Lecture - 1

Let A, B be sets:

We denote the cardinality of the sets to be $|A|$.

Notation $[N] := \{1, 2, 3 \dots N\}$

$\mathbb{N} = \{1, 2, 3 \text{---} N\}$ set of natur-al numbers

$\mathbb{Z}$ set of integers

$\mathbb{Q}$ set of rational numbers.

$\mathbb{Z}_+ = \{0, 1, 2 \dots\}$

$\mathbb{R}_+$ = +ve real numbers.

Any set A which is bijective to a subset of $\mathbb{N}$ is called countable. (one one & onto)

Any set which has a finite cardinality is called a countably finite set.

Any set which is bijective to the set of natural numbers $\mathbb{N}$ is called a countably infinite set.

Let A, B be sets.

We denote a function $f : B \to A$ by $f \in A^B$ where $(b, f(b)) \in B \times A$ and for each $b \in B$, there is only one value $f(b) \in A$.

The set B is called the domain of fn f and the set $\{f(b) \in A : b \in B\}$ is called the range of function f.

The collection of all A valued func with domain A is called A^B.

Ex:- $\underset{B}{\{a, b, c, d\}} \to \underset{A}{\{1, 2, 3\}}$ $|B| = 4$, $|A| = 3$

$|B \times A| = 12$

$A^B = \{ ((a,1),(b,1),(c,1),(d,1)), \ldots$

So on all sets of functions that are possible.

$|A|^{|B|} = 3^4$ because each element in the domain has 3 values possible, i.e. $3 \times 3 \times 3 \times 3$.

$$A^{[N]} = \{ f : [N] \to A \mid f \text{ is fn} \}$$
$$= \{ (\underset{\in A}{f(1)}, \ldots, \underset{\in A}{f(N)}) : f \text{ is a fn.} \}$$
$$= \{ N\text{-length seq. } A \text{ valued} \}$$

$A^{[N]}$ is collection of A valued N-length sequences N-length sequences.

$A^{\mathbb{N}}$ is a collection of all A-valued countably infinite sequences.

$a \in A^{\mathbb{N}}$, $a_1 \in A$, $a_2 \in A$, ... $a_n \in A$

$f \in A^{\mathbb{N}}$: $f(1) \in A$, $f(2) \in A$, ... $f(N) \in A$

2 - Sample Space

Coin toss where the outcomes are heads and tails and denoted by H & T respectively. $\{H, T\}$

Defn. { The set of all possible outcomes of of a random experiment is called sample space, and is denoted by Ω. }

Ex. 1. Single coin toss. $\Omega = \{H, T\}$

2 - [Finite coin toss] : Toss N times)

$\Omega = \{H, T\}^{[N]}$ $\quad \omega \in \Omega$ $\quad \omega_1 \in \{H, T\}$ $\quad \omega_2 \in \{H, T\}$ .... $\omega_n \in \{H, T\}$

3. countably ∞ coin tosses

$\Omega = \{H, T\}^{\mathbb{N}}$ $\quad w_i \in \{H, T\}$ $\forall i \in \mathbb{N}$

4. [Point on a +ve real line]

$\Omega = \mathbb{R}_+$

5. [Countable points on a +ve real line]

$\Omega = \mathbb{R}_+^{\mathbb{N}}$

$\omega \in \Omega$, then $\omega = (\omega_1, \omega_2 \ldots)$

$\omega_i \in \mathbb{R}_+$ for all $i \in \mathbb{N}$.

3. Event - Space

Defn: A collection of subsets of sample space Ω is called an event space if it is a σ-algebra over subsets of Ω, and is denoted by $\mathcal{F}$.

In other words the collection $\mathcal{F}$ satisfies the following conditions

1. $\underbrace{\Omega}_{\text{sure / certain event}} \in \mathcal{F}$
2. If $A \in \mathcal{F}$, $A^c \in \mathcal{F}$
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$.

The elements of the event space are called events.

Remark 1. $\mathcal{F} \subseteq 2^{\Omega} = \{A : A \subseteq \Omega\}$

2. $\phi \in \mathcal{F}$

$A = \{1, 2, 3\}$ $\quad 2^{\Omega} = \{\phi, \{1, 2, 3\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$

Remark 2: $\phi \in \mathcal{F}$

Remark 3: If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$

Remark 4: If $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcap_{i \in \mathbb{N}} A_i \in \mathcal{F}$

Ex. - $\mathcal{F} : \{\phi, \Omega\}$ it is an eventspace

2 - (Single coin toss) $\Omega = \{H, T\}$

$\mathcal{F} = \{\phi, \{H, T\}, \{H\}, \{T\}\}$

$= 2^{\Omega}$

3. {Finite coin tosses} $\Omega = \{H, T\}^{[N]}$

$\mathcal{F} = 2^{\Omega} = \{A : A \subseteq \Omega\}$

Remark: - It is possible that $\mathcal{F} \neq 2^{\Omega}$

i.e there can be a $\mathcal{F} \subset 2^{\Omega}$

i.e There exists $A \subset \Omega$, $A \notin \mathcal{F}$

Defn. consider a sample space Ω and a family $F \subseteq 2^{\Omega}$ of subsets of Ω.

Then the event space generated by F is the smallest event space containing each element of F, and is denoted $\sigma(F)$.

Refer to PPs lecture notes.

Lecture 2

Disjoint sets/events: Let $(\Omega, \mathcal{F})$ be a pair of sample space & event space, and a sequence of events $(A_n \in \mathcal{F} : n \in \mathbb{N})$

The sequence is mutually disjoint if $A_n \cap A_m = \emptyset \ \forall \ n \neq m \in \mathbb{N}$

- If $\bigcup_{n \in \mathbb{N}} A_n = \Omega$ then $(A_n : n \in \mathbb{N})$ is a partition of sample space.

Ex: $(\Omega, \mathcal{F})$ pair of sample & event space.
If $(A_n \in \mathcal{F} : n \in \mathbb{N})$ be a seq. of events.
Sequence of events means that it is countably ∞.

1. $\mathbb{1}\{x_n \in \bigcap_{n \in \mathbb{N}} A_i\} = \prod_{i \in \mathbb{N}} \mathbb{1}\{x_n \in A_i\}$

2. $\mathbb{1}\{x_n \in \bigcup_{n \in \mathbb{N}} A_i\} = \sum_{i \in \mathbb{N}} \mathbb{1}\{x_n \in A_i\}$

only if its disjoint. (mutually)

If it isn't $\mathbb{1}_{\{x_n \in \bigcup_{n \in \mathbb{N}} A_i\}} < \sum_{i \in \mathbb{N}} \mathbb{1}_{\{x_n \in A_i\}}$

Example: For any event $A \in \mathcal{F}$, let $N(A)$ denote the number of times the event A occurs in N trials.

$$N(A) = \sum_{i=1}^{N} \mathbb{1}(x_i \in A)$$

Relative frequency of $\frac{N(A)}{N} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{x_n \in A\}}$

1. $\forall A \in \mathcal{F}: 0 < \frac{N(A)}{N} \leq 1$

2. $\forall A$ $(A_i \in \mathcal{F} : i \in \mathbb{N})$ mutually disjoint

$$\frac{N(\bigcup_{i \in \mathbb{N}} A_i)}{N} = \sum_{i=1}^{\mathbb{N}} \frac{N(A_i)}{N}$$

3. For the certain event Ω, $\frac{N(\Omega)}{N} = 1$

$0 \leq \frac{N(A)}{N} \leq 1$ $N(A)$ increases with N

$\lim_{N \to \infty} \frac{N(A)}{N}$ may exist.

Axioms of Probability

We define a probability measure P

$P: \underset{\text{event space}}{\mathcal{F}_e} \to [0,1]$ that satisfies the following axioms.

(i) (non-negativity) $A \in \mathcal{F}_e$, $P(A) \geq 0$

(ii) (σ-additivity) For an ∞ seq. $(A_n \in \mathcal{F} : n \in \mathbb{N})$ of mutually disjoint events:

$$P(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} P(A_n)$$

(iii) (certainity) $P(\Omega) = 1$

Define (Probability Space) A sample space Ω, an event space $\mathcal{F} \subseteq 2^{\Omega}$, and a probability measure $P: \mathcal{F} \to [0,1]$, together define a prob. space $(\Omega, \mathcal{F}, P)$.

"Probability is never of outcomes but of events"

2. Properties of Probability

[sample space = outcome space]

Thm: For any probability space $(\Omega, \mathcal{F}, P)$, the following prop. of P hold true.

(i) impossibility $P(\phi) = 0$

(ii) (finite additivity) For any mutually disjoint $(A_1, A_2, \ldots, A_n) \in \mathcal{F}^n$ we have $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

(iii) (monotonicity) $A, B \in \mathcal{F}$

$A \subseteq B$ then $P(A) \leq P(B)$

(iv) (inclusion-exclusion) $A, B \subseteq \mathcal{F}$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(v) (continuity) $(A_i \in \mathcal{F} : i \in \mathbb{N})$ s.t. $\lim_n A_n$ exists then $P(\lim_n A_n) = \lim_n P(A_n)$

Proof 1 (Impossibility) $(E_i : i \in \mathbb{N}) \in \mathcal{F}$

$E_1 = \Omega$, $E_2 = E_3 = \ldots E_n = \phi$

$\Rightarrow P(\bigcup_{i \in \mathbb{N}} E_i) = \sum_{i \in \mathbb{N}} P(E_i)$ (From σ-additivity)

$$\Rightarrow \quad P(\Omega) = P(\Omega) + n \sum P(\phi)$$

$$P(\phi) = 0$$

(ii) (Finite additivity) : $(E_i : i \in \mathbb{N}) \subseteq \mathcal{F}$ mutually disjoint.

s.t. $E_i = A_i$, $i \in [n]$, $E_i = \phi \; \forall i > n$

$$P(\bigcup_{i=1}^{n} A_i) = P(\bigcup_{i=1}^{\mathbb{N}} E_i) = \sum_{i=1}^{n} P(E_i) + \sum_{i>n} P(E_i)$$

$$= \sum_{i=1}^{n} P(A_i) + 0$$

(iii) (monotonicity) Let $A, B \in \mathcal{F}$ $A \subseteq B$

$$E_1 = A \quad E_2 = B \cap A^c$$

$$P(B) = P(E_1 \cup E_2) = P(A) + \underbrace{P(B \setminus A)}_{\geq 0}$$

$$\therefore P(B) \geq P(A)$$

(iv) (inclusion-exclusion) Let $A, B \in \mathcal{F}$

$$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(B \cap A^c)$$

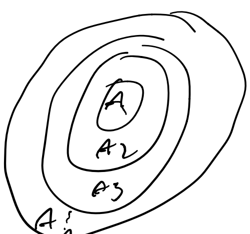
$$= P(A) + P(B) - 2 \times P(A \cap B) + P(A \cap B)$$

$\therefore$ Done

(v) Continuity.

Limit of sets

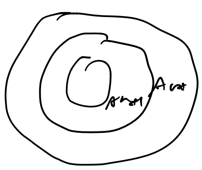
$[A_i \subseteq A_{i+1} \ \forall i \in \mathbb{N}]$



If $(A_n \in \mathcal{F} : n \in \mathbb{N})$ is a seq. of monotone non decreasing sets.

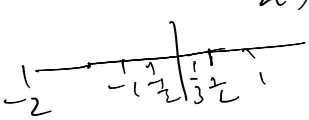
The $\lim_{n \to \infty} A_n := \bigcup_{n \in \mathbb{N}} A_n$

If $A_n \in \mathcal{F} : n \in \mathbb{N})$ is a monotone non increasing seq. of sets. For each $i \in \mathbb{N}$, $A_i \supseteq A_{i+1}$



Then $\lim_{n \to \infty} A_n := \bigcap_{n \in \mathbb{N}} A_n$

Ex: $A_n = [-2, -\frac{1}{n})$ $\quad B_n = [-2, \frac{2}{n})$



A_n limit is increasing
B_n lt. is decreasing.

$\lim_n A_n = [-2, 0)$ $\quad \lim_n B_n = [-2, 0]$

Defn (limits of sets)

$$\limsup_n A_n = \bigcap_{n \in N} \underbrace{\bigcup_{m \geq n} A_m}_{E_n \downarrow} = \lim_n E_N \qquad E_n := \bigcup_{m \geq n} A_m$$

$$\liminf_n A_n := \bigcup_{n \in N} \underbrace{\bigcap_{m \geq n} A_m} = \lim_n F_n \qquad F_n := \bigcap_{m \geq n} A_m$$

In general $\liminf_n A_n \subseteq \limsup_n A_n$

$$F_{n_0} = \bigcap_{m \geq n_0} A_m \qquad F_{n_0} \subseteq A_m \ \forall m \geq n_0$$

$$\bigcup_{n \in N} F_n \subseteq \underbrace{\bigcup_{m \geq n} A_m}_{E_n} \text{ for}$$

each $n \in N$

$F_1 \subseteq A_m \quad \forall \ m \geq 1$

$F_2 \subseteq A_m \quad \forall \ m \geq 2$

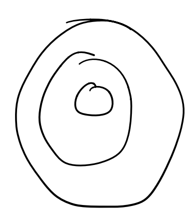
When $\liminf_n A_n = \limsup_n A_n$ then we say that limit of A_n exists and is defined as $\lim A_n = \limsup A_n = \liminf A_n$

3. Proof of continuity

[Increasing sequence of events]

$(A_n \in \mathcal{F} : n \in N) \uparrow$

$A_n = A_1 \cup A_2 \dots \cup A_n$



$\lim_n A_n = \bigcup_{i \in N} A_i$

$A_1, A_2 \cap A_1^c, A_3 \cap A_2^c$ ---- are disjoint

$P(A_i \cap A_{i+1}^c) = P(A_i) - P(A_{i-1})$

$$P(A_n) = P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(A_1 \bigcup_{i=2}^{n} (A_i \setminus A_{i-1})\right)$$

$$= P(A_1) + \sum_{i=2}^{n} \left[P(A_1) - P(A_{i-1})\right]$$

$$P\left(\lim_{n\to\infty} A_n\right) = P(A_1) + \sum_{i\geq 2} \left[P(A_i) - P(A_{i-1})\right]$$

$$= P(A_1) + \lim_{n\to\infty} \sum_{i=2}^{n} \left[P(A_i) - P(A_{i-1})\right]$$

$$= \lim_{n\to\infty} P(A_n)$$

Lecture 03:

- Independence of Events

1) Outcome space: Ω
2) Event space: $\mathcal{F}$

- Consider N trials of an experiment over $(\Omega, \mathcal{F}_e)$
- Let x_n be the outcome of n^{th} experiment.
- Let $A, B \in \mathcal{F}$

$$N(A) = \sum_{i=1}^{N} \mathbb{1}(x_i \in A) \qquad N(B) = \sum_{i=1}^{N} \mathbb{1}(x_i \in B)$$

$$N(A \cap B) = \sum_{i=1}^{N} \mathbb{1}(x_i \in A \cap B)$$

- Relative freq. of A, B & $A \cap B$ are

$$\frac{N(A)}{N}, \quad \frac{N(B)}{N}, \quad \frac{N(A\cap B)}{N} \text{ respectively}$$

- Rel. freq. where A happened where B happened $\dfrac{N(A\cap B)}{N(B)}$

Defn conditional prob: For any event $B \in \mathcal{F}_e$ $P(B) > 0$ we can define the conditional prob. function

$P(\cdot \mid B) : \mathcal{F} \to [0,1]$ for any event $A \in \mathcal{F}$ conditioned on the event B as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Lemma: For any event $B \in \mathcal{F}_{\epsilon}$ s.t $P(B) > 0$, the conditional probability $P(\cdot \mid B) : \mathcal{F}_{\epsilon} \to [0,1]$ is a "probability measure" on space $(\Omega, \mathcal{F}_{\epsilon})$

Proof: 1) Non negativity : For any event $A \in \mathcal{F}_{\epsilon}$

$$P(A \mid B) \geq 0 \quad \text{since } P(A \cap B) \geq 0$$

2) σ-additivity: Let $(A_i \in \mathcal{F}_{\epsilon} : i \in \mathbb{N})$ be mutually disjoint.

Then
$$P((\cup_{i \in \mathbb{N}} A_i) \cap B) = P(\cup_{i \in \mathbb{N}} (A_i \cap B))$$
$$= \sum_{i \in \mathbb{N}} P(A_i \cap B)$$

σ additivity.

Dividing by $P(B)$ on both sides. we get σ-additivity.

3) Certainity: $P(\Omega \mid B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

2. Law of Total Probability

Theorem: Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider a partition of the outcome space Ω, $B = (B_n \in \mathcal{F}_{\epsilon} : n \in \mathbb{N})$

Then for any event $A \in \mathcal{F}_{\epsilon}$, we have

$$P(A) = \sum_{n \in \mathbb{N}} P(A \cap B_n)$$

Proof: For any event $A \in \mathcal{F}_{\epsilon}$ and partition B of Ω, $A = A \cap \Omega = A \cap (\cup_{n \in \mathbb{N}} B_n)$

$$= \bigcup_{n \in \mathbb{N}} (A \cap B_n)$$

$$P(A) = \sum_{n \in \mathbb{N}} P(A \cap B_n)$$

Remark: if $P(B_n) \geq 0 \ \forall n \in \mathbb{N}$, for partition B of Ω,

then $P(A) = \sum_{n \in \mathbb{N}} P(A | B_n) P(B_n)$

7. Independence:

Defn. (indep. of events) For a prob. space $(\Omega, \mathcal{F}, P)$, a family of events $(A_i \in \mathcal{F}; i \in I)$ is said to be indep., if for any finite set $F \subseteq I$, we have

$$P\left(\bigcap_{i \in F} A_i\right) = \prod_{i \in F} P(A_i)$$

Remark: The certain event Ω and impossible event ϕ are independent to every event. $A \in \mathcal{F}$.

Ex. [Countably infinite coin tosses]

$\Omega = \{H, T\}^{\mathbb{N}}$ $\quad \mathcal{F} = \sigma(\mathcal{F}_n : n \in \mathbb{N})$

where $\mathcal{F}_n := \{\omega \in \Omega : (\omega_1, \omega, \ldots \omega_n) \in \{H,T\}^n\}$

$\{(H, x x x x)\} \in \mathcal{F}_1 \quad \{(T, x x x)\} \in \mathcal{F}_1$

$\mathcal{F}_1 = \{(H, xxx), (T, xxx)\}$

$\mathcal{F}_2 = \{(H, T, xxx), (H, H, xxxx), (T, H, xxx), (T, T, xxx)\}$

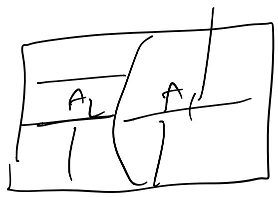
$A_1 = \{\omega \in \Omega : \omega_1 = H\}$ $\quad A_2 = \{\omega \in \Omega ; \omega_1 = T\}$

$\mathcal{F}_1 = \{A_1, A_2\}$ similarly, $\mathcal{F}_2 = \{A_1^{(2)}, A_2^{(2)}, A_3^{(2)}, A_4^{(2)}\}$

$\mathcal{F}_n = \sigma(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \ldots)$

$\omega \in \Omega$: $\quad k_n = \sum_{i=1} \mathbb{1}_{\{\omega_i = H\}}$

$$\sum_{i=1}^{n} \mathbb{1}_{\{\omega_i = T\}} = \sum_{i=1}^{n} \left(1 - \mathbb{1}_{\{\omega_i = H\}}\right) = n - k_n$$



$$\mathcal{F}_1 = \{(\phi, A_1^{(1)}, A_2^{(1)}, \Omega)\}$$

$$\mathcal{F}_2 = \{\phi, A_1^{(2)}, A_2^{(2)}, A_3^{(2)}, A_4^{(2)}\}$$

$$A_1^{(2)} \cup A_2^{(2)} = A_1^{(1)}$$

$$\therefore \mathcal{F}_1 \subset \mathcal{F}_2$$

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \ldots \subseteq \mathcal{F}_n \subseteq \ldots$$

coarser $\qquad\qquad$ finer.

Let $A \in \mathcal{F}_n$ $\quad P(A) = \sum_{\omega \in A} p^{k_n} (1-p)^{n-k_n}$

$$E_i = \{\omega \in \Omega : \omega_i = H\} \in \mathcal{F}_i \in \mathcal{F}$$

To show $P\left(\bigcap_{i=1}^{N} E_i\right) = \prod_{i=1}^{n} (P(E_i))$

1. $(E_1 \cap E_2 \cap \ldots]$

$$P(E_1 \cap E_2 \ldots \cap E_n) = P(\{H, H, H, \ldots \in \mathcal{F}_n = p^n \ldots)$$

An event $E \in \sigma(\mathcal{F})$ if and only if you can write it as countable unions, intersections and complements. ...]

$$P(E_i) = P(\{\omega \in \Omega : (\omega_1, \ldots \omega_i) \in \{H, T\}^i, \omega_i = H\})$$

$$= \sum_{k=0}^{i-1} p \binom{i-1}{k} p^k (1-p)^{i-1-k}$$

$$= p$$

This implies that $P(\bigcap_{i=1}^{n} E_i) = \prod_{i=1}^{n} P(E_i)$

$\Omega = \{H, T\}$ $\mathcal{F}_t = \sigma(E_i : i \in \mathbb{N})$

$P = ?$ $P(E_1 \cap E_2 \cap E_3 \ldots \cap E_n)$

Check that this defn. leads to the same prob $= p^n$ as defined 2.

6. Conditional Independence.

Def-n. For a prob. space $(\Omega, \mathcal{F}, P)$ a family of events $(A_i \in \mathcal{F} : i \in I)$ is said to be conditionally independent given an event $C \in \mathcal{F}$, s.t. $P(C) > 0$, if for any finite set $F \subseteq I$, we have

$$P(\bigcap_{i \in F} A_i \mid C) = \prod_{i \in F} P(A_i \mid C)$$

Remarks: 1) $A, B, C \in \mathcal{F}$ $P(C) > 0$ A & B are conditionally independent given C if $P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$

2) A & B are indep iff they are conditionally indep. given Ω.

3) 2 events may be indep but not conditionally independent.

2 events may be conditionally indep but not indep.

Lecture 4

Random Variables.

- Sample space : Ω
- Event space : $\mathcal{F} \subseteq 2^{\Omega}$ $\begin{cases} 1.\ \Omega \in \mathcal{F} \\ 2.\ A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \\ 3.\ (A_i \in \mathcal{F} : i \in \mathbb{N}) \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F} \end{cases}$

3. Probability function $P : \mathcal{F} \longrightarrow [0,1]$

$$\begin{cases} 1-\ P(A) \geq 0 \\ 2\ \ P\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} P(A_i) \text{ if } (A_i \in \mathcal{F} : i \in \mathbb{N}) \text{ disjoint} \\ 3-\ P(\Omega) = 1 \end{cases}$$

4. Properties of Probability
   - $P(\phi) = 0$; $-P(A \cup B) = P(A) + P(B) - P(A \cap B)$
   - $P(A) \leq P(B)$ if $A \subseteq B$; $-\lim_n P(A_n) = P(\lim_n A_n)$ for $(A_n \in \mathcal{F} : n \in \mathbb{N}) \in \mathcal{F}^{\mathbb{N}}$

5. Independence of Events
   - $(A_i \in \mathcal{F} : i \in I)$: indep if $P\left(\bigcap_{i \in F} A_i\right) = \prod_{i \in F} P(A_i)$ for any $F \subseteq I$ finite
   - $A, B \in \mathcal{F}$ and $P(B) > 0$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

   - $A, B, C \in \mathcal{F}$ and $P(C) > 0$

     A, B conditionally indep given C if

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C)$$

1) Random Variable

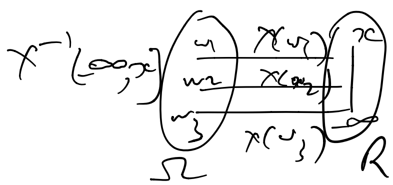
Defn: Consider a Prob space $(\Omega, \mathcal{F}, P)$

Ω → outcomes, $\mathcal{F}$ → events, P → Prob. Space.

A random variable X is a mapping $X : \Omega \to \mathbb{R}$ is a real valued function from the sample (outcome) space to real numbers; s.t. for each $x \in \mathbb{R}$ the set $A_x = \{ \omega \in \Omega : X(\omega) \le x \} = X^{-1}\underbrace{(-\infty, x]}_{B_x} \in \mathcal{F}$

We say that the random variable is X is $\mathcal{F}$ measurable and ~~the~~ prob. of this event is denoted by

$$F_X(x) := P(A_x) = P(\{X \le x\}) = P_o X^{-1}(-\infty, x].$$



The function $F_X : R \to [0,1]$ is called the "distribution function" (CDF) of a random variable X.

Remark 1: 1) Since $\omega \in \Omega$ is random, this implies $X(\omega)$ is random.

2) Prob. is defined only for "events" not for random variables.

The events of interest for random variables are often level sets $A_x := X^{-1}(-\infty, x]$ for any real x.

$$P(\{X \in A\}) = P_o X^{-1}(A) \quad \text{for } A \in \mathcal{B}(R)$$

Defn:- $\mathcal{B}(\mathbb{R}) = \sigma((-\infty, x] : x \in \mathbb{R})$

Remark: The event space generated by a random variable X is collection of inverse of Borel sets.

$\mathcal{F} \to$ event space $\quad$ σ-algebra formed from half open sets $(-\infty, x]$ are Borel algebras.

$X^{-1}\underbrace{(-\infty, x]}_{B_x} \in \mathcal{F}_e$

$B(\mathbb{R}) = \sigma(B_x : x \in \mathbb{R})$

$B \in \mathcal{B}(\mathbb{R}) \qquad X^{-1}(B) \in \mathcal{F}_e$

$X^{-1}(\bigcup_{i \in \mathbb{N}} B_{x_i}) = \bigcup_{i \in \mathbb{N}} X^{-1}(B(x_i)) \in \mathcal{F}_e$

$\sigma(X) := \sigma(X^{-1}(B_x) : x \in \mathbb{R})$

Lemma: [Prop. of dist function] $F_X : \mathbb{R} \to [0, 1]$ for any r.v X

1- is monotonically non-decreasing

2- is right continuous at all points $x \in \mathbb{R}$

3- has the upper limit $\lim_{x \to \infty} F_X(x) = 1$

the lower limit $\lim_{x \to -\infty} F_X(x) = 0$.

Proof:— Let X be a random variable defined on prob. space $(\Omega, \mathcal{F}, P)$

1- We have to show $F_X(x_1) \leq F_X(x_2)$ for $x_1 \leq x_2$. By defn.

$F_X(x_1) = P(X^{-1}(-\infty, x_1]) = P(A_{x_1})$

$F_X(x_2) = P(X^{-1}(-\infty, x_2]) = P(A_{x_2})$

Since, $A_{x_1} \subseteq A_{x_2}$ it implies that $F_X(x_1) \leq F_X(x_2)$

[from monotonicity of problem]

[If $\omega \in A_{x_1}$ then $X(\omega) \leq x_1 \leq x_2$ $\omega \in A_{x_2}$.

Hence, $A_{x_1} \subseteq A_{x_2}$

2- Right continuity $(x_n : n \in \mathbb{N}) \subseteq \mathbb{R}$ s.t

$\lim_n x_n = x \in \mathbb{R}$

$$\underset{n}{\text{Lt}}\, A_{x_n} = A_x = x^{-1}(-\infty, x]$$
$$\underset{n}{\text{Lt}}\, A_{x_n} = x^{-1}(-\infty, x_n]$$

$$P(\underset{n}{\text{Lt}}\, A_{x_n}) = P(A_x) \underset{\text{cty. of prob.}}{=} \underset{n}{\text{Lt}}\, P(A_{x_n})$$

$$F_x(x) = \lim_{n\to\infty} F_x(x_n) = \underset{x_n \to x}{\text{Lt}} F_x(x_n)$$

[To show right

Lecture 5

Random vectors, joint distribution and independence

1. Random Vectors: For a vector $x \in \mathbb{R}^n$, we can [Projection] define a map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the "projection" of an n-length vector onto its i^{th} component such that $\pi_i(x) = x_i$

Ex: $f : \Omega \to \mathbb{R}^n$ then $f(\omega) \in \mathbb{R}^n$

$(f_1(\omega), f_2(\omega), \ldots f_n(\omega))$ where $f_i(\omega) = (\pi_i \circ f)(\omega)$
$f_i = (\pi_i \circ f)$

For the function f, we can write the inverse image of a set $B \subseteq \mathbb{R}^n$, as

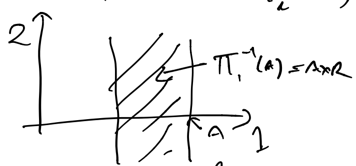
$$\begin{aligned} f^{-1}(B) &= \{\omega \in \Omega : f(\omega) = B\} \\ &= \{\omega \in \Omega : \pi_i(f(\omega)) = \pi_i(B), \forall i \in [n]\} \\ &= \bigcap_{i \in [n]} \{\omega \in \Omega : (\pi_i \circ f)(\omega) = \pi_i(B)\} \end{aligned}$$

$$= \bigcap_{i \in [n]} F_i^{-1}(\pi_i(B)) \quad \blacksquare$$

Remark: For a subset $A \subseteq \mathbb{R}$ and projection $\pi_i : \mathbb{R}^n \to \mathbb{R}$

$$\pi_i^{-1}(A) = \{x \in \mathbb{R}^n : \pi_i(x) \in A\}$$

$$= \mathbb{R} \times \cdots \underset{\uparrow\, i^{th}}{A} \times \cdots \times \mathbb{R}$$



Exercise: Show that $F_i^{-1}(\pi_i(B)) = \mathbb{R} \times \cdots \pi_i(f^{-1}(B)) \times \cdots \times \mathbb{R}$ ($\omega \in \Omega$; $\uparrow$ i^{th} dimension)

[Defn] Random Vectors: Consider a prob. space $(\Omega, \mathcal{F}_e, P)$.

For a finite $n \in \mathbb{N}$, a random vector $X : \Omega \to \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, the following subset

$$\xrightarrow{\text{event}} A(x) := \{\omega \in \Omega : X_i(\omega) \leq x_i \ \forall\ i \in [n]\} \in \mathcal{F}_e$$

$$= \bigcap_{i \in [n]} X_i^{-1}(-\infty, x_i]$$

We say that the random vector X is $\mathcal{F}$-measurable, and $P(A(x)) = F_X(x)$

$$= F_{X_1, X_2, \ldots X_n}(\{X_1 \leq x_1, X_2 \leq x_2, \ldots X_n \leq x_n\})$$

$$= P\left(\bigcap_{i=1}^{n} X_i^{-1}(-\infty, x_i]\right)$$

The function $F_X : \mathbb{R}^n \to [0,1]$ is called the joint distribution of a random vector X.

Remark: (a) If X is a random vector, then X_i are random variables $\forall\ i \in [n]$

(b) If (X_i) are random variables, then X is a random vector.

a) $\bigcap_{i \in [n]} X_i^{-1}(-\infty, x_i] \in \mathcal{F}_e$

$$x = (\infty, \ldots x_i, \ldots \infty)$$

$$X_j^{-1}(x_j) = \Omega \qquad i \neq j$$

$$X_j^{-1}(\mathbb{R}) = \Omega$$

$$X_i^{-1}(-\infty, x_i] \cap \Omega \in \mathcal{F}$$

b) $x \in \mathbb{R}^n$ $\bigcap_{i=1}^{n} X_i^{-1}(-\infty, x_i] \in \mathcal{F}$

$$A_x = \bigcap_{i=1}^{n} A_i(x_i) \text{ where, } A_i(x_i) = X_i^{-1}(-\infty, x_i]$$

Lemma [Marginal Distribution] $X: \Omega \to \mathbb{R}^n$ defined on $(\Omega, \mathcal{F}, P)$, with joint dist. $F_X: \mathbb{R}^n \to [0,1]$.
The dist. of $(\pi_i \circ X)$ is called the i^{th}-marginal dist. and can be obtained

$$F_{X_i}(x_i) = \lim_{\substack{x_j \to \infty \\ \forall j \neq i}} F_X(x)$$

1.4 Independence of Random variables:

Defn: A random vector $X: \Omega \to \mathbb{R}^n$ defined on $(\Omega, \mathcal{F}, P)$ is called "independent" if $F_X(x) = \prod_{i=1}^{n} F_{X_i}(x_i)$ $\forall x \in \mathbb{R}^n$

"Inverse images are more important than functions & intersections are more important than unions".

$$F_X(x) = P(A(x)) = P\left(\bigcap_{i=1}^{n} A_i(x_i)\right) = \prod_{i=1}^{n} P(A_i(x_i))$$

That is, for all $x \in \mathbb{R}^n$, $(A_1(x_1), A(x_2), A(x_3), \dots A_n(x_n))$ are mutually indep events]

Defn: An event is said to be identically distributed if $F_{X_i} = F_{X_1}$ $\forall i \in [n]$.

[Defn: An event space generated by a random vector $X: \Omega \to \mathbb{R}^n$ is denoted by $\sigma(X) \subseteq \mathcal{F}$]

Exercise: $\sigma(X) = \sigma(X_1, \dots X_n)$

Defn: A family of collection of events $(\mathcal{A}_i \subseteq \mathcal{F} : i \in I)$ is called indep if for any finite set $F \subseteq I$ and $A_i \in \mathcal{A}_i$ $\forall i \in F$, $P(\bigcap_{i \in F} A_i) = \prod_{i \in F} P(A_i)$

Remark: If two collections of events are mutually independent then event space generated by them are also mutually independent [Dynkin's π-λ theorem]

$\Rightarrow (\sigma(\mathcal{A}_i) \subseteq \mathcal{F} \ : i \in I)$.

Theorem: For an indep. random vector $X: \Omega \to \mathbb{R}^n$ $(\sigma(X_i) : i \in [n])$ are indep.

Proof: For any Borel sets $A_1, \dots A_n \in \mathcal{B}(\mathbb{R})$