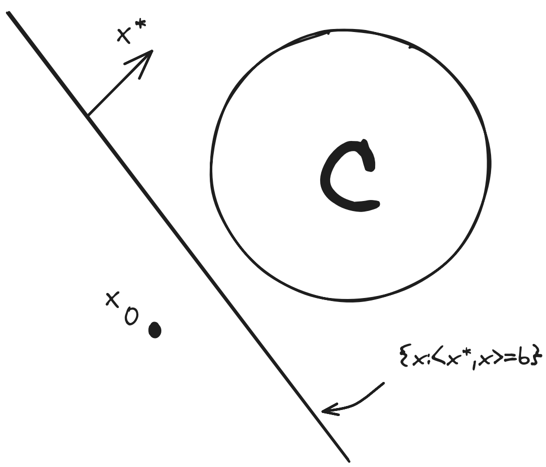
Lecture 19 More Duality

Note by Samion Suwito on 1/4/25

Convexity and Duality

Base Result: If $C \subset X$ is a closed convex set,
$x_0 \notin C \implies \exists x^* \in X^* \text{ s.t.} \langle x^*, x\rangle < \langle x^*, x_0\rangle \forall x \in C$



Separation Theorem
A closed set means the set contains a boundary, for example [-5,0) is a convex set but not a closed for the right side is not a distinct. So it's a set + its accumulation points.

Consequence of base result: $C =$ closed convex set $\iff$ $C =$ intersection of (closed) half spaces.

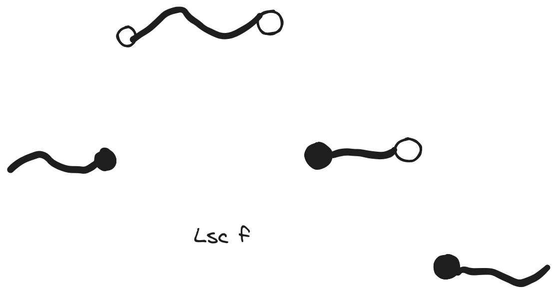
$$C = \bigcap_{x^* \in X^*} \{x : \langle x^*, x\rangle = h_c(x^*)\}$$

where $h_c(x^*) = \max_{x\in C}\langle x^*, x\rangle$

Connections to Convex Functions

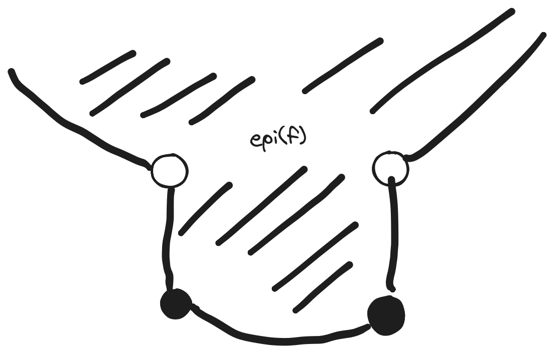
A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is **lower semicontinuous** (lsc) if $\forall$ convergent $(x_n)_{n\geq 1} \subset X$

$$\lim_{n\to\infty}\inf \geq f(\lim_n x_n)$$



Every jump is continuous on the lower.

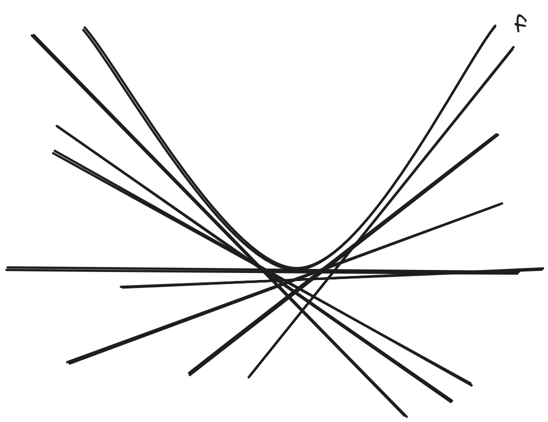
Claim: A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is lsc $\iff$ $\text{epi}(f)$ is closed



Notice if the higher point in the jump was solid then the epigraph would have a dotted line and not be closed therefore.

Theorem: If $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex, lsc, then

$$f(x) = \sup_{\text{affine } a \leq f} \{a(x)\} \quad x \in X = \mathbb{R}^n$$



(a bunch of affine functions under a convex function)

Moral: “every” convex function looks max affine

Duality in Optimisation

Convex Conjugate

For $f : X \to \mathbb{R} \cup \{+\infty\} (\text{dom } f \neq 0)$, define $f^* : X^* t_9 \mathbb{R} \cup \{+\infty\}$ by
$f^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$
We consider this to be the "convex conjugate of f" or the "Legandre-Fenchel Transform"
$\sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$ is always convex and lsc

Economic Interpretation

$x = (x_1, \ldots, x_n)$, $x_i =$ quantity of good i produced.
$f(x)$ is cost to produce items $x_1, \ldots x_n$
$x_i^* =$ price of item i
Then $\langle x^*, x \rangle - f(x)$ is revenue - cost
So then by taking the sup as in $\sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}$ we get the maximum revenue

Examples

Example: For set $K \subset X$, define indicator

$$I_K(x) = \begin{cases} 0 \text{ if } x \in K \\ +\infty \text{ if } x \notin K \end{cases}$$

The complex conjugate

$$\begin{aligned} I_K^*(x^*) &= \sup_{x \in X} \{\langle x^*, x \rangle - I_K(x)\} \\ &= \sup_{x \in K} \langle x^*, x \rangle = h_K(x^*) \end{aligned}$$

Where h is the support function

Example: If $a : X \to \mathbb{R}$ is affine. $a(x) = \langle x_a^*, x \rangle + b$ for some $x_a^* \in X^*$, $b \in \mathbb{R}$

$$\begin{aligned} a^*(x^*) &= \sup_{x \in X} \{\langle x^*, x \rangle - \langle x_a^*, x \rangle - b\} \\ &= \sup_{x \in X} \{\langle x^* - x_a^*, x \rangle - b\} \\ &= \begin{cases} +\infty \text{ if } x^* \neq x_a^* \\ -b \text{ if } x^* = x_a^* \end{cases} \end{aligned}$$

Properties

f, f^* are defined on different spaces, so it doesn't make sense to compare them directly in general. Nevertheless f, f^* satisfy following **Fenchel's Inequality**:

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*) \quad \forall x \in X, x^* \in X^*$$

This comes from the definition as
$f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\} \geq \langle x^*, x\rangle - f(x)$

Property 2:
Conjugation is order-reversing:

$$\underbrace{f \leq g}_{f(x)\leq g(x)\ \forall x\in X} \quad \implies g^* \leq f^*$$

$f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\} \geq \sup_{x\in X}\{\langle x^*, x\rangle - g(x)\} = g^*(x)$
Property 3
To get back to something comparable to f, take conjugate of f^*:

$$f^{**}(x) = \sup_{x^*\in X^*}\{\langle x^*, x\rangle - f^*(x^*)\}$$

This is called the "biconjugate" order preservation as you order reverse twice:
$f \leq g \implies f^{**} \leq g^{**}$
Example: $a(x) = \langle x_a^*, x\rangle + b$

$$a^*(x^*) = \begin{cases} +\infty \text{ if } x^* \neq x_a^* \\ -b \text{ if } x^* = x_a^* \end{cases}$$

$a^{**}(x) = \sup_{x^*\in X^*}\{\langle x^*, x\rangle - a^*(x^*)\} = \langle x_a^*, x\rangle + b = a(x)$

Weak Duality

$f^{**} \leq f$
Proof

$$f^{**}(x) = \sup_{x^*\in X^*}\{\underbrace{\langle x^*, x\rangle - f^*(x^*)}_{\leq f(x) \text{ Fenchel's inequality}}\} \leq f(x)$$

Strong Duality

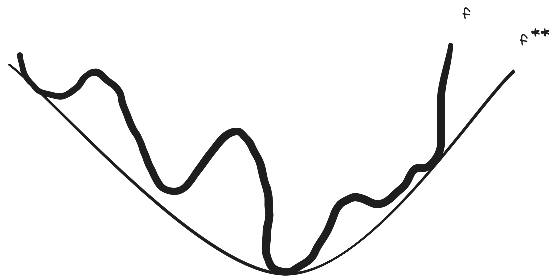
Theorem (Fenchel-Moreau) Let $f : X \to \mathbb{R} \cup (+\infty)$ (no convexity assumption)
$f = f^{**} \iff f$ is convex, lsc
Proof: Already know $f^{**} \leq f$ by weak duality

If $a \leq f, a$ affine $\implies a = a^{**} \leq f^{**}$ (order preservation)

$$f(x) = \sup_{\text{affine } a \leq f} \{ \underbrace{a(x)}_{= a^{**} \leq f^{**}} \} \leq f^{**}(x)$$

Only have to prove this direction since f^{**} is convex and lsc

Consequence: f^{**} is the pointwise-greatest convex lsc, function that lies below f



While f is not convex we f^{**} shares the global minima making it sound like all not convex functions are easy but f^{**} may be hard to compute or not share the same feasible set

If g is convex, lsc, $g \leq f \implies g = g^{**} \leq f^{**}$

Take the epigraph of f and then the closure of its convex hull

Primal and Dual optimisation Problems

Consider objective $f : X \to \mathbb{R} \cup \{+\infty\}$ and the optimisation problem $\inf_{x \in X} f(x)$ this looks unconstrained but we could take $f(x) = f_0(x) + I_K(x)$ where $K =$ feasible set. This is the primal problem and the dual will be covered next time.

Good practice: write $p^\star$ not p^* for optimal value