Recall:

- $\dot{x}(t) = h(x(t))$, $x(0) = \bar{x} \in \mathbb{R}^d$.

- The ode is "well-posed" if for any $\bar{x} \in \mathbb{R}^d$, the solution $x(\cdot)$ is unique and further,

$$\bar{x} \in \mathbb{R}^d \longmapsto x(\cdot) \in C([0,\infty), \mathbb{R}^d) \text{ is continuous}$$

- $$\rho(x(\cdot), y(\cdot)) = \sum_{T=1}^{\infty} \left(\|x(\cdot)|_T - y(\cdot)|_T\|_T \wedge 1 \right) 2^{-T}$$

Remark: Changing 2^{-T} to l^{-T}, $l \geq 1$, does not change the topology.

A sufficient condition for well-posedness:

Thm: Let h be Lipschitz. Then the ode $\dot{x}(t) = h(x(t))$, $x(0) = \bar{x}$ is well-posed.

Remark: h is Lipschitz if $\exists\, L > 0$, finite such that $\forall x, y \in \mathbb{R}^d$,

$$\|h(x) - h(y)\| \leq L\|x - y\| \quad (\text{eg: } e^x \text{ is } \underline{\text{not}} \text{ Lipschitz})$$

Counter-example: $\dot{x}(t) = \sqrt{x(t)}$, $x(0) = 0$, $\mathbb{R}^1$

$h(x) = \sqrt{x}$

This has 2 solutions:

① $\frac{dx}{dt} = \sqrt{x}$, $x(0)=0$

$\Rightarrow \quad x(t) = \frac{t^2}{4}$, $t \geq 0$

② $x(t) \equiv 0$, $\forall t \geq 0$

The ode is therefore not well-posed.

<u>Ingredient 1</u>:

<u>Lemma</u>: (Gronwall Inequality)

Suppose that $\quad u(t) \leq C + K \int_0^t u(s)v(s)\,ds, \quad t \in [0,T]$

for non-negative fns $u(\cdot), v(\cdot)$ and scalars $C, K, T \geq 0$.

Then

$$u(t) \leq C e^{K\int_0^t v(s)ds}, \quad t \in [0,T].$$

<u>Proof</u>: Write $f(t) = \int_0^t u(s)\,v(s)\,ds$

$$\begin{aligned} \dot{f}(t) &= u(t)v(t) \\ &\leq (C + Kf(t))v(t) \end{aligned}$$

Now, $$\frac{d}{dt}\left[f(t)\, e^{-K\int_0^t v(s)ds}\right] = \dot{f}(t)\, e^{-K\int_0^t v(s)ds} - Kf(t)v(t)\, e^{-K\int_0^t v(s)ds}$$

$$= \left[\dot{f}(t) - Kf(t)v(t)\right] e^{-K\int_0^t v(s)ds}$$

$$\leq C v(t)\, e^{-K\int_0 v(s)ds}$$

$$= \frac{-C}{K} \frac{d}{dt}\left[e^{-K\int_0^t v(s)ds}\right]$$

Integrating,

$$f(t)\, e^{-K\int_0^t v(s)ds} - \underset{(\text{at } t=0)}{0} \leq -\frac{C}{K}\left[e^{-K\int_0^t v(s)ds} - 1\right]$$

$$\therefore f(t) \leq \frac{C}{K}\left[e^{K\int_0^t v(s)ds} - 1\right]$$

So, $u(t) \leq C + Kf(t)$

$$= C e^{K\int_0^t v(s)ds} \quad \blacksquare$$

<u>Lemma</u> (Discrete Gronwall Inequality)

Suppose that

$$x_{n+1} \leq C + K\sum_{m=0}^{n} x_m a(m) \quad , n \geq 0 \quad \boxed{(x_0 \leq C)}$$

for non-negative $\{x_n\}$, $\{a(n)\}$, and scalars $C, K \geq 0$.

Then, with $t(n) = \sum_{m=0}^{n-1} a(m)$, we have

$$x_{n+1} \leq C e^{Kt(n+1)} \quad , n \geq 0.$$

$f(n) - f(n-1)$

$\leq (C + Kf(n-1))\, a(n)$

$f(n) \leq f(n-1)(Ka(n)+1)$

<u>Proof</u>: Let $f(n) = \sum^{n} x_m a(m)$

$m=0$ $\qquad + C a(n).$

$$f(n+1) - f(n) = x_{n+1}\, a(n+1) \leq \big(C + K f(n)\big) a(n+1)$$

$f(0) = x_0 a(0).$

$$f(n+1) \leq C a(n+1) + f(n)\,[1 + K a(n+1)]$$

$$\leq C a(n+1) + C a(n)\,[1 + K a(n+1)] + f(n-1)\,[1+K a(n+1)][1+K a(n)]$$

$K \sum_{K+1}^{n+1} a(m)$

e

$t(n+1) - t(K)$

$$\vdots$$

$$\leq C \sum_{K=1}^{n+1} a(K) \prod_{m=K+1}^{n+1} [1 + K a(m)] + f(0) \prod_{m=1}^{n+1} [1 + K a(m)]$$

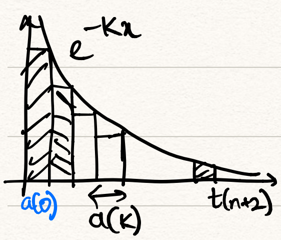
(Using $e^x \geq 1 + x$)

$$\leq C \sum_{K=1}^{n+1} a(K)\, e^{K(t(n+2) - t(K+1))} + f(0)\, e^{K(t(n+2) - t(1))}$$

$$\leq C e^{K t(n+2)} \Big[\sum_{K=1}^{n+1} a(K)\, e^{-K t(K+1)} + \underbrace{x_0}_{\leq C} a(0)\, e^{-K t(1)} \Big]$$

$$\leq C e^{K t(n+2)} \Big[\sum_{K=0}^{n+1} a(K)\, e^{-K t(K+1)} \Big]$$

(★) Step.



$$\leq C e^{K t(n+2)} \Big[\frac{-1}{K} \Big(e^{-K t(n+2)} - e^{-K t(0)} \Big) \Big]$$

Must be $t(1)$?

[Note: $t(0) = 0$]

$$f(n+1) \leq \frac{C}{K} \Big[-1 + e^{K t(n+2)} \Big]$$

$$x_{n+1} \leq C + K f(n)$$

$$\leq C e^{K t(n+1)}$$

∎

Ingredient 2: Fixed point theorems:

Suppose (S, ρ) is a complete metric space, i.e., every Cauchy sequence converges.

Def^n: Let $f: S \to S$. We say $x^* \in S$ is a fixed point of f if $f(x^*) = x^*$.

We say f is a contraction if $\rho(f(x), f(y)) \leq \alpha \rho(x,y), \forall x, y \in S$,

where $\alpha \in [0,1)$.

$f: x \mapsto e^x$ is not a contraction mapping.

$e^x - e^y$

$e^x(1 - e^{y-x})$

$\geq e^x(1$

$e^y(e^{x-y} - 1)$

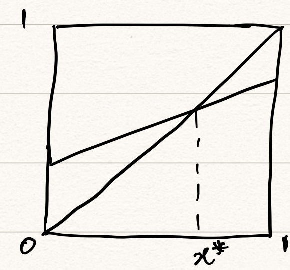
$\geq e^y(x-y)$

$\geq x - y$, if y & x large enough

eg. $S = [0,1]$

$\rho =$ usual distance

$f(x) = \frac{1}{2} + \frac{x}{4}$



Thm: Let f be a contraction on a complete metric space. Then there exists a unique fixed point x^* of f. Further, for any $x_0 \in S$, the iteration

$$x_{n+1} = f(x_n), \quad n \geq 0$$

satisfies $\rho(x_{n+1}, x^*) \leq \alpha^{n+1} \rho(x_0, x^*)$

[i.e., $x_n \to x^*$ exponentially fast]

Pf: $x_n = f^n(x_0)$

Claim: The sequence $\{x_n\}$ is Cauchy.

$$\rho(x_n, x_m) \leq \alpha^m \rho(f^{n-m}(x_0), x_0)$$

[Use the fact that
$\rho(f^{n-m}(f^m(x_0)), f^m(x_0))$
$\leq \alpha^m \rho(f^{n-m}(x_0), x_0)$
[Pulling $f^m(\cdot)$ out]]

$$\leq \alpha^m \Big[\rho(f^{n-m}(x_0), f^{n-m-1}(x_0)) + \rho(f^{n-m-1}(x_0), f^{n-m-2}(x_0)) + \dots + \rho(f(x_0), x_0)\Big]$$

$$\leq \alpha^m \left[\alpha^{n-m-1} + \alpha^{n-m-2} + \dots + \alpha + 1\right] \rho(f(x_0), x_0)$$

$$\leq \frac{\alpha^m}{1-\alpha} \rho(x_1, x_0) \xrightarrow[m \to \infty]{} 0$$. Hence Cauchy.

Now, let $x_n \to x^*$

$\Rightarrow f(x_n) \to f(x^*)$

But both the sequences $\{x_n\}$ and $\{f(x_n)\}$ are just shifts of one another

$\Rightarrow f(x^*) = x^*$.