Thm.: Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$-sub-martingale satisfying $\sup_n \mathbb{E}[X_n^+] < \infty$.

Then $\lim_{n\to\infty} X_n = X_\infty$ exists a.s.

Pf.: $\overline{X} = \limsup_{n\to\infty} X_n$, $\underline{X} = \liminf_{n\to\infty} X_n$.

Fix N, fix $a<b$.

$$\mathbb{E}[\beta_N(a,b)] \leq \frac{\mathbb{E}[X_N^+] + |a|}{b-a}$$

Let $\beta_\infty(a,b) = \lim_{N\to\infty} \beta_N(a,b) = \#$ of upcrossings.

$$\mathbb{E}[\beta_\infty(a,b)] \underset{\text{Fatou's Lemma}}{\leq} \liminf_{N\to\infty} \mathbb{E}[\beta_N(a,b)] \leq \frac{B+|a|}{b-a}$$

, where $B := \sup_n \mathbb{E}[X_n^+]$

$$\Rightarrow \beta_\infty(a,b) < \infty \text{ a.s.}$$

Now, $\{\overline{X} > \underline{X}\} = \bigcup_{a<b,\ a,b \to \text{rational}} \{\overline{X} > b > a > \underline{X}\}$

$$\therefore P(\overline{X} > \underline{X}) \leq \sum_{a<b,\ a,b \text{ rational}} P[\overline{X} > b > a > \underline{X}]$$

$= 0$

$\Rightarrow \overline{X} = \underline{X}$ a.s., $\{X_n\}$ has a limit a.s. with $\pm\infty$ as possible value.

- $X_\infty < \infty$ a.s. S'pose $P[X_n \to \infty] > 0$.

$$\infty > \limsup_{n\to\infty} \mathbb{E}[X_n^+] \geq \liminf_{n\to\infty} \mathbb{E}[X_n^+]$$

$$\overset{\text{(Fatou's Lemma)}}{\geq} \mathbb{E}\left[\liminf_{n\to\infty} X_n^+\right]$$

$$\geq \mathbb{E}\left[\liminf_{n\to\infty} X_n\right]$$

$$= \infty$$

- $X_n > -\infty$ a.s.

$$\mathbb{E}[X_n] = \mathbb{E}[X_n^+] - \mathbb{E}[X_n^-] \overset{\text{sub-mart.}}{\geq} \mathbb{E}[X_0]$$

$$\sup_n \mathbb{E}[X_n^-] \leq B + |\mathbb{E} X_0|$$

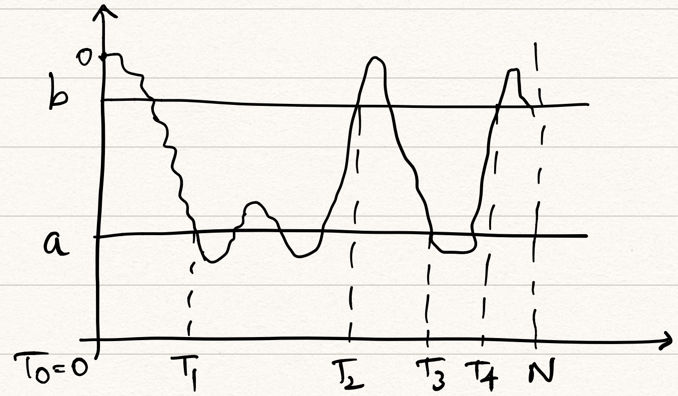
(Exercise: follow the same procedure to show $P[X_n \to -\infty] = 0$)

- Corollary: Every martingale which is $\mathcal{L}_1$-bounded, $\sup_n \mathbb{E}|X_n| < \infty$, converges.

Pf: $\mathbb{E}[|X_n|] = \mathbb{E}X_n^+ + \mathbb{E}X_n^- = 2\mathbb{E}X_n^+ - \mathbb{E}X_0$

& $\mathbb{E}X_n = \mathbb{E}X_0$ ▨

Qn:



T_5 onwards $= N$.

Now, $$X_N - X_0 = \underbrace{\sum_{m\geq 0}\left(X_{T_{2m+1}} - X_{T_{2m}}\right)}_{\mathbb{E}[X_{T_{2m+1}}] \geq \mathbb{E}[X_{T_{2m}}]} + \underbrace{\sum_{m\geq 1}\left(X_{T_{2m}} - X_{T_{2m-1}}\right)}_{\beta_N(a,b)(b-a)}$$

"$X_{T_{2m+1}}$ is below a, $X_{T_{2m}}$ is above b, but how?"

Ans is → ∵ there is a good chance that T_{2m} is N itself (it never down-crosses).

Discussion: $\sup_n \mathbb{E}|X_n| < \infty$ or $\sup_n \mathbb{E}X_n^+ < \infty$.

We know $\mathbb{E}|X_n| < \infty$, but $\sup_n \mathbb{E}|X_n|$ could be ∞.

We will work with martingales with $\mathbb{E}X_n^2 < \infty$, for all n.
Such X_ns are in $\mathcal{L}_2$, but need not be $\mathcal{L}_2$-bounded.

Thm: Doob decomposition:

Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$-submartingale. Then $X_n = M_n + A_n$, $n \geq 0$,

where $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale with zero-mean and

$\{A_n\}$ is an increasing process, adapted to $\{\mathcal{F}_{n-1}\}$, (predictable)
where $\mathcal{F}_{-1} = \{\phi, \Omega\}$.

Furthermore, this decomposition is unique (a.s.)

Pf: Write
$$M_n = \sum_{i=0}^{n} \left(X_i - \mathbb{E}[X_i | \mathcal{F}_{i-1}]\right),$$
$$A_n = \sum_{i=1}^{n} \left(\mathbb{E}[X_i | \mathcal{F}_{i-1}] - X_{i-1}\right) + \mathbb{E}X_0 \quad , n \geq 0.$$

(1) $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale

L_1

Adapted

$\mathbb{E}[M_{n+1} | \mathcal{F}_n]$
$= M_n + \mathbb{E}[X_{n+1} - \mathbb{E}[X_{n+1}|\mathcal{F}_n] | \mathcal{F}_n]$

(2) $A_{n+1} \geq A_n$ a.s.
A_n is Predictable $(\in \mathcal{F}_{n-1})$

Defn: $\mathcal{L}_2$ martingales or sq-integrable martingales: Martingales with $\mathbb{E}[X_n^2] < \infty, \forall n$.

Properties:

- $\{X_n^2\}$ is an $\mathcal{F}_n$- sub-martingale.

- Let $X_n^2 = M_n + A_n$ be its Doob decomposition.

$$A_n = \mathbb{E}X_0^2 + \sum_{i=1}^{n} \mathbb{E}[X_i^2 | \mathcal{F}_{i-1}] - X_{i-1}^2$$

$$= \mathbb{E}X_0^2 + \sum_{i=1}^{n} \mathbb{E}[(X_i - X_{i-1})^2 | \mathcal{F}_{i-1}]$$

"Quadratic variation process"

- $A_\infty = \lim_{n\to\infty} A_n$ (with $+\infty$ as a possible value for the limit)

Thm: Let $\{X_n\}$ be a square integrable $\{\mathcal{F}_n\}$-martingale. Let $\{M_n\}, \{A_n\}, A_\infty$ be as defined. Then $\{X_n\}$ converges a.s. on the set $\{A_\infty < \infty\}$ and $X_n = o(A_n)$ on $\{A_\infty = \infty\}$.

$$\left(X_n = o(A_n) \Leftrightarrow \lim_{n\to\infty} \frac{|X_n|}{A_n} = 0 \right)$$

Pf: Fix $a > 0$.

T_a = hitting time of (a^2, ∞) of the process $\{A_n\}$.

<u>Note</u>: T_a is a stopping time.

<u>Claim</u>: $X_{T_a \wedge n}$ is a sq- integrable $\{\mathcal{F}_n\}$ – martingale with quadratic variation process. $A_{T_a \wedge n}$.

$$\mathbb{E}[|X_{n\wedge T_a}|] \leq \left(\mathbb{E}[X^2_{n\wedge T_a}]\right)^{1/2}$$

$$= \left(\mathbb{E}[A_{n\wedge T_a}]\right)^{1/2}$$