Lecture 21 Lagrange Duality

Note by Samion Suwito on 4/8/25

Last Time

General Weak Duality

$F(x, y)$ s.t. $F(x, 0) = f(x)$

$$p^\star = \inf_{x \in X} f(x) \geq \sup_{y^* \in Y^*} \{-F^*(0, y^*)\} = d^\star$$

Theorem: If F is convex and $0 \in \text{ri}(\text{dom } v)$ (relative interior), where $v(y) = \inf_{x \in X} F(x, y)$. Then strong duality holds ($p^\star = d^\star$). Moreover, dual optimum attained by some $y^* \in Y^*$
(whether small perturbations maintain feasibility.)

Optimal problem of the dual problem always provides lower bound on original problem

Lagrange Duality

Lagrange Duality: A well developed, useful instance of duality
Primal Problem (no convexity assumption):

$$\begin{aligned} \min_{x \in X = \mathbb{R}^n} \; & f_0(x) \\ \text{s.t. } & f_i(x) \leq 0, i = 1 \dots m \end{aligned}$$

Define the **Lagrangian**:

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \quad x \in X, \lambda \in \mathbb{R}^m$$

The Lagrange *Dual Problem* is:

$$d^\star = \sup_{\lambda \geq 0} \underbrace{\left(\inf_{x \in X} \mathcal{L}(x, \lambda)\right)}_{(1)} = \text{max of concave fn in } \lambda$$

$\lambda =$ "dual variable"

(1) Unconstrained problem in x = infimum of affine (in λ) functions = concave function in λ. ($\lambda \mapsto \inf_x \mathcal{L}(x, \lambda)$ is concave)

Weak duality holds:

$$\begin{aligned} p^\star \inf_x \{f_0(x) : f_i(\lambda) \geq 0\} &\geq \inf_x \left\{ f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) : f_i(x) \leq 0 \right\} \\ &\geq \inf_{x \in X} \mathcal{L}(x, \lambda) \\ &\implies p^\star \geq \sup_{\lambda \geq 0} \inf_{x \in X} \mathcal{L}(x, \lambda) = d^\star \end{aligned}$$

Relating Lagrange Duality to the general duality

Define $G(x) : X = \mathbb{R}^n \to Y = \mathbb{R}^m$,

$$G(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

$K = \{y \in Y : y \leq 0\}$.

Introduce perturbation function

$F(x, y) := f_0(x) + I_K(G(x) + y)$

$v(y) = \min_x f_0(x)$ s.t. $f_i(x) + y_i \leq 0 \; i = 1 \ldots m$. Same as the primal problem if $v(0)$ so $v(0) = p^\star$

Evaluate:

$$
\begin{aligned}
F^*(0,y^*) &= \sup_{x,y}\{\langle y^*, y\rangle - f_0(x) - I_K(G(x)+y)\}\\
&= \sup_{x,y}\{\langle y^*, y\rangle + \langle y^*, G(x)\rangle \underbrace{-\langle y^*, G(x)\rangle - f_0(x)}_{\mathcal{L}(x,y^*)} - I_K(G(x)+y)\}\\
&= \sup_{x,y}\{\langle y^*, G(x)+y\rangle - \mathcal{L}(x,y^*) - I_k(G(x)+y)\}\\
&= \sup_{x,y}\{\langle y^*, G(x)+y\rangle - I_k(G(x)+y) - \mathcal{L}(x,y^*)\}\\
&= \sup_{x}\sup_{y}\{\underbrace{\langle y^*, G(x)+y\rangle - I_k(G(x)+y)}_{I_K^*(y^*)} - \mathcal{L}(x,y^*)\}\\
&= I_K^*(y^*) - \inf_{x\in X}\mathcal{L}(x,y^*)
\end{aligned}
$$

Continue to compute

$$
F^*(0,y^*) = I_K^*(y^*) - \inf_{x\in X}\mathcal{L}(x,y^*)
$$

$$
\begin{aligned}
I_K^*(y^*) &= \sup_{y\in Y}\{\langle y^*, y\rangle - I_K(y)\}\\
&= \sup_{y:y\leq 0}\langle y^*, y\rangle = \begin{cases} 0 \text{ if } y^*\geq 0\\ +\infty \text{ otherwise}\end{cases}\\
d^\star &= \sup_{y^*\in Y^*}\{-F^*(0,y^*)\}\\
&= \sup_{y^*\in Y^*}\{\inf_{x\in X}\mathcal{L}(x,y^*) - I_K^*(y^*)\}\\
&= \sup_{y^*\geq 0}\{\inf_{x\in X}\mathcal{L}(x,y^*)\}
\end{aligned}
$$

Now that we see Lagrange Duality special case, we have sufficient conditions for strong duality to hold.
Namely:

- If $f_i : X \to \mathbb{R}\cup\{+\infty\}$ is convex for each i then F is convex
- If the Primal Problem (P) remains feasible for small enough perturbation y
  Then strong duality holds and dual optimum is achieved.

Slater's Condition: Let $D = \bigcap_{i=0}^m \operatorname{dom} f_i$. If f_i is convex for each $i = 0\dots m$ and $\exists x_0 \in D$ s.t. $f_i(x_0) < 0, i = 1\dots m$ then a Strong (Lagrange) Duality holds, and dual optimum is achieved.

If there is a point that is strictly feasible then small perturbations will make it such that $f_i(x_0 + y) \leq 0$ is still feasible.

When one exists, the optimal dual variable $\lambda^\star$ is called a **Lagrange Multiplier**.
Lagrange Multipliers characterise sensitivity of solutions. Consider:

$$\begin{aligned} &\min_x \; f_0(x) \\ &\text{s.t. } f_i(x) \leq -\delta_i \quad i = 1 \dots m \end{aligned}$$

"Tightening the constraint by δ_i"

Theorem: If strong duality holds, for unperturbed problem, and dual optimum achieved by $\lambda^\star \geq 0$ then: $f_0(x) \geq p^\star + \langle \delta, \lambda^\star \rangle \;\; \forall x$ satisfying the constraints above.

Proof (An opportunity to see duality in action):
For x satisfying the above constraints, weak duality holds

$$\begin{aligned} f_0(x) &\geq \sup_{\lambda \geq 0} \inf_{x \in X} \left\{ f_0(x) + \sum \lambda_i (f_i(x) + \delta_i) \right\} \\ &\geq \inf_{x \in X} \left\{ f_0(x) + \sum \lambda^\star f_i(x) \right\} + \langle \delta, \lambda^\star \rangle \\ &= \inf_{x \in X} \underbrace{\mathcal{L}(x, \lambda^\star)}_{d^\star = p^\star} + \langle \delta, \lambda^\star \rangle \end{aligned}$$

Could you have two lagrangian with different functions