Functional Analysis (Contd...) :

Are there good sufficient conditions for relative compactness in the weak topology on $\mathcal{L}_2[0,T]$?

<u>Thm</u>: Let $\mathcal{F} \subset \mathcal{L}_2[0,T]$ be a family that is $\|\cdot\|_2$-bounded. Then $\mathcal{F}$ is relatively (sequentially) compact in the weak topology on $\mathcal{L}_2[0,T]$.

Remark: In particular, every bounded energy sequence of functions in $\mathcal{L}_2[0,T]$ has a convergent subsequence.

(Special case of a theorem called Banach-Alaoglu Theorem).

<u>Thm</u>: (Banach-Saks) If $f_n \xrightarrow{w} f$ in $\mathcal{L}_2[0,T]$, then there is a subsequence $\{n_k\}$ such that

$$\left\| \frac{1}{N} \sum_{k=1}^{N} f_{n_k} - f \right\|_2 \xrightarrow[N\to\infty]{} 0.$$

The space $\mathcal{C}([0,\infty);\mathbb{R}^d) \equiv \mathcal{C}([0,\infty))$; we already put a metric on this space. This induces a topology (τ).

We can come to this topology in a different way.

$$f \in \mathcal{C}[0,\infty) \xrightarrow[\text{restriction to } T]{} f\big|_T \in \mathcal{C}[0,T].$$

$f_n \xrightarrow{\tau'} f$ iff $\forall T$, $f_n\big|_T \xrightarrow[n\to\infty]{} f\big|_T$ in $C[0,T]$.

(i.e.), $\left\| f_n\big|_T - f\big|_T \right\| \xrightarrow[n\to\infty]{} 0$, i.e., $\sup_{t\in[0,T]} \|f_n(t) - f(t)\| \xrightarrow[n\to\infty]{} 0$,

(i.e.) iff $f \longmapsto f\big|_T$ is continuous for every T.

In the exercise, we showed that $f_n \xrightarrow{\tau'} f$ $\boxed{\text{iff}}$ $\rho(f_n, f) = \sum_{T=1}^{\infty} 2^{-T} \left\| f_n\big|_T - f\big|_T \right\| \xrightarrow[n\to\infty]{} 0.$

Hence, $\tau' = \tau$.

<u>Thm</u>: $\mathcal{Y} \subset C[0,\infty)$ is relatively compact in the topology induced τ iff it is equicontinuous and pointwise bounded.
(Arzela-Ascoli for $C[0,\infty)$).

$$X = \{ f : [0,\infty) \longrightarrow \mathbb{R}^d, \text{ m'ble}, \ \|f\big|_T\|_{\mathcal{L}_2[0,T]} < \infty, \text{ for every } T>0 \}.$$
(Analogue of $\mathcal{L}_2[0,T]$).

Endow it with a topology that makes the following maps continuous.

$\forall T>0$, $\forall g \in \mathcal{L}_2[0,T]$, $f \longmapsto \langle f\big|_T, g \rangle_{\mathcal{L}_2[0,T]}$

$$= \int_{[0,T]} \langle f(t), g(t) \rangle_{\mathbb{R}^d} \, dt$$

is continuous.

(Analogue of weak topology on $\mathcal{L}_2[0,T]$).

<u>Thm</u>: A set $\mathcal{Y} \subset X$ is relatively (sequentially) compact in the weak topology on X if

$$\sup_{f \in \mathcal{Y}} \|f|_T\|_{\mathcal{L}_2[0,T]} \leq B_T < \infty, \ \forall T > 0.$$

What is the difficulty here? Given a sequence $f_n \in \mathcal{Y}$, we can (by the theorem for relative compactness in $\mathcal{L}_2[0,T]$) get a g_T, with

$$f_{n_k} \longrightarrow g_T \quad (n_k \text{ may depend on } T).$$

But we need a <u>single g</u>, the restrictions of which are the above limit f.

<u>Thm</u>: (Extension of Banach-Saks) Suppose $f_n \longrightarrow f$ in X with the indicated topology. Then for each $T>0$, $\exists \{n_k\}$ s.t.

$$\left\| \frac{1}{N} \sum_{k=1}^{N} f_{n_k}|_T - f|_T \right\|_{\mathcal{L}_2[0,T]} \xrightarrow[N\to\infty]{} 0.$$

Stochastic Recursive Inclusions :

Consider the generalization:

$$x_{n+1} = x_n + a(n)\left[y_n + M_{n+1}\right]$$

- $a(n)$, as before, satisfies (A2).
- $\{M_n\}$ martingale diff. sequence w.r.t.
$$\mathcal{F}_n = \sigma(x_m, y_m, M_m, m \leq n, n \geq 0).$$
- $y_n \in h(x_n)$ $\forall n$.

(A1') Assumptions on $h: \mathbb{R}^d \to$ subsets of $\mathbb{R}^d$.

Set-valued function.

Marchaud.

(i) For each $x \in \mathbb{R}^d$, $h(x)$ is convex and compact.

(ii) For all $x \in \mathbb{R}^d$, $\sup_{y \in h(x)} \|y\| \leq K(1 + \|x\|)$ for some $K > 0$.

(iii) h is upper semi-continuous. (USC), i.e., $x_n \to x$, $y_n \to y$, $y_n \in h(x_n) \Rightarrow y \in h(x)$.

Discussion:

$$\frac{dx(t)}{dt} = h(x(t)),$$

$$h(x) = \begin{cases} -1, & x > 0, \\ 5, & \text{if } x = 0. \end{cases}$$

This does not admit a "classical solution".

So we make it to be: $\dfrac{dx(t)}{dt} \in \bar{h}(x(t))$, $\bar{h}(x) = \begin{cases} \{-1\}, & \text{if } x > 0, \\ [-1, 5], & \text{if } x = 0. \end{cases}$