Lecture 20 Examples of Duality

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Last Time

Tools for working with Convexity

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - f(x)\}$$
$$f^{**}(x) = \sup_{x^* \in X^*} \{\langle x^*, x\rangle - f^*(x^*)\}$$

Weak duality: $f^{**} \leq f$, f^{**} is greatest convex lsc lower bound on f, “convex relaxation”

Primal-Dual Optimisation Problems

Primal Problem

Consider function $f : X \to \mathbb{R} \cup \{+\infty\}$, and optimisation (primal) problem $p^* = \inf_{x\in X} f(x)$. This is why we call it p^* since it’s the optimal primal value (Includes constraints e.g. $f(x) = f_0(x) + I_k(x)$ for $K = \{x : f_i(x) \leq 0, i = 1 \dots m\}$)

$$I_K(x) = \begin{cases} 0 \text{ if } x \in K \\ +\infty \text{ if } x \notin K \end{cases}$$

$\implies$

$$\begin{aligned} p^* = \min_x \; & f_0(x) \\ \text{s.t. } & f_i(x) \leq 0 \quad i = 1 \dots m \end{aligned}$$

Dual problem

For another finite dimensional vector space $Y = \mathbb{R}^m$, define a “**perturbation function**” $F : X \times Y \to \mathbb{R} \cup \{+\infty\}$ s.t. $F(x, 0) = F(x)$.

Example: $f(x) = f_0(x) + I_K(x)$,$K = \{x : f_i(x) \leq 0, i = 1 \dots m\}$

$$F(x, y) = f_0(x) + I_{K(y)}(x), K(y) = \{x : f_i(x) \leq y_i, i = 1 \dots m\}$$

We can perturb the constraints of the original problem essentially by tightening or loosening with y

Define **value function** $v(y) = \inf_{x \in X} F(x, y)$
At $v(0) = p^*$
Observe:

$$
\begin{aligned}
v^*(y^*) &= \sup_{y \in Y} \{\langle y^*, y\rangle - \underbrace{v(y)}_{\inf_x F(x,y)} \} \\
&= \sup_{(x,y) \in X \times Y} \{\langle y^*, y\rangle - F(x, y)\} \\
&= F^*(0, y^*)
\end{aligned}
$$

The biconjugate is typically more useful to compare with:

$$
\begin{aligned}
v^{**}(y) &= \sup_{y^* \in Y^*} \{\langle y^*, y\rangle - v^*(y^*)\} \\
&= \sup_{y^* \in Y^*} \{\langle y^*, y\rangle - F^*(0, y^*)\}
\end{aligned}
$$

We can evaluate the biconjugate at 0 to remove the first term and by weak duality

$$
\implies p^* = v(0) \geq v^{**}(0) = \sup_{y^* \in Y^*} \{-F^*(0, y^*)\}
$$

Perhaps in principle p^* may be difficult to solve however maximising the concave function on the right gives a nice lower bound that can be computed like a convex problem. The $\sup$ on the right is called the dual problem and we call the optimal value d^*

$$
\sup_{y^* \in Y^*} \{-F^*(0, y^*)\} = d^*
$$

Primal-Dual Relationship

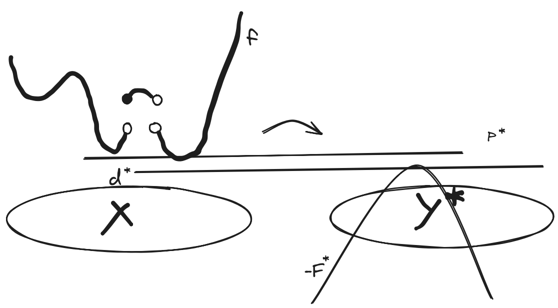
By *Weak duality*: $p^* \geq d^*$ will always have this relationship
Primal Problem: $p^* = \inf_{x \in X} f(x)$
Dual Problem: $d^* = \sup_{y^* \in Y^*} \{-F^*(0, y^*)\}$
If x_0, y_0^* satisfy $f(x_0) \leq -F^*(0, y_0^*) \implies x_0$ is primal optimal and y_0^* is dual optimal, if this inequality is true we have a "**Certificate of Optimality**"

If we instead have $f(x_0) \leq -F^*(0, y_0^*) + \epsilon$ then we have **epsilon optimal** where we know p^* is as most as far as ϵ.

Duality gap: $f(x) - (-F^*(0, y^*))$ (always ≥ 0 by weak duality)
We can only drive the duality gap to 0 when we have strong duality.

Question: When does "strong duality" ($p^* = d^*$) hold? (meaning we can solve for d^* instead)
Theorem: If $f : X \to \mathbb{R} \cup \{+\infty\}$ has $\operatorname{dom} f \neq 0$, F is convex lsc, and $0 \in \operatorname{int}(\operatorname{dom} v)$, then strong duality holds, and dual optimality is achieved.

Conditions which ensure strong duality are generically referred to as "constraint qualifications" normally involve: convexity & some modest technical assumption (regularity)



On the left is a non convex primal problem with a optimal solution of p^* and on the right is a dual problem with optimal solution of d^*

Max min inequality

$$\max_y \min_x \Phi(x, y) \leq \min_x \max_y \Phi(x, y)$$

Example: Sion's minimax theorem: If $\Phi(x, y)$ is convex lsc on $x \in X$ (if you fix y and vary x), concave usc in $y \in Y$ (if you fix x and vary y) where X and Y are convex bounded then:

$$\underbrace{\min_x \max_y \Phi(x, y)}_{LHS} = \underbrace{\max_y \min_x \Phi(x, y)}_{RHS}$$

Most important result in game theory. Consider x and y to be two players and $\Phi(x, y)$ is the payoff when both plays an action. If we find these properties we find a saddle point, a Nash equilibrium where neither players need to diverge from their strategies.

Proof: define $F(x, y) = \sup_{y^* \in Y^*} \{\Phi(x, y^*) + \langle y^*, y \rangle\}$

$$LHS = v_0 = v^{**}(0) = \sup_{y^* \in Y^*} \{-F^*(0, y^*)\} = RHS$$

Example: Von Neumann's minimal theorem: $A \in \mathbb{R}^{m \times n}$. Let
$Y = \{y \in \mathbb{R}^m : y \geq 0, \sum y_i = 1\}$, $X = \{x \in \mathbb{R}^n : x \geq 0, \sum x_i = 1\}$

$$\min_{x \in X} \max_{y \in Y} y^\top Ax = \max_{y \in Y} \min_{x \in X} y^\top Ax$$

The i, jth entry of A would correspond to the ith action of y and jth action of x

Rock paper scissors example:

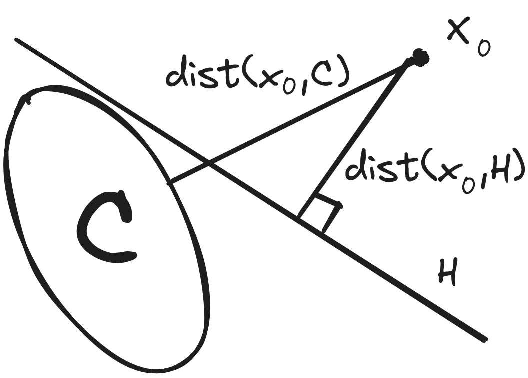
$$A = \begin{array}{c|ccc} - & R & P & S \\ \hline R & \\ P & \\ S & \end{array} \hspace{-4.2em} \begin{array}{c} \\ \left[\begin{array}{ccc} 0 & +1 & -1 \\ -1 & 0 & +1 \\ +1 & -1 & 0 \end{array}\right] \end{array}$$

The optimal strategy would be therefore $\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]^\top$.

C closed convex, $x_0 \notin C$

$$\min_{x \in C} \|x - x_0\|_2 = \max \operatorname{dist}(x, H)$$

Where H is a hyper plane separating C from x_0



$$F(x, y) = \|x - x_0\|_2 + I_C(x + y)$$

If you plug in the primal and dual problem with this perturbation function you will get the original statement.

Lagrange Duality

Without any convexity assumption, consider the following primal problem:

$$
\begin{aligned}
&\min_{x \in X} f_0(x) \\
&\text{s.t. } f_i(x) = 0 \quad i = 1 \dots m
\end{aligned}
$$

Define the Lagrangian with $x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m$:

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$$

The Lagrangian dual problem is:

$$d^* = \sup_{\lambda \geq 0} \inf_{x \in X} \mathcal{L}(x, \lambda)$$

Which is a min max problem. Continue Lagrange duality next time.