Functional Analysis (Contd...) :

A special Banach space $\mathcal{L}_2[0,T] = \left\{ f : [0,T] \rightarrow \mathbb{R}^d \text{ s.t. } \int_{[0,T]} \|f(t)\|^2 dt < \infty \right\}$.

$\because$ we can define an inner product :

$$\langle f, g \rangle = \int_{[0,T]} \langle f(t), g(t) \rangle \, dt$$

$\langle \cdot, \cdot \rangle$ is an inner product if (i) symmetric
(ii) Linear in first argument
(iii) $\langle f, f \rangle \geq 0$, with equality iff $f = 0$ a.e.

Metric: $\sqrt{\langle f-g, f-g \rangle}$. This norm makes $\mathcal{L}_2[0,T]$ a complete space.

Linear functionals on $\mathcal{L}_2[0,T]$.

$$T : \mathcal{L}_2[0,T] \longrightarrow \mathbb{R}$$

$$f \longmapsto T(f) \quad \text{s.t.} \quad T(f+g) = T(f) + T(g)$$
$$T(\alpha f) = \alpha T(f)$$

T is continuous if $T^{-1} O$ is open in $\mathcal{L}_2[0,T]$
O: open in $\mathbb{R}$.

Examples : $d=1$

1. $T_1(f) = \int_{[0,T]} f(t) \, dt$

2. $T_2(f) = \int_{[0,T/2]} f(t) \, dt = \int_{[0,T]} f(t) \mathbb{1}_{[0,T/2)}(t) \, dt$

3. $T_g(f) = \int f(t) g(t) \, dt$, let $g \in \mathcal{L}_2[0,T]$.

Can check that these functionals are continuous.

<u>Thm</u>: (Riesz representation theorem).

Any continuous linear functional T on $\mathcal{L}_2[0,T]$ can be written as a T_g, where $g \in \mathcal{L}_2[0,T]$.

Counter example: Consider the following $g \notin \mathcal{L}_2[0,T]$

$$g(t) = \begin{cases} 1/t & , 0 \leq t \leq T \\ 0 , & t = 0 \end{cases}$$

Check that T_g with this g is not continuous.

Instead of functions on $[0,T]$, consider functions on $\{1, 2, \dots, n\}$.
$\mathcal{L}_2(\{1,2,\dots,n\})$ is just $\mathbb{R}^n$. (Can view it as vectors).

$$T_g(f) = \begin{bmatrix} g_1 & g_2 & \cdots & g_n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Linear transformations $T: \mathcal{L}_2\{1,2,\dots,n\} \longrightarrow \mathbb{R}^m$.

$$T_g(f) = \begin{matrix} [g_{11} & g_{12} & \cdots & g_{1n}] \\ [g_{21} & g_{22} & \cdots & g_{2n}] \\ & \vdots & & \\ [g_{m1} & \cdots & & g_{mn}] \end{matrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \quad \left(\text{Matrix product}\right).$$

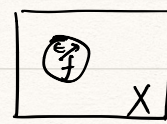
Topology in a <u>Banach</u> Space:

$(X, \|\cdot\|)$

$\|\cdot\|$ defines a metric $d(f,g) = \|f-g\|$.

In $\mathcal{L}_2[0,T]$, $d(f,g) = \sqrt{\langle f-g, f-g\rangle}$

Open balls in this metric space define a topology. (Include $X, \phi, \overbrace{B_\epsilon(f)}^{\text{Base of the topology}}$ $\forall f$, $\forall \epsilon > 0$. Arbitrary unions, finite intersections).



Strong <u>topology</u> (τ)

When we say T is continuous, one has verified that $T^{-1}O \in \tau$ for every O open in $\mathbb{R}$.

As an aside, consider τ_T (top. generated by a cont. T).

$$\text{Base}(\tau_T) = \{T^{-1}O : O \text{ is open in } \mathbb{R}\}.$$

This is the smallest topology that makes T a continuous function.

τ_w: Weak <u>Topology</u>:

Topology generated by $T_g^{-1}O$ for O open in $\mathbb{R}$, $g \in \mathcal{L}_2[0,T]$. This is the "coarsest" topology that every such T_g continuous.

Convergence in this topology : We say $f_n \xrightarrow{w.} f$ if $T_g(f_n) \longrightarrow T_g(f)$ <u>for every g</u> in $\mathcal{L}_2[0,T]$.

This is a <u>weaker</u> notion of convergence. Suppose $f_n \longrightarrow f$ in $\mathcal{L}_2[0,T]$, then by the usual def^n,

$$d(f_n, f) = \sqrt{\langle f_n - f, f_n - f\rangle} = \left(\int_{[0,T]} |f_n(t) - f(t)|^2 dt\right)^{1/2} \xrightarrow{n\to\infty} 0.$$

<u>Claim</u>: If so, then $f_n \xrightarrow{w.} f$.

<u>Pf</u>:

Take any $g \in \mathcal{L}_2[0,T]$. We have to show $T_g(f_n) \longrightarrow T_g(f)$.

$T_g(f_n) = \langle g, f_n \rangle$.

$$|T_g(f_n) - T_g(f)| = |\langle g, f_n - f\rangle| \le \|g\|_{\mathcal{L}_2[0,T]} \|f_n - f\|_{\mathcal{L}_2[0,T]} \xrightarrow{n\to\infty} 0$$

But $f_n \xrightarrow{w.} f$ for more sequences!

<u>Example</u>: $f_n(t) = \sin\frac{2\pi n t}{T}$, $t \in [0,T]$

$$f_n \not\to 0$$

since $d(f_n, f) = \|f_n\|^2 = T \not\to 0$.

For every $g \in \mathcal{L}_2[0,T]$, $T_g(f_n) = c\int_0^T g(t)\sin\frac{2\pi n t}{T}\,dt \xrightarrow[n\to\infty]{} 0$,

since g has a Fourier series : $a_g(0)$, $\underbrace{a_g(n)}_{\text{cosines}}$, $\underbrace{b_g(n)}_{\text{sines}}$;

$$a_g(0)^2 + \sum_{n\geq 1}\left(a_g^2(n) + b_g^2(n)\right) < \infty$$

So $a_g(n) \xrightarrow{n\to\infty} 0$!!

So $f_n \xrightarrow{w.} f$!!