Samip Shah
samipjanish.shah@mavs.uta.edu

(1) Express the function $n^3/1000 - 100n^2 - 100n + 3$ in terms of ,Θ-notation.

- Let $f(n) = n^3/1000 - 100n^2 - 100n + 3$
  We consider the highest power of the equation and ignore the constant associated with that term. Thus, $f(n) = \Theta(n^3)$

(2) Let f(n) and g(n) be asymptotically nonnegative functions. Using the basic definition of , Θ -notation, prove that max(f(n) , g(n)) = Θ(f(n) + g(n))

- We can assume that there exist n_1,n_2 such that

  $$f(n) >= 0 \; ; n>n_1$$
  $$g(n) >= 0 \; ; n>n_2$$

  Now, let $n_0 = \max(n_1,n_2)$; $n>n_0$

  $$f(n)=< \max(f(n) , g(n))$$
  $$g(n)=< \max(f(n) , g(n))$$
  $$[f(n) + g(n)] =< 2 [\max(f(n) , g(n))]$$
  $$\max(f(n) , g(n)) =< f(n) + g(n)$$

  Thus we can prove that

  $$0=< (1/2) [f(n) + g(n)] =< \min(f(n) , g(n)) =< f(n) + g(n) \; ; n>n_0$$

  This is definition of $\Theta(f(n) + g(n))$ where $C_1= (½)$, $C_2= 1$

(3) Show that for any real constants a and b, where b > 0, $(n+a)^b = \Theta(n^b)$

- We need to find three constants C_1, C_2, n_0 such that

  $$0=<C_1n^b =< (n+a)^b =< C_2n^b \; ; n>n_0$$

  As we know that n>|a| we have,

  $$n+a =< n+|a| < 2n$$

  As n >= 2|a|, we have

  $$n+a >= n -|a| > (1/2)n$$

  Thus, when n >= 2|a|, we have

  $$0 =< (1/2)n =< n+a =< 2n$$

  As we know that b is positive constant

  $$0 =< ((1/2)n)^b =< (n+a)^b =< (2n)^b$$
  $$0 =< (1/2)^b n^b =< (n+a)^b =< 2^b n^b$$

  So, C1 = (½), C2= 2^b $n_0 = 2|a|$
  It satisfies the definition

(4) Let f(n) and g(n) be asymptotically positive functions. Prove or disprove each of the following conjectures.

- **f(n) + g(n) = Θ(min f(n) , g(n))**
  Let f(n) = n and g(n) = n^2
  We know that $n^2 + n \neq \Theta(\min (n^2, n)) = \Theta(n)$
  Thus INCORRECT
- **f(n) + o(f(n)) = Θ (f(n))**
  According to definition,
  0 =< o(f(n)) =< f(n), Hence f(n) =< f(n) + o(f(n)) =< 2(f(n))

Therefore, we can say that f(n) + o(f(n)) = Θ (f(n)). CORRECT

(5) Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = T(n/2) + n^2$. Use the substitution method to verify your answer.

- Rate of increase in number of subproblems in each recursion = 1
  Rate of decrease in subproblem size = 2
  Hence in each level of the tree, there is only one node of cost $c(n/2^i)2$ at depth i=0,1,2,…,lgn.
  Hence, total cost of the tree is:

$$T(n) = \sum_{i=0}^{\lg n} c\ (n/2^i)^2$$
$$= cn^2 \sum_{i=0}^{\lg n} (1/4)^i$$
$$= cn^2 \sum_{i=0}^{\infty} (1/4)^i$$
$$= O(n^2)$$

Now the sum is independent of n, resulting in a constant factor.
We must prove the same using substitution method. Thus

$$T(n) = T(n/2) + n^2$$
$$=< d(n/2)^2 + n^2$$
$$= d(n^2/4) + n^2$$
$$= n^2\ ((d/4)+1)$$
$$=< dn^2$$

The last step is true if d >= (d/(4+1)), i.e. d>= (4/3)

(6) Use a recursion tree to determine a good asymptotic upper bound on the recurrence T(n) = 2T (n – 1) + 1. Use the substitution method to verify your answer.

- The problem size at depth i is n-i.
  Thus total cost at level i would be i=0,1,2,3,4…..n-1 ie 2^i

$$T(n) = \sum_{i=0}^{n-1} 2^i$$
$$= (2^{n-1} - 1)/(2-1)$$
$$= 2^{n-1} - 1$$
$$= \Theta(2^n)$$

We guess $T(n) =< c2^n + n$

$$T(n) =< (2)(\ c\ 2^{n-1}) + (n-1) + 1$$
$$= c2^n + n$$
$$= O(2^n)$$

(7) Can the master method be applied to the recurrence $T(n) = 4T\ (n/2) + n^2 \lg n$? Why or why not? Give an asymptotic upper bound for this recurrence.

- Here in this problem we have a=4, b=2, f(n)= $n^2\lg n \neq O(n^{2-\epsilon}) \neq \Omega(n^{2-\epsilon})$
  So, we cannot apply master method.
  Let's assume $T(n) =< cn^2\lg^2 n$,

$$T(n) =< 4T\ (n/2) + n^2 \lg n$$
$$= 4c(n/2)^2\lg^2(n/2) + n^2\lg n$$
$$= cn^2\ lg(n/2)\ \lg(n) - cn^2\ lg(n/2)\ \lg(2) + n^2\lg(n)$$
$$= cn^2\lg^2 n - cn^2\lg(n)\ \lg(2) - cn^2\ lg(n/2) + n^2\lg(n)$$
$$= cn^2\lg^2 n + (1-c)n^2 lg(n) - cn^2\ lg(n/2) (c>1)$$
$$=< cn^2\lg^2 n - cn^2\ lg(n/2)$$

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$$=< cn^2 \lg^2 n$$

(8) Give asymptotic upper and lower bounds for T(n) in each of the following recurrences. Assume that T(n) is constant for n =< 2. Make your bounds as tight as possible and justify your answers.

- The master method applies to recurrences of the form
  T(n) = aT(n/b) + f(n) -----→ Compare Following equation with all
  where a≥1, b> 1, and f is asymptotically positive.

- **T(n) = 2T(n/2) + n^4.**
  By master theorem, T(n) = $\Theta(n^4)$
- **T(n) = T(7n/10) + n.**
  By master theorem, T(n) = $\Theta(n)$
- **T(n) = 16T(n/4) + n^2.**
  By master theorem, T(n) = $\Theta(n^2 \lg n)$
- **T(n) = 7T(n/3) + n^2.**
  By master theorem, T(n) = $\Theta(n^2)$
- **T(n) = 7T(n/2) + n^2.**
  By master theorem, T(n) = $\Theta(n^{\lg 7})$
- **T(n) = 2T(n/4) + $n^{1/2}$.**
  By master theorem, T(n) = $\Theta(n^{1/2} \lg n)$
- **T(n) = T(n-2) + n^2.**
  Let d=m mod 2

$$T(n) = \sum_{j=1}^{j=(n/2)} (2j+d)^2$$
$$= \sum_{j=1}^{(n/2)} 4j^2 + 4jd + d^2$$
$$= [((n)(n+2)(n+1))/6] + [(n(n+2)d)/2] + [(d^2n)/6]$$
$$= \Theta(n^3)$$