Lock-in Probabilities (Contd...)

Recall that:

$$\Pr\{\rho_m \geq \delta \mid B_{m-1}\} \leq \Pr\left\{\sup_{n_m \leq k \leq n_{m+1}} \|\delta_{n_m,k}\| < \frac{\delta}{2e^{LT}} \,\Big|\, B_{m-1}\right\}$$

$$\leq \frac{\mathbb{E}\left[\sup_{n_m \leq k \leq n_{m+1}} \|\delta_{n_m,k}\|^2 \,\Big|\, B_{m-1}\right]}{\left(\frac{\delta}{2e^{LT}}\right)^2} \quad \left[\begin{array}{l}\text{Note that:}\\ \delta_{n_m,n_m} = 0\end{array}\right]$$

$$\leq \frac{c(2)}{\left(\frac{\delta}{2e^{LT}}\right)^2} \sum_{k=n_m+1}^{n_{m+1}} (a(k-1))^2 \, \mathbb{E}\left[\|M_k\|^2 \mid B_{m-1}\right] \longrightarrow (*)$$

Claim: $\exists C_2$ s.t. $\mathbb{E}\left[\|M_k\|^2 \mid B_{m-1}\right] \leq C_2$ a.s. $, \forall k = n_m+1, \ldots, n_{m+1}$.

Modulo the claim,

$$(*) \leq \frac{c(2)C_2}{(\delta/2e^{LT})^2} \left(b(n_m) - b(n_{m+1})\right)$$

Sum the RHS for $m = 0$ to ∞,

$$\Rightarrow (*) \leq \frac{c(2)C_2}{(\delta/2e^{LT})^2} b(n_0) \xrightarrow{n_0 \to \infty} 0 \quad \blacksquare$$

Proof of Claim:

LHS: $\mathbb{E}\left[\mathbb{E}\left[\|M_k\|^2 \mid \mathcal{F}_{k-1}\right] \mid B_{m-1}\right]$, $k \in \{n_m+1, \ldots, n_{m+1}\}$.

$$\leq \mathbb{E}\left[K\left(1+\|x_{k-1}\|^2\right)\middle|\mathcal{B}_{m-1}\right]$$

$$\leq K\left(1+\sqrt{\mathbb{E}\left[\|x_{k-1}\|^2 \middle| \mathcal{B}_{m-1}\right]}\right)^2$$

Re-indexing, $k = n_m, \dots, n_{m+1}-1$. We will bound $\mathbb{E}\left[\|x_k\|^2 \middle| \mathcal{B}_{m-1}\right]$.

Let $\|x_k\|_* = \sqrt{\mathbb{E}\left[\|x_k\|^2 \middle| \mathcal{B}_{m-1}\right]}$. Employing Δ-ineq on $x_{k+1} = x_k + a(k)\left[h(x_k)+M_{k+1}\right]$, we get:

$$\|x_{k+1}\|_* \leq \|x_k\|_* + a(k)K_1\left(1+\|x_k\|_*\right) + a(k)\underbrace{\|M_{k+1}\|_*}_{\leq \sqrt{K}\left(1+\|x_k\|_*\right)}$$

$$\leq \|x_k\|_*\left(1+a(k)K_2\right) + a(k)K_2$$

$$\leq \|x_k\|_*\left(e^{a(k)K_2}\right) + a(k)K_2$$

$$\leq \left(\|x_{k-1}\|_* e^{K_2 a(k-1)} + a(k-1)K_2\right)e^{K_2 a(k)} + a(k)K_2$$

$$\vdots$$

$$\leq \underbrace{\|x_{n_m}\|_*}_{\in B} e^{K_2\left(a(n_m+1)+\dots+a(n_{m+1})\right)} + K_2 \overbrace{\left(a(n_m+1)+\dots+a(n_{m+1})\right)}^{(T_{m+1}-T_m \leq T+1)} \times e^{\underbrace{K_2\left(a(n_m+1)+\dots+a(n_{m+1})\right)}_{\leq (T+1)}}$$

$$\leq C_2$$

- Other Martingale Inequalities :

Suppose $\mathbb{E}[\,\|M_{n+1}\|^2 \mid \mathcal{F}_n] \leq K(1+\|x_n\|^2)$ a.s., $\forall n \geq 0$

is replaced by $\|M_{n+1}\| \leq K(1+\|x_n\|)$ a.s., $\forall n \geq 0$.

To bound

$Pr\left\{ \max_{n_m < k \leq n_{m+1}} \|\delta_{n_m,k}\| > \frac{\delta}{2e^{LT}} \,\middle|\, B_{m-1} \right\}$, we used Chebyshev and Burkhölder.

Now we shall use McDiarmid's inequality.

Use

$$\|x_{k+1}\| \leq \|x_k\| + a(k)(1+\|x_k\|)K_1 \quad [\text{Not the } * \text{ norm}]$$
$$+\, a(k)K(1+\|x_k\|)$$
$$= \|x_k\|\Big(1+a(k)\overbrace{(K+K_1)}^{K_2}\Big) + a(k)(K+K_1)$$
$$\leq \underbrace{\|x_{n_m}\|}_{\in B} e^{K_2(T+1)} + K_2(T+1)\,e^{K_1(T+1)}$$
$$\leq C_2 \quad \text{a.s.} \qquad [\text{Since under } B_{m-1},\ x_{n_m} \in B]$$

Thus, $\|M_{k+1}\| \leq K_1(1+C_2) = \overline{K}$ a.s. (by the replacement assumption from (A3))

Thm: (McDiarmid's Inequality)

Let $\{S_n\}$ be an $\{\mathcal{F}_n\}$-martingale with $b_k \leq S_k - S_{k-1} \leq c_k$. Then,

$$P\left[\max_{1\le k\le n} |\delta_k| \geq t\right] \leqslant 2e^{-2t^2/\sum_{k=1}^{n}(c_k-b_k)^2}$$

Apply conditioned McDiarmid, conditioned on B_{m-1},

$$\Pr\left\{\max_{n_m<k\le n_{m+1}} \|\delta_{n_m,k}\| > \frac{\delta}{2e^{LT}} \,\Big|\, B_{m-1}\right\} \leqslant \frac{C_2 e^{-2C_3\left(\frac{\delta}{2e^{LT}}\right)^2}}{C_4\sum_{k=n_{m+1}}^{n_{m+1}}(a(k))^2}$$

[Use the fact that $\|u\|_\infty \le \|u\| \le \sqrt{d}\,\|u\|_\infty$]

$$= C_2 e^{-\left[\left(\frac{\delta}{e^{LT}}\right)^2 C_5 \cdot \frac{1}{b(n_{m+1})-b(n_m)}\right]}$$

Sum up the RHS over $m\geqslant 0$, we get:

$$\sum_{m\geqslant 0} C_2 e^{-\frac{\delta^2}{e^{2LT}}\frac{C_5}{[b(n_{m+1})-b(n_m)]}} \leqslant e^{-C_6/b(n_0)} = o(b(n_0))$$

This requires a calculation

$\xrightarrow[n_0\to\infty]{} 0$ [Faster rate than $Cb(n_0)$]