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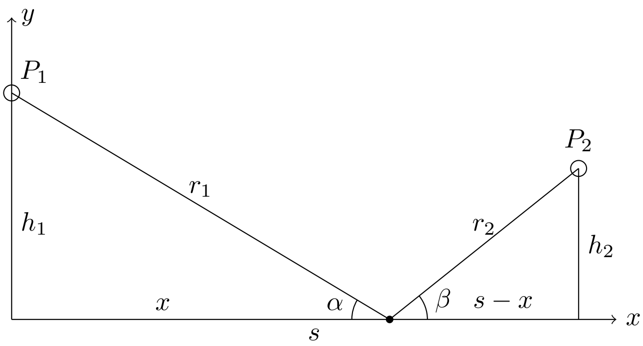
I Introduction

This project looks to investigate the nature of illumination experienced by a horizontal road by two lamps at either side of the road (the lights being opposite one another). Knowing how high to place the lamps as well as their optimal power output are crucial for achieving a homogeneous illumination of the road. Achieving such homogeneous illumination is crucial, principally for concerns of safety. Wide roads often suffer from inhomogenous illumination and this investigation looks to model the situation to obtain insights into optimal lamp placement and power.

II Overview of the Problem

We assume the road in question to be entirely horizontal with no curvature in question. Parallel to the direction of the road we impose a two dimensional Cartesian coordinate system as shown in the schematic. The roadside lamps

The roadside lamps have both an illumination power P_i and a height from the surface of the road h_i. The width of the road is denoted by s and so with respect to the coordinate system, the position of lamp one is at (0,h_1) and at (s,h_2). We let X be a general point along the width of the road with coordinates (x,0).



Cartesian coordinate system with labeled axis above.

III Minimum Illumination

We look to find a treatment for the problem such as to find the point on the road which is minimally illuminated. From elementary geometry we can find the values for r_1 and r_2 in terms of other variables:

$$r_1^2 = h_1^2 + x^2 \tag{1}$$

$$r_2^2 = h_2^2 + (s-x)^2 \tag{2}$$

The light intensity originating from a point source is dependant on the inverse value of the square of the distance along with the impact angle of the light waves. As a result, the total illumination at a point X across the horizontal road is given by:

$$Z(x) = I_1(x) sin\alpha + I_2(x) sin\beta \tag{3}$$

The light intensities $I_1(x)$ and $I_2(x)$ at a point x along the road are given by

$$I_1(x) = \frac{P_1}{r_1^2} = \frac{P_1}{h_1^2 + x^2} \tag{4}$$

$$I_2(x) = \frac{P_2}{r_2^2} = \frac{P_1}{h_2^2 + (s-x)^2} \tag{5}$$

where (1) and (2) are used in place of r_1 and r_2. The sine of α and β are found through elementary trigonometry

$$sin\alpha = \frac{h_1}{\sqrt{h_1^2 + x^2}} \tag{6}$$

$$sin\beta = \frac{h_2}{\sqrt{h_2^2 + (s-x)^2}} \tag{7}$$

As a result a final expression for Z(x) yields

$$Z(x) = \frac{P_1 h_1}{\sqrt{(h_1^2 + x^2)^3}} + \frac{P_2 h_2}{\sqrt{(h_2^2 + (s-x)^2)^3}} \tag{8}$$

Finding the Minimally Illuminated Point
To obtain the point of minimal illumination we look to take the derivative of $Z(x)$ and find the roots between $0 \le x \le s$. $Z'(x)$ can be found by hand and is:

$$Z'(x) = -3\frac{P_1 h_1 x}{(h_1^2 + x^2)^{\frac{5}{2}}} - \frac{3}{2}\frac{P_2 h_2(-2s + 2x)}{(h_2^2 + (s-x)^2)^{\frac{5}{2}}} \tag{9}$$

Considering the following physical situation where: $P_1 =$ 2000W, $P_2 =$ 3000W, $h_1 =$ 5m, $h_2 =$ 6m and $s =$ 20m then one can plot the total illumination, the derivative of the illumination and the illumination attributed to just source one and source two independently on a graph [fig.1].

Root Bisection
From inspection of the graph one can determine a reasonable interval between where the root of the derivative of the illumination should lie. This interval was set to be between 7m and 13m and using an interval bisection algorithm the root was numerically evaluated to be $x = 9.34m$ and as a result the minimally illuminated point of the road with the physical parameters given above is $9.34m$ from the origin. The tolerance of the bisection scheme was set as $\epsilon = 0.001$ such that

$$|x_2 - x_1| \le \epsilon \tag{10}$$

We see thus that the original interval Δx is reduced to $\Delta x/2$ after a single bisection and thus after two bisections it is further reduced to $\Delta/2^2$. Hence after n bisections $\frac{\Delta x}{2^n}$. Setting $\frac{\Delta x}{2^n} = \epsilon$ and solving for n yields:

$$n = \frac{ln\frac{\Delta x}{\epsilon}}{ln2} \tag{11}$$

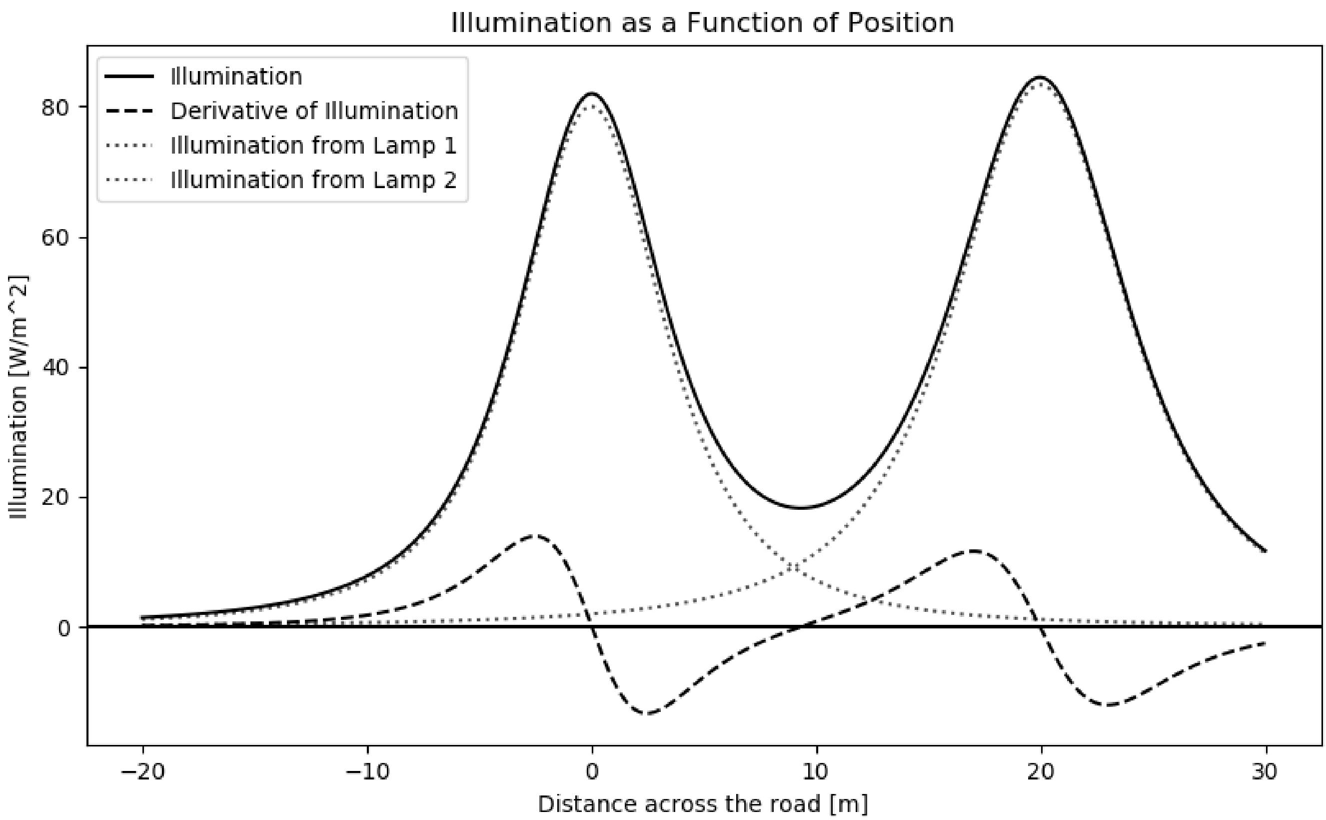


Figure 1: Illumination as a function of position

Varying h_2 to Maximise Illumination
The problem is now extended such that we now consider the optimum height of the second lamp and as a result, the function for illumination is now a function of two variables $Z(x, h_2)$.
We can find the minimally illuminated point as h_2 is varied by applying the bisection approach to a range of values for h_2. The results of which are found in fig2.

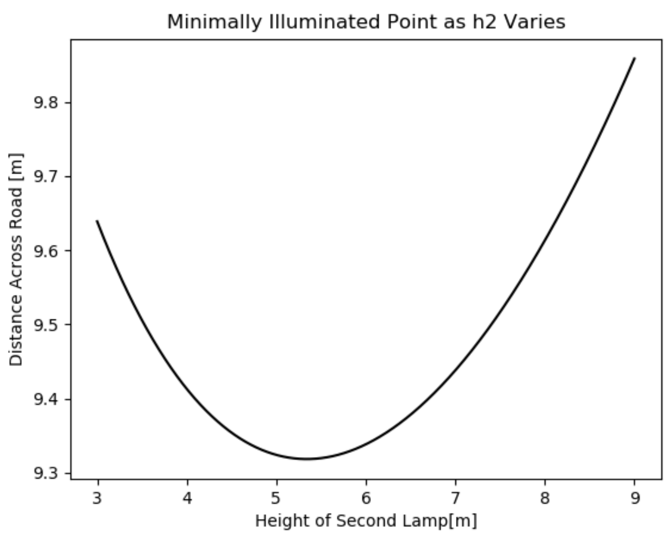


Fig2. A plot of the minimally illuminated point as h_2 varies.

Finding the Point of Maximal Illumination To find the point of maximal illumination of $Z(x, h_2)$ we search for the point where the gradient of $Z(x, h_2)$ becomes zero. The gradient can be calculated by hand yielding:

$$\nabla g = -30000\frac{x}{(25+x^2)^{\frac{5}{2}}} - 4500\frac{h_2(-40+2x}{(h_2^2+(20-x)^2)^{\frac{5}{2}}}\vec{i}, \\ \frac{3000}{(h_2^2+(20-x)^2)^{\frac{3}{2}}} - 9000\frac{h_2^2}{(h_2^2+(20-x)^2)^{\frac{5}{2}}}\vec{j} \tag{12}$$

To find the root of the gradient function for this system of equations we turn to a Newton Raphson implementation for a system of equations. Upon solving we obtain a value of $x = 9.50m$ and $h_2 = 7.42m$. To verify that these are the maximum we can analyse the eigenvalues of the Hessian of $Z(x, h_2)$ to verify a maximum has been found among all minimally illuminated points $(x(h_2), 0)$

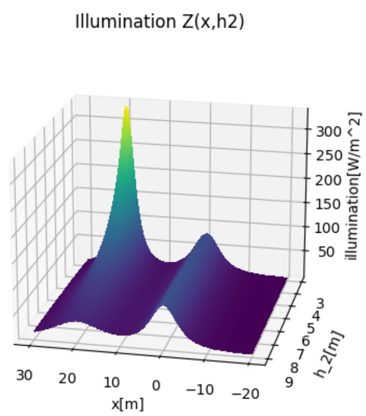


Fig3. Plot of illumination as a function of x and h_2.