Recall:

Prop 1: There is a $c_0 > 1$, $T > 0$, such that for all initial conditions x with $\|x\| \leq 1$,

$$\|\phi_t^c(x)\| < \frac{1}{4}, \text{ for } t \in [T, T+1], \ c > c_0.$$

Prop 2: Assume (A1)–(A3) and (A5).

$\dot{x}(t) = h_c(x(t))$ where c is a constant (that rescale $\bar{x}$ to $\hat{x}$ at time T_m)

$$c = r(m)$$

(a) $\lim_{m \to \infty} \sup_{t \in [T_m, T_{m+1})} \|\hat{x}(t) - x^{T_m}(t)\| = 0$ a.s., where $x^{T_m}(t)$, $t \geq T_m$, is the solution to the ode above with $c = r(m)$.

(b) For each m, for each k such that $n_m \leq k \leq n_{m+1}$,

$$\|\hat{x}(t(k))\| \leq K^* \quad \text{a.s.}$$

Prop 3: $\sup_{\substack{t \in [0, T+1] \\ \|\bar{x}(0)\| \leq c_0}} \|\bar{x}(t)\| \leq \bar{B}$ a.s.

Pf of Propos^n 2: • First show boundedness of $\mathbb{E}[\|\hat{x}(t)\|^2]$. Then show convergence of rescaled $\hat{\xi}_n$. Then follow the steps of BCL.

Lemma: $\sup_t \mathbb{E}[\|\hat{x}(t)\|^2] < \infty$

Pf: Let m be fixed. Consider $n_m \le k \le n_{m+1}$.

$$\bar{x}(t(k+1)) = \bar{x}(t(k)) + a(k)\left[h(\bar{x}(t(k))) + M_{k+1}\right]$$

$$\Rightarrow \hat{x}(t(k+1)) = \hat{x}(t(k)) + a(k)\left[h_{r(m)}(\hat{x}(t(k))) + \hat{M}_{k+1}\right]$$

$$\left\{ \text{Since } \frac{h(\bar{x}(t(k)))}{r(m)} = \frac{h(r(m)\,\hat{x}(t(k)))}{r(m)} = h_{r(m)}(\hat{x}(t(k))) \right\}$$

$$\text{(A3)} \Rightarrow \mathbb{E}\left[\|\hat{M}_{k+1}\|^2 \mid \mathcal{F}_k\right] \le \frac{K(1+\|x_k\|^2)}{r(m)^2}$$

$$\le K(1+\|\hat{x}(t(k))\|^2)$$

[So (A3) is available for scaled iterates]
(with the same K)

$$\text{So } \mathbb{E}\left[\|\hat{M}_{k+1}\|^2 \cancel{\mid \mathcal{F}_k}\right] \le K\left(1+\mathbb{E}\left[\|\hat{x}(t(k))\|^2\right]\right)$$

$$\|\hat{M}_{k+1}\|_* := \sqrt{\mathbb{E}\left[\|\hat{M}_{k+1}\|^2\right]} \le \sqrt{K}\left(1+\|\hat{x}(t(k))\|_*\right)$$

[Since $\sqrt{1+y^2} < 1+y$]

(b) (A1) $\Rightarrow$ $\|h_c(x)\| \le K_1(1+\|x\|)$

(c) Put (a) & (b) together:

$$\|\hat{x}(t(k+1))\|_* \le \|\hat{x}(t(k))\|_* + a(k)\left(K_1(1+\|\hat{x}(t(k))\|_*)\right)$$

Key step.

$$+a(k)\left(\sqrt{K}\left(1+\|\hat{x}(t(k))\|_*\right)\right)\|$$

$$= \|\hat{x}(t(k))\|_* \left(1+a(k)\left(K_1+\sqrt{K}\right)\right) + a(k)\left(K_1+\sqrt{K}\right)$$

$$\leq \underbrace{\underbrace{\|\hat{x}(t(n_m))\|_*}_{\leq 1}\, e^{(K_1+\sqrt{K})(T+1)}}_{\text{Note the form}} \left[1+\left(K_1+\sqrt{K}\right)(T+1)\right]$$

[By chaining, as we'd done earlier]

So, $\|\hat{x}(t(k+1))\|_*$ is bounded. Then $\sup_t \mathbb{E}[\|\hat{x}(t)\|^2] < \infty$.

<u>Lemma</u>: $\hat{\xi}_n = \sum_{k=0}^{n-1} a(k)\hat{M}_{k+1}$ converges a.s.

Pf: As before, it is enough to prove that $\sum_k a(k)^2\, \mathbb{E}[\|\hat{M}_{k+1}\|^2 / \mathcal{F}_k] < \infty$ a.s.

So it suffices to show that $\mathbb{E}\left[\sum_k a(k)^2\, \mathbb{E}[\|\hat{M}_{k+1}\|^2 | \mathcal{F}_k]\right] < \infty$

Now, $$\sum_k a(k)^2\, \mathbb{E}[\|\hat{M}_{k+1}\|^2] \leq \underbrace{\left(\sum_k a(k)^2\right)}_{<\infty} \underbrace{\left(K\left(1+(K_1+\sqrt{K})(T+1)+1\right) e^{(K_1+\sqrt{K})(T+1)}\right)}_{K^*}$$

$$< \infty$$

Hence, $\hat{\xi}_n$ converges.

Finishing the proof <u>of Prop 2</u>:

$$\dot{x}^{T_m}(t) = h_{a(m)}(x^{T_m}(t)), \quad x^{T_m}(T_m) = \hat{x}(T_m), \quad t \geq T_m,$$

$t < T_{m+1}$ ↳ To avoid jumps.

What all did we need for BCL ?

(1) $h_{r(m)}(\cdot)$ should be Lipschitz, with a constant independent of m. ✓

(2) $\|\hat{x}(t(n_m+k))\| < K^*$ a.s., K^* is indep of m.
(eqvt. of (A4)).

(3) $\sup_{k\geq 0} \|\hat{\delta}_{l,l+k}\| = \sup_{k \geq 0} \left\|\sum_{j=l}^{l+k} a(j)\, \hat{M}_{j+1}\right\| \xrightarrow[l\to\infty]{} 0$ a.s. ✓

Verifying (2):

$$\|\hat{x}(t(n_m+k))\| \leq \|\hat{x}(t(n_m))\| + \sum_{j=0}^{k-1} a(n_m+j)\left(\|h_{r_m}(0)\| + L\|\hat{x}(t(n_m+j))\|\right) + \underbrace{B_{n_m}}_{\text{Bound on the noise.}}$$

$$\leq \|\hat{x}(t(n_m))\| + \sum_{j=0}^{k-1} a(n_m+j)\left(\|h(0)\| + L\|\hat{x}(t(n_m+j))\|\right) + B_{n,m}$$

[Since $\|h_{r_m}(0)\| \leq \|h(0)\|$]

$$\leq \left(1 + B_{n_m} + \|h(0)\|\underbrace{(T+1)}_{\substack{\text{since} \\ T+1 > T_{m+1}-T_m}}\right) + L\sum_{j=0}^{k-1} a(n_m+j)\, \|\hat{x}(t(n_m+j))\|$$

$$\leq \left(1 + B_{n_m} + \|h(0)\|(T+1)\right) e^{L(T+1)}, \quad n_m \leq k < n_{m+1}.$$

↳ discrete Gronwall

$\leq B$, i.e., iterates are bdd.

This proves the entirety of Propos^n 2.

Proof of Propos^n 3 :

$$\|\bar{x}(t(n_m+K+1))\| \leq \|\bar{x}(t(n_m))\| + \sum_{j=0}^{K-1} a(n_m+j)\left(\|h(0)\|+L\|\bar{x}(t(n_m+j))\|\right)$$

Must be K.

$$+\sum_{j=0}^{K-1} a(n_m+j) M_{n_m+j+1}$$

Must be K.

Just use $\|\bar{x}(t(n_m))\| \leq C_0$. Every other step is the same.

(Why don't we just use chaining, instead of discrete Gronwall?) ⊙ ch.