Introduction via two examples :

- The urn model :
  - Initially empty urn. Balls, red or black, are added one at a time.

y_n = # of red balls at time n.

$x_n = \frac{y_n}{n}$, $x_0 = 0$ (convention)

Let $p: [0,1] \to [0,1]$, $p(x_n)$ = prob. that the $(n+1)^{st}$ ball is red.

(We assume $p(0) = 0$) $\to$ so in this setting, we'll have only black balls all through.

Polyá urn scheme : Start with 1 black & 1 red.

$y_0 = 1$, $x_0 = \frac{1}{2}$,

and at any stage, $x_n = \frac{y_n}{n+2}$.

Further, $p(x) = x$. (Add red ball if you pick red & black, o.w.)

(Back to original setting) :

So we setup a recursion :

$$y_{n+1} = y_n + \xi_{n+1} ,$$

where $\xi_{n+1} = \mathbb{1}\{(n+1)^{st} \text{ ball is red}\}$.

Recursion for x:

$$x_{n+1} = \frac{n x_n + 1 \cdot \xi_{n+1}}{n+1}$$

$$= x_n + \frac{1}{n+1} \underbrace{\left(p(x_n) - x_n\right)} + \frac{1}{n+1}\left(\xi_{n+1} - p(x_n)\right)$$

[After centering]

$$= x_n + \frac{1}{n+1}\left(h(x_n)\right) + \frac{1}{n+1} M_{n+1},$$

(*)

where $M_{n+1} = \xi_{n+1} - p(x_n)$, $n \geq 0$.

This is a seq. of zero-mean random variables.

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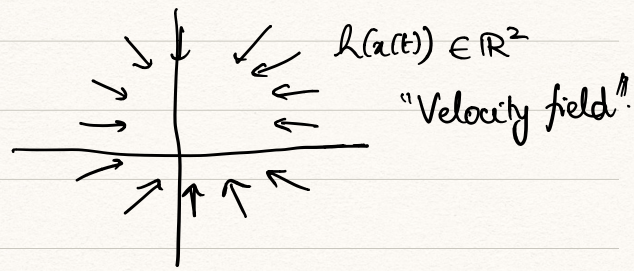
Further, $\mathbb{E}[M_{n+1} \mid \xi_m, m \leq n] = 0$. This is a "Martingale-difference sequence"; it is uncorrelated with the past.

(i.e.), $\mathbb{E}[M_{n+1} \cdot f(x_1, \ldots, x_n)] = 0$, since $(x_1, \ldots, x_n) = g(\{\xi_m\}_{m \leq n})$ & take outer expectation in 1.

This reminds us of Euler's scheme for "ode"s.

$\dot{x}(t) = h(x(t))$. For instance, if $x \in \mathbb{R}^2$,



In the discretized setting,

$$x_{n+1} = x_n + a\,h(x_n) \;; \qquad a = \Delta t \;\&\; n \to n\Delta t.$$

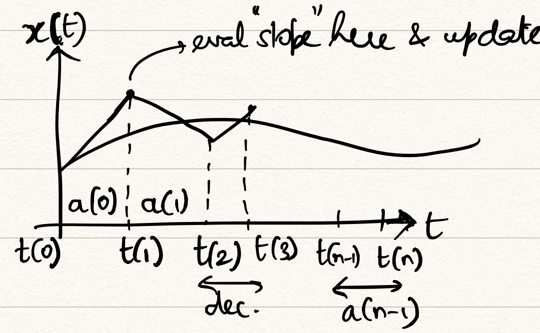
Differences between ↑ and (*) we were working with:

① a is replaced by $a(n) = \frac{1}{n+1}$ (non-uniform step-sizes).

and $a(n) \xrightarrow[n\to\infty]{} 0$

Further, we'd like that $\sum_{n=1}^{\infty} a(n) = \infty$, to cover all times $[0, \infty)$

Notation:



The hope is that the iterates will track the ODE.

② $h(x_n)$ is replaced by $h(x_n) + M_{n+1}$, noisy.

We need to find the Total accumulated noise from x_{n+1} onwards:

$$x_{n+K} = x_n + \begin{pmatrix} \text{updates as per the} \\ \text{Euler scheme} \end{pmatrix} + \sum_{m=n}^{n+K-1} a(m) M_{m+1}$$

So we'd like $\displaystyle\lim_{n\to\infty} \sum_{m=n}^{\infty} a(m) M_{m+1} = 0$ a.s. One

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sufficient cond^n is for $\displaystyle\lim_{n\to\infty} \sum_{m=0}^{\infty} a(m) M_{m+1} < \infty$ a.s.

In our example,

$$\lim_{K\to\infty} \mathbb{E}\left[\left(\sum_{m=0}^{K} a(m) M_{m+1}\right)^2\right] = \lim_{K\to\infty} \sum_{m=0}^{K} (a(m))^2 \, \mathbb{E}[M_{m+1}^2]$$

(Exercise) using Mart. diff. property.

$$\leq \sum_{m=0}^{\infty} a(m)^2 \quad \left[\text{Since } M_{m+1} = \xi_{m} - p(x_n)\right]$$

So if $\displaystyle\sum_{m=0}^{\infty} a(m)^2 < \infty$, then $\displaystyle\sum_{m=0}^{\infty} a(m) M_{m+1} < \infty$ a.s.

"So noise doesn't accumulate, so long as we start out far off in time (n is large)"

We can go a little further in the specific example:

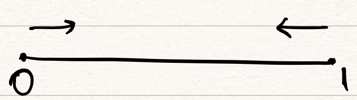
Take $p(\cdot)$ to be Lipschitz continuous, i.e.,

$$\exists L \text{ s.t. } \quad |p(x)-p(y)| \leq L|x-y|.$$

Then the o.d.e. $\dot{x}(t) = h(x(t)) = p(x(t)) - x(t)$ has a unique solution (well-posed $\longrightarrow$ two classes from now).

At $x=0$, $h(0) \geq 0$ $\}$ since $x(0)=0$ is the initial cond^n.
$x=1$, $h(1) \leq 0$

$x(t) \in [0,1] \ \forall t \geq 0.$



- If $x(t)$ is increasing, then $x(t)$ converges.
- If $x(t) \downarrow$, then, again, $x(t)$ converges.
- If neither, then $x(t) \uparrow$ and then $\downarrow$ or vice-versa,

  i.e., $\frac{dx(t)}{dt}$ changes sign & <u>must cross zero</u> (continuity of $h(\cdot)$)

|| $x(t)$ then converges because x does not change once it ||

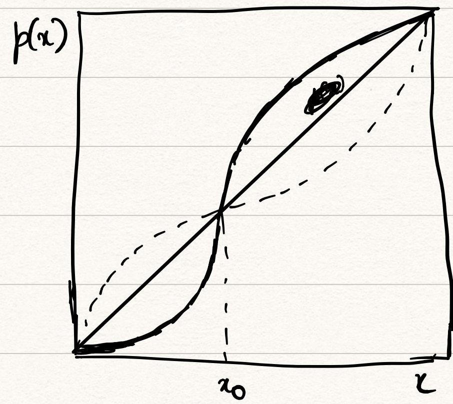
reaches the zero-velocity point.

We will see that $x_n \longrightarrow H = \{x : h(x) = 0\}$ a.s.

This set is the set of "rest points" for the dynamics given by the ODE.

Since $h(x) = (p(x) - x)$, the "rest points" are precisely the fixed points of $p(\cdot)$.

As an example,



Consider the solid black line. $h(x) = p(x) - x$ for $x < x_0$ is $-$ve, so