# Lecture 16 Convexity

*Note by Samion Suwito on 3/13/25*

## Convex Optimisation

Optimisation problem of form:

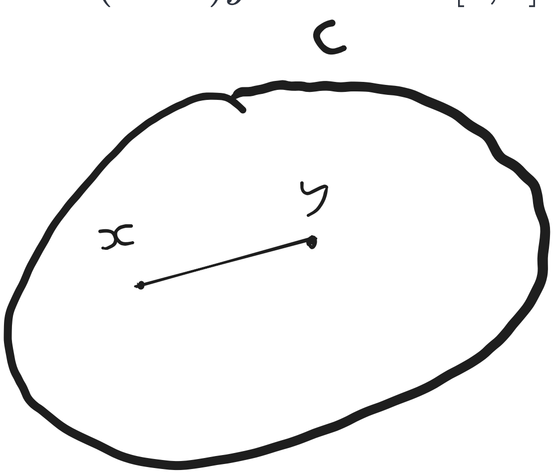
$$\begin{aligned} &\min_x f_0(x) \\ &\text{s.t. } f_i(x) \le 0 \ \ \forall i = 1 \dots m \end{aligned}$$

Where $f_i$s are convex functions

## Convexity

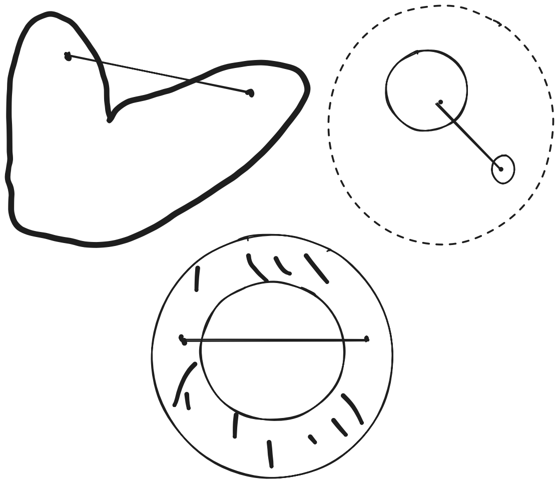
*A geometric concept*

**Definition**: A convex set $C \subset \mathbb{R}^n$ is one of the property that if $x, y \in C$ then $\theta x + (1-\theta) y \in C \ \ \forall \theta \in [0, 1]$




Any two points can be joined by a line segment and will be in C

Here are a few examples of non-convex sets.



Connection between convexity & linear algebra is very deep (**duality**). Duality relates one problem to another.

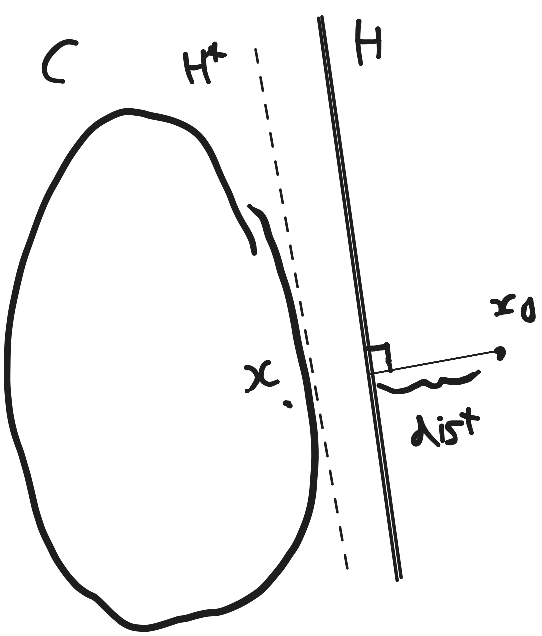
## Preview of Duality

$$\min_{x \in C} \|x - x_0\|_2$$

Where $x_0 \notin C$
Find closest $x \in C$ to $x_0$
We can reform this problem as

$$\max \operatorname{dist}(x_0, H)$$

where $H$ is a hyperplane separating $x_0$ from $C$.




Duality is important as it *certifies* optimality!

For any $x \in C$ and any $H$ that separates $x_0$ from $C$ we get:

$$\|x - x_0\| \geq \text{dist}(x_0, H)$$

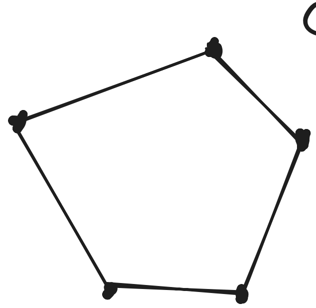
If we can find both of these are equal we therefore know that it's equal. A compute creates a dual problem and solves that too to Find the optimal value and creates a **certificate of optimality**. We consider the difference between the LHS and RHS to be the **duality gap**.

# Back to Convexity

## Examples of Convex Sets

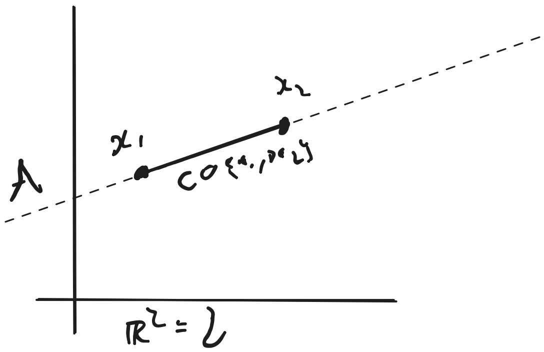
1. Given a set of points $\{x_1, \ldots, x_n\}$, their *Linear Hull* $\mathcal{L} = \{\sum \lambda_i x_i; \lambda_i \in \mathbb{R}\} = \text{Sp}\{x_1, \ldots, x_n\}$ is a convex set.
   Proof:
   $$x = \sum \lambda_i x_i, \quad y = \sum \mu_i x_i \in \mathcal{L}$$
   $$\theta x + (1-\theta) y = \sum (\theta \lambda_i + (1-\theta)\mu_i) x_i \in \mathcal{L}$$
2. *Affine Hull*: $\mathcal{A} = \{\sum \lambda_i x_i; \sum \lambda_i = 1\}$ is a convex set
   Proof:
   $$x = \sum \lambda_i x_i, \quad y = \sum \mu_i x_i \in \mathcal{A}$$
   $$(\theta \lambda_i + (1-\theta)\mu_i) = \xi_i$$
   $$\sum \xi_i = \theta \sum \lambda_i + (1-\theta) \sum \mu_i = 1$$
3. *Convex hull*: $\text{co}\{x_1 \ldots x_n\} = \{\sum \lambda_i x_i : \sum \lambda_i = 1, \lambda \geq 0\} =$ smallest convex set containing $\{x_1 \ldots x_n\}$

   co{$x_1$, ... $x_n$}

   We can do the exact same proof as above but with $\lambda_i, \mu_i \geq 0$ showing how $\sum \lambda_i = \sum \mu_i = 1$

Difference between hulls:




4. *Conic Hull* = $\text{conic}\{x_1, \ldots x_n\} = \{\sum \lambda_i x_i, \lambda_i \geq 0\}$ where cone $C : x \in C \implies \alpha x \in C \;\; \forall \alpha \geq 0$ Smallest cone containing all of $x_i$

*Examples*:

- Convex Hull of $\{vv^\top : v \in \mathbb{R}^n\}$ $=$ all PSD matrices on $\mathbb{R}^n$
- Conic Hull of $\{vv^\top : v \in \mathbb{R}^n, \|v\|_2 = 1\}$ meaning the set of 1D projections (rank 1 matrices). It turns out to also be all PSD matrices when you write out the spectral decomposition.
- Linear Hull of $\{vv^\top : v \in \mathbb{R}^n, \|v\|_2 = 1\}$ $=$ all symmetric $n \times n$ matrices.

# Operations that preserve Convexity:

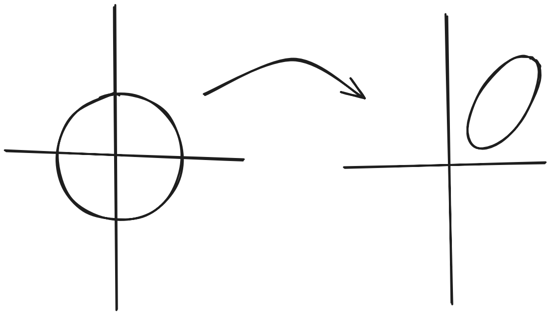
## Intersection.

If $\mathcal{A} =$ index set. $C_\alpha$ is convex $a \in \mathcal{A}$ then $\bigcap_{\alpha \in \mathcal{A}} C_\alpha$ is convex.
*Example 1:* Every half space $\mathcal{H}_{a,b} = \{x : a^\top x \leq b\}$ is convex
$\implies$ Every polytope is convex $P = \bigcap_{i=1}^m \mathcal{H}_{a,b}$
Every convex set can be represented as an infinite intersection of half spaces.
*Example 2*: SOC $= \{(x,t) \in \mathbb{R}^{n+1} : \|x\|_2 \leq t\}$
$\|x\|_2 \leq t \iff u^\top x \leq t \;\; \forall u : \|u\| \leq 1$

$$\begin{aligned} SOC &= \{(x,t) \in \mathbb{R}^{n+1} : u^\top x \leq t \;\; \forall u : \|u\|_2 \leq 1\} \\ &= \bigcap_{u : \|u\|_{2 \leq 1}} \mathcal{H}_{u,t} \end{aligned}$$

## Affine Transformations

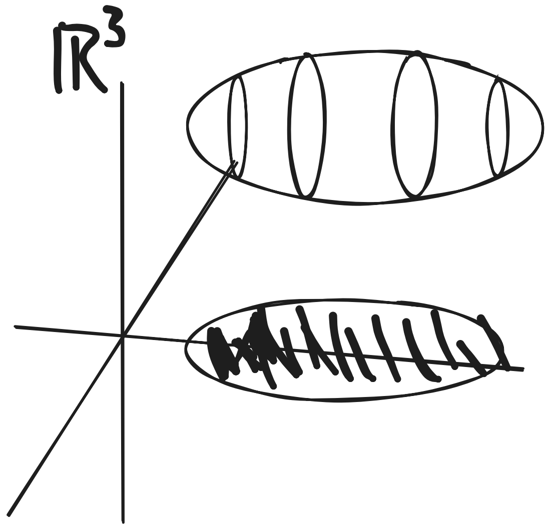
Affine transformations of Convex sets are convex.
$C$ convex, $AC + b = \{Ax + b : x \in C\}$



If $x, y \in AC + b \implies x = Ax' + b, y = Ay' + b$ for some $x', y' \in C$.
$\theta x + (1-\theta)y = A(\theta x' + (1-\theta)y') + b \in AC + b$

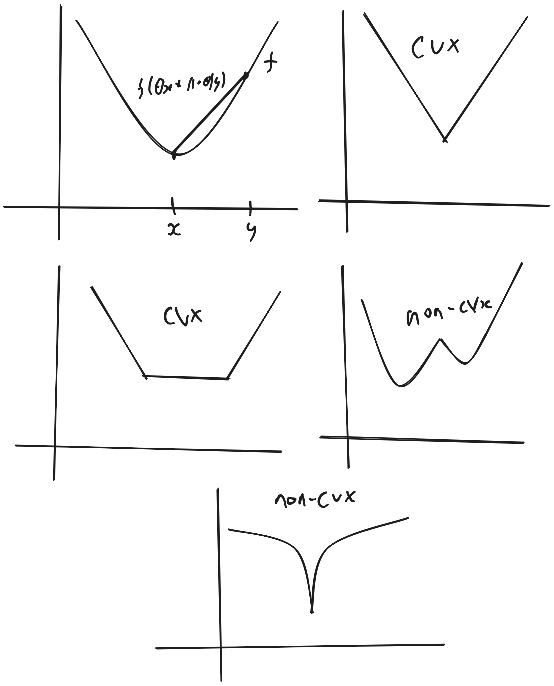
## Projection

Projections of Convex sets are convex. “Convex sets have shadows.”




# Convex Functions

In optimisation, we often consider functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$.
Effective domain: $\operatorname{dom} f = \{x : |f(x)| < \infty\}$
A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex if
$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y) \;\; \forall x, y, \theta \in [0,1]$
Can’t put $-\infty$ or else we would have a situation like $+\infty + -\infty$.
Convex $f$




## Examples

*Example 1*: Every norm $\|\cdot\|$ is convex. $\|\theta x + (1-\theta) y\| \leq \theta\|x\| + (1-\theta)\|y\|$
*Example 2*: $f(x) = \log\frac{1}{x}, x > 0$
$\operatorname{dom} f = \mathbb{R}_{>0}$
Sometimes defined as

$$f(x) = \begin{cases} \log\frac{1}{x} > 0 \\ +\infty \leq 0 \end{cases}$$

*Recall*: $\operatorname{epi} f = \{(x,t) : f(x) \leq t\}$ this is a set in $\mathbb{R}^{n+1}$
*Claim*: $f$ is convex $\iff$ $\operatorname{epi} f$ i a convex set.
If $(x_1, t_1), (x_2, t_2) \in \operatorname{epi} f$

$$\begin{aligned} &\implies \theta t_1 + (1-\theta) t_2 \geq \theta f(x_1) + (1-\theta) f(x_2) \\ &\quad \geq f(\theta x_1 + (1-\theta) x_2) \\ &\implies (\theta x_1 + (1-\theta) x_2, \theta t_1 + (1-\theta) t_2) \in \operatorname{epi} f \end{aligned}$$

*Example*: $f(x) = \log\left(\sum e^{x_i}\right) \approx \max_i x_i$ (log-sum-exp function)
$\operatorname{epi} f = \left\{(x,t) : \sum e^{x_i} \leq e^t\right\} = \left\{(x,t) : \sum e^{(x_i - t)} \leq 1\right\}$
The above is therefore convex as $z \to e^z$ is convex.

# Convexity Preserving Operations on Functions

- If $f, g$ are convex $\implies \alpha f + \beta g$ is convex for $\alpha, \beta \geq 0$
  *Example:* $H(p) = \sum p_i \log \frac{1}{p_i}$ (entropy function with probability distribution p)
  *Example* Pointwise max of convex functions in convex
  $h(x) = \max\{f(x), g(x)\}, x \in \mathbb{R}^n$ as this is taking the intersection of their epigraphs.`