Lecture 25 Algorithms 2

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Last Time

Constrained Problem:

$$\begin{aligned} &\min_x f_0(x) \\ &\text{s.t. } f_i(x) \le 0 \quad i = 1 \dots m \end{aligned} \tag{P}$$

Assume: Strictly feasible convex optimisation problem (i.e. Slater)

Phase I-II approach:

Phase I: Find strictly feasible x_0
Phase II: Solve (P) starting from x_0

Interior Point Method:

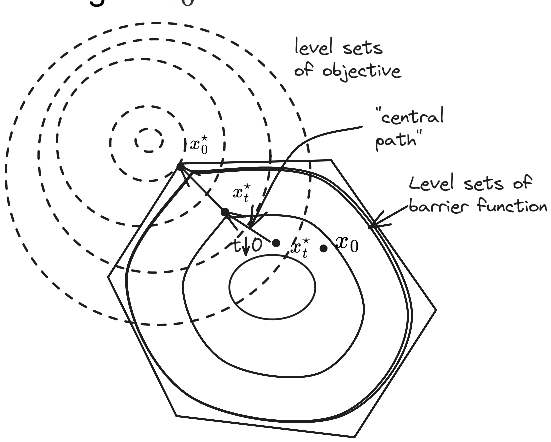
For $t > 0$,

$$\phi(t) = \begin{cases} \log \frac{1}{t} & t > 0 \\ +\infty & t \le 0 \end{cases}$$

Solve:

$$\min_x f_0(x) + t \sum \phi(-f_i(x))$$

starting at x_0. This is an unconstrained problem.



Solution $x_t^\star$ satsfies, for $\lambda_i(t) = \frac{t}{-f_i(x_t^\star)}$
The interior point method is essentialy tryying to satisfy KKT Conditions

- $x_t^\star$ primal feasible and $\lambda_i(t)$ dual feasible (as $-f_i(x_t^\star) > 0$)
- $\nabla f_0(x_t^\star) + \sum_{i=1}^m \lambda_i(t) \nabla f_i(x_t^\star) = 0$
- $f_i(x_t^\star)\lambda_i(t) = -t \quad i = 1 \dots m$

Also: $f_0(x_t^\star) \leq p^\star + mt$ sub-optimality is directly proportional to t and the number of constraints.

General Approach to solving (P) within ε-sub-optimality

1. Find strictly feasible point x_0
2. Interior Point Method: For $t = t_0$, solve unconstrained problem $\min_x f_0(x) + t\sum \phi(-f_i(x))$ starting from x_0
3. $x_0 \leftarrow x_t^\star, t \leftarrow \alpha t$, where e.g. $\alpha = \frac{1}{10}$ is a parameter repeat step 2.
   Quit when $tm < \epsilon$
   SUMT - Sequential Unconstrained Minimisation Technique

How to execute Phase I?

Consider:

$$\min_{s \in \mathbb{R}, x \in \mathbb{R}^n} s$$
$$\text{s.t. } f_i(x) \leq s \quad i = 1 \dots m$$

Easy to find a strictly feasible point, given $x_0 \in \bigcap_{i=0}^m \operatorname{dom} f_i$
Just consider point (x_0, s_0), where $s_0 = 1 + \max_{i=1\dots m} f_i(x_0)$.

If we optimise the problem above (using SUMT?), we can get $s^\star < 0$ because of the given strict feasibility of (P) that we assumed and therefore we can get a strictly feasible point.

Solution: $(\tilde{x}^\star, s^\star)$ to this problem has property that $\tilde{x}^\star$ is strictly feasible for (P).

Essentially Phase I and Phase II both involve optimising a constrained optimisation problem making it of similar difficulty but Phase I's optimisation problem is easier to solve.

Remark: Generally much easer to find a point $x_0 \in \bigcap_{i=0}^m \operatorname{dom} f_i$, (a defined point) then it is to find a strictly feasible point.

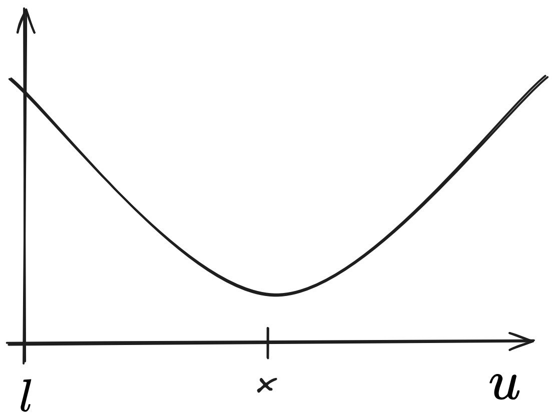
- *Ex.* LP constraints: $Ax \leq b$, $x_0 = 0$
- *Ex.* SOCP Constraints $||A_i x - b_i|| + C_i x + d_i = f_i(x)$ then $x_0 = 0$ works

Unconstrained 1D Minimisation

We've reduced constrained minimisation problems to unconstrained minimisation problems

We still need to understand how to solve a given unconstrained convex minimisation problem. Start with functions of a single variable:

Let $f : \mathbb{R} \to \mathbb{R} + \cup \{+\infty\}$ be a convex differentiable function:
Assume we have an interval $[l, u]$ that contains an optimal point. (i.e.
$f'(l) \leq 0, f'(u) \geq 0$ looks like a bowl)



Bisection to find $x^\star$:

1. Set $x = \frac{1}{2}(l + u)$
2. Check direction $f'(x)$,
    1. If $f'(x) > 0, u \leftarrow x$
    2. If $f'(x) < 0, l \leftarrow x$
    3. If $f'(x) = 0, x^\star \leftarrow x$
3. Repeat until $|f'(x)||u - l| \leq \epsilon$

By first-order characterisation of convexity:
$f(x^\star) \geq f(x) + f'(x)(x^\star - x) \geq f(x) - \epsilon$
Since $(x^\star - x) = |u - l|$ at its maximum
$\implies f(x) \leq f(x^\star) + \epsilon$

Remark: Some optimisation problems can be reduced to 1-dimensional problems (e.g. by duality)

Example of Remark

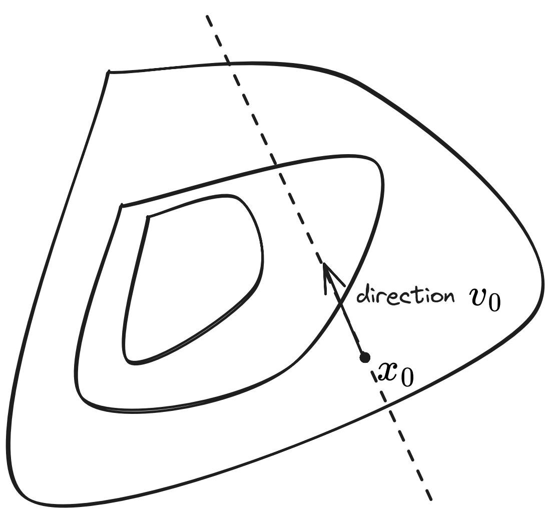
$$\min_x \sum_{i=1}^n f_i(x)$$
$$\text{s.t. } a^\top x = b$$

Assume f_i's convex and Slater's condition holds therefore strong duality holds.

$$\begin{aligned} p^\star &= \max_{\mu \in \mathbb{R}} g(\mu) \\ g(\mu) &= \inf_x \sum f_i(x_i) - \mu(a^\top x - b) \\ &= \mu b - \sum_{i=1}^n \max_{x_i}(\mu a_i x_i - f_i(x_i)) \\ &= \mu b - \sum_{i=1}^n f_i^*(\mu a_i) \end{aligned}$$

We can distribute the computing of the f_i^* (convex conjugates).

In general, 1D optimisation is a useful tool/subroutine for unconstrained convex optimisation.



In the picture above e only search in the one direction leading to:

$$\min_{t \in \mathbb{R}} f(x_0 + tv_0)$$

Similar to gradient descent in gradient descent but v_0 points towards negative gradient.