Lecture 12 SOCP

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Last Time:

$$\min_x \; f_0(x)$$
$$\text{s.t. } Ax \leq b$$

LP: $f_0(x) = c^\top x$
Linear objective, linear constraints
QP: $f_0(x) = \frac{1}{2}x^\top Hx + c^\top x, H \geq 0$
Convex quadratic objective

Examples

LASSO

$$\min_x \; \|Ax - b\|_2^2$$
$$\text{s.t. card}(x) \leq k$$

Through *relaxation*:

$$\min_x \; \|Ax - b\|_2^2 + \lambda\|x\|_1$$

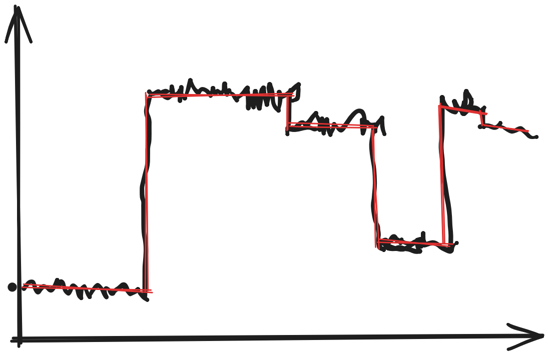
Where $\lambda \geq 0$ (LASSO ℓ_1 regularisation)
Claim: The above is a QP

$$\begin{aligned}\min_{x,t} \; & \|Ax - b\|_2^2 + \lambda \sum t_i \\ \text{s.t. } & x_i \leq t_i \\ & -x_i \leq t_i\end{aligned}$$

$\forall i = 1 \dots n$
To solve, we reduce t_i until we get the absolute value of x_i. Now expand the norm through foiling to get the quadratic form to put into a solver.

Piecewise constant fitting

We are given this noisy data and we want to fit the red line such that we have constant values between intervals.



Goal: Minimise complexity of output function through minimising number of times values are switched as it's constant in the intervals

The black noisy data is y and the constant red function is x

$$\min_x \|x-y\|_2^2$$

However we need to add constraints to limit the switching

$$D = \begin{bmatrix} 1 & -1 & 0 & \dots \\ 0 & 1 & -1 & \dots \\ 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & -1 \end{bmatrix}, Dx = \begin{bmatrix} x_1 - x_2 \\ \vdots \\ x_{n-1} - x_n \end{bmatrix}$$

Then we can use the cardinality of Dx to constraint how much it switches through $\mathrm{card}(Dx) \leq k$. Afterwards similar to the earlier example we *relax*:

$$\min_x \|x-y\|_2^2 + \lambda \|Dx\|_1$$

Applications of LASSO

Anywhere we want to do "fitting" subject to *sparsity constraints*.

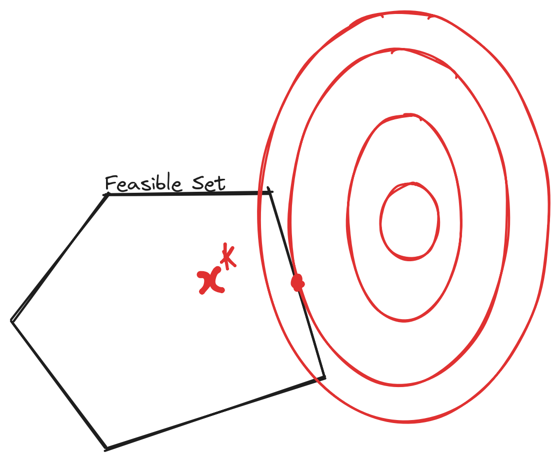
Image Compression

$$\min_x \|Ax-b\|_2^2 + \lambda \|x\|_1$$

Where the columns of A are the wavelet bases, where we try to add less of the bases. Generally, **no closed form solution to QP**.

General Picture of QP

Feasible set is still a polytope $\{x : Ax \leq b\}$. The objective function is a bowl meaning the level sets are eliptical.



Not a helpful picture to draw on the exam

We know that x^* is not on a vertex but still on an edge unlike the LP meaning we can't use the simplex algorithm of hopping corner to corner. If the optimum is achieved on the inside of the set then we can take the derivative and solve for 0 instead.

Ex: Index Tracking

How to allocate portfolio to track stock index as closely as possible.
Let $y_k =$ index at time k
Let $x_i =$ fraction of portfolio invested in asset i
Let $[R]_{ki} =$ return of asset i at time k

$\sum x_i = 1$ (All portfolio invested)
$x_i \geq 0$ (non negative fraction)

$$\begin{aligned} \min \ & \|Rx - y\|_2^2 \\ \text{s.t.} \ & \sum x_i = 1 \\ & x_i \geq 0 \end{aligned}$$

Try to allocate portfolio that invested in matched the market as closely as possible.
We cannot turn this problem into least squares as the second constraint splits the subspace/affine set property.
The first constraint still keeps this subspace property which we can think of it as a linearly constrained LS.

Second Order Cone Program (SOCP)

$LP \subset QP$ where $H = 0$.
$LP \subset QP \subset SOCP$

Standard form:

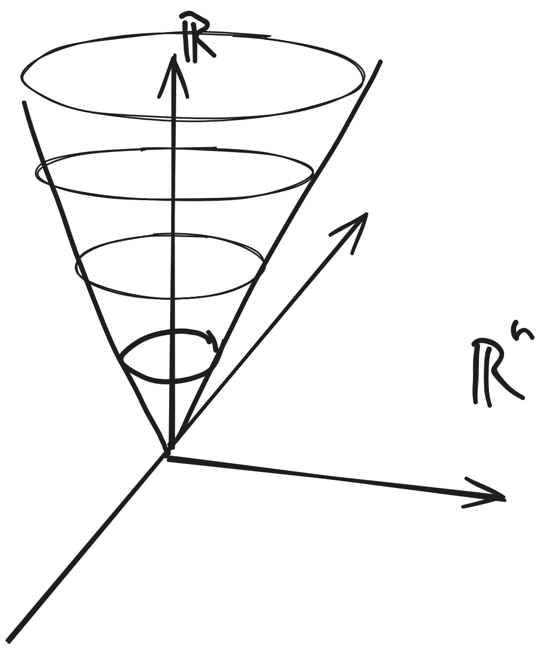
$$\begin{aligned} &\min_x \; c^\top x \\ &\text{s.t. } \|A_i x - b_i\|_2^2 \leq c_i^\top x + d_i \end{aligned}$$

$\forall i = 1 \ldots m$
Where c, c_i, A_i, b_i, d_i are all given.

What is a second order cone?

In $\mathbb{R}^{n+1}$, the “second order cone” is the set

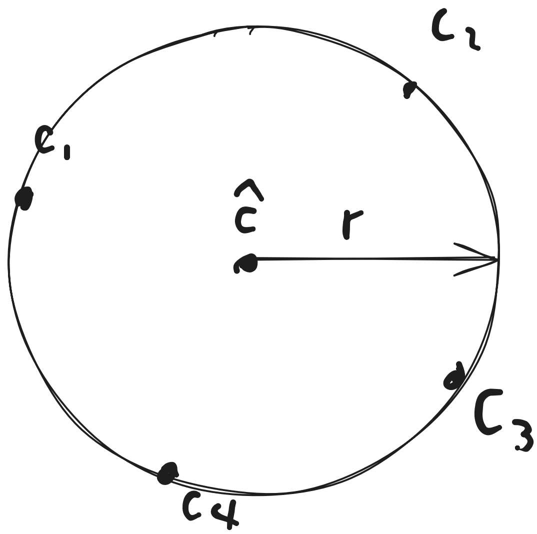
$$\{(y,t) : y \in \mathbb{R}^n, t \in \mathbb{R}, \|y\|_2 \leq t\}$$



$\|A_i x - b_i\|_2$ is y_i
$C_i^\top x + d_i$ is t_i
Constraint $\iff$ $(y_i, t_i) \in$ SOCP

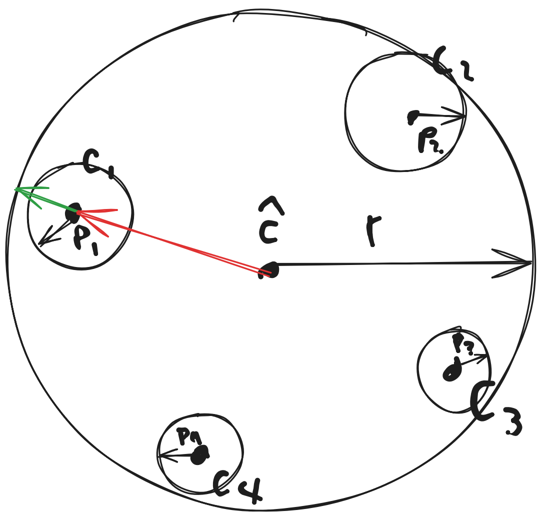
Example

Find ball of minimum radius that encloses c_i



Find $\hat{c}$ and r. This problem is an LP

Now instead think of this problem with people moving around.



Now find ball of min radius that encloses $B(c_i, \rho_i), i = 1 \ldots 4$ where B is a ball.

$$\min_{\hat{c}, r} \; r$$
$$\text{s.t. } \|\hat{c} - c_i\|_2 + \rho_i \leq r$$

This constraint is shown in the diagram above.

SOCP Constraints

$$\|A_i x - b_i\|_2 \leq c_i^\top x + d_i$$

Remember no square in standard form

Examples

Affine constraints: $Cx \leq r$,
$-c_i^\top$ is its row of C, $r_i = d_i$
$A_i = b_i = 0$
Giving $0 \leq -Cx + r_i$

Quadratic Constraints: $x^\top Qx + c^\top x$ s.t. $Q \geq 0$

$$\left\| \begin{bmatrix} \sqrt{2}Q^{1/2} \\ -C^\top \end{bmatrix} x + \begin{bmatrix} 0 \\ t - \frac{1}{2} \end{bmatrix} \right\| \leq t - c^\top x + \frac{1}{2}$$

Very not obvious (no need to do ever).
Example: $\|Rx\|_2^2 \leq r^\top x$
$\rightarrow x^\top R^\top Rx - r^\top x \leq 0$