Recall:

① Example for Attractor $\not\Rightarrow$ Liapunov stable

$$\dot{r} = r(1-r)$$
$$\dot{\phi} = r(1-\cos\phi)$$

We can convert this to Cartesian co-ordinates:

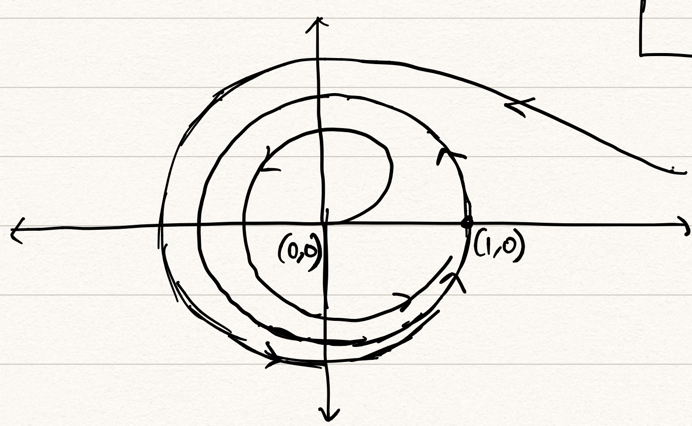
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(1-r) - y(r-x) \\ y(1-r) + x(r-x) \end{pmatrix}$$

[Verify this] ✓

$$\dot{x} = \dot{(r\cos\phi)} = \dot{r}\cos\phi - r(\sin\phi)\dot{\phi}$$
$$= r(1-r)\cos\phi - y\,r(1-\cos\phi)$$
$$= x(1-r) - y(r-x)$$

& IIIly for $\dot{y}$

$$= h\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$$



$\{(1,0)\}$ is an attractor, since we can choose <u>any</u> open ball around it that <u>excludes</u> $(0,0)$. (since starting from $(0,0)$, we stay at $(0,0)$).

So O could be $\mathbb{R}^2 \setminus \{(0,0)\}$. ||

Definitions (Contd...):

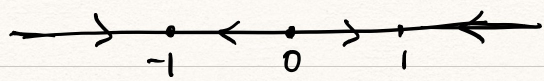
(d) A compact invariant set M is asymptotically stable if it is Liapunov stable and an attractor.

(e) A compact invariant set M is globally asymp. stable if it is asymp. stable with domain of attraction $\mathbb{R}^d$.

Eg. (i) $\dot{x}(t) = Hx(t)$, $d=2$, $Re(\lambda_i)<0$

$\{(0,0)\}$ is globally asymp. stable.

(ii) $\dot{x}(t) = x(1-x^2)$, $x \in \mathbb{R}$.



$M_1 = [-1, 1]$ is compact, invariant, attractor, Liapunov stable, and hence asymp. stable

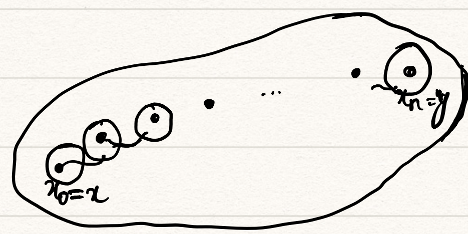
(infact globally asymp. stable)

$M_2 = [0,1]$ is compact, invariant, not an attractor, not Liapunov stable.

(f) A $\boxed{\text{compact invariant set}}$, M, is internally chain transitive (ICT) (or ict) if $\forall x, y \in M$, $\forall \epsilon > 0$, $\forall T > 0$ there exist $n \geq 1$ and $n+1$ points,

$$x = x_0, x_1, x_2, \cdots, x_n = y$$

such that a trajectory initiated at x_i meets the ϵ-nbd of x_{i+1} at time $t_i \geq T$, $i = 0, 1, \ldots, n-1$.

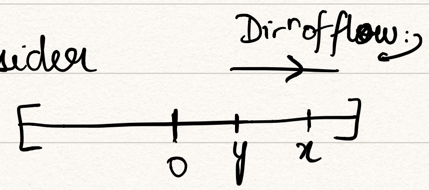


(Jumps of size ϵ are allowed)

Examples: $\dot{x}(t) = x(t)\left(1 - (x(t))^2\right)$

(a) $M = [-1, 1]$ is globally asymp. stable. But it is not ict.

To see this, consider



x, T chosen very large
ϵ chosen very small

(b) $\{0\}$ is ict. So is $\{-1\}$, $\{+1\}$.

(c) $\{-1, 0, 1\}$ is not ict (by choosing $\epsilon < 1$, we can't make any jumps inside M.)

Compact ✓
invar ✓
Attractor ✓ , with domain of attraction $\mathbb{R}^d$.
Liapunov stable ✗

Asymp. stable ✗

Connected ✗ (As in path-connectedness) → Will be detailed out later

(Back to Stoch. Approx. Scheme)

<u>Thm</u>: Recall assumptions (A1), (A2), (A3), (A4).

Under these assumptions, $x_n \longrightarrow A$ a.s.

where A is a possibly sample path dependent connected ict set for the dynamics $\dot{x}(t) = h(x(t))$.

eg. Consider

$$h(x) = \begin{cases} x(1-x^2), & |x| < \sqrt{2} \\ \text{linear}, & x > \sqrt{2} \\ \text{linear}, & x < -\sqrt{2} \end{cases}$$

So (A1) holds. Assume (A2)–(A4). Then x_n converges a.s.

$x_n \to +1$ or $x_n \to 0$ or $x_n \to -1$ a.s.