$A = \bigcap_n C_n$,

C_ns are non-empty, compact, connected, nested decreasing.

Then A is connected.

<u>Proof</u>: Suppose A is not connected.

There exist U, V, open, disjoint such that

$$U \cap A \neq \phi, \quad V \cap A \neq \phi.$$

In particular, $U \cap C_n \neq \phi$, $V \cap C_n \neq \phi$, $\forall n \geq 0$.

Consider $\{C_n \setminus U \cup V\}$ $\longrightarrow$ nested decreasing, $\boxed{\text{compact.}}$

$$\bigcap_n \{C_n \setminus U \cup V\} = \left(\bigcap_n C_n\right) \setminus U \cap V = A \setminus U \cup V = \phi$$

So, $\exists N$ for which $C_N \setminus U \cup V = \phi$. C_N, however, is connected.

$$\left[\; C_N \subset U \cup V \; \right]$$

Thm: Under (A1)–(A4), $x_n \xrightarrow{n\to\infty} A$ a.s., where A is some connected ict set.

Liapunov functions:

Def: $V: \mathbb{R}^d \to [0,\infty)$ is a Liapunov function for the dynamics $\dot{x}(t) = h(x(t))$ if it satisfies:

(i) V is C^1, (continuously diff'ble).

$$\lim_{\|x\|\to\infty} V(x) = \infty.$$

In particular, lower level sets of V, $V^{-1}([0,a]) = \{x\in\mathbb{R}^d : V(x)\le a\}$, are compact.

(ii) $H := \{x\in\mathbb{R}^d : V(x)=0\}$ is non-empty.

(iii) $\langle h(x), \nabla V(x)\rangle \le 0$ $\forall x\in\mathbb{R}^d$, with equality iff $x\in H$.

Discussion: 1) $V(x(t))$ along the dynamics $x(0)=\bar{x}\in\mathbb{R}^d$.

$$\frac{dV(x(t))}{dt} = \left\langle \nabla V(x(t)), \frac{dx(t)}{dt}\right\rangle$$

$$= \langle \nabla V(x(t)), h(x(t))\rangle \le 0$$

2) Ideal situation is when we can argue $V(x(t)) \downarrow 0$.

3) $\boxed{h(x) = -\nabla g(x)}$ for some g, bounded from below; let g be C^1, $\boxed{g(x) \to \infty \text{ as } \|x\| \to \infty.}$ Then, we claim that we can take $V = g$.

(i) $\to$ holds by assumption

(iii) $\langle h(x), \nabla V(x) \rangle = \langle -\nabla g(x), \nabla g(x) \rangle$
$$= -\|\nabla g\|^2 \leq 0,$$

with eq iff $\nabla g = 0$.

(ii) Holds if $\boxed{\nabla g = 0 \text{ iff } g = 0.}$

Thm: Assume (A1)–(A4). Let V be a Liapunov function for its dynamics and let H be its zero set.

$$x_n \xrightarrow[n\to\infty]{} A \text{ a.s.}$$ where A is a connected ict subset of H.

Examples and non-examples:

e.i) $\dot{x} = -x$, $x \in \mathbb{R}^d$.
$V(x) = \frac{1}{}\|x\|^2$

(Note: $\nabla V(x) = x$
$\dot{x} = -\nabla(x)$)

So this is a Liapunov fn.

2) $h(x) = \begin{cases} x(1-x^2), & |x| \leq \sqrt{2} \\ -5x+4\sqrt{2}, & x > \sqrt{2} \\ -5x-4\sqrt{2}, & x < -\sqrt{2} \end{cases}$

$$\int_{-\infty}^{-\sqrt{2}} (+5x+4\sqrt{2})\,dx$$
$$= \left[\frac{5x^2}{2}\right]_{-\infty}^{-\sqrt{2}} + 4\sqrt{2}[x]_{-\infty}^{-\sqrt{2}}$$
$$+ \underbrace{\int_{-\sqrt{2}}^{0} -t(1-t^2)\,dt}_{= \left[\frac{-t^2}{2}\right]_{-\sqrt{2}}^{0} + \left[\frac{t^4}{4}\right]_{-\sqrt{2}}^{0}}$$
$= 5 - 8 - 1 - 1$?

$\dot{x} = h(x)$.

Is it a gradient system?

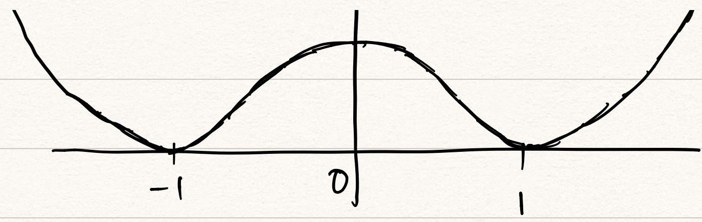
$h(x) = -\nabla V(x)$. Integrate the -ve of $h(x)$.

$x > 0$: $V(x) = \begin{cases} C + \int_0^x -t(1-t^2)\,dt = \frac{1}{4} + \frac{x^4}{4} - \frac{x^2}{2}, \quad \leq \sqrt{2} \\ \qquad (\text{taking } C = 1/4) \text{?} \\ C + \int_0^{\sqrt{2}} -t(1-t^2)\,dt + \int_{\sqrt{2}}^{x} -(-5t+4\sqrt{2})\,dt \\ \qquad = \frac{1}{4} + \frac{5}{2}(x^2-2) - 4\sqrt{2}(x-\sqrt{2}) \\ \qquad x > \sqrt{2}. \end{cases}$

How did this come?

$V \in \mathcal{C}^1$.

Further, (i) $\lim_{\|x\| \to \infty} V(x) = \infty$.



(ii) $H = \{-1, +1\}$ $\longrightarrow$ non-empty.

(iii) $\langle \nabla V(x), h(x) \rangle \leq 0$

However, $\langle \nabla V(x), h(x) \rangle = 0$ for points other than those in H.

Since $h(x) = 0$ at $x = 0$, where $V(x) \neq 0$, it is <u>not</u> a Liapunov fn.

Now, consider

(3) $$V_1(x) = \begin{cases} V(x), & |x| \geq 1 \\ 0, & |x| < 1 \end{cases}$$

This V_1 is a Liapunov function. (Check!). OK.

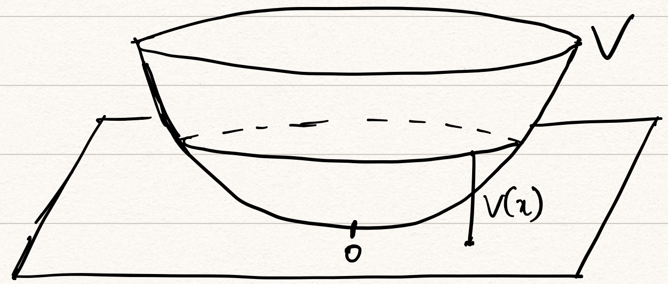
What are connected ict subsets of $H_1 = V_1^{-1}(\{0\})$
$= [-1, 1]$?

$= \{-1\}, \{0\}, \{+1\}.$

We can conclude x_n converges, and the converged value is -1 or 0 or 1.

(4) Typically, $\text{dist}^2(x, A)$ works (but not useful, since we already know A).

(5) say, V look like:



Say $\langle \nabla V(x), h(x) \rangle = 0$. So $V(x(t))$ does not change.

$$\frac{dV(x(t))}{dt} = \langle \nabla V(x(t)), h(x(t)) \rangle = 0$$

Yeah

Since dynamics preserves $V(x(t))$, this is not a Liapunov function. (Condn (iii) is violated since $V(x) \neq 0$ at such x).

Limitation of our theory:

1) Consider the foll. dynamics:

$$\dot{r} = \begin{cases} 0, & r \leq 1 \\ -(r-1)^2, & 1 \leq r \leq 2 \\ \alpha r + \beta, & r \geq 2. \end{cases}$$

(to ensure Lipschitz prop of h)

[α must be -ve]

$$\dot{\phi} = \omega.$$

(*)

$\Omega_a = \{x \in \mathbb{R}^2 : \|x\| = \|a\|\}, \quad \|a\| \leq 1.$

Note

$\Omega_a = \{x \in \mathbb{R}^2, \|x\| = 1\}, \quad \|a\| > 1.$

Is this the zero set of the Liapunov fn for the dynamics?

Connected ict sets of $A = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$

OK.

Ω_a is a connected ict set for each a.

A itself is a connected ict set. Our theory fails to identify the specific connected ict subset of A.