I Introduction

Over the course of this investigation we look at the dynamics of a tennis ball traveling through air which we consider to be an ideal fluid for the purposes of the investigation. The ball is assumed to be travelling near to the surface of the Earth. We wish to model the flight of the tennis ball from a specific starting point as well as model the impact of "topspin" as the ball travels through the air. Topspin is a phenomenon (with which any keen tennis player is well acquainted with) where by the motion of the air around the rotating ball generates an additional force, the Magnus force, which further impacts the dynamics of the motion [1] . We shall investigate this in section II.

II Physical Scenario

The investigation pivots on the ball in question which we state to have the following key characteristics:

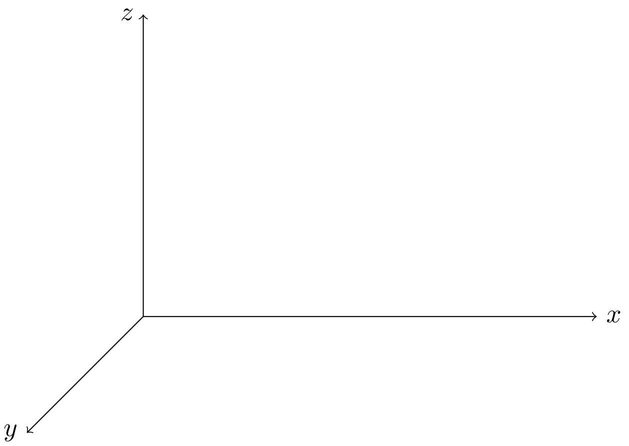
- Mass, m
- Diameter, d
- Motion is assumed to be near to the surface of the Earth

The typical mass of a tennis ball is 0.05kg, the typical diameter is taken to be 0.063**cm as stated by the international tennis federation (ITF) [2].

The tennis ball is further taken to spin with an angular velocity $\vec{\omega}$:

- $\vec{\omega}$ has the direction of the axis of rotation
- $\vec{\omega}$ has a magnitude of $\omega = \dot{\phi}(t)$ where $\dot{\phi}(t) = \frac{d\omega}{dt}$ and $\phi(t)$ is the angle of rotation.

To serve as our space for the investigation we impose a Cartesian coordinate system (x,y,z) on the surface of the Earth (such that the curvature of the Earth is considered to be negligible).



Cartesian coordinate system with labeled axis above.

III Forces

The ball traveling freely in air from a set starting point is subject to the following three key forces.

Weight Force
In our treatment of the problem we consider the weight force to be a vector quantity as a consequence of the gravitational attraction due to the nearby Earth.

$$\vec{G} = m\vec{g} \tag{1}$$

where $\vec{g} = (0, 0, -g)$ is the vector representing gravitational acceleration.

Drag Force
As the ball travels through the air there is a force opposing its forward motion due to the air pressure in front of the ball generating a reaction force. As the air pressure ahead of the ball is greater than behind the ball this force counters the forward motion and is known as the drag force. The drag is in the opposite direction to velocity $\vec{v}$:

$$\vec{D} = \frac{-D_L(v)\vec{v}}{v} \tag{2}$$

Due to the rough surface on a standard tennis ball the flow of air around the ball is considered to be turbulent at all speeds. As a result the drag coeficient can be considered to be constant at all times. From the theory of ideal fluids (Richardson et al.)[3] we obtain:

$$D_L(v) = C_D \frac{1}{2} \frac{\pi d^2}{4} \rho v^2 \tag{3}$$

where ρ is the air density which is assumed to be $1.29kgm^{-3}$. The coefficient C_D is fully analysed in section III however for the time being we shall simply state that it is dependant on the tennis ball revolution and the material on the surface of the ball.

Magnus Force
Since a key part of the investigation is considering how the spin of the ball impacts the trajectory we must factor in the Magnus force. As the ball spins in the air, the air around the ball is forced into motion. If a ball has topspin the air pressure above is decreased. A horizontally travelling tennis ball will thus experience a force downward in addition to the other previously discussed forces.

$$M_L(v) = C_M \frac{1}{2} \frac{\pi d^2}{4} \rho v^2 \tag{4}$$

Again like C_D the constant C_M depends on the rotation of the ball and the surface material and will be addressed in section IV.

IV Finding Coefficients

The coefficients C_D and C_M are dependant on the velocity of the tennis ball in the fluid (in our case air) as well the revolution of the ball as it passes through the air. The final factor impacting these coefficients is the material at the surface of the ball which impacts the balls rotation and motion through the air.

For a typical velocity range of $v \in [13.5, 30]ms^{-1}$ [4] and a ball revolution rate of $n \in [800, 3300]rpm$ [4] the coefficients C_D and C_M are variable on only v/ω.

For a standard tennis ball covered (for the majority of the surface) in a standard felt like nylon compound the following expressions represent C_D and C_M.

$$C_D = 0.508 + \left(\frac{1}{22.053 + 4.196(\frac{v}{\omega})^{\frac{5}{2}}}\right)^{\frac{2}{5}} \tag{5}$$

$$C_M = \frac{1}{2.022 + 0.981(\frac{v}{\omega})} \tag{6}$$

The constants in the expression are chosen in accordance with experimental results [reference].Finally to simplify things we neglect the deceleration of the ball's revolution as it travels through the air and treat ω as a constant.

V Governing Equations

To mathematically describe the motion of the ball as it passes through the air we turned to the trusted and ever reliable Newtonian mechanics to obtain our governing equations for the system. From Newtons second law for the position vector as a function of time $\vec{r}(t)$:

$$m\frac{d^2\vec{r}(t)}{dt^2} = -m\vec{g} - D_L\frac{\vec{v}}{v} + M_L\frac{\vec{\omega}}{\omega} \times \frac{\vec{v}}{v} \tag{7}$$

with the accompanying initial conditions for position and for velocity:

$$\vec{r}(0) = \vec{r_0} \tag{8}$$

$$\frac{d\vec{r}}{dt}(0) = \vec{v_0} \tag{9}$$

The equation proposed in (7) is a nonlinear system of three differential equations. These equations possess no analytic solution of value and as a result we turn to numerical methods to solve the equations.

Simplifications

Since in our treatment consider exclusively the impact of the presence of topspin we can make a further simplification. A ball during a topspin lob has its vector of angular velocity lying in the horizontal plane and this vector additionally is orthogonal to the velocity vector $\vec{v}(t)$ for all $t \geq 0$. As a result the flight path of the ball lies entirely in the vertical plane, allowing us to reduce the number of differential equations in our system by one.

The final form of the governing equations are then:

$$\ddot{x} = -C_D\alpha v\dot{x} + \eta C_M\alpha v\dot{z} \tag{10}$$

$$\ddot{z} = -g - C_D\alpha v\dot{z} + \eta C_M\alpha v\dot{x} \tag{11}$$

with $v = \sqrt{\dot{x}^2 + \dot{z}^2}$ and $\alpha = \frac{\rho\pi d^2}{8m}$ which accounts for the air pressure, dimensions and mass of the ball. η describes the direction of rotation for the ball with $\eta = 1$ modelling topspin. The initial conditions at t = 0 are:

$$x(0) = 0, \quad z(0) = h, \quad \dot{x}(0) = v_0cos(\theta), \quad \dot{z}(0) = v_0sin(\theta)$$

We define v_0 as the magnitude of the initial velocity and θ as the angle between the vector v_0 and the x axis.

VI Numerical Approach

To numerically solve the system obtained in section V we turn to a work horse of numerical methods, the family of Runge-Kutta methods. Specifically the fourth order Runge-Kutta method (RK4) suits our needs well, with its strong trade off on accuracy compared to its computational expense. The procedure is outlined below:

$$\vec{y}(x+h) = \vec{y}(x) + \frac{1}{6}(\vec{K_0} + 2\vec{K_1} + 2\vec{K_2} + \vec{K_3}) \tag{12}$$

with the expressions for K given by

$$\begin{aligned} \vec{K_0} &= h\vec{F}(x, \vec{y}), \\ \vec{K_1} &= h\vec{F}(x + \frac{h}{2}, \vec{y} + \frac{\vec{K_0}}{2}) \\ \vec{K_2} &= h\vec{F}(x + \frac{h}{2}, \vec{y} + \frac{\vec{K_1}}{2}) \\ \vec{K_3} &= h\vec{F}(x + h, \vec{y} + \vec{K_2}) \end{aligned} \tag{13}$$

where h is the step size of the simulation. The expression $\mathbf{F}(x, \mathbf{y})$ is obtained from the differential equation we are looking to solve:

$$\mathbf{y'} = \mathbf{F}(x, \mathbf{y}) \tag{14}$$

We immediately note that the RK4 method requires first order differential equations. At the present moment our system as outlined by equations (10) and (11) are two second order differential equations and as a result we need to reduce this to a system of four first order differential equations. We do this my making the substitutions:

$$v_x = \dot{x}, \quad v_z = \dot{z}$$

Following through with the substitution we get the following set of four first order differential equations which can then be numerically solved using the RK4 approach. The system now is:

$$\dot{x} = v_x \tag{15}$$

$$\dot{v_x} = -C_D\alpha v v_x + \eta C_M\alpha v v_z \tag{16}$$

$$\dot{z} = v_z \tag{17}$$

$$\dot{v_z} = -g - C_L\alpha v v_x - \eta C_M\alpha v v_x \tag{18}$$

with the initial conditions having the form:

$$x(0) = 0, \quad z(0) = h, \quad v_x(0) = v_0cos(\theta), \quad v_z(0) = v_0sin(\theta).$$

The values of the parameters for the simulation are as follows: $g = 9.81ms^{-1}$, $d = 0.063m$, $m = 0.05kg$, $\rho = 1,29kgm^{-3}$. The initial conditions are then: $h = 1m$, $v_0 = 25ms^{-1}$, $\theta = 15°$.

VII Results

For the simulation a step size of 0.01 was chosen and the trajectory of the ball with and without topspin was simulated. The results can be found in fig 1.

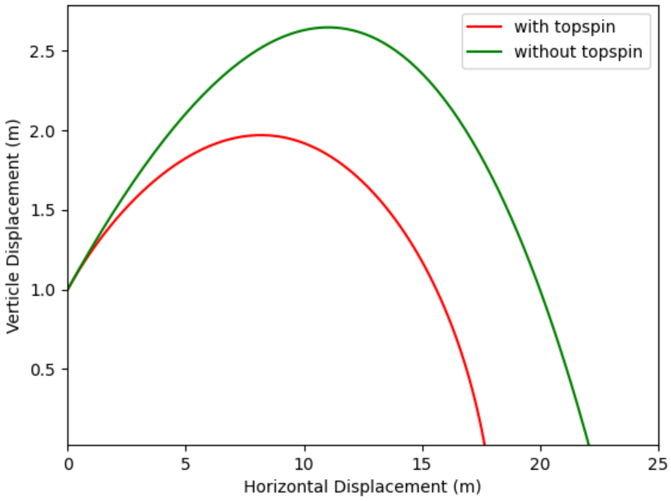


Fig.1 Trajectories with and without topspin.

From the graph we see the predicted effect of the magnus force on the ball as it passes through the air. We see that the magnus force acts downwards on a ball with topspin, translating to a shorter displacement reached before the ball hits the ground. Another insight from the model is that the angle the ball makes with the court is steeper in the case of topspin as compared to the absence of topspin.

Error

The standard implementation of the RK4 numerical solution suffers from the fact that we struggle to estimate the size of the truncation error in our calculations. As the step size was small this should not severely impact our insights from the model. It however is always prudent to ensure the system under investigation is stable such that the local errors do not accumulate catastrophically and that the global error remains abounded. An way to give improved confidence in our model would be to turn to an adaptive Runge Kutta implementation allowing us to evaluate the truncation error in each step and alter the step size to compensate.[5]

References

[1] P. Denis3 R. Cayzac E. Carette2 and P. Guillen. "MAGNUS EFFECT: PHYSICAL ORIGINS AND NUMERICAL PREDICTION". In: *26TH INTERNATIONAL SYMPOSIUM ON BALLISTICS MIAMI, FL,* (SEPTEMBER 12–16, 2011). DOI: 10.1115/1.4004330.

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[5] Jaan Kiusalaas. "Numerical Methods in Engineering with Python 3". In: (2013).