Defn: A set 'A' is *positively invariant* if $a \in A$ implies that the solution $x(\cdot)$ with $x(0) = a$ has $\boxed{x(t) \in A}$ for all $t \geq 0$.

$(\phi_t(a) \in A \quad , \forall t \geq 0)$. (or, $\phi_t(A) \subset A$)

(||ly, *negatively invariant*).

The set A is *invariant* if it is both positively and negatively invariant.

• *Examples* (a) $\dot{x}(t) = Hx(t) \quad , d = 2, \quad Re(\lambda_i) < 0 \quad , i = 1,2.$
$Im(\lambda_i) = 0 \quad , i = 1,2.$

Then $A = \{b : ||b|| < 1\}$ is positively invariant.

Note: It is NOT negatively invariant, $\because \dot{x}(-t) = -\dot{x}(t)$
$= -Hx(-t)$

[(So the eigenval have +ve real parts)
$\Rightarrow$ NOT negatively invar.]

(b) $A = \{0\}$ is BOTH positively and negatively invariant.

(c) $A = R^2$ is also invariant.

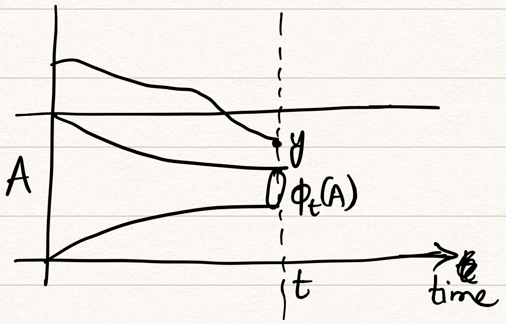
Lemma: A is invariant if and only if $\phi_t(A) = A, \forall t \in \mathbb{R}$.

Pf: ($\Leftarrow$): follows from definitions.

($\Rightarrow$): We will prove this for all $t \geq 0$.
Negative t can be similarly handled.

We shall show the contrapositive:

Fix $t \geq 0$. Let $\phi_t(A) \subsetneq A$.



Then there exists $y \in A \setminus \phi_t(A)$ such that $\phi_{-t}(y) \notin A$.

Then, since $y \in A$, $\phi_{-t}(A) \not\subset A \Rightarrow$ A is not negatively invariant. $\blacksquare$

Lemma: Ω_a is closed and invariant.
(Closure follows, since Ω_a is an intersection of closed sets)

Pf: (i) $\Omega_a = \emptyset$

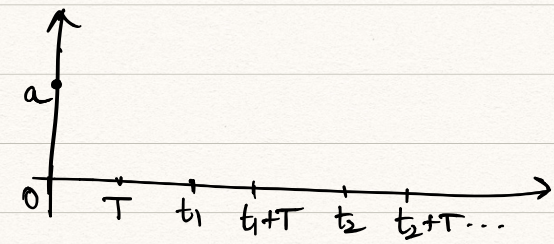
$\phi_t(\emptyset) = \emptyset$, invariant.

(ii) $\Omega_a \neq \emptyset$.

Let $b \in \Omega_a$. It suffices to show that $\phi_T(b) \in \Omega_a$, for all $T \in \mathbb{R}$.

Now, since $b \in \Omega_a$, $\exists \{t_k\} \uparrow \infty$, with $x(0) = a$, $x(t_k) \to b$, $\phi_{t_k}(a) \longrightarrow b$.

Now, consider



$$\phi_{t_k+T}(a) = \phi_T(\phi_{t_k}(a)) \xrightarrow{k \to \infty} \phi_T(b) \quad \text{[By continuity of } \phi_T \text{ map]}$$

Thus, we have a sequence of time points $s_k = t_k + T$, $s_k \uparrow \infty$,

$$x(s_k) = \phi_{s_k}(a) = \phi_{t_k+T}(a) \longrightarrow \phi_T(b) \in \Omega_a. \quad \blacksquare$$

(The case where $T<0$ is ~~not very clear~~)

Wait until the first time $t_k + T > 0$ & start seq. from here.

Defn: The set of <u>eqbm points</u> or <u>rest points</u> of the ode are:

$$\{b : h(b) = 0\}.$$

Remarks: a) S'pose b is an equilibrium point.

Ω_b : $x(0) = b \Rightarrow x(t) = b \quad \forall t \geq 0.$

Hence $\Omega_b = \{b\}$. $\|$

(b) S'pose $\{b\}$ is invariant.

3 notions:
Ω_a, invar set (A),
Eqbm set (E).

$\Omega_a \to$ invar.
If $A = \{b\}$, then $b \in E$.
$\Longrightarrow$
If $b \in E$, then $\Omega_b = \{b\}$.

If $x(0) = b$, then for all $t \geq 0$, $x(t) = b$ by +ve invariance.

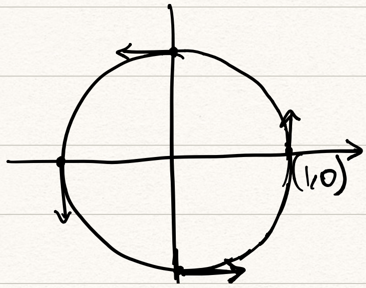
Then $\frac{dx(t)}{dt} = 0 = h(x(t)) = h(b)$.

Hence, b is a rest point.

This line of thought suggests checking if $\{b : h(b) = 0\}$ is a singleton. Then perhaps the ode may end up there (Intuition).

Caution: This need not necess. hold.

Example: $\dot{x}(t) = Hx(t)$; $H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda_1, \lambda_2 = \pm j$.



Again $\{\underline{b} : H\underline{b} = \underline{0}\} = \{\underline{0}\}$. Yet, trajectories don't converge to it.

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Stability:

(a) <u>Defn</u>: A compact invariant set M is called an <u>attractor</u> if there exists an open nbd, O, of M s.t. any trajectory starting in O remains in O, and converges to M.

<u>Examples</u>: (i) $\dot{x}(t) = Hx(t)$, $d=2$, $Re(\lambda_1) = Re(\lambda_2) < 0$

$M = \{0\}$ is an attractor.

(ii) $H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda_1, \lambda_2 = \pm j$.

$M = \{b : \|\underline{b}\| \le 1\}$ is compact & invariant.

But M is NOT an attractor, since trajectories with $\|x(0)\| = 1+\epsilon$ don't converge to M.

(b) <u>Defn</u>: For an attractor M, the largest such O is the domain of attraction of M, or the <u>basin</u> of M.

(Example: Such an O for eg(i) above, is $\mathbb{R}^2$)

(c) Defn: A compact invariant set M is Liapunov stable if

$\boxed{\forall \epsilon > 0, \exists \delta > 0}$ s.t. $\phi_t(N_\delta(M)) \subset N_\epsilon(M), \forall t \geq 0$.

(Note that $N_\delta(M) \subset N_\epsilon(M)$).

(Any trajectory initiated in the δ- nbd of M remains in its ϵ- nbd.)

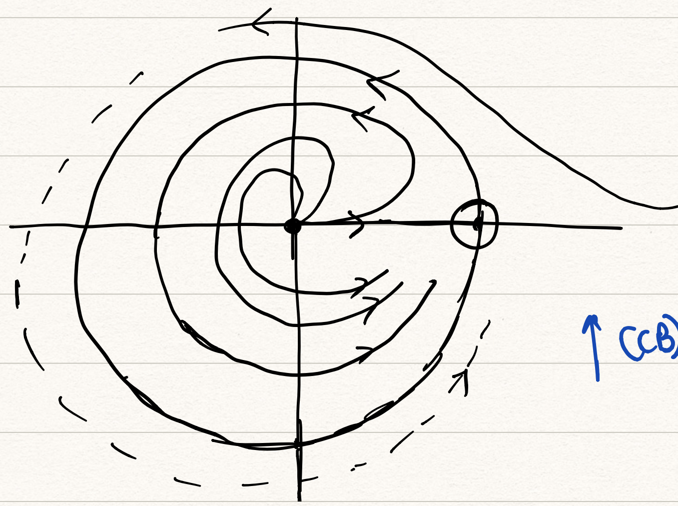
Remark: (a) Example (ii) is Liapunov stable. (with $\epsilon = \delta$)

But Liapunov stable $\not\Rightarrow$ Attractor.

(b) Also, attractor $\not\Rightarrow$ Liapunov stable.

Example: In polar co-ordinates,

$$\dot{r} = r(1-r) \qquad [r \text{ gets attracted to } r=1.]$$
$$\dot{\phi} = r(1-\cos\phi).$$



(1,0) is an attractor, but is not Liapunov stable.

↑ (CB) to understand this better

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