \sum_{i=1}^m t \underbrace{\frac{1}{-f_i(x_t^\star)}}_{>0} \nabla f_i(x_t^\star) = 0$$

First notice $x_t^\star$ is strictly feasible as if it wasn't the objective would be $+\infty \implies -f_i(x_t^\star) > 0$ This looks like KKT conditions which above is stationarity and the latter is feasibility.
Define $\lambda_i(t) = \frac{t}{-f_i(x_t^\star)} > 0$ which is feasible for lagrange dual (D) to (P)

Note: $x_t^\star$ minimises $\mathcal{L}(x, \lambda(t))$, where $\mathcal{L}$ is lagrangian for (P) because
$\nabla_x \mathcal{L}(x, \lambda(t))|_{x_t^\star} = 0$
$\implies p^\star \geq d^\star \geq \inf_x \mathcal{L}(x, \lambda(t)) = \mathcal{L}(x_t^\star, \lambda(t))$
$= f_0(x_t^\star) + \sum_{i=1}^m \lambda_i(t) f_i(x_t^\star) = f_0(x_t^\star) - mt$
$\implies x_t^\star$ is feasible for $(P) \implies f_0(x_t^\star) \leq p^\star + tm$
In this case tm represents the duality gap and therefore we can track the sub-optimality with each iteration as we shrink t.

Example of "interior point method"



"central path"
$x_t^\star$ as $t \downarrow 0$