**Thm:** Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$-sub-martingale satisfying $\sup_n \mathbb{E}[X_n^+] < \infty$.

Then $\lim_{n\to\infty} X_n = X_\infty$ exists a.s.

**Pf:** $\overline{X} = \limsup_{n\to\infty} X_n$, $\underline{X} = \liminf_{n\to\infty} X_n$.

$(X_N - a)^+ = \begin{cases} X_N - a, & \text{if } X_N \geq a \\ 0, & \text{o.w.} \end{cases}$

Fix $N$, fix $a<b$.

$\frac{\mathbb{E}[(X_N - a)^+]}{b-a} \overset{?}{\leq} \frac{\mathbb{E}[X_N^+] + |a|}{b-a}$

$X_N^+ + |a| = \begin{cases} X_N + |a|, & \text{if } X_N \geq 0 \\ |a|, & \text{o.w.} \end{cases}$

$$\mathbb{E}[\beta_N(a,b)] \leq \frac{\mathbb{E}[X_N^+] + |a|}{b-a}$$ (Yeah, OK)

Let $\beta_\infty(a,b) = \lim_{N\to\infty} \beta_N(a,b) = \#$ of upcrossings.

$$\mathbb{E}[\beta_\infty(a,b)] \leq \liminf_{N\to\infty} \mathbb{E}[\beta_N(a,b)] \leq \frac{B + |a|}{b-a}$$ , where $B := \sup_n \mathbb{E}[X_n^+]$

Fatou's Lemma

$$\Rightarrow \beta_\infty(a,b) < \infty \text{ a.s.}$$

Now, $$\{\overline{X} > \underline{X}\} = \bigcup_{a<b,\ a,b \to \text{rational}} \{\overline{X} > b > a > \underline{X}\}$$

$$\therefore P(\overline{X} > \underline{X}) \leq \sum_{a<b,\ a,b \text{ rational}} P[\overline{X} > b > a > \underline{X}]$$ (Countable sum of 0s is 0)

(But why is each 0??)

$= 0$

$\Rightarrow \overline{X} = \underline{X}$ a.s., $\{X_n\}$ has a limit a.s. with $\pm\infty$ as possible values.

- $X_\infty < \infty$ a.s. S'pose $P[X_n \to \infty] > 0$.

$$\infty > \limsup_{n\to\infty} \mathbb{E}[X_n^+] \geq \liminf_{n\to\infty} \mathbb{E}[X_n^+]$$

$$\underset{\text{(Fatou's Lemma)}}{\geq} \mathbb{E}\left[\liminf_{n\to\infty} X_n^+\right]$$

$$\geq \mathbb{E}\left[\liminf_{n\to\infty} X_n\right]$$

$$= \infty \quad \therefore \boxed{\text{Contradiction}}$$

- $X_n > -\infty$ a.s.

$$\mathbb{E}[X_n] = \mathbb{E}[X_n^+] - \mathbb{E}[X_n^-] \overset{\text{sub-mart.}}{\geq} \mathbb{E}[X_0]$$

$$\sup_n \mathbb{E}[X_n^-] \leq B + |\mathbb{E} X_0|$$

$\infty = \mathbb{E}[\liminf X_n^-] \leq \liminf_{n\to\infty} \mathbb{E}[X_n^-] \leq \limsup_{n\to\infty} \mathbb{E}[X_n^-] < \infty$ (since $X_n \to \infty$ with +ve prob)

(Exercise: follow the same procedure to show $P[X_n \to -\infty] = 0$)

✓ (Yeah the exact same procedure holds)

- Corollary: Every martingale which is $\mathcal{L}_1$-bounded, $\sup_n \mathbb{E}|X_n| < \infty$, converges.

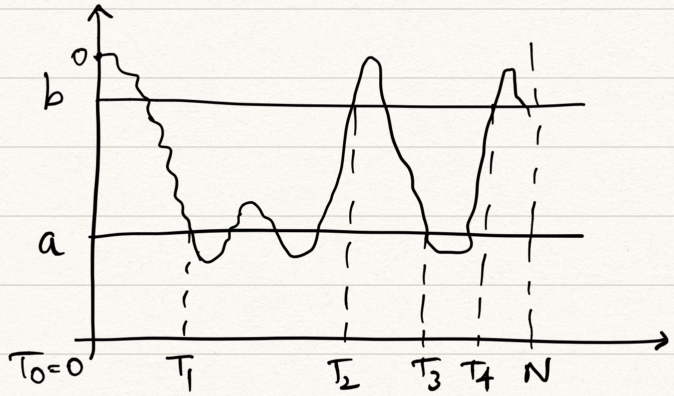
$\mathbb{E}[X_0] = \mathbb{E} X_n = \mathbb{E}X_n^+ - \mathbb{E}X_n^-$

**Pf:** $\mathbb{E}[|X_n|] = \mathbb{E}X_n^+ + \mathbb{E}X_n^- = 2\mathbb{E}X_n^+ - \mathbb{E}X_0$ $\forall n$ $\mathbb{E}X_n^+ - \mathbb{E}X_0$.

$\& \ \mathbb{E}X_n = \mathbb{E}X_0$ $\blacksquare$

(So if $\sup_n \mathbb{E}|X_n|$ is bounded, then so is $\sup_n \mathbb{E}X_n^+$.).

**Qn:**




$T_5$ onwards $= N$.

Now, $X_N - X_0 = \underbrace{\sum_{m\geq 0}\left(X_{T_{2m+1}} - X_{T_{2m}}\right)} + \underbrace{\sum_{m\geq 1}\left(X_{T_{2m}} - X_{T_{2m-1}}\right)}_{\beta_N(a,b)(b-a)}$

$$\downarrow$$

$$\underbrace{\mathbb{E}[X_{T_{2m+1}}] \geq \mathbb{E}[X_{T_{2m}}]}$$

"$X_{T_{2m+1}}$ is below a, $X_{T_{2m}}$ is above b, but how?"

Ans is $\rightarrow$ $\because$ there is a good chance that $T_{2m}$ is $N$ itself (it never down-crosses).

**Discussion:** $\sup_n \mathbb{E}|X_n| < \infty$ or $\sup_n \mathbb{E}X_n^+ < \infty$.

We know $\mathbb{E}|X_n| < \infty$, but $\sup_n \mathbb{E}|X_n|$ could be $\infty$.

We will work with martingales with $\mathbb{E}X_n^2 < \infty$, for all $n$.
Such $X_n$s are in $\mathcal{L}_2$, but need not be $\mathcal{L}_2$-bounded.

$(*) \nearrow \quad \infty > \left(\mathbb{E}X_n^2\right)^{1/2} \geq \left(\mathbb{E}[|X_n|]\right)$

**Thm:** Doob decomposition:

Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$-submartingale. Then $X_n = M_n + A_n$, $n \geq 0$,

where $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale with zero-mean and

$\{A_n\}$ is an increasing process, adapted to $\{\mathcal{F}_{n-1}\}$, (predictable)
where $\mathcal{F}_{-1} = \{\phi, \Omega\}$.

Furthermore, this decomposition is unique (a.s.)

**Pf:** Write $M_n = \sum_{i=0}^{n} \left(X_i - \mathbb{E}[X_i | \mathcal{F}_{i-1}]\right)$,

$$A_n = \sum_{i=1}^{n} \left(\mathbb{E}[X_i | \mathcal{F}_{i-1}] - X_{i-1}\right) + \mathbb{E}X_0 \quad , n \geq 0.$$

(1) $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale

- $L_1$
- Adapted
- $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n + \mathbb{E}[X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{F}_n] | \mathcal{F}_n]$

(2) $A_{n+1} \geq A_n$ a.s. ✓

$A_n$ is Predictable ($\in \mathcal{F}_{n-1}$)

**Defn:** $\mathcal{L}_2$ martingales or sq-integrable martingales: Martingales with $\mathbb{E}[X_n^2] < \infty$, $\forall n$.

**Properties:**

- $\{X_n^2\}$ is an $\mathcal{F}_n$- sub-martingale.

- Let $X_n^2 = M_n + A_n$ be its Doob decomposition.

$$A_n = \mathbb{E}X_0^2 + \sum_{i=1}^{n} \left( \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] - X_{i-1}^2 \right) \quad \checkmark$$

$$= \mathbb{E}X_0^2 + \sum_{i=1}^{n} \mathbb{E}\left[ (X_i - X_{i-1})^2 | \mathcal{F}_{i-1} \right]$$

$\mathbb{E}[X_i^2|\mathcal{F}_{i-1}] + (X_{i-1})^2 - 2X_{i-1} \underbrace{\mathbb{E}[X_i|\mathcal{F}_{i-1}]}_{=X_{i-1}}$

"Quadratic variation process"

- $A_\infty = \lim_{n\to\infty} A_n$ (with $+\infty$ as a possible value for the limit)

(It is an increasing process, & thus has a limit)

**Thm:** Let $\{X_n\}$ be a square integrable $\{\mathcal{F}_n\}$- martingale (not sub-mart.). Let $\{M_n\}$, $\{A_n\}$, $A_\infty$ be as defined. Then $\{X_n\}$ converges a.s. on the set $\{A_\infty < \infty\}$ and $X_n = o(A_n)$ on $\{A_\infty = \infty\}$.

$$\left( X_n = o(A_n) \Leftrightarrow \lim_{n\to\infty} \frac{|X_n|}{A_n} = 0 \right)$$

(Class #8)
(16/9/19)

**Pf:** Fix $a>0$. $\quad$ $\min\{n \mid A_{n+1} > a^2\}$, if $A_\infty > a^2$

(1) $T_a = \begin{cases} \ldots \\ \infty, \text{ if } A_\infty \le a^2 \end{cases}$

**Note:** $T_a$ is a stopping time. (despite $A_{n+1}$)

**Claim:** $X_{T_a \wedge n}$ is a sq-integrable $\{\mathcal{F}_n\}$-martingale with quadratic variation process $A_{T_a \wedge n}$.

$A_n = \sum_{i=1}^{n} E\left[\left((H.X)_i - (H.X)_{i-1}\right)^2 \middle| \mathcal{F}_{i-1}\right] + E[(H.X)_0^2]$

Hint to prove this: $H_n = \mathbb{1}\{T_a \geq n\}$

$X_{n \wedge T_a} = \{(H.X)_n\}$,

where $(H.X)_n = \sum_{i=1}^{n} H_i (X_i - X_{i-1}) + X_0$.

Identify its quadratic variation, and verify that this quadratic variation is $\{(H.A)_n\} = A_{n \wedge T_a}$. yes.

So now, $E[|X_{n \wedge T_a}|] \leq \left(E[X^2_{n \wedge T_a}]\right)^{1/2}$

$= \left(E[A_{n \wedge T_a}]\right)^{1/2}$

Using the fact that: $X_n^2 = M_n + A_n$, $A_n = \sum_{i=1}^{n} \left(E[X_i^2 | \mathcal{F}_{i-1}] - X_{i-1}^2\right) + X_0^2$ so $E[A_n] = E[X_n^2]$

$\leq a$ (By def$^n$ of $T_a$).

$\Rightarrow X_{n \wedge T_a}$ converges a.s.

This means that $X_n$ converges a.s. on $\{T_a = \infty\} \equiv \{A_\infty \leq a^2\}$

So now, if we consider $\{A_\infty < \infty\}$, we note that

$$\bigcup_{a=1}^{\infty} \{A_\infty \leq a^2\} \quad \text{covers} \quad \{A_\infty < \infty\}.$$

② We intend showing that $X_n = o(A_n)$ a.s. on $\{A_\infty = \infty\}$

$$Z_n = \sum_{m=0}^{n-1} \left( \frac{X_{m+1} - X_m}{1 + A_{m+1}} \right) + \frac{X_0}{1 + A_0}$$

We can easily verify that $(Z_{n+1} - Z_n)$ is a martingale difference sequence. (cond'ed on $\mathcal{F}_n$, $A_{n+1}$ is measurable).

$\Rightarrow \{Z_n\}$ is an $\{\mathcal{F}_n\}$-martingale.

- This is sq-integrable $(EZ_n^2 < \infty$, for each $n)$
- Let $\{B_n\}$ be the quadratic variation process for $\{Z_n\}$.

We'll show that $\{B_\infty < \infty\}$ a.s. —— (To be shown)

Under $\{B_\infty < \infty\}$, $\{Z_n\}$ converges a.s.

Take $\quad u_m = X_{m+1} - X_m$, and $v_m = 1 + A_{m+1}$

Thus, $$\left| \sum_{m=0}^{n-1} \frac{u_m}{v_m} \right| < \infty \quad \text{a.s.}$$ [As a consequence of the proof of the first part]

then, $$\frac{\sum_{m=0}^{n-1} u_m}{v_{n-1}} \xrightarrow{n\to\infty} 0$$ (Kronecker's lemma) (To be shown)

$$\frac{X_n}{1+A_n} \xrightarrow{n\to\infty} 0$$ (which is the same as $X_n = o(A_n)$)

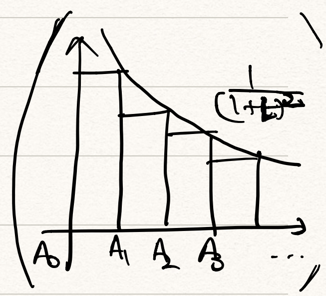
$\downarrow$ (CB)

Proof of $\{B_\infty < \infty\}$ a.s. :

$$B_N - B_0 = \sum_{n=0}^{N-1} (B_{n+1} - B_n) = \sum_{n=0}^{N-1} \mathbb{E}\left[(Z_{n+1} - Z_n)^2 \mid \mathcal{F}_n\right]$$

$$= \sum_{n=0}^{N-1} \frac{\mathbb{E}\left[(X_{n+1} - X_n)^2 \mid \mathcal{F}_n\right]}{(1+A_{n+1})^2}$$

$$= \sum_{n=0}^{N-1} \frac{A_{n+1} - A_n}{(1+A_{n+1})^2}$$




$$\leq \int_0^{A_N} \frac{1}{(1+t)^2}\, dt$$

$$\leq \int_0^{\infty} \frac{1}{(1+t)^2}\, dt < \infty$$

Take $N \to \infty$, & we get that $\{B_\infty < \infty\}$ a.s.

Proof of Kronecker's Lemma:

Let $w_n = \sum_{m=0}^{n-1} \frac{u_m}{v_m}$, where $v_m \to \infty$, $v_m \geq 0$
and $w_n \longrightarrow w^*$ as $n \to \infty$.

$$w_{n+1} - w_n = \frac{u_n}{v_n} \Rightarrow u_n = (w_{n+1} - w_n) v_n.$$

Thus,
$$\frac{\sum_{m=0}^{n-1} u_m}{v_{n-1}} = \frac{\sum_{m=0}^{n-1} (w_{m+1} - w_m) \cdot v_m}{v_{n-1}}$$

$$= \left(w_n \cdot \frac{v_{n-1}}{v_{n-1}}\right) - \frac{\sum_{m=0}^{n-2} w_m (v_{m+1} - v_m)}{v_{n-1}}$$

$$= w_n - \sum_{m=0}^{n-2} w_m \boxed{\frac{(v_{m+1} - v_m)}{v_{n-1}}}$$

The sum over m of these guys is 1
(So it's an "average" of the $w_m$'s)

$$\xrightarrow{n \to \infty} w^* - w^* = 0$$
(Césaro mean)

$\uparrow$ (CB)

# II Convergence Analysis:

SA scheme: $x_{n+1} = x_n + a(n)\left[h(x_n) + M_{n+1}\right]$,

$x_0 = \bar{x}_0 \in \mathbb{R}^d$.

Assumptions:

(A1): $h: \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is Lipschitz with constant $L$.

(A2): $\sum_n a(n) = \infty, \quad \sum_n a^2(n) < \infty$.

(A3): Define $\mathcal{F}_n = \sigma(x_m, M_m, m \leq n)$

$= \sigma(x_0, M_1, M_2, \ldots, M_n)$, $n \geq 0$

(a) $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale difference sequence, i.e.,

$$\mathbb{E}\left[M_{n+1} - M_n \mid \mathcal{F}_n\right] = 0, \text{ a.s.}, \; n \geq 0.$$

(b) $\{M_n\}$ is square-integrable with the following growth-bound

$$\mathbb{E}\left[\|M_{n+1}\|^2 \mid \mathcal{F}_n\right] \leq K(1 + \|x_n\|^2) \quad \text{a.s.}, \; n \geq 0$$

where $K < \infty$.

Remarks:

- (A1) $\Rightarrow$ $h$ grows at most linearly.

Fix some point $x_0$ (maybe $x_0 = 0$)

Then
$$\begin{aligned}\|h(x)\| &= \|h(x) - h(x_0) + h(x_0)\| \\ &\leq L\,\|x - x_0\| + \|h(x_0)\| \\ &\leq L\,\|x\| + L\|x_0\| + \|h(x_0)\| \\ &\leq K_{x_0}\left(1 + \|x\|\right) \qquad , \forall x \in \mathbb{R}^d.\end{aligned}$$

- (Square integrability)

(A1) and (A3) $\Rightarrow$ $\mathbb{E}[\|x_n\|^2]$ and $\mathbb{E}[\|M_n\|^2]$ are finite for each $n$.

On account of (A3)(b), it suffices to show that $\mathbb{E}[\|x_n\|^2]$ is finite $\forall n$.

Observation: $\mathbb{E}[\|x_n\|^2]^{1/2}$ is a norm in an appropriate space: $\mathcal{L}_2(\Omega, P)$

$$= \left( \int_\Omega \|x_n(\omega)\|^2 dP(\omega) \right)^{1/2} =: \|x_n\|_*$$

Using induction, $(n=0)$ is straightforward: $\|x_0\|_* = \|\bar{x}_0\| < \infty$.

$$\|x_{n+1}\|_* \leq \|x_n\|_* + a(n)\|h(x_n)\|_* + a(n) \cdot \|M_{n+1}\|_*$$

$$\leq \|x_n\|_* + a(n) K_{x_0}(1 + \|x_n\|_*)$$

$$+ a(n)\sqrt{K}\left(\sqrt{1+\|x_n\|_*^2}\right) < \infty$$

---

Claim: $X_{T_a \wedge n}$ is a sq-integrable $\{\mathcal{F}_n\}$-martingale with quadratic variation process $A_{T_a \wedge n}$.

$$A_n = \sum_{i=1}^{n} \mathbb{E}\left[\left((H.X)_i - (H.X)_{i-1}\right)^2 \middle| \mathcal{F}_{i-1}\right] + \mathbb{E}[(H.X)_0^2]$$

Hint to prove this: $H_n = \mathbb{1}\{T_a \geq n\}$

$$X_{n \wedge T_a} = \{(H.X)_n\},$$

where $$(H.X)_n = \sum_{i=1}^{n} H_i(X_i - X_{i-1}) + X_0.$$

Identify its quadratic variation, and verify that this quadratic variation is $[(H \circ A)_n] = A_{n \wedge T_a}$.

$$Q_n = \sum_{i=1}^{n} \mathbb{E}\left[\left(\sum_{j=1}^{i} H_j(X_j - X_{j-1}) - \sum_{j=1}^{i-1} H_j(X_j - X_{j-1})\right)^2 \Big/ \mathcal{F}_{i-1}\right] + 0$$

$$= \sum_{i=1}^{n} H_i \mathbb{E}\left[(X_i - X_{i-1}))^2 / \mathcal{F}_{i-1}\right] \overset{?}{=} \sum_{i=1}^{n} H_i (A_i - A_{i-1}),$$

Yes.

$A_n \rightarrow$ Quad variation process of $X_n^2$

$$\therefore A_i - A_{i-1} = \mathbb{E}\left((X_i - X_{i-1})^2 / \mathcal{F}_{i-1}\right)$$ OK.