# SLAM Geometry Notes

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## 1 Coordinate Frames

A coordinate frame consists of an origin and a set of three orthogonal unit axes.

### World / Global Frame (W)

The world frame (or global frame) is fixed and generally serves as the reference frame.

### Camera / Body Frame (C/B)

The camera or body frame moves with the robot and expresses points relative to the robot's position.

### Landmark Frame (L)

A local frame used for describing static scene landmarks.

## 2 Transformations and Lie Groups

### Rigid Transformations

We denote $T_{AB}$ as the transformation from frame $B$ to $A$. A pose is a pair $(R, t)$, where $R$ is the rotation and $t$ is the translation.

Switching between world and camera coordinates:

$$p_C = R_{CW}(p_W - t_{CW}), \qquad p_W = R_{WC}p_C + t_{CW}, \qquad R_{WC} = R_{CW}^T$$

### Definition of a Lie Group

A Lie group $G$ is both a group and a smooth manifold with smooth group operations:

$$(g_1, g_2) \mapsto g_1 g_2, \qquad g \mapsto g^{-1}$$

### Definition of a Lie Algebra

A Lie algebra is a vector space equipped with a bilinear, antisymmetric bracket $[a, b] = ab - ba$ satisfying the Jacobi identity:

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

Examples:

$$SO(3) \leftrightarrow \mathfrak{so}(3), \qquad SE(3) \leftrightarrow \mathfrak{se}(3)$$

The lowercase symbols denote the Lie algebras (tangent spaces at the identity), while uppercase symbols denote the Lie groups (manifolds of transformations).

# 3 The Special Orthogonal Group $SO(3)$

### Definition

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I,\ \det(R) = 1\}$$

It represents all 3D rotations. Each $R \in SO(3)$ preserves lengths and angles.

### Associated Lie Algebra $\mathfrak{so}(3)$

The Lie algebra of $SO(3)$ is

$$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$$

Every element of $\mathfrak{so}(3)$ corresponds to a vector $\phi = (\phi_x, \phi_y, \phi_z)^T \in \mathbb{R}^3$ via the **hat operator**:

$$\phi^\wedge = \begin{bmatrix} 0 & -\phi_z & \phi_y \\ \phi_z & 0 & -\phi_x \\ -\phi_y & \phi_x & 0 \end{bmatrix}, \qquad (\phi^\wedge)^\vee = \phi$$

The hat operator maps a vector to its corresponding skew-symmetric matrix such that $\phi^\wedge v = \phi \times v$.

### Exponential Map on $\mathfrak{so}(3)$

The exponential map links infinitesimal rotations in the Lie algebra $\mathfrak{so}(3)$ to finite rotations in the Lie group $SO(3)$. For a rotation vector $\phi \in \mathbb{R}^3$ with skew-symmetric form $\phi^\wedge \in \mathfrak{so}(3)$,

$$\exp : \mathfrak{so}(3) \to SO(3), \qquad R(\phi) = \exp\big(\phi^\wedge\big)$$

This expression arises by integrating the differential equation:

$$\dot{R}(t) = \phi^\wedge R(t), \qquad R(0) = I$$

whose unique solution is $R(t) = \exp(t\,\phi^\wedge)$. The trajectory $R(t)$ forms a smooth one-parameter subgroup of $SO(3)$, representing continuous rotation with constant angular velocity $\phi$. Evaluating at $t = 1$ gives the finite rotation $R(\phi)$, corresponding to a rotation by angle $\|\phi\|$ about the unit axis $\hat{u} = \phi/\|\phi\|$.

### Rodrigues' Formula

Let $\theta = \|\phi\|$ and $\hat{u} = \phi/\theta$. The exponential of a skew-symmetric matrix admits a closed form:

$$R(\phi) = I + \frac{\sin\theta}{\theta}\phi^\wedge + \frac{1-\cos\theta}{\theta^2}(\phi^\wedge)^2$$

which is known as Rodrigues' rotation formula. Expanding $\exp(\phi^\wedge)$ via the Taylor series and using $(\hat{u}^\wedge)^3 = -\hat{u}^\wedge$ gives the above closed form.

### Axis–Angle Representation

A rotation can equivalently be described by an angle $\theta$ about a unit axis $\hat{u}$:

$$R(\hat{u}, \theta) = I + \sin\theta\, \hat{u}^\wedge + (1 - \cos\theta)(\hat{u}^\wedge)^2$$

and inversely,

$$\theta = \cos^{-1}\left(\frac{\mathrm{trace}(R) - 1}{2}\right), \quad \hat{u} = \frac{1}{2\sin\theta}\begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

## 4 Quaternion Representation of Rotation

A unit quaternion $q = [q_w, q_x, q_y, q_z]^T$ encodes rotation as:

$$q_w = \cos\left(\frac{\theta}{2}\right), \quad \mathbf{q_v} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} = \hat{u}\sin\left(\frac{\theta}{2}\right)$$

Quaternion conjugate and inverse:

$$q^* = [q_w, -q_x, -q_y, -q_z]^T, \qquad q^{-1} = q^*$$

### Applying a Quaternion Rotation

Represent a vector $v$ as a pure quaternion $[0, \mathbf{v}]$. The rotated vector is given by:

$$v' = qvq^{-1}$$

### Quaternion to Matrix and Back

Rotation matrix from quaternion:

$$R(q) = \begin{bmatrix} 1 - 2(q_y^2 + q_z^2) & 2(q_x q_y - q_z q_w) & 2(q_x q_z + q_y q_w) \\ 2(q_x q_y + q_z q_w) & 1 - 2(q_x^2 + q_z^2) & 2(q_y q_z - q_x q_w) \\ 2(q_x q_z - q_y q_w) & 2(q_y q_z + q_x q_w) & 1 - 2(q_x^2 + q_y^2) \end{bmatrix}$$

To recover a quaternion from $R$:

$$\theta = 2\cos^{-1}(q_w), \quad \hat{u} = \frac{1}{\sin(\theta/2)}\begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}, \quad \sin\left(\tfrac{\theta}{2}\right) \neq 0$$

## 5 The Special Euclidean Group $SE(3)$

### Definition

The group of rigid-body motions in 3D:

$$SE(3) = \left\{ \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \middle| R \in SO(3),\ t \in \mathbb{R}^3 \right\}$$

### Homogeneous Coordinates

A point $p_W$ in homogeneous form:

$$\tilde{p}_W = \begin{bmatrix} p_W \\ 1 \end{bmatrix}, \qquad \tilde{p}_C = T_{CW}\tilde{p}_W$$

Inverse transform:

$$T_{WC} = T_{CW}^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

## 6 The Lie Algebra $\mathfrak{se}(3)$ and Exponential Map

### Elements and Hat Operator

An element of $\mathfrak{se}(3)$ is a 6D twist:

$$\xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix}, \quad \rho, \phi \in \mathbb{R}^3$$

with hat representation:

$$\xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ 0 & 0 \end{bmatrix}$$

### Exponential Map $\exp : \mathfrak{se}(3) \to SE(3)$

$$T(\xi) = \exp\bigl(\xi^\wedge\bigr) = \begin{bmatrix} R(\phi) & J(\phi)\rho \\ 0 & 1 \end{bmatrix}, \qquad R(\phi) = \exp\bigl(\phi^\wedge\bigr)$$

where $J(\phi)$ is the **left Jacobian** of $SO(3)$.

### Jacobian Definition and Closed Form

$$J(\phi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\phi^\wedge)^n = I + \frac{1-\cos\theta}{\theta^2}\phi^\wedge + \frac{\theta - \sin\theta}{\theta^3}(\phi^\wedge)^2, \qquad \theta = \|\phi\|$$

### Pose Updates on $SE(3)$

Small motion increments in 3D space are represented in the Lie algebra $\mathfrak{se}(3)$ and applied to a current pose $T \in SE(3)$ via the exponential map.

Given a perturbation $\xi = [\rho^T, \phi^T]^T \in \mathbb{R}^6$, the pose update is written as:

$$T \leftarrow \exp\bigl(\xi^\wedge\bigr)\, T$$

This update composes a small transformation on the manifold while preserving the $SE(3)$ group structure.

The rotational and translational components evolve as:

$$R \leftarrow \exp\bigl(\phi^\wedge\bigr)R, \qquad t \leftarrow \exp\bigl(\phi^\wedge\bigr)t + J(\phi)\rho$$

For small angles ($\theta \to 0$), the first-order approximations hold:

$$R \approx (I + \phi^\wedge)R, \qquad t \approx t + \rho + \tfrac{1}{2}\phi^\wedge \rho$$

or equivalently,

$$\exp(\xi^\wedge) \approx \begin{bmatrix} I + \phi^\wedge & \rho + \frac{1}{2}\phi^\wedge \rho \\ 0 & 1 \end{bmatrix}$$