Multiple Time Scales:

x_n : faster time scale

y_n : slower time scale

$$x_{n+1} = x_n + a(n)\left[h(x_n, y_n) + M^{(1)}_{n+1}\right]$$

$$y_{n+1} = y_n + b(n)\left[g(x_n, y_n) + M^{(2)}_{n+1}\right]$$

$$\left.\begin{aligned} h &: \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^d \\ g &: \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^k \end{aligned}\right\} \text{Lipschitz}$$

$M^{(i)}_{n+1}$: mart. diff. sequences wrt $\mathcal{F}_n = \sigma(x_m, y_m, M^{(1)}_m, M^{(2)}_m, m \le n)$

Assume : $\sum_n a(n) = \sum_n b(n) = \infty$, $\sum_n \left[a^2(n) + b^2(n)\right] < \infty$, $\frac{b(n)}{a(n)} \xrightarrow[n\to\infty]{} 0$

(A1), (A2), (A3).

$$\mathbb{E}\left[\|M^{(i)}_{n+1}\|^2 \mid \mathcal{F}_n\right] \le K\left(1 + \|x_n\|^2 + \|y_n\|^2\right), \text{ a.s.}, \ i = 1, 2.$$

Examples : $a(n) = \frac{1}{n}$, $b(n) = \frac{1}{n \log n}$

$a(n) = \frac{1}{n^{2/3}}$, $b(n) = \frac{1}{n}$.

Intuition : $\frac{dx}{dt} \sim \frac{x_{n+1} - x_n}{b(n)} \sim \frac{a(n)}{b(n)} h(x_n, y_n)$

$\frac{dy}{dt} \sim \frac{y_{n+1} - y_n}{b(n)} \sim g(x_n, y_n)$

Anticipate : $dx = h(x(t), y(t)) \cdot \frac{1}{}$

$$\left.\begin{aligned} \frac{dx}{dt} &= \ldots \frac{1}{\epsilon} \\ \frac{dy}{dt} &= g(x(t), y(t)) \end{aligned}\right\} \epsilon \downarrow 0 \quad \text{(singularly perturbed ode)}$$

While $\underbrace{\text{analyzing } x}_{\Downarrow}$, y is quasi static.

$h(x(t), y(t)) = \Theta(1)$
$\Rightarrow \Delta t \approx \Theta(\varepsilon)$

Rescale time $s = t/\varepsilon$.

$$\bar{x}(s) = x(s\varepsilon)$$

and $$\frac{d\bar{x}(s)}{ds} = \left(\frac{dx(s\varepsilon)}{ds}\right)\varepsilon = h(x(s\varepsilon), y(s\varepsilon)) \cdot \frac{1}{\cancel{\varepsilon}} \times \cancel{\varepsilon}$$
$$= h(\bar{x}(s), y(s\varepsilon))$$

Assumptions:

(B1) a) $\dot{x}(s) = h(x(s), y)$

quasistatic $\longrightarrow$ static

y is now viewed as a parameter.

This system has a globally asymptotically stable eqbm $\lambda(y)$.

b) $\lambda : \mathbb{R}^k \longrightarrow \mathbb{R}^d$ is Lipschitz

As y varies slowly, $x(\cdot)$ would have equilibriated at $\lambda(y)$.

Go to y-dynamics

$$\frac{dy}{dt} = g(\lambda(y(t)), y(t)) = \tilde{g}(y(t))$$

(B2) $\dot{y} = g(\lambda(y), y) = \tilde{g}(y)$ has y^* as its globally asymp. stable eqbm.

(B3) $\sup_n \left(\|x_n\| + \|y_n\|\right) < \infty$ a.s.

[We are headed towards $(x_n, y_n) \xrightarrow[n\to\infty]{} (\lambda(y^*), y^*)$]

<u>Lemma</u>: $(x_n, y_n) \xrightarrow[n\to\infty]{} \{(\lambda(y), y) : y \in \mathbb{R}^k\}$ a.s.

Pf: Look @ the fast timescale

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + a(n)\left[\begin{pmatrix} h(x_n, y_n) \\ \frac{b(n)}{a(n)} g(x_n, y_n) \end{pmatrix} + \begin{pmatrix} M^{(1)}_{n+1} \\ \frac{b(n)}{a(n)} M^{(2)}_{n+1} \end{pmatrix}\right]$$

$$= \begin{pmatrix} x_n \\ y_n \end{pmatrix} + a(n)\left[\begin{pmatrix} h(x_n, y_n) \\ 0 \end{pmatrix} + M_{n+1} + \begin{pmatrix} 0 \\ \varepsilon_n \end{pmatrix}\right],$$

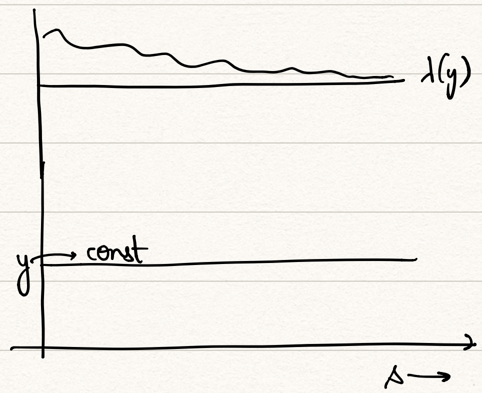
$$\varepsilon_n = \frac{b(n)}{a(n)} g(x_n, y_n)$$

Generalized BCL told us we could handle $\varepsilon_n \to 0$.

$\begin{pmatrix} x_n \\ y_n \end{pmatrix} \xrightarrow[n\to\infty]{}$ some connected ict set for the dynamics $\begin{pmatrix} dx(s)/ds \\ dy(s)/ds \end{pmatrix}$

$$= \begin{pmatrix} h(x(s), y(s)) \\ 0 \end{pmatrix}$$

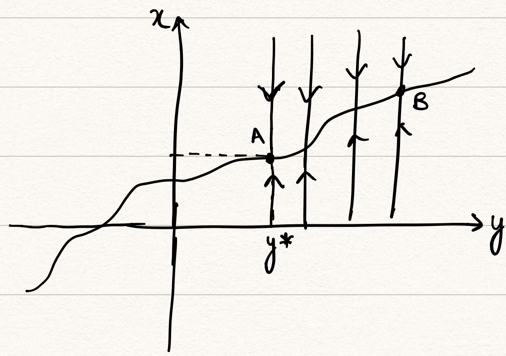
Connected ict sets for this dynamics must have y = some constant.



The connected ict sets are, for the x-alone $\{\lambda(y) : y \in \mathbb{R}^k\}$

$\{(\lambda(y), y) : y \in \mathbb{R}^k\}$ is connected.

[Follows from continuity of $\lambda(\cdot)$]

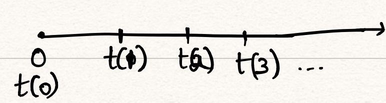


(For any initial y, y remains fixed. So we can make ϵ-jumps to get from pt A to pt B.)

<u>Thm</u>: $(x_n, y_n) \xrightarrow[n\to\infty]{} (\lambda(y^*), y^*)$ a.s.

<u>Pf</u>: Look at slower timescale.

<u>Recall</u>: $t(0) = 0, \quad t(n) = \sum_{m=0}^{n-1} b(m)$



$\bar{y}(t)$ = piecewise linear cts. interpolation of the y_n's.

$$\xi_n = \sum_{m=0}^{n-1} b(m) M^{(2)}_{m+1} \quad , n \geq 1.$$

$[t]$ = embedded time point closest to & smaller than t.

$$\bar{y}(t(n+m)) = \bar{y}(t(n)) + \sum_{k=0}^{m-1} b(n+k)\left[g(x_{n+k}, y_{n+k}) - g(\lambda(y_{n+k}), y_{n+k})\right]$$

$$+ \sum_{k=0}^{m-1} \int_{t(n+k)}^{t(n+k+1)} \left[g(\lambda(y_{n+k}), y_{n+k}) - g(\lambda(\bar{y}(t)), \bar{y}(t))\right] dt$$

Tracking error

Discretization error.

$$+ \sum_{k=0}^{m-1} \int_{t(n+k)}^{t(n+k+1)} g(\lambda(\bar{y}(t)), \bar{y}(t))\, dt$$

$$+ \delta_{n,n+m}$$

And $$y^{t(n)}(t(n+m)) = y^{t(n)}(t(n)) + \int_{t(n)}^{t(n+m)} g(\underbrace{\lambda(y^{t(n)}(t))}_{\text{How?}}, y^{t(n)}(t))\, dt$$

Mimic the BCL proof. (We can handle Martingale noise, Disc. error & the diff bet.)

Tracking error:

$$\|\cdot\| \leq \sum_{k=0}^{m-1} b(n+k) L \|x_{n+k} - \lambda(y_{n+k})\|$$

$\underset{\text{Lipschitz}}{\overset{y \mapsto}{}}$ $\leq L \sum_{k=0}^{m-1} b(n+k) \sup_{l \geq n} \|x_n - \lambda(y_l)\|$

$\leq LT \sup_{l \geq n} \|x_l - \lambda(y_l)\|$ [x_l has already gone to $\lambda(y_l)$.]

$\xrightarrow{n \to \infty} 0$

BCL applies $\Rightarrow$ y_n's converge to connected ict set of $\dot{y}(t)$
$= g(\lambda(y(t)), y(t)),$ $t \geq 0$
$= \tilde{g}(y(t))$

Ok what happens to Discretiz^n error? not clear.

This connected ict set must be $\{y^*\}$.

<u>FACT</u> (Can be verified) S'pose x^* is g.a.s. eqbm. Then the only connected ict set must be $\{x^*\}$.