Recall: $T \to$ fixed. $T_0 = 0$,

$$T_{m+1} = \min\{t(n) : t(n) \geq T_m + T\}$$

$$\sup_n a(n) \leq 1$$

$\bar{x}(\cdot) \longrightarrow$ interpolated trajectory

$\hat{x}(\cdot)$, with $\hat{x}(t) = \dfrac{\bar{x}(t)}{r(m)}$, $t \in [T_m, T_{m+1})$,

where $r(m) = \|\bar{x}(T_m)\| \vee 1$.

Thm: Under (A1)–(A3) and (A5), we have $\sup_n \|x_n\| < \infty$ (Stability)

Prop 1: ("ϕ_t^c is like ϕ_t^∞ for large c")

There is a $c_0 > 1$, $T > 0$, such that for all initial conditions x_0 with $\|x_0\| \leq 1$, we have

$$\|\phi_t^c(x_0)\| < 1/4, \quad \text{for } t \in [T, T+1], \quad c > c_0.$$

Prop 2: Assume (A1)–(A3) and (A5). Let $\dot{x}(t) = h_c(x(t))$ with solution $x^{T_m}(t)$, $t \geq T_m$ and initial condition $x^{T_m}(T_m) = \hat{x}(T_m)$

(a) $\displaystyle\lim_{m \to \infty} \sup_{t \in [T_m, T_{m+1})} \|\hat{x}(t) - x^{T_m}(t)\| = 0$ a.s.

(b) For each m, for each k s.t. $n_m \leq k \leq n_{m+1}$, we have

$T_m = t(n_m)$

$\|\hat{x}(t(k))\| \leq K^*$ $\longleftarrow$

<u>Prop3</u>: $\sup_{t \in [0, T+1]} \|\bar{x}(t)\| \leq \bar{B}$ a.s.

$\|\bar{x}(0)\| \leq C_0$

Pf of <u>theorem</u>:

(a) Suppose $\exists T_{m_k}, k \geq 1, r(m_k) := \|\bar{x}(T_{m_k})\| \xrightarrow[k \to \infty]{} \infty$

If $r(m) > C_0$ $(C_0 > 1)$, then $\|\hat{x}(T_m)\| = 1 = \|x^{T_m}(T_m)\|$.

By Prop1, $T_{m+1} \in T_m + [T, T+1]$, $r(m) > C_0$, the c for rescaling is $r(m)$,

we have that $\|x^{T_m}(T_{m+1})\| < 1/4$ (Note: This is not the $\hat{x}(\cdot)$ trajectory)

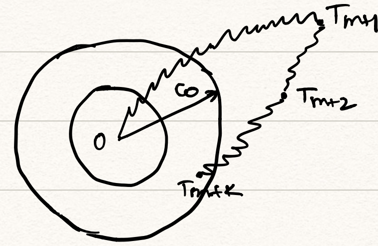
By Prop 2, for suff. large m,

$$\begin{aligned}\|\hat{x}(T_{m+1}^-)\| &\leq \|x^{T_m}(T_{m+1}^-)\| + 1/4 \\ &= \|x^{T_m}(T_{m+1})\| + 1/4 \\ &\leq 1/2 \quad \text{, a.s.}\end{aligned}$$

If $r(m) > C_0 \Rightarrow \dfrac{\|\hat{x}(T_{m+1}^-)\|}{\underbrace{\|\hat{x}(T_m)\|}_{=1}} = \dfrac{\|\bar{x}(T_{m+1}^-)\|}{\|\bar{x}(T_m)\|} = \dfrac{\|\bar{x}(T_{m+1})\|}{\|\bar{x}(T_m)\|} \leq 1/2$, a.s.

Therefore, so long as we are outside the C_0-ball, distance to origin continues to halve until we enter C_0.

$\overset{x(\cdot)}{\text{we are}}$



The only way $r(m_k) \to \infty$ is by having longer & longer jumps as $k \to \infty$, i.e.

But by Prop 3, we can't have these long jumps. Contradiction.

So $\nexists T_{m_k}$ with $r(m_k) \to \infty$.

So we can conclude that $\sup_m \|\bar{x}(T_m)\| = \bar{C} < \infty$ a.s.

By Prop. 2(b), $\sup_k \|\hat{x}(t(k))\| \leq K^*$, $\forall m$, $\forall k \in \{n_m, n_m+1, \ldots, n_{m+1}\}$

$$\text{But} \quad \bar{x}(t(k)) = \hat{x}(t(k)) \cdot r(m)$$
$$\leq K^* \bar{C}, \ \forall k$$
$$\Rightarrow \|x_k\| \leq C_1 \text{ a.s.} \qquad \blacksquare$$

Proofs:

Prop 1. There is a $C_0 > 1$, $T > 0$, s.t. for all initial conditions x_0 s.t. $\|x_0\| \leq 1$, we have

$$\|\phi_t^c(x_0)\| \leq \frac{1}{\ }\ \text{for all} \ t \in [T, T+1], \ c > C_0.$$

<u>Lemma 1</u>: (ϕ_t^∞ is "well-behaved")

$\exists T>0$ s.t. for all initial conditions in the unit ball we have

$$\|\phi_t^\infty(x_0)\| \leq \frac{1}{8}, \quad \forall t \geq T.$$

Pf: 1) Asymptotic stability $\Rightarrow$ Liapunov stability.
$\exists \delta>0$ s.t. if $\|x_0\| < \delta$, then $\|\phi_t^\infty(x_0)\| < 1/8$ for all $t \geq 0$.

2) Asymp. stability $\Rightarrow$ $\{0\}$ is an attractor.
For any x_0 in the unit ball, $\exists T_{x_0} > 0$ s.t. $\|\phi_{T_{x_0}}^\infty(x_0)\| \leq \frac{\delta}{2}$

3) Suppose x_0, y_0 are two starting points for the ode.

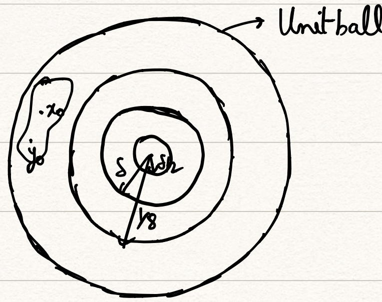
$$x(t) = x_0 + \int_0^t h_\infty(\phi_s^\infty(x_0))\, ds$$

$$y(t) = y_0 + \int_0^t h_\infty(\phi_s^\infty(y_0))\, ds$$

$$\|\phi_t^\infty(x_0) - \phi_t^\infty(y_0)\| = \|x(t) - y(t)\|$$
$$\leq \|x_0 - y_0\| + L\int_0^t \|\phi_s^\infty(x_0) - \phi_s^\infty(y_0)\|\, ds$$

By Gronwall,
$$\|\phi_t^\infty(x_0) - \phi_t^\infty(y_0)\| \leq \|x_0 - y_0\| e^{LT}, \quad t \in [0,T].$$

4)



Choose the nbd around x_0, U_{x_0} so that

$$y \in U_{x_0} \Rightarrow \|\phi_t^\infty(y_0) - \phi_t^\infty(x_0)\| \leq \delta/2$$

So $\quad \|\phi_{T_{x_0}}^\infty(y_0)\| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta$

So for all $y_0 \in U_{x_0}$, T_{x_0} works, $\phi_t^\infty(y_0)$ never exits the $1/8$-ball for all $t \geq T_{x_0}$.

For each $x_0 \in$ unit ball, consider U_{x_0}. Open cover for the compact unit ball. Find a finite subcover, $U_{x_0^1}, U_{x_0^2}, \ldots, U_{x_0^n}$.

Take $T = \max\{T_{x_0^1}, \ldots, T_{x_0^n}\}$. $\boxtimes$

Extend to h_c for $c >$ some $c_0 > 1$.

$\downarrow$

<u>Lemma 2</u>: $\quad K \subset \mathbb{R}^d \quad , T > 0.$

(1) $\quad \|\phi_t^c(y_0) - \phi_t^\infty(x_0)\| \leq \left[\|x_0 - y_0\| + \varepsilon(c)\, T\right] e^{LT}$

where $\varepsilon(c) \xrightarrow[c\to\infty]{} 0$, depends on K, but is indep of $x_0 \in K$.

(2) If $x_0 = y_0$, then, upper bound is $\varepsilon(c)\, T e^{LT}$.

Pf: $\|\phi_t^c(y_0) - \phi_t^\infty(x_0)\|$

$$\leq \|y_0 - x_0\| + \int_0^t \underbrace{\|h_c(\phi_s^c(y_0)) - h_\infty(\phi_s^\infty(x_0))\|}\,ds$$

$$\leq \|h_c(\phi_s^c(y_0)) - h_c(\phi_s^\infty(x_0))\|$$

$$+ \|h_c(\phi_s^\infty(x_0)) - h_\infty(\phi_s^\infty(x_0))\|$$

$$\leq L\,\|\phi_s^c(y_0) - \phi_s^\infty(x_0)\| + \underbrace{\varepsilon(c)}_{\xrightarrow{c\to\infty} 0}$$

holds if we can show $\phi_s^\infty(x_0)$ comes from some compact set if $x_0 \in K$, i.e., $\phi_{[0,T]}^\infty(K)$ is compact.

Then Gronwall gives $\|\phi_t^c(y_0) - \phi_t^\infty(x_0)\| \leq (\|x_0 - y_0\| + \epsilon(c)T)e^{LT}$

The second part is easy.

Pf of Prop 1: Choose T as in Lemma 1. Apply Lemma 2, but with $T+1$ (instead of T). Pick c_0 large enough so that $\varepsilon(c)(T+1)e^{L(T+1)} \leq \frac{1}{8}$

By Δ-ineq, $\|\phi_t^c(x_0)\| \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$ $(c \geq c_0)$.