$$N(A) := \sum_{n \in \mathbb{N}} \mathbb{1}\{S_n \in A\}$$

$$N(\phi) = 0$$

Then $N : \Omega \to \mathbb{Z}_+^{B(x)}$ is called a counting process

$$\omega \mapsto (B(x) \to \mathbb{Z}_+)$$

Defn: A counting process is simple if the underlying pt. process is simple

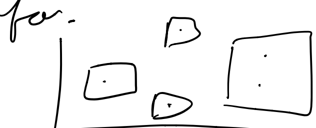
Ex:- $\mathcal{X} = \mathbb{R}_+$ $\quad B(x) = \sigma((0,t]; t \in \mathbb{R}_+)$

$$N(A) = N(0,t] = \sum_{n \in \mathbb{N}} \mathbb{1}\{S_n \in (0,t]\}$$

$$N(t) = \sum_{n \in \mathbb{N}} \mathbb{1}\{S_n \le t\}$$

$$N : \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$$

Remark: (i) Pt. processes and counting process N have same info.



(ii) The dist. of point processes S is characterized by the finite-dim dist. of $(N(A_1), N(A_2), N(A_3) \dots N(A_k))$ where $A_1, A_2 \dots A_k$ are bounded for some finite $k \in \mathbb{N}$.

$$P(N(A_1) \dots\dots N(A_k)) = ?$$

Thm: Dist. of S is completely characterized by void prob. $P\{N(A) = 0\}$ $\forall A \in B(x)$.

$\{0,1\}$

2. Poisson Point processes.

Defn:- $N : \Omega \to \mathbb{Z}^+$ is poisson if

$$P\{N = n\} = e^{-\lambda} \frac{\lambda^n}{\lfloor n}, \text{ for } \lambda > 0, n \in \mathbb{N}$$

Rem:- $\mathbb{E}N = \text{Var } N = \lambda$

$$M_n(t) = \mathbb{E}e^{tN} = e^{-\lambda}\sum_{n\in\mathbb{N}} e^{nt}\frac{\lambda^n}{\underline{n}}$$

$$= e^{-\lambda}\sum_{n\in\mathbb{N}}\frac{(\lambda e^t)^n}{\underline{n}}$$

$$= e^{\lambda(e^t-1)}$$

iii) $|A| = \int_{x\in A} dx = \int_{x\in A} dx_1 dx_2 \dots dx_d$

Vol. of the sol. $A \in B(\mathbb{R}^d)$

$$\underbrace{\Lambda(A)}_{\text{intensity measure}} = \int_{x\in A} \underbrace{\lambda(x)}_{\text{intensity density}} dx = \int_{x\in A}\lambda(x)\, dx_1 \dots dx_d$$

$\lambda : \mathbb{R}^d \to \mathbb{R}_+$

If $\lambda(x) = \lambda \ \forall x \in \mathbb{R}^d$, then $\Lambda(A) = \lambda|A|$

Check further from notes.

Defn:- The poisson pt. process $S : \Omega \to x^{\mathbb{N}}$ of intensity measure $\Lambda : B(\mathbb{R}) \to \mathbb{R}_+$ is defined by the finite dim. dist. of $N : S \mapsto \mathbb{Z}_+^{B(x)}$

$$P\{N(A_1) = n_1, \dots, N(A_k) = n_k\} = \prod_{i=1}^{k} e^{-\Lambda(A_i)}\frac{\Lambda(A_i)^{n_i}}{n_i!}$$

for any $k \in \mathbb{Z}_+$, $(n_1, \dots n_k) \in \mathbb{Z}_+^k$

$\forall$ bounded & mutually disjoint $A_1, \dots A_n \in B(x)$

If $\Lambda(A) = \lambda|A|$ then S is a "homogeneous Poisson Point process" and λ is its intensity

3. Equivalent characterizations

Defn: A counting process $N : \Omega \to \mathbb{Z}_+^{B(x)}$ has "complete indep. prop." (CIP) if for any collection of finite disjoint & bounded sets $A_1, A_2 \dots A_k \in B(x)$,

the vector $(N(A_1) \dots, N(A_k))$

Theorem [Equivalence] Following are equivalent for a simple counting process $N : \Omega \to \mathbb{Z}_+^{\mathcal{B}(\mathbb{R})}$

i) N is Poisson with locally finite measure Λ.

ii) For each $A \in \mathcal{B}(\mathbb{R})$ $P\{N(A) = 0\} = e^{-\Lambda(A)}$

iii) " " " $N(A) \sim \text{Poisson}(\Lambda(A))$

(iv) N has Complete Independence property and $\mathbb{E} N(A) = \Lambda(A)$

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii) $P\{N(A) = 0\} = e^{-\Lambda(A)}$

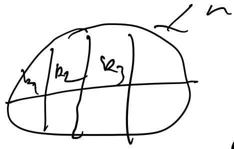
(ii) $\Rightarrow$ (iii) void probabilities give the entire distribution

$$1 - P\{N(A) = 0\} = \sum_{k \in \mathbb{N}} P\{N(A) = k\}$$

$$= 1 - e^{-\Lambda(A)} = e^{-\Lambda(A)} \sum_{k \in \mathbb{N}} \frac{\Lambda(A)^k}{k!}$$

(iii) $\Rightarrow$ (iv) Mean: $\mathbb{E} N(A) = \Lambda(A)$

CIP: $A_1, \ldots, A_k$ disjoint & bounded $\in \mathcal{B}(\mathbb{R})$



$$A = \bigcup_{i=1}^{k} A_i$$

$$N(A) = \sum_{i=1}^{k} N(A_i)$$

$s_1, s_2, \ldots s_n \in A$ $$\mathbb{E} N(A) = \sum_{i=1}^{k} \mathbb{E} N(A_i)$$

$$\Lambda(A) = \sum_{i=1}^{k} \Lambda(A_i)$$

$$P\{N(A)=n\} = P\left(\bigcup \{N(A_i)=n_i\}\right)$$

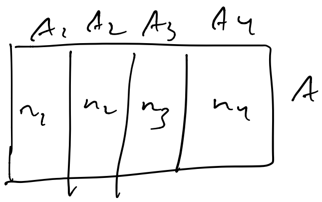
Lec: 25 Poisson Processes: Conditional Distribution

Let $x = \mathbb{R}^d$ d-dim Euclidean space

Prop:

Let $k \in \mathbb{N}$. For a poisson pt-process $S: \Omega \to x^{\mathbb{N}}$ with

consider a bdd subset $A \in \phi(x)$ $A_1, A_2, \dots A_n \in \phi(x)$



Proof: $$LHS \equiv \frac{P(\{N(A_1) = n_1, \dots N(A_k) = n_k, N(A) = n\})}{P\{N(A) = n\}}$$

$A \subseteq B$
$[A \cap B = A]$

$$= \frac{P\left(\bigcap_{i=k}^{k} \{N(A_i) = n\}\right)}{e^{-\Lambda(A)} \frac{\Lambda(A)^n}{\lfloor n}}$$

$$= \frac{\lfloor n \left(\prod_{i=1}^{k} e^{-\Lambda(A_i)} \frac{\Lambda(A_i)^{n_i}}{\lfloor n} \right)}{e^{-\Lambda(A)} \Lambda(A)^n}$$

$$= \binom{n}{n_1 n_2 n_3 - n_k} \prod_{i=1}^{k} \left[\frac{\Lambda(A_N)}{\Lambda(A)} \right]$$

Do all this from the notes again.

Remarks:

1. $N(A) = \sum_{i=1}^{k} (N(A_i))$

$$\mathbb{E}\, N(A) = \sum_{i=1}^{k} \mathbb{E}\, N(A_i)$$

$$\Lambda(A) = \sum_{i=1}^{k} \Lambda(A_i)$$

2 $\quad p_i := \dfrac{\Lambda(A_i)}{\Lambda(A)} \in [0, 1]$

$$(p_1, \dots, p_k) \in \mathcal{M}([k])$$

a. $P\left(\{N(A_i) = 1\} \mid \{N(A) = 1\}\right) = \dfrac{\Lambda(A_i)}{\Lambda(A)}$

$$S = (S_1, \dots, S_n \dots)$$

$$S \equiv \{S_1, \dots \dots S_n \dots\}$$

$$P\left(\{|S \cap A_i| = 1\} \mid \{|S \cap A| = 1\}\right) = \frac{\Lambda(A_i)}{\Lambda(A)}$$

$$P\left(\{N(A_i) = 1\} \mid \{N(A) = 1\}\right)$$

Let's call the pt. of S in A to be in S_1

then, $\quad p_i = P\left(\{S_1 \in A_i\} \mid \{S_1 \in A\}\right) = \dfrac{\Lambda(A_i)}{\Lambda(A)}$

Let $\{N(A) = n_i\}$ we denote $S_1, \dots S_{n_i}$ be the points in A,

then $\quad P\left(\{N(A_i) = n_i\} \mid \{N(A) = n_i\}\right) = \left[\dfrac{\Lambda(A_i)}{\Lambda(A)}\right]^{n_i}$

$$P\left(\{|S \cap A_i| = n_i\} \mid \{|S \cap A| = n_i\}\right) = p_i^{n_i}$$

$$P\left(\bigcap_{j=1}^{n_i} \{S_j \in A_i\} \mid \{S_1, \dots S_{n_i} \in A\}\right) = p_i^{n_i}$$

$$= \prod_{j=1}^{n_i} P\left(\{S_j \in A_i\} \mid \{S_j \in A\}\right)$$

Diff from CIP which says that

$N(A_1), N(A_2) \dots N(A_k)$ are indep.

Pts. in A_i are indep placed. given that they are in A.

v. $\quad P\left(\bigcap_{i=1}^{k} \underset{\{|S \cap A_i| = n_i\}}{\underset{\shortparallel}{\{N(A_i) = n_i\}}} \mid \{N(A) = n\}\right) = \binom{n}{n_1, \dots n_k} p_1^{n_1} \cdots p_k^{n_k}$

Let's call pt.s in A $S_1, \dots S_n$

Let's call $E_1, E_2 \dots E_k$ be partitions of $S \cap A$

s.t $|E_i| = n_i$

Let $E := S \cap A \subseteq N$ s.t $|E| = n$

$$P\left(\bigcap_{i=1}^{k} \{ S \cap A_i = E_i \} \,\middle|\, \{ S \cap A = E \} \right)$$

$$\left[\mathbb{1}^{\,n}_{n_1 + n_2 + \dots n_k = n} \right]$$

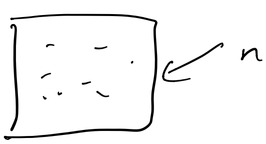
$[S_1\ S_2\ S_3\ S_4]$

$\{S_1, S_2\}$ $\{S_3, S_4\}$

$\vdots$ $\{S_2, S_4\}$

$$\sum_{(E_1, \dots E_k)} P\left(\bigcap_{i=1}^{k} \{ S \cap A_i = E_i \} \,\middle|\, \{ S \cap A = E \} \right) = P\left(\bigcap_{i=1}^{k} \{ N(A_i) = n_i \} \,\middle|\, \{ N(A) = n \} \right)$$

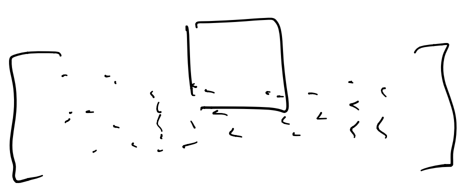
[Given that there are n points in A, the location of these points are iid in A with distribution $\frac{\Lambda(\cdot)}{\Lambda(A)}$]



$$\begin{bmatrix} P(X^{-1}(-\infty, x]) \\ P\{ X \le x \} \\ P\{ s_i \in A_i \} = \frac{\Lambda(A_i)}{\Lambda(A)} \end{bmatrix}$$

viii- If the Poisson process is homogeneous, then the dist. is uniform

ix- Poisson process with Λ



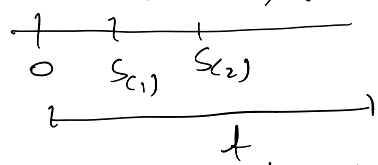
$A \in \mathcal{B}(\mathcal{X})$ a − $N(A) \sim \text{Poisson}(\Lambda(A))$

b − Given $N(A)$, each pt. is i.i.d with density $\frac{\lambda(x)}{\Lambda(A)}$, $x \in A$

$$\Lambda(A) = \int_{x \in A} \lambda(x)\, dx$$

Corollary :- $\mathcal{X} := \mathbb{R}_+$. Ordered set of points

$(S(1), S(2), \ldots\ldots : n \in \mathbb{N})$



Given $N(t) = k$, the density of $(S(1), \ldots S(k))$ is

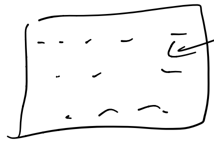
$$f_{S(1), \ldots S(k) \mid N(t) = k}(t_1, \ldots t_k) = k! \, \frac{\mathbb{1}\{0 < t_1 < t_2 \ldots < t_k\}}{t}$$

Proof: Check from notes.

Lecture 26

1. Laplace Functionals

$L_S(f) = \mathbb{E}$



$S : \Omega \to x^{\mathbb{N}}$ simple

$dx = ?$ $\quad df(x) =$

$|[x, x+dx)|$ $\quad |f(x+dx), f(x)|$

$$dN(x) = N(x+dx) - N(x) = N([x, x+dx))$$

$$dN(x) = \begin{cases} 0 & x \notin S \\ \delta_x & x \in S \end{cases} \qquad S = \{S_1, S_2, \dots, S_n\}$$

For a function: $f : \underbrace{\mathbb{R}^d}_{x} \to \mathbb{R}$ $\qquad$ A is a finite Area

$$\int_{x \in A} f(x)\, dN(x) = \sum_{i \in \mathbb{N}} f(S_i) \mathbb{1}_{\{S_i \in A\}}$$

Defn: "Laplace functional" $\mathcal{L}$ of a point process $S : \Omega \to x^{\mathbb{N}}$ and associated counting process $N : \mathcal{B}(x) \to \mathbb{Z}_+$ is defined $\forall f : x \to \mathbb{R}_+$ as

$$\mathcal{L}_S(f) = \mathbb{E} \exp\left(-\int_x f(x)\, dN(x)\right)$$

Remark: For simple fn. $f(x) = \sum_{i=1}^{k} t_i \mathbb{1}_{\{x \in A_i\}}$

$$\mathcal{L}_S(f) = \mathbb{E} \exp\left(-\sum_{i=1}^{k} t_i \int_x \mathbb{1}_{\{x \in A_i\}} dN(x)\right)$$

$$= \mathbb{E} \exp\left(-\sum_{i=1}^{k} t_i N(A_i)\right)$$

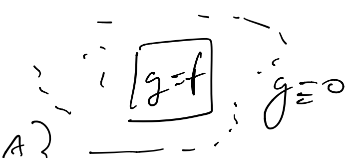
$$= \mathbb{E} \exp\left(-\langle (t_1, \dots, t_k), (N(A_1) \dots N(A_k))\rangle\right)$$

Prop: For poisson process $S: \Omega \to \mathcal{X}^{\mathbb{N}}$ with intensity measure $\Lambda : \mathbb{B}(\mathcal{X}) \to \mathbb{R}_+$

$$\mathcal{L}_S(f) = \exp\left(-\int_{\mathcal{X}} (1 - e^{-f(x)})\, d\Lambda(x)\right)$$

Proof :- Let $A \in \mathcal{B}(\mathcal{X})$ bounded.

$$g : \mathcal{X} \to \mathbb{R}_+$$
$$x \longmapsto f(x)\, \mathbb{1}_{\{x \in A\}}$$



$$\mathcal{L}_S(g) := \mathbb{E} \exp\left(-\int_{\mathcal{X}} g(x)\, dN(x)\right)$$
$$= \mathbb{E} \exp\left(-\int_A f(x)\, dN(x)\right)$$
$$= \mathbb{E}\left\{ \exp\left(-\sum_{i \in \mathbb{N}} f(S_i)\, \mathbb{1}_{\{S_i \in A\}}\right)\right]$$

$$P\{N(A) = n\} = e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!}$$

$$f_{S_1, S_2 \cdots S_n | N(A) = N}(x_1, x_2 \cdots x_n) = \prod_{i=1}^{n} \left[\frac{d\Lambda(x_i)}{\Lambda(A)}\right] \mathbb{1}_{\{x_i \in A\}}$$

$$\mathcal{L}_S(g) \underset{\text{tower prop}}{=} \mathbb{E}\left\{ \mathbb{E}\left[\exp\left(-\sum_{i \in \mathbb{N}} f(S_i)\, \mathbb{1}_{\{S_i \in A\}}\right) \Big| N(A)\right]\right]$$

$$= \sum_{n \in \mathbb{Z}^+} P\{N(A) = n\}\, \mathbb{E}\left[\exp\left(-\sum_{i \in \mathbb{N}} f(S_i)\, \mathbb{1}_{\{S_i \in A\}}\right) \Big| N(A) = n\right]$$

$$= \sum_{n \in \mathbb{Z}^+} e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!}\, \mathbb{E}\left[\prod_{i=1}^{n} e^{-f(S_i)} \Big| S \cap A = \{S_1, \cdots S_n\}\right]$$

$$\overset{\text{cdnl indep}}{=} \sum_{n \in \mathbb{Z}^+} e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!} \prod_{i=1}^{n} \mathbb{E}\left[e^{-f(S_i)} \,|\, S_i \in A\right]$$

$S_1, S_2, \cdots S_n$
$|S \cap A| = n$

$$\left(f_{S_1, \cdots S_n | S \cap A = \{S_1 \cdots S_n\}} = \prod_{i=1}^{n} f_{S_i | S \in A}\right)$$

$$= \sum_{n\in\mathbb{Z}_+} e^{-\Lambda(A)} \frac{\Lambda(A)^n}{n!} \left[\int e^{-f(x)} \frac{d\Lambda(x)}{\Lambda(x)}\right]$$

$$= \sum_{n\in\mathbb{Z}_+} e^{-\Lambda(x)} \frac{\beta_f(A)^n}{n!} \left[\beta_f(A) := \int_{x\in A} e^{-f(x)}\, d\Lambda(x)\right]$$

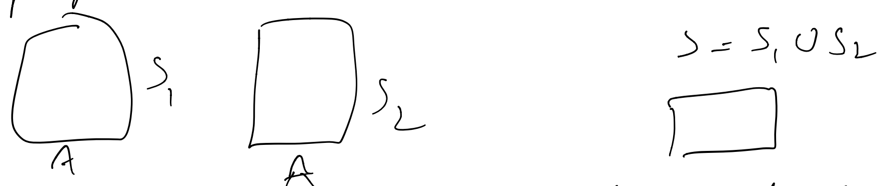
$$= e^{-\Lambda(A)+\beta_f(A)}$$

$$= \exp\left(-\int_A \left(1-e^{-f(x)}\right) d\Lambda(x)\right)$$

Result follows from taking $A_k \uparrow X$ & M.C.T

$\left(\int e^{-f(x)}\, dN(x) \Rightarrow \int e^{-f(x)}\, d\Lambda(x)\right)$

1.1 Superposition



$\bigcup_{N\in\mathbb{N}} S$ $\quad S_1, S_2, \ldots S^{(k)} \ldots$ are simple point processes

$\sum_{i\in\mathbb{N}} N_i \in N_1, N_2 - N_k \ldots$

$\sum_{i\in\mathbb{N}} \Lambda_i \in (\Lambda_1 --- \Lambda_k ---$

Thm: Superposition of indep Poisson point process with (Λ_k) is Poisson pt. process with $\sum_k \Lambda_k$ iff $\sum_k \Lambda_k$ is locally finite.

Proof: $\mathcal{L}_S(f) = \mathbb{E}\exp\left(-\int_X f(x) \sum_k dN_k(x)\right)$

$$= \mathbb{E}\left[\prod_k \exp\left(-\int_X f(x)\, dN_k(x)\right)\right]$$

$$\underset{\uparrow\ \text{independence}}{=} \prod_k \underbrace{\mathbb{E} \exp\left(-\int_x f(x)\, dN_k(x)\right)}_{\mathcal{L}_{\Phi_k}(f)}$$

$$= \prod_k \exp\left(-\int_x \left(1 - e^{-f(x)}\right) d\Lambda_k(x)\right)$$

$$= \exp\left(-\int_x \left(1 - e^{-f(x)}\right) d\underbrace{\sum_k \Lambda(x)}_{d\Lambda}\right) \blacksquare$$

1.2 Thinning

Probabilistic retention function : $p: \mathcal{X} \to [0,1]$

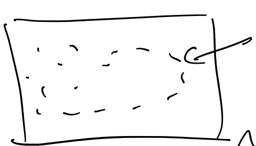
Simple point process $S: \Omega \to \mathcal{X}^{\mathbb{N}}$

The thinning of simple point process $S: \Omega \to \mathcal{X}^{\mathbb{N}}$ with ret. function p is a simple pt. process $S^p: \Omega \to \mathcal{X}^{\mathbb{N}}$

$$S^p = (S_n \in S : Y(S_n) = 1)$$

where $(Y(S_n) : n \in \mathbb{N})$ indep. r.v.s with

$$\mathbb{E}[Y(S_n) \mid S_n] = p(S_n)$$



$Y(S_n)$ is the indicator of retention of point S_n in S^p

Thm: For a Poisson pt. process $S: \Omega \to \mathcal{X}^{\mathbb{N}}$ with intensity measure $\Lambda: \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$, and ret. prob. fn. $p: \mathcal{X} \to [0,1]$ the S^p is Poisson with intensity measure

$$\Lambda^p : \mathcal{B}(\mathcal{X}) \to \mathbb{R}_+$$

$$A \longmapsto \Lambda^p(A) = \int_A p(x)\, d\Lambda(x)$$

Proof:- $\mathcal{L}_S(f) = \mathbb{E} \exp\left(-\int_x f(x)\, dN^p(x)\right)$

$A \in \mathcal{B}(\mathcal{X})$ bdd $\quad N^p(A) = \sum_{i \in \mathbb{N}} \mathbb{1}_{\{S_i \in A\}} Y(S_i)$

$$N^p(A) = \sum_{S_i \in A} Y(S_i) \qquad dN^p(x) = \begin{cases} 0 & x \notin S \\ \delta_x Y(x) & x \in S \end{cases}$$

$$\int_A f(x)\, dN^p(x) = \sum_{S_i \in A} f(S_i)\, Y(S_i)$$

$$\mathcal{L}_S(g) = \mathbb{E}\left[e^{-\int_A f(x)\, dN^p(x)} \right] \quad [g = f 1_A]$$

$$\underset{\text{tower property}}{=} \mathbb{E}\left[\mathbb{E}\left[e^{-\int_A f(x)\, dN^p(x)} \,\middle|\, N(A) \right] \right]$$

$$= \sum_{n \in \mathbb{Z}_+} P\{N(A) = n\}\, \mathbb{E}\left[e^{-\sum_{i=1}^{n} f(S_i)\, Y(S_i)} \,\middle|\, N(A) = n \right]$$

$$= \sum_{n \in \mathbb{Z}^+} P\{N(A) = n\} \prod_{i=1}^{n} \mathbb{E}\left[e^{-f(S_i) Y(S_i)} \,\middle|\, S_i \in A \right]$$

$$= \sum_{n \in \mathbb{Z}^+} P(N(A) = n) \prod_{i=1}^{n} \mathbb{E}\left[e^{-f(S_i)} p(S_i) + (1 - p(S_i)) \,\middle|\, S_i \in A \right]$$

$$\left[e^{-f(S_i) \cdot 1} p(S_i) + e^{-f(S_i) \cdot 0} (1 - p(S_i)) \right]$$

$$= e^{-\Lambda(A)} \sum_{n \in \mathbb{Z}_+} \frac{\Lambda(A)^n}{L n} \left[\int_A \left[e^{-f(x)} p(x) + (1 - p(x)) \right] \frac{d\Lambda(x)}{\Lambda(A)} \right]$$

$$= e^{-\Lambda(A)} + \int_A \left[e^{-f(x)} p(x) + (1 - p(x)) \right] d\Lambda(x)$$

$$= \exp\left(- \int_A (1 - e^{-f(x)})\, \underbrace{p(x)\, d\Lambda(x)}_{d\Lambda^p(x)} \right)$$

Lecture 27: Poisson Processes on half-line

L Simple point process on the half-line.

Defn: A st. process on the half-line $N: \Omega \to \mathbb{Z}^{\mathbb{R}_+}$ is a "counting process" if

(i) $N_0 = 0$

(ii) for each $\omega \in \Omega$, N is non-decreasing, integer-valued & right continuous with left limit.

Lemma: A counting process has finitely many discontinuities in $(0,t]$

Defn: A simple counting process has unit jumps a.s.

Remark: Following discussion doesn't apply to higher dimensions.

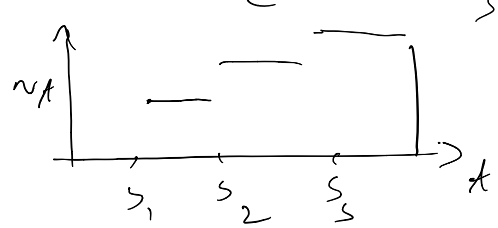
Defn: n^{th} arrival instant $[S_0 = 0]$

$S_n := \inf\{ t > 0 : N_t \geq n \}$

Inter arrival time $X_n = S_n - S_{n-1}$

Simple pt. process: $P\{X_n = 0\} = P\{X_n \leq 0\} = 0$

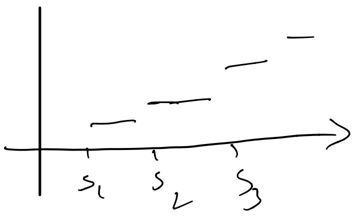
Lemma: $\{S_n \leq t\} = \{N_t \geq n\}$



Proof: Let $\omega \in \{S_n \leq t\}$ then $N_{S_n} = n \leq N_t$ ($S_n \leq t$)

1st part proven: $\{S_n \leq t\} \subseteq \{N_t \geq n\}$

Conversely: Let $\omega \in \{N_t \geq n\}$, then $S_n \leq t$



$$S_n := \inf\{ t \in \mathbb{R}_+ : N_t \geq n \} \leq t$$

$$\{N_t \geq n\} \leq \{S_n \leq t\}$$

Corollary: $\{S_n \leq t, \ S_{n+1} \geq t\} = \{N_t = n\}$

Proof: from the notes

Lemma: $P_n(t) := P\{N(t)=n\} = \underbrace{P\{S_n \leq t\}}_{F_n(t)} - \underbrace{P\{S_{n+1} \leq t\}}_{F_{n+1}(t)}$

Proof:- $\{S_n \leq t\} = \{S_n \leq t, S_{n+1} \leq t\} \cup \{S_n \leq t, S_{n+1} > t\}$

$$= \{S_{n+1} \leq t\} \cup \{S_n \leq t, S_{n+1} > t\}$$

$$F_n(t) = F_{n+1}(t) + P\{N(t) = n\} \quad ∎$$

2- IID exp. inter-arrival time characterization

$$P\{N(A_1) = k_1, N(A_2) = k_2\} = \binom{k_1+k_2}{k_1, \ k_2} e^{-\Lambda(A_1)} \frac{\Lambda(A_1)^{k_1}}{k_1!} e^{-\Lambda(A_2)} \frac{\Lambda(A_2)^{k_2}}{k_2!}$$

Prop: The counting process associated with a Poisson pt. process on half-time is Markov.

Proof: $\mathcal{F} = (\mathcal{F}_t \subseteq \mathcal{F} : t \in \mathbb{R}_+)$

$$\mathcal{F}_t = \sigma(N_s : s \leq t)$$

Let $H_s \in \mathcal{F}_s$, $P(\{N_t = n\} \mid H_s \cap \{N_s = k\})$

$$\overset{\text{indep increments}}{=} P(\{N_t - N_s = n-k\} \mid H_s \cap \{N_s = k\})$$

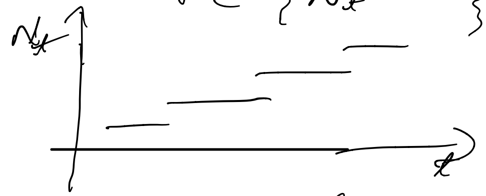
$$= P(\{N_t - N_s = n-k\})$$

$= P(\{N_t = n\} \mid \{N_s = k\})$

$N(s,t)$ is independent of $N(0,s]$

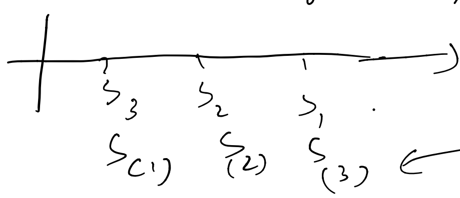
For a homogeneous Poisson pt. process

$$P(\{N_t = n\} \mid \{N_s = k\}) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{n-k}}{\lfloor n-k}$$



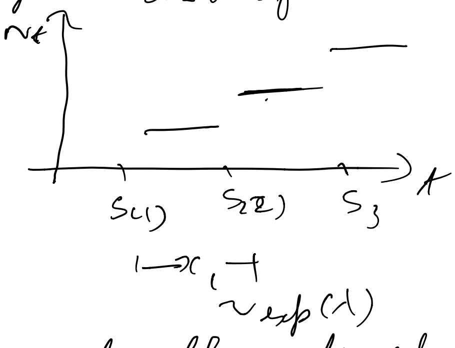
Thm: The counting process associated with a Poisson pt. process on half-line is Strongly Markov.

[Remark: The n^{th} arrival time is the n^{th} order statistic of the point process]



all S_t above are actually $S_{(t)}$, the S_t given here is different.

Prop: A simple counting process $N: \Omega \to \mathbb{Z}_+^{\mathbb{R}_+}$ is a homog Poisson process with a finite pos. rate λ, iff the inter-arrival time seq. $X: \Omega \to \mathbb{R}_+^{\mathbb{N}}$ is iid with expon. dist. of rate λ.



Proof: ($\Leftarrow$) $P\{N_t = 0\} = e^{-\lambda t}$ then N is the counting process associated with homog. Poisson Process.

Since, $\{N_t = 0\} = \{X_1 > t\}$

$P\{N_t = 0\} = P\{X_1 > t\} = e^{-\lambda t}$

($\Rightarrow$) It suffices to show that X is IID:

[Since $X_1 \sim exp(\lambda)$ from homog. Poisson Process]

(i) N is strongly Markov

(ii) $S_{(1)}, S_{(2)}, \ldots$ are all stopping times.

From SMP, $(N_{S_{(n)}+t} - N_{S_{(n)}} : t \geq 0)$ indep of $\sigma(N_s : s \leq S_{(n)})$

$$S_{(n+1)} = \inf\{t > 0 : N_t \geq n+1\}$$
$$= \inf\{t > S_{(n)} : N_t \geq n+1\}$$
$$= S_{(n)} + \inf\{t > 0 : N_{S_{(n)}+t} \geq n+1\}$$
$$= S_{(n)} + \inf\{t > 0 : N_{S_{(n)}+t} - N_{S_{(n)}} \geq 1\}$$

$$X_1 = \inf\{t > 0 : N_t \geq 1\}$$

(i) X_{n+1} is indep. of $\sigma(N_s : s \leq S_{(n)})$

(ii) X_{n+1} is identically dist. to X_1

$[N_{S_{(n)}+t} - N_{S_{(n)}} \stackrel{d}{=} N_t]$ stationary of incr.

$[S_{(n)} = \sum_{i=1}^{n} X_i]$

Lemma 2.2 For any time $t > 0$, a Poisson Process is finite a.s.

Proof: $\lim_{n\to\infty} \frac{1}{n} S_{(n)} = \mathbb{E}X_1 = \frac{1}{\lambda}$ a.s SLLN

Proof from notes. Rest from Lecture Notes.