Lecture 17 Characterisations of Convexity

Note by Samion Suwito on 3/18/25

Midterm Review

Question 1

Find compact SVD & Rank of A

$$
\begin{aligned}
A = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix} &= \begin{bmatrix} 0 & U_r \Sigma_r V_r \\ V_r \Sigma_r U^\top & 0 \end{bmatrix} \\
&= \begin{bmatrix} U_r & 0 \\ 0 & V_r \end{bmatrix} \begin{bmatrix} 0 & \Sigma_r {V_r}^\top \\ \Sigma_r U_r^\top & 0 \end{bmatrix} \\
&= \begin{bmatrix} U_r & 0 \\ 0 & V_r \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & \Sigma_r \end{bmatrix} \begin{bmatrix} 0 & V_r^\top \\ U_r^\top & 0 \end{bmatrix}
\end{aligned}
$$

We know the rank is therefore $2r$
Eigenvalues:

$$
A \begin{bmatrix} u_i \\ s_i v_i \end{bmatrix} = \begin{bmatrix} 0 + s_i B v_i \\ B^\top u_i + 0 \end{bmatrix} = \begin{bmatrix} s_i \sigma_i u_i \\ \sigma_i v_i \end{bmatrix} = \sigma_i \begin{bmatrix} u_i \\ s_i v_i \end{bmatrix}
$$

Question 2

Projection of x onto $\mathcal{R}(A)$ is $A(A^\top A)^{-1} A^\top x$ in ($\| \cdot \|_2$):

$$
\begin{aligned}
\arg \min_{a \in \mathcal{R}(A)} \|a - x\|_2^2 &= \min_z \|Az - x\|_2^2 \\
&= Az^* = A(A^\top A)^{-1} A^\top x
\end{aligned}
$$

Same Q in $\|z\|_c^2 = z^\top C z$:

$$
\begin{aligned}
&\min_z \|C^{1/2} A z - C^{1/2} x\|_2^2 \\
&Az^* = A(A^\top C A)^{-1} A^\top C x
\end{aligned}
$$

Question 3

$$\|M\|_1 = \max_{x:\|x\|_1=1} \|Mx\|_1 =? \max_j \|m_j\|_1$$

$$\|Mx\|_1 = \|\sum x_j a_j\|_1 \leq \sum |x_j| \|m_j\|_1$$
$$= \max_j \|m_j\|_1$$

Form:

$$\min_x \|M + \sum_{i=1} x_i B_i\| \text{ as LP}$$

$$= \min_x \max_j \|m_j + \sum x_p b_j(p)\|$$
$$= \min_{x,u} u \text{ s.t.} \left\|m_j + \sum x_p b_j(p)\right\|_1 \leq u$$
$$= \min_{x,u,t} u \text{ s.t. } \pm\left(m_{ij} + \sum x_i b_{ij}(p)\right) \leq t_i$$
$$\sum t_{ij} \leq u \forall j = 1 \ldots n$$

Where $b_j(p)$ is the jth column of B_p

Question 4

Solve LP:

$$\min c^\top x \text{ s.t. } Ax \leq b$$

where we have $\|c - c_0\|_2 \leq r$

$$\min_x \max_{c:\|c-c_0\|\leq r} c^\top x \text{ s.t. } Ax \leq b$$
$$= \max_{u:\|u\|_2\leq 1} c_0^\top x + ru^\top x$$
$$= \min c_0^\top x + rt \text{ s.t. } Ax \leq b, \|x\|_2 \leq t$$

What if $\|c_0\|_2 \leq r$ and $b = 0$

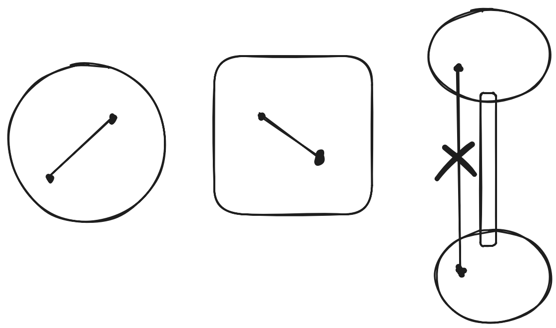
$$\max_{c:\|c-c_0\|\leq r} c^\top x \geq 0 \implies p^* \geq 0$$

$$p^* \min_x \max_{c:\|c-c_0\|_2\leq r} c^\top x \leq \max_{c:\|c-c_0 \leq r\|} c^\top 0 = 0$$

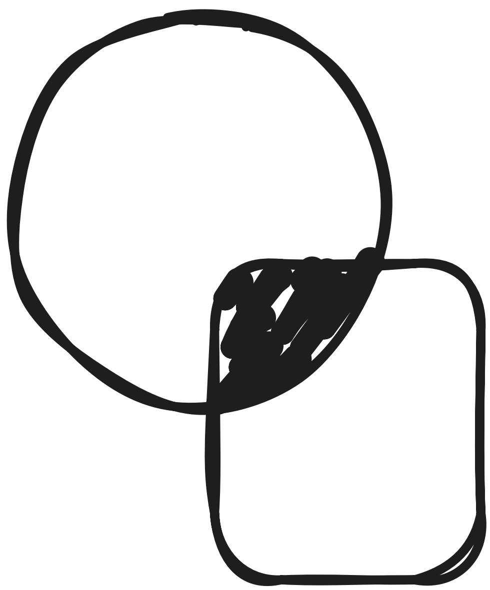
$x^* = 0$ given $p^* = 0$ and check if it’s feasible.

Convexity

A set $C \in \mathbb{R}^n$ is convex if $\forall x_0, x_1 \in C, \theta x_0 + \bar{\theta} x_1 \in C \; \theta \in [0,1], \bar{\theta} = (1-\theta)$



Intersection of two convex sets are still convex



$f : \mathbb{R}^n \to \mathbb{R}$ is convex if $\operatorname{epi} f$ is a convex set
Jensen's inequality: $f(\theta x + \bar{\theta} y) = \theta f(x) + \bar{\theta} f(y) \forall x, y, \theta \in [0,1]$
Maximum of two functions preserve convexity

If f is convex then $x \mapsto f(Ax+b)$is convex:

$$\begin{aligned} f(A(\theta x + \bar{\theta} y) + b) &= f(\theta(Ax+b) + \bar{\theta}(Ay+b)) \\ &\leq \theta f(Ax+b) + \bar{\theta} f(Ay+b) \end{aligned}$$

Example

Say $\mathcal{L}$ is a convex function. Then the problems

$$\min_w \sum_i \mathcal{L}(w^\top x + b_i)$$

Has a convex objective.

Differential Characterisation of Convexity

Claim: If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is differentiable on open domain then f is convex if and only if $f(y) \geq f(x) + \nabla f(x)^\top f(y-x) \forall x, y \in \operatorname{dom} f$
Looks like first order taylor series of f around x.

If f convex,

$$\begin{aligned} \frac{f(x+\lambda(y-x)) - f(x)}{\lambda} &\leq \frac{\lambda(f(y) + (1-\lambda)f(x) - f(x))}{\Lambda} \\ &= f(y) - f(x) \end{aligned}$$

$\lambda \in (0,1]$ $f(x+\lambda(y-x)) = \lambda y + (1-\lambda)x$
Directional derivative $\nabla f(x)^\top (y-x)$

Conversely, Fix x, y, λ, $z = \lambda x + (1-\lambda)y$
$f(x) \geq f(z) + \nabla f^\top(z)(x-z)$
$f(y) \geq f(z) + \nabla f^\top(z)(y-z)$
$\implies \lambda f(x) + (1-\lambda)f(y) \geq f(z)$

Second Order Condition

Claim: If f is twice differentiable then f is convex if and only if the $\nabla^2 f(x) \geq 0$ (psd) $\forall x \in \operatorname{dom}(f)$

Example: e^x is convex
$f(x) = e^x \implies f''(x) = e^x \geq 0$
Example 2: $-\log x$ is convex on $x > 0$
$f(x) = -\log x$
$f'(x) = -\frac{1}{x}$
$f''(x) = \frac{1}{x^2} > 0$

Convexity along a line

f is convex if and only if $t \geq 0 \mapsto f(x_0 + tz)$ is convex $\forall t \geq 0$ and sufficiently small s.t. $x_0 + tz \in \operatorname{dom}(f)$

Example: $f(x) = -\log \det(x)$ is convex on $x > 0$

Continue next lecture.