Lecture 14 Review

Note by Samion Suwito on 3/6/25

Last Time: Robust Optimisation

Robust Optimisation = Optimisation where cost/constraints are uncertain. (typically for floating point precision etc.)
Robust LP:

$$\begin{aligned} &\min_x c^\top x \\ &\text{s.t. } a_i^\top x \le b_i \;\; \forall a_i \in \mathcal{U}_i, i = 1 \dots m \end{aligned}$$

$\iff$

$$\begin{aligned} &\min_x c^\top x \\ &\text{s.t. } \phi_{\mathcal{U}_i}(x) \le b_i \; i = 1 \dots m \end{aligned}$$

Where **support function** $\phi_{\mathcal{U}}(x) := \max_{a \in \mathcal{U}} a^\top x$
We change it as we don't know what a_i is exactly but we are able to find the support function (ex. LP balls).

Typical Examples of $\mathcal{U}$

ℓ^p balls

$\mathcal{U} = \{a : \|a - a_0\| \le t\}$

$$\phi_{\mathcal{U}}(x) = \max_{a \in \mathcal{U}} a^\top x$$

$$a = a_0 + tu \text{ for} \|u\| \le 1$$

$$a_0^\top x + t \max_{u : \|u\| \le 1} u^\top x = a_0^\top x + t\|x\|^*$$

Example: If $\|\cdot\| = \|\cdot\|_p$, then $\|\cdot\|^* = \|\cdot\|_q$ where $\frac{1}{p} + \frac{1}{q} = 1, p \ge 1$

Examples

ℓ^∞ example

$$\begin{aligned}\min\ & c^\top x \\ \text{s.t. } & a_i^\top x \le b_i \ \ \forall a_i : \|a_i - \hat{a}_i\|_\infty \le \epsilon\end{aligned}$$

The maximum machine quantisation error is at most ϵ due to finite precision. We can reformulate as:

$$\begin{aligned} & \min\ c^\top x \\ & \text{s.t. } \hat{a}_i^\top x + \epsilon\|x\|_1 \le b_i \\ \iff & \min_{x,u}\ c^\top x \\ & \text{s.t. } a_i^\top x + \epsilon \sum u_i \le b_i \\ & \quad x_i \le u_i \\ & \quad -x \le u_i \end{aligned}$$

ℓ^1 example

$$\begin{aligned} & \min\ c^\top x \\ & \text{s.t. } a_i^\top x \le b_i \ \ \forall a_i : \|a_i - \hat{a}_i\|_1 \le \epsilon \\ \iff & \min\ c^\top x \\ & \text{s.t. } \hat{a}_i^\top x + \epsilon\|x\|_\infty \le b_i \\ \iff & \min\ c^\top x \\ & \text{s.t. } \hat{a}_i^\top x + \epsilon u \le b_i \\ & \quad -x_i \le u \\ & \quad x_i \le u \end{aligned}$$

ℓ^2 example

$$\begin{aligned} & \min\ c^\top x \\ & \text{s.t. } a_i^\top x \le b_i \ \ \forall a_i : \|a_i - \hat{a}_i\|_2 \le \epsilon \\ \iff & \min\ c^\top x \\ & \text{s.t. } \hat{a}_i^\top x + \epsilon\|x\|_2 \le b_i \end{aligned}$$

Now an SOCP Problem

Ellipsoid Uncertainty

$$
\begin{aligned}
\mathcal{U} &= \{a : (a-a_0)^\top P^{-1}(a-a_0) \leq 1\} \\
&= \{a_0 + P^{1/2}u : \|u\|_2 \leq 1\}
\end{aligned}
$$

Makes an ellipsoid where the eigenvectors of P are the axis and the lengths are the eigenvalues
Elilipsoid uncertainty, $P \geq 0$ (PSD)

$$
\begin{aligned}
\phi_{\mathcal{U}}(x) = \max_{a \in U} a^\top x &= \max_{u:\|u\|_2 \leq 1} a_0^\top x + (P^{1/2}u)^\top x \\
&= a_0^\top x + \max_{u:\|u\|_2 \leq 1} u^\top (P^{1/2}x) \\
&= a_0^\top x + \|P^{1/2}x\|_2
\end{aligned}
$$

Reparameterisation of $\mathcal{U}$

$$
\begin{aligned}
\mathcal{U} &= \{a : (a-a_0)^\top P^{-1}(a-a_0) \leq 1\} \\
&= \{a_0 + v : v^\top P^{-1} v \leq 1\} \\
&= \{a_0 + P^{1/2}u : u^\top u \leq 1\} \\
&= \{a_0 + P^{1/2}u : \|u\|_2 \leq 1\}
\end{aligned}
$$

Robust LS

$$
\min_x \|Ax - b\|_2
$$

Let’s say we don’t know A exactly instead we have the robust version:

$$
\min_x \max_{\Delta:\|\Delta\|_2 \leq \rho} \|(A+\Delta)x - b\|_2
$$

We can consider the first line to be the ideal model while the one above is a model that better captures reality.

We want to compute maximum over Δ:

$$
\begin{aligned}
&\max_{\Delta:\|\Delta\|_2 \leq \rho} \|(A+\Delta)x - b\|_2 \\
&\qquad \leq \|Ax - b\|_2 + \max_{\Delta:\|\Delta\|_2 \leq \rho} \|\Delta x\|_2 \\
&\qquad \leq \|Ax - b\|_2 + \max_{\Delta:\|\Delta\|_2 \leq \rho} \|\Delta\|_2 \|x\| \\
&\qquad = \|Ax - b\|_2 + \rho \|x\|_2
\end{aligned}
$$

Then the Δ that gives equality throughout:

$$\Delta = \frac{\rho}{\|Ax-b\|_2\|x\|_2}(Ax-b)x^\top$$

Returning to the original problem:

$$\begin{aligned}&\min_x \max_{\Delta:\|\Delta\|_2\le\rho} \|(A+\Delta)x-b\|_2 \\ &= \min \|Ax-b\|_2+\rho\|x\|_2\end{aligned}$$

Which we recognise as an SOCP (not LS due to missing square)

$$\begin{aligned}&\min_{x,t,u} t+\rho u \\ &\text{s.t. } \|Ax-b\|_2\le t \\ &\quad\;\; \|x\|_2\le u\end{aligned}$$

Review for Midterm

We've seen several structured classes of optimisation problems:

$$LP \subset QP \subset SOCP$$

Where LS is in QP.

LP

Standard Form:

$$\begin{aligned}&\min_x c^\top x \\ &\text{s.t. } Ax\le b\end{aligned}$$

Where the objective function is a **linear cost** and the constraints are **affine functions**
LP is the class to be most familiar with. Make sure to format solution with linear cost and affine functions and if stated to put in standard form, write in as above
Examples:

- Resource/budget allocation
- Network flows
- ℓ_1/ℓ_∞ regressions, introduce slack variables to turn into a linear program.
- Linear cost with ℓ_1, ℓ_∞ regularisation (encouraging sparse solutions etc.)

QP

Standard Form:

$$\min_x \frac{1}{2}x^\top Hx + c^\top x \quad H \geq 0$$
$$\text{s.t. } Ax \leq b$$

Where there is a **convex quadratic objective** and affine constraints
Examples:

- Problems where we want to minimise energy/variance with constraints (e.g. subject to a controller).
- Risk and volatility in a portfolio
- Index tracking/portfolio allocation.
- LASSO ℓ^1-regularised least squares (e.g. piecewise constant fitting like wavelet compression)
- LS w/ ℓ^∞ regularisation

SOCP

Standard Form:

$$\min_x c^\top x$$
$$\text{s.t. } \|A_i x - b_i\|_2 \leq c_i^\top x + d_i \;\; i = 1 \ldots m$$

Where there is a linear objective and SOCP constraints. Remember no square on the ℓ^2 norm. c_i not related to c.

SOCP constraints are *very flexible*, they include:

- Any convex quadratic constraint (no need to convert the QP constraint to SOCP, too complicated)
- Hyperbolic constraints, handles $\frac{1}{x_i}$,
- **Lecture 13 Robust Optimisation**

Examples

- Problems naturally involving distance (in the ℓ^2 sense):
- Robust Optimisation w/ ℓ^2 uncertainty (idk about otherℓ)

- ℓ^1 squared regularisation term

SDP

Purely for your own information

$$LP \subset QP \subset SOCP \subset SDP$$

SDP = Semidefinite Programs:

$$\begin{aligned} \min_x \ & \mathrm{Tr}(C^\top X) = \langle C, X \rangle \\ \text{s.t. } & \mathrm{Tr}(A_i X) \leq b_i i = 1 \ldots m \\ & X \geq 0() \end{aligned}$$