(From last time around):

<u>Thm</u>: Let h be Lipschitz. Then the ode $\dot{x}(t) = h(x(t))$ is well-posed.

<u>Lemma</u>: (Gronwall Inequality) Suppose that $u(t) \leq C + K\int_0^t u(s)v(s)ds$, $t \in [0,T]$, for non-negative $u(\cdot), v(\cdot)$ and scalars $C, K, T \geq 0$. Then

$$u(t) \leq Ce^{K\int_0^t v(s)ds}, t \in [0,T].$$

<u>Lemma</u>: (Discrete Gronwall Inequality) Suppose that $x_{n+1} \leq C + K\sum_{m=0}^{n} x_m a(m)$, ~~as~~ $x_0 \leq C$, for non-negative $\{x_n\}, \{a(n)\}$ and scalars $C, K \geq 0$.

Then $x_{n+1} \leq Ce^{Kt(n+1)}$, $n \geq 0$, where $t(n) = \sum_{m=0}^{n-1} a(m)$, with $t(0) = 0$.

<u>Thm</u>: Let (S, ρ) be a complete metric space. Let $f: S \longrightarrow S$ be a contraction with contraction parameter $\alpha \in [0,1)$, i.e., $\rho(f(x), f(y)) \leq \alpha \rho(x,y) \; \forall x, y \in S$. Then there exists a unique fixed point, x^*, of f. Further, for any $x_0 \in S$, the iteration

$$x_{n+1} = f(x_n), n \geq 0 \text{ satisfies}$$

$$\rho(x_n, x^*) \leq \alpha^n \rho(x_0, x^*), n \geq 0.$$

Proof of the well posedness theorem:

$$x(0) = \bar{x} \in \mathbb{R}^d.$$

$$x(t) = \bar{x} + \int_0^t h(x(s))ds \quad . \text{ Fix } T > 0.$$

For a $y(\cdot) \in C([0,T])$, consider the following mapping:

$$y(\cdot) \longmapsto z(\cdot) = F(y(\cdot))$$

$$z(t) = \bar{x} + \int_0^t h(y(s))ds.$$

$$F : C([0,T]) \longrightarrow C([0,T]).$$

The solution to the ode is a fixed pt of this mapping.

FACT: $C([0,T])$ is a complete metric space.

Let $z_1(\cdot), z_2(\cdot)$ be $F(x_1(\cdot)), F(x_2(\cdot))$ respectively.

So $$\|z_1(\cdot) - z_2(\cdot)\|_T = \sup_{t\in[0,T]} \|z_1(t) - z_2(t)\|$$

$$= \sup_{t\in[0,T]} \left\| \int_0^t (h(x_1(s)) - h(x_2(s)))ds \right.$$

$$\leq \sup_{t\in[0,T]} \int_0^t \| h(x_1(s)) - h(x_2(s))\| ds.$$

$$\leq LT \|x_1(\cdot) - x_2(\cdot)\|_T \quad \text{[Employing Lipschitz property \& } \Delta\text{-ineq]}$$

So now pick T s.t. $LT < 1$.

So by the contraction theorem, $\exists$ a unique fixed pt.

So we can patch up the solutions ~~from~~ $[0,T]$, $[T,2T]$ & so on.

We now need to show continuity in the initial condition:

$\bar{x}_1, \bar{x}_2 \longrightarrow$ two initial condns with solutions $x_1(\cdot)$ & $x_2(\cdot)$. $\in C[0,\infty)$

$$\|x_1(t) - x_2(t)\| = \left\| \bar{x}_1 + \int_0^t h(x_1(s))ds - \bar{x}_2 - \int_0^t h(x_2(s))ds \right\|$$

$$\leq \|\bar{x}_1 - \bar{x}_2\| + L\int_0^t \|x_1(s) - x_2(s)\| ds$$

(Gronwall ineq)

[For $t \in [0,T]$]

$$\leq \|\bar{x}_1 - \bar{x}_2\| e^{LT}$$

So $\|x_1(\cdot) - x_2(\cdot)\| \leq \sum_{T=1}^{\infty} 2^{-T} \|\bar{x}_1 - \bar{x}_2\| e^{LT}$

$\rho(x_1(\cdot), x_2(\cdot))$

$$= \|\bar{x}_1 - \bar{x}_2\| \sum_{T=1}^{\infty} 2^{-T} e^{LT},$$ which is summable so long as $\frac{e^L}{2} < 1$.

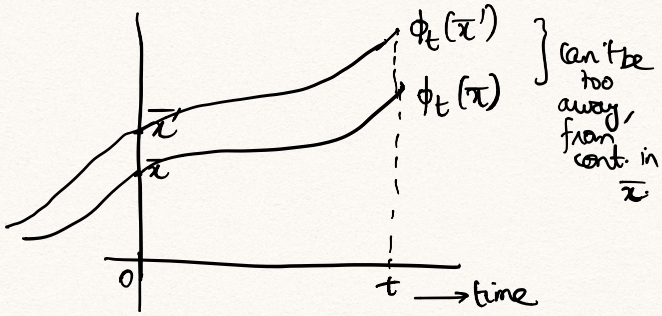
In fact, to ensure that ✓ is summable, instead of 2^{-T}, put some l^{-T}, s.t. $\frac{e^L}{l} < 1$

Addnl Remarks:

1. Can extend solution to negative t as well. Work with $[-T, 0]$ with 0 as the endpoint, and $\bar{x}$ as the terminal condition.

2. Consider $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$, $t \in \mathbb{R}$.

$\bar{x} \longmapsto \phi_t(\bar{x})$



Further, $\phi_t(\bar{x}') \neq \phi_t(\bar{x})$, for any t, by uniqueness of the solution for a given $\bar{x}$.

From uniqueness, both forward & backward ϕ_t are invertible (with inverse ϕ_{-t}).

3. $\dot{x}(t) = h(x(t))$, $x(T) = \bar{x}$. We don't have $h(t, x(t))$ (Time invariance)

4. $\boxed{\phi_t \text{ is Lipschitz, with constant } e^{Lt}}$.

Likewise, $\phi_t^{-1} = \phi_{-t}$.

So ϕ_t is a "homeomorphism" (cont. fn whose inverse is also cont.)

Note that $\phi_0 = Id$ (Identity fn)

5. $\phi_s(\phi_t(\bar{x})) = \phi_{s+t}(\bar{x}) = \phi_t(\phi_s(\bar{x}))$

So $\{\phi_t\}$ is a group of homeomorphisms.

<u>Martingales</u> (An introduction):

$(\Omega, \mathcal{F}, P)$: Probability space. Filtration is an increasing family of σ-fields $\{\mathcal{F}_n\}_{n\geq 0}$ $(\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots)$

(P-complete, i.e., contains all P-null events).

$\{X_n\}_{n\geq 0}$, ~~is~~ a sequence of random variables on $(\Omega, \mathcal{F}, P)$, is said to be adapted to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$, if X_n is $\mathcal{F}_n$-measurable, for all $n\geq 0$.

Natural filtration associated $\{X_n\}_{n\geq 0}$ is $\mathcal{F}_n^X = \sigma(X_m, m\leq n)$, (smallest σ-field...) $n\geq 0$.

<u>Defn</u>: Given $\{\mathcal{F}_n\}_{n\geq 0}$, a sequence of integrable random variables $\{X_n\}_{n\geq 0}$ which is adapted to $\{\mathcal{F}_n\}_{n\geq 0}$, we say $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is a martingale if

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_n \quad , \forall n\geq 0$$

Remark: $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ is a sub-martingale if '=' is replaced by '$\geq$'.