Asynchronous Stochastic Approximation:

$$x_{n+1}(i) = x_n(i) + a(\nu(i,n))\,\mathbb{1}\{i \in Y_n\}\left[h_i(x_n) + M_{n+1}(i)\right]$$

$Y_n \subset \{1,2,\dots,d\}$ subset of components that are updated at time n.

$$\nu(i,n) = \sum_{m=0}^{n} \mathbb{1}\{i \in Y_m\} = \#\text{ of updates of the } i^{th} \text{ agent.}$$

Updates are done comparably often: $\liminf_{n\to\infty} \frac{\nu(i,n)}{n} > 0$ a.s.

$$\bar{a}(n) = \max_{i \in Y_n} a(\nu(i,n)) \;;\; t(n) = \sum_{m=0}^{n-1} \bar{a}(m)$$

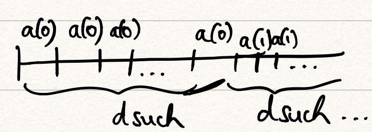
$$x_{n+1}(i) = x_n(i) + \bar{a}(n)\left[\frac{a(\nu(i,n))}{\bar{a}(n)}\,\mathbb{1}\{i \in Y_n\}\left[h_i(x_n) + M_{n+1}(i)\right]\right]$$

We anticipate that we will track:

$$\frac{dx(i)(t)}{dt} \sim \lambda_i(t)\, h_i(x(t))$$

$$\text{So,} \quad \frac{dx(t)}{dt} = \begin{pmatrix} \lambda_1(t) & & & \\ & \lambda_2(t) & & \\ & & \ddots & \\ & & & \lambda_d(t) \end{pmatrix} h(x(t))$$

For eg., for Round-Robin rule of updating $\bar{a}$:

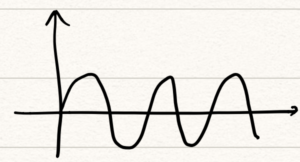


(We anticipate that $\lambda_i(t) = 1 \;, \forall i$)

Thm: Assume (A1)–(A4), and that the updates happen comparably often, a.s. Then any limit point of $\{\bar{x}(s+\cdot), s \in \mathbb{R}_+\}$ in $C([0,\infty), \mathbb{R}^d)$ as $s \to \infty$ is a solution to

$$\dot{x}(t) = \Lambda(t) h(x(t)),$$

where $\Lambda(t)$ is a $d \times d$ diagonal-matrix-valued measurable function, with $\Lambda_{ii}(t) \in [0,1]$, $\forall i, \forall t$.

Remark: For example, if $\bar{x}(\cdot)$ is the sine fn, 

the limit pts in $C([0,\infty), \mathbb{R}^1)$ are really $\{\sin(t+T), T \in (0, 2\pi)\}$.

(More) Preliminaries:

Functional Analysis Preliminaries:

- $C([0,T], \mathbb{R}^d)$: Space of continuous functions from $[0,T] \to \mathbb{R}^d$.

$$\|f\| := \sup_{t \in [0,T]} \|f(t)\|, \text{ norm in } C([0,T], \mathbb{R}^d).$$

- Banach Space $(X, \|\cdot\|)$:
  (1) Vector space over reals
  (2) $\|\cdot\| : X \to [0,\infty)$ is a norm
   (a) $\|f\| \geq 0$ with equality iff $f = 0$.

(b) $\|\alpha f\| = |\alpha| \, \|f\|$, $\alpha \in \mathbb{R}$

(c) $\|f+g\| \leq \|f\| + \|g\|$,

(3) $\|\cdot\|$ is complete, i.e., every Cauchy sequence converges

(More explicitly, $\{f_k\} \subset X$, $\|f_m - f_n\| \xrightarrow{m,n\to\infty} 0$, then there is an $f \in X$ s.t. $\|f_k - f\| \xrightarrow{k\to\infty} 0$)

Note

• **Examples**:

1) $\left(\mathcal{C}([0,T], \mathbb{R}^d), \|\cdot\|\right)$ is a Banach space.

2) $\left(\mathcal{L}_p([0,T]), \|\cdot\|_p\right)$ is also a Banach space, when $1 \leq p < \infty$.

$= \{f : [0,T] \to \mathbb{R}$, s.t. $\int_{[0,T]} |f(t)|^p \, dt < \infty\}$

$$\|f\|_p = \left(\int_{[0,T]} |f(t)|^p \, dt\right)^{1/p}$$

These are defined only upto equivalences. $f = g$ in $\mathcal{L}_p$ means $\int_{[0,T]} |f(t) - g(t)|^p \, dt = 0$ $\Leftrightarrow$ $f = g$ almost everywhere in $[0,T]$

3) $\mathcal{C}([0,\infty), \mathbb{R}^d)$ is not a Banach space (Defn. of norm itself is an issue). But we can define a metric (a distance), here.

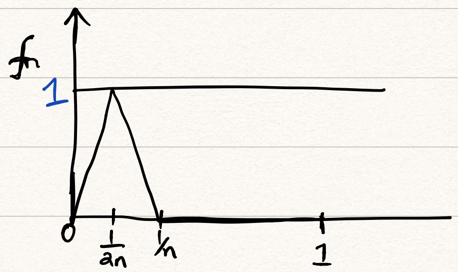
4) $\mathcal{L}_p([0,T])$ is not a Banach space for $0 < p < 1$.

Not Banach spaces, but metric spaces.

Banach space is a metric space with distance $d(f,g) = \|f - g\|$.

We'd like to hunt for convergent subsequences (conditions for...)

- A sequence that takes values on a compact set, but doesn't have a convergent subsequence in $\mathcal{C}([0,1])$.



$\|f_n\| \equiv 1$, for all n.

f_n converges pointwise to $f \equiv 0$. This is the natural candidate for the limit.

But $\|f_n - f\| = \|f_n\| = 1$, $\forall n$
$\not\to 0$.

This holds for any sub-sequence. No sub-sequential limit. $\{f_n\}$ is not <u>relatively compact</u>.

<u>Relatively compact</u>: A set A is relatively compact if $\bar{A}$ is compact. What are relatively compact subsets of $\mathcal{C}([0,T], \mathbb{R}^d)$? How do we define compactness here?? Ok

- Equicontinuity at a point $t \in [0,T]$ of a family $A \subset \mathcal{C}([0,T])$
$\equiv (\mathcal{C}([0,T]; \mathbb{R}^d))$

A is equicontinuous if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

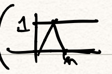
$$|s-t| < \delta \Rightarrow \sup_{f \in A} \|f(s) - f(t)\| < \epsilon.$$

- Equicontinuous, if the family is equicontinuous at all $t \in [0,T]$.

  (**) → [The δ can depend on t]

- Point-wise bounded if $\sup_{f \in A} \|f(t)\| < \infty$ for each t.

- Arzela-Ascoli Thm: A set $A \subset \mathcal{C}([0,T])$ is relatively compact if and only if it is equicontinuous and point-wise bounded.

Our fn () is pt-wise bdd. So check for eq. cont. @ 0.

Is it that $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$|s - 0| < \delta \Rightarrow \sup_n |f_n(s) - f_n(0)| < \epsilon$$

$$\Rightarrow \sup_n |f_n(s)| < \epsilon.$$

We cannot find a δ s.t. the above holds.

A suff. cond^n for equicontinuity ⟶ A fly of Lipschitz fns with same const.