Martingales:

$(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow$ Probability Space

$\{\mathcal{F}_n\}_{n\geq 0}$, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \rightarrow$ filtration (sub σ-fields of $\mathcal{F}$).

X_n is adapted to $\mathcal{F}_n$ if X_n is $\mathcal{F}_n$-measurable.

<u>Defn</u>: $\{X_n\}$ is an $\{\mathcal{F}_n\}$ martingale if
- X_n is integrable for every n.
- $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n$ a.s.
- X_n is adapted to $\mathcal{F}_n$.

(Submartingale if $\mathbb{E}[X_{n+1} | \mathcal{F}_n] \geq X_n$ a.s.)

<u>Egs</u>: ① Random Walk: Let $\{\xi_n\}$ be indep. RVs., with $\mathbb{E}\xi_n = 0$.

Let $\mathcal{F}_n = \sigma\{\xi_1, \ldots, \xi_n\}$, $n \geq 0$

<u>Def</u>: $\mathcal{F}_0 = \{\phi, \Omega\}$

$\mathbb{E}[\xi_n^+] = \mathbb{E}[\xi_n^-]$ both finite

$\Rightarrow \mathbb{E}[\xi_n^+ + \xi_n^-] < \infty$

<u>Def</u>: $S_0 = 0$, and $S_n = \xi_1 + \cdots + \xi_n$, $n \geq 1$.

- S_n is adapted to $\mathcal{F}_n$
- S_n is integrable, since $\mathbb{E}[|S_n|] \leq \mathbb{E}[|\xi_1| + \cdots + |\xi_n|] < \infty$ ✓
- Further, $\mathbb{E}[S_{n+1} | \mathcal{F}_n] = S_n + \mathbb{E}[\xi_{n+1} | \mathcal{F}_n]$

$= \mathbb{E}[\xi_{n+1}] = 0.$

Note that: S_n is a sub-martingale if $\mathbb{E}[\xi_n] \geq 0$.

② Product Martingale: Let $\{\xi_n\}$ be indep non-neg RVs with $\mathbb{E}\,\xi_n = 1$, $n \geq 1$. Again, $\mathcal{F}_n$ is as before.

Define $X_0 = 1$, and $X_n = \xi_1 \dots \xi_n$

$\rightarrow$ Then, $\{X_n\}$ is an $\{\mathcal{F}_n\}$- martingale.
It is a sub-martingale if $\mathbb{E}[\xi_n] \geq 1$, $n \geq 1$.

③ Doob Martingale: Any integrable random variable can be put at the end of some martingale.

Let X be integrable. Let $\{\mathcal{F}_n\}$ be any filtration.

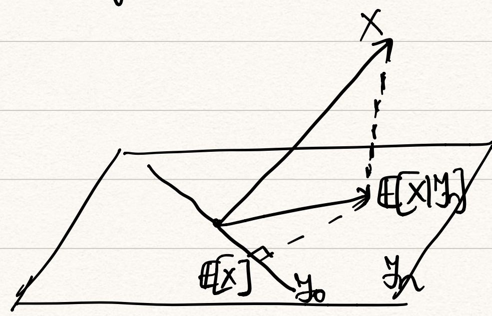
Def: $X_n = \mathbb{E}[X \mid \mathcal{F}_n]$, $n \geq 0$.

Then, $\{X_n\}$ is an $\{\mathcal{F}_n\}$- martingale.

- X_n is adapted to $\mathcal{F}_n$. (Since X_n is a "fn" of $\mathcal{F}_n$).
- X_n has finite mean $\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{F}_n]]$
  $= \mathbb{E}[X]$.

  So if X is integrable, so is X_n. ||

Remarks: Interpreting $\mathbb{E}[X|\mathcal{F}_n]$



Iterated expectations procedure

$$\mathbb{E}[X|\mathcal{F}_n] = \underset{Y \in \mathcal{F}_n}{\operatorname{argmin}} \; \mathbb{E}\left[(X-Y)^2\right]$$

Therefore, $\mathbb{E}[X] = \underset{Y \in \mathcal{F}_0}{\operatorname{argmin}} \; \mathbb{E}\left[(X-Y)^2\right]$, but $\mathcal{F}_0 = \{\phi, \Omega\}$

$\Rightarrow Y$ is a <u>constant</u> RV.

Now,

- $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}\left[\,\mathbb{E}[X|\mathcal{F}_{n+1}]\,|\mathcal{F}_n\right]$

$= \mathbb{E}[X|\mathcal{F}_n] = X_n$ (Transitivity, or the "tower property" of conditional expectations).

We could consider $X = f(\xi_1, \xi_2, \dots, \xi_m)$, $\{\xi_n\}$ iid.

$\mathcal{F}_n = \sigma\{\xi_1, \dots, \xi_n\}$, $0 \leq n \leq m$.

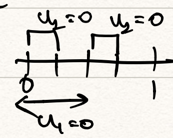
Evolution $X_n = \mathbb{E}[X|\mathcal{F}_n] = \mathbb{E}[X|\xi_1, \dots, \xi_n]$

is the evolution of X obtained by revealing
$\xi_1, \xi_2, \dots, \xi_n$, with X at the very end.

- U uniform on $[0,1]$.

$U = 0.u_1 u_2 u_3 \dots$ binary expansion

Note that : $u_1, u_2, \dots$ are iid $\mathrm{Ber}(1/2)$.



$\mathcal{F}_n = \sigma\{u_1, u_2, \dots, u_n\}$, $X_n = \mathbb{E}[U \mid \mathcal{F}_n]$. (Doob martingale again)

④ Polya urn: An urn contains b_0 black balls and r_0 red balls at $t=0$. A ball is drawn unif. at random from the urn & it is returned to the urn with an addnl ball of the same color.

$B_n = \#$ of black balls at time n.
$R_n =$ —"— red —"— .

For $n=0$, $B_0 = b_0$, $R_0 = r_0$

For $n \geq 1$, def: $\xi_n = \mathbb{1}\{n^{th}$ draw is black$\}$

$\therefore B_n = B_{n-1} + \xi_n$, $R_n = R_{n-1} + 1 - \xi_n$

$\mathcal{F}_n = \sigma\{\xi_1, \dots, \xi_n\}$

$X_n = B_n$, $\{X_n\}$ is $\{\mathcal{F}_n\}$-martingale.

b_0+r_0+n

- X_n is $\mathcal{F}_n$-measurable ✓
- X_n is integrable, since $X_n \le 1$.

- $$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \frac{\mathbb{E}[B_n + \xi_{n+1}|\mathcal{F}_n]}{b_0+r_0+n+1}$$

$\longrightarrow$ We know $\xi_1, \dots, \xi_n$ $\Rightarrow$ We know B_n.

$$= \frac{B_n + \mathbb{E}[\xi_{n+1}|\mathcal{F}_n]}{b_0+r_0+n+1}$$

$$= \frac{B_n + \frac{B_n}{b_0+r_0+n}}{b_0+r_0+n+1} = X_n.$$

5 New martingales from old:

Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$-martingale. Let $\phi: \mathbb{R} \to \mathbb{R}$ be convex & integrable.

Let $\mathbb{E}[|\phi(X_n)|]$ finite for all n.

Claim: $\{\phi(X_n)\}$ is an $\{\mathcal{F}_n\}$ martingale

- Adapted to ~~$\mathcal{B}$~~ $\mathcal{F}_n$ ✓ $\longrightarrow$ Full pf: For any $B \subset \mathbb{R}$, $\phi^{-1}(B)$ is measurable
- Integrable

$\cdot\ \mathbb{E}[\phi(X_{n+1})|\mathcal{F}_n]$

$\geq \phi(\mathbb{E}[X_{n+1}|\mathcal{F}_n]) = \phi(X_n)$

[ϕ is convex & hence continuous & hence measurable]

$\Rightarrow X_n^{-1}(\phi^{-1}(B)) \in \mathcal{F}_n$.

Further, if $\{X_n\}$ is a sub-martingale & ϕ is convex & increasing, then $\{\phi(X_n)\}$ is also a sub-martingale.

Optional stopping or Sampling:

Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$-submartingale. Consider $n_0 \leq n_1 \leq n_2 \leq \dots$

Consider $\{X_{n_k}\}_{k\geq 0}$. It is easy to see that $\mathbb{E}[X_{n_{k+1}}|\mathcal{F}_{n_k}] \geq X_{n_k}$.

So $\{X_{n_k}\}_{k\geq 0}$ is an $\{\mathcal{F}_{n_k}\}$-submartingale.

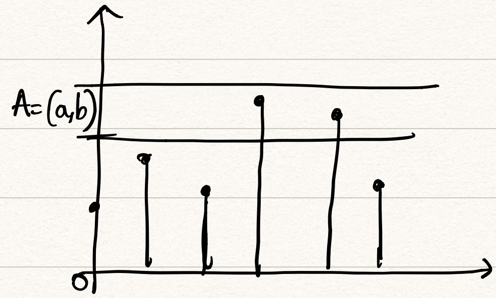
Recall:

Defn: (Stopping time) A $\{0,1,2,\dots,\infty\}$-valued RV T is a stopping time wrt a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ if for each $n = \{0,1,2,\dots,\infty\}$, $[T\leq n]$ is $\mathcal{F}_n$-measurable, with $\mathcal{F}_\infty = \bigvee_{n=1}^{\infty} \mathcal{F}_n$

eg: $\{X_n\} \rightarrow$ sequence of real-valued RVs.

$\{\mathcal{F}_n^X\} \to$ natural filtration

$A \to$ Borel — smallest set made up of open sets in $\mathbb{R}$.



$$T = \begin{cases} \min\{n \geq 0 \mid X_n \in A\}, & \text{if this set is non-empty} \\ \infty, & \text{if this set is empty.} \end{cases}$$

(First hitting time of A)   T is a stopping time.