Lecture 13 Robust Optimisation

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Last Time

SOCP

Standard form

$$
\begin{aligned}
&\min_x c^\top x \\
&\text{s.t. } \|Ax_i - b_i\|_2 \le c_i^\top x + d_i \;\; i = 1 \ldots m
\end{aligned}
$$

The first line is the linear objective and the second are the SOCP constraints

Examples of SOCP Constraints

1. Affine constraints: $Ax \le b \implies LP \subset SOCP$
2. Quadratic constraints: $x^\top Qx + c^\top x \le t, Q \ge 0 \implies QP \subset SOCP$
   Rewrite the constraints as

$$
\left\| \begin{bmatrix} \sqrt{2}Q^{1/2} \\ -C^\top \end{bmatrix} x + \begin{bmatrix} 0 \\ t - \frac{1}{2} \end{bmatrix} \right\| \le t - c^\top x + \frac{1}{2}
$$

   Example:

$$
\begin{aligned}
&\min_x x^\top Hx + c^\top x, H \ge 0 \\
&\text{s.t. } Ax \le b
\end{aligned}
$$

$$\implies$$

$$
\begin{aligned}
&\min_{t,x} t + c^\top x \\
&\text{s.t. } Ax \le b \\
&\qquad x^\top Hx \le t
\end{aligned}
$$

   Where the last constraint is equivalent to:

$$\begin{pmatrix} x \\ t \end{pmatrix}^\top \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix} - \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \leq 0$$

Which we can transform back to the constraints written at the top.

3. Hyperbolic constraints below
4. $$|w|^p \leq t, p \geq 1, \in \mathbb{Q}$$

   Can be cast as SOCP

QCQP ("Quadratically constrained" QP) $\subset SOCP$

Typical Examples of SOCP

Example 1

$$\min_x \sum_{i=1}^k \|A_i x - b_i\|_2$$

Important Note No "2" in the exponent.
Why is it an SOCP:

$$\begin{aligned} &\min_{x,t} \sum t_i \\ &\text{s.t. } \|A_i x - b_i\|_2 \leq t_i, \; i = 1 \ldots m \end{aligned}$$

Facility Placement
$A_i = I, b_i =$ Location of ith facility. Problem: find x in "center" of facilities where you find a b that minimises the distance between x and all b_i

Example 2: Minmax Regression

Equivalent to minimising error in the worst case scenario

$$\min_x \max_{i=1\ldots k} \|A_i x - b_i\|_2$$

Transform to:

$$\begin{aligned} &\min_{x,u} u \\ &\text{s.t. } \|A_i x - b_i\|_2 \leq u \;\; \forall i = 1 \ldots k \end{aligned}$$

Example 3: Sums of different norms

$$\min_x \sum_{i=1}^{k} \|A_i x - b_i\|_2 + \lambda\|x\|_1 + \mu\|x\|_\infty \;\; \lambda, \mu \geq 0$$

Transform to:

$$\begin{aligned} &\min_{x,t,u,v} \sum t_i + \lambda u + \mu v \\ &\text{s.t. } \|A_i x - b_i\|_2 \leq t_i \\ &\quad x_i \leq u_i \\ &-x_i \leq u_i \\ &\quad x_i \leq v \\ &-x_i \leq v \end{aligned}$$

Where the constraints regarding u_i is $\|x\|_1$ and v is $\|x\|_\infty$

Inventory Control (Harris)

$$\begin{aligned} &\min h^\top x + \sum_{i=1}^{n} \frac{c_i d_i}{x_i} \\ &\text{s.t. } b^\top x \leq b_0 \\ &\qquad \ell_i \leq x_i \leq u_i \end{aligned}$$

Where:
$x_i =$ quality of item i in inventory
$h_i =$ cost of holding item i/unit
$c_i =$ delivery cost of item i
$d_i =$ demand o f item i
$b_i =$ space required to store item i/unit
$b_0 =$ space available
$\ell_i, u_i =$ lower/upper bounds
So the first term is to minimise to hold the item and the second term is from finding that delivery costs is inversely proportional to items

Can be written as **Hyperbolic Constraints** used when the variable is inverse.

Hyperbolic Constraints

$$\|x\|_2^2 \leq yz,\ y, z \geq 0$$
$$\Longleftrightarrow$$
$$\left\|\begin{pmatrix} 2x \\ y - z \end{pmatrix}\right\|_2 \leq y + z$$

We can rewrite it as

$$\left\|\begin{pmatrix} 2x \\ y - z \end{pmatrix}\right\|_2^2 \leq (y + z)^2 = y^2 + z^2 - 2yz$$

and on the left hand we can finally rewrite it to formulate the SOCP as

$$4\|x\|_2^2 + y^2 + z^2 - 2yz$$

Return to Inventory Control

$$\begin{aligned}
&\min h^\top x + \sum_{i=1}^{n}(c_i d_i)y_i \\
&\text{s.t. } b^\top x \leq b_0 \\
&\qquad \ell \leq x \leq u \\
&\qquad x_i y_i \geq 1 \\
&\qquad y_i \geq 0 \\
&\qquad x_i \geq 0
\end{aligned}$$

We can then transform the $x_i y_i \geq 1$ constraint using the hyperbolic to

$$\left\|\begin{pmatrix} 2 \\ y_i - x_i \end{pmatrix}\right\|_2 \leq y_i + x_i,\quad i = 1 \ldots n$$

Creating the SOCP Constraints.

Robust Optimisation

Robust Optimisation: Incorporating uncertainty about cost and or constraints into the optimisation problem to be solved.

Examples

Typical examples of uncertainty set:
$\mathcal{U} =$ finite set of vectors

$\mathcal{U} = \ell^p$ ball for some $p \geq 1$
$p = 2$ is the Euclidean ball
$p = \infty$ is the box uncertainty

Ax=b

$$Ax = b \implies x = A^{-1}b$$

What if A_δ is s.t. $\|A - A_\delta\| \leq \delta$ = perturbation of A
Solution $x_\delta = A_\delta^{-1}b$ could be drastically different from x
A_δ could be the A that can be stored into the computer instead with floating point precision
Robust Version:

$$\arg\min_x \max_{A_\delta \in \mathcal{U}} \|A_\delta x - b\|_2$$

where $\mathcal{U}$ is some uncertainty

Robust LP

With constraint uncertainty

$$\begin{aligned} &\min c^\top x \\ &\text{s.t. } a_i^\top x \leq b_i \ \ \forall a_i \in \mathcal{U}_i, i = 1 \ldots m \end{aligned}$$

Definition of U

For A given set $\mathcal{U} \in \mathbb{R}^n$ its **support function** is defined as

$$\phi_{\mathcal{U}}(x) = \max_{a \in \mathcal{U}} a^\top x$$

Robust Constraint:

$$a^\top x \leq b \ \ \forall a \in \mathcal{U} \iff \phi_u(x) \leq b$$

Example

$\mathcal{U} = \{a : \|a - a_0\| \leq t\}$ machine is incapable to hold a_0

$$\phi_{\mathcal{U}}(x) = \max_{a : \|a - a_0\| \leq t} a^\top x$$

we can reparameterize:

$a = a_0 + tu$ where $\|u\| \leq 1$
to get:

$$\begin{aligned} \max_{u:\|u\|\leq 1} a_0^\top x + tu^\top x &= a_0^\top x + t \max_{u:\|u\leq 1\|} u^\top x \\ &= a_0^\top x + t\|x\|^* \end{aligned}$$

Which is equivalent to the constraints.
$a^\top x \leq b \forall a \in \mathcal{U} \implies a_0^\top x + t\|x^*\| \leq b$