<u>Recall</u>:

$$H \subset \underset{\hookrightarrow \text{open}}{B} \subset \overline{B} \subset G$$

We had fixed $0 < \epsilon_1 < \epsilon$ such that $H^{\epsilon_1} \subset H^{\epsilon} \subset B$. Fix $\delta > 0$ such that

$N_\delta(H^{\epsilon_1}) \subset H^{\epsilon} \subset N_\delta(H^{\epsilon}) \subset B$. We wish to show that this is indeed possible.

Firstly, we wish to show that we can pick ϵ s.t. $H^{\epsilon} \subset B$.

Now, s'pose that $H^{\eta} \not\subset B$ for all $\eta > 0$. $\Rightarrow$ For each $n \in \mathbb{N}$, $H^{1/n} \not\subset B$,
i.e., $\exists y_n \notin B$ s.t. $V(y_n) < \frac{1}{n}$.

Since y_n's come from $V^{-1}([0,1])$, lower level set of V assoc with level 1, which is compact.

Hence, we can find some subsequence $y_{n_k} \xrightarrow{k \to \infty} y$ and $V(y) = 0 \Rightarrow y \in H$

Since $y \in H \subset B$, $\exists N_\epsilon(\{y\}) \subset B$. But $y_{n_k} \xrightarrow{k\to\infty} y$, yet are not in B.
This is a contradiction. ▨

So there exists some $\epsilon > 0$ s.t. $H^{\epsilon} \subset B$. In particular, $H^{\epsilon/2} \subset B$.

Now, from the uniform continuity of V, given $\epsilon/2$, $\exists \delta > 0$ s.t. $\forall x, y$ satisfying $\|x - y\| < \delta$, we have $|V(x) - V(y)| < \epsilon/2$.

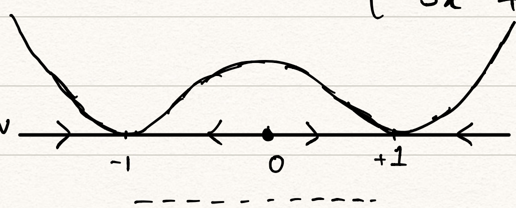
Now consider $N_\delta(H^{\epsilon/2})$. For any $y \in N_\delta(H^{\epsilon/2})$, $\exists x \in H^{\epsilon/2}$ s.t. $\|x-y\| < \delta$. So $V(y) \le V(x) + \epsilon/2 \le \epsilon/2 + \epsilon/2 = \epsilon$.

Hence, $y \in B$, which implies that $N_\delta(H^{\epsilon/2}) \subset B$. ▨

Avoidance of traps (An overview):

Consider the setting
$$\dot{x} = \left.\begin{cases} x - x^3, & |x| \le \sqrt{2} \\ -5x + 4\sqrt{2}, & x > \sqrt{2} \\ -5x - 4\sqrt{2}, & x < -\sqrt{2} \end{cases}\right] h(x)$$

Recall → a "non-Liapunov fn"



Note that $\{0\}$ is a saddle point.

actual Liapunov fn (shunted to zero)

$x_{n+1} = x_n + a(n)\left[h(x_n) + M_{n+1}\right]$. We already know that $x_n \to -1$ or $x_n \to 0$ or $x_n \to +1$

Thm: Assume $\forall n \ge 0$, $\underbrace{\mathbb{E}\left[M_{n+1} M_{n+1}^T \mid \mathcal{F}_n\right] = Q(x_n)}_{\text{Noise covariance matrix}}$. Now, if $\Lambda_- I_d < Q(x) < \Lambda_+ I_d$, $\forall x \in \mathbb{R}^d$, $0 < \Lambda_- < \Lambda_+ < \infty$, then $x_n \to +1$ or $x_n \to -1$ a.s. "If the noise is rich enough, then we escape local maxima & saddlepoints".

(where '<' ordering implies that RHS − LHS is PSD).
(or) $y^T Q(x)\, y \ge \Lambda_- \|y\|^2$

Chapter 4:

Stability:

S'pose $h(x) = -x + g(x)$, $g(\cdot)$ is Lipschitz and bounded. For very large x, we anticipate dynamics to be governed by $h_\infty(x) = -x$. Since $h_\infty(\cdot)$ has the origin as g.a.s. eqbm, we anticipate stability.

Goal: Formalize this idea.

(A5) Define $h_c(x) = \dfrac{h(cx)}{c}$, $c \geq 1$. Assume

(a) $h_c(\cdot) \xrightarrow{c \to \infty} h_\infty(\cdot)$, uniformly on compacts.

(b) The ode $\dot{x}(t) = h_\infty(x(t))$ has the origin as its g.a.s. eqbm.

Thm: Under (A1)–(A3) and (A5), then $\overbrace{\sup_n \|x_n\| < \infty \text{ a.s}}^{(A4)}$.

Some observations:

- Suppose h is Lipschitz with constant L. Then so are h_c and h_∞. ✓

Thus, $\{h_c\}_{c \geq 1}$ is equicontinuous. [Equicont. @ x means that the same δ works for all members of the family. "Equicont", means that the δ doesn't depend on x.]

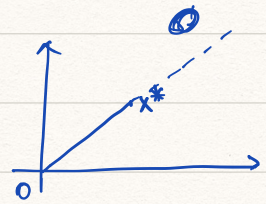
(Comes from Lipschitz property) ✓

[This equicontinuity is sufficient to show that $h_c(\cdot) \longrightarrow h_\infty(\cdot)$ u.o.c., by "Arzela-Ascoli"]

- $h_\infty(ax) = a h_\infty(x)$ [Follows from $h_c(x) = \frac{h(cx)}{c}$, $c \geq 1$]

- If $\dot{x}(t) = h_\infty(x(t))$ has an isolated equilibrium. Then that eqbm is the origin.

Pf is:



S'pose x^* is an isolated eqb^m

$\Rightarrow \exists N_\delta(x^*)$ s.t.

$\exists$ no eqbm $\in N_\delta(x^*)$

$\Rightarrow h_\infty(x^*) = 0$

But from prev. pt, $h_\infty(ax^*) = 0$.

So all pts on the line connecting origin & x^*, are eqb equilibria. Contradiction. $\boxtimes$

Notation: Fix $T > 0$. $T_0 = 0$,

$$T_{m+1} = \min\{t(n) : t(n) \geq T_m + T\}, \quad m \geq 0.$$

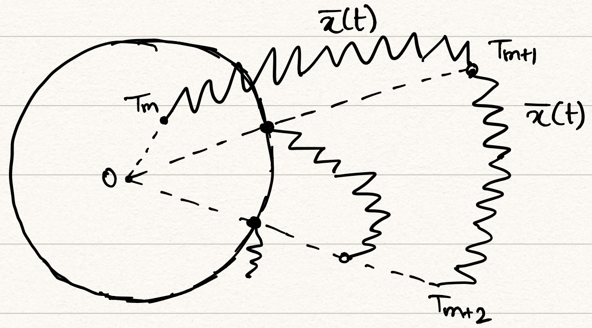
For simplicity, assume $a(n) \leq 1 \;\; \forall n \;\; \left(\text{eg. } \frac{1}{n+1}\right)$. So,

$$T_{m+1} \leq T_m + T + 1, \quad \forall m \geq 0.$$

$\bar{x}(\cdot)$: interpolated trajectory

$\hat{x}(t)$, $t \geq 0$ is defined via $\hat{x}(t) = \frac{\bar{x}(t)}{r(m)}$, $t \in [T_m, T_{m+1})$

where $r(m) = \|\bar{x}(T_m)\| \overset{\max}{\vee} 1$



So, $\hat{x}(T_{m+1}^-) = \dfrac{\bar{x}(T_{m+1}^-)}{r(m)} = \dfrac{\bar{x}(T_{m+1})}{r(m)}$ [Since $\bar{x}$ is continuous]

and $\hat{x}(T_{m+1}) = \dfrac{\bar{x}(T_{m+1})}{r(m+1)}$