Recall:

Thm: Assume (A1)–(A4). Assume that a Liapunov function, V, is available for the dynamics and let H be its zero set. Then,

$$x_n \longrightarrow A \quad \text{a.s., where } A \text{ is a connected ict subset of } H.$$

Proof: $C_0 := \sup_n \|x_n\|$, $M := \sup_{x : \|x\| \le C_0} V(x)$

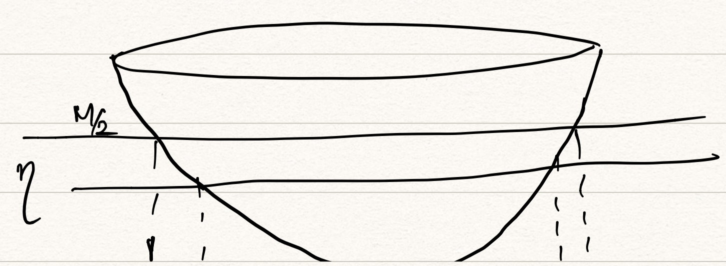
$H^\eta = \{x \in \mathbb{R}^d : V(x) < \eta\}$, open since V is continuous.

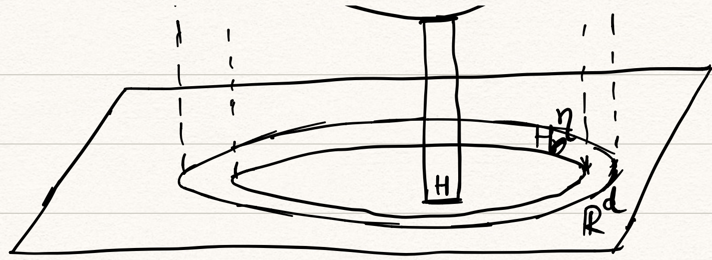
$\overline{H}^\eta = \{x \in \mathbb{R}^d : V(x) \le \eta\}$, closed (closure of H^η).

Fix $0 < \eta < \frac{M}{2}$. We will show $\exists t_1$ such that $\forall t \ge t_1$, $\overline{x}(t) \in H^{2\eta}$

[i.e., $\overline{x}(t) \longrightarrow H$].

Picture:





- $\Delta := \min\limits_{\substack{x \in \underbrace{\underbrace{\overline{H^M}}_{\substack{\text{compact}\\\text{(lower levelset)}}} \setminus H^\eta}_{\text{compact}}}} \left| \langle \nabla V(x), h(x) \rangle \right|$ [we're cutting away the set (H) where $\langle \nabla V(x), h(x) \rangle = 0$]

So $\Delta > 0$ (strictly) (min over compact sets is attained).

So $\dfrac{dV(x(t))}{dt} \leq -\Delta$.

Now, if $T > \dfrac{M}{\Delta}$, then we would have entered H^η, starting from any point in $\overline{H^M}$.

Reasoning: If not $V(x(T)) - V(x(0)) = \displaystyle\int_0^T \frac{dV(x(t))}{dt}\, dt$

$$\leq -\Delta T < -M.$$

$\Rightarrow V(x(0)) > V(x(T)) + M$

$\Rightarrow V(\cancel{x}(0)) > M$, a contradiction.

So we know for any $\eta > 0$, any initial condition in $\overline{H^M}$,

$x(t) \longrightarrow H^\eta$!

(Not $\bar{x}(t)$)

- Continuous functions on compact sets (here $\overline{H^M}$) are uniformly continuous.

(Recall) $\forall \eta > 0, \exists \delta > 0$ s.t. $\forall x, y \in \overline{H^M}$, $\|x-y\| < \delta$

$$\Rightarrow |V(x) - V(y)| < \eta.$$

- By the BCL, $\exists s_0 > 0$ s.t. $\forall s \geq s_0$,

$$\sup_{t \in [s, s+T]} \|\bar{x}(s+t) - x^s(s+t)\| < \delta.$$

Same δ as the one above

In particular, $\|\bar{x}(s+T) - x^s(s+T)\| < \delta$.

- Put these together :

$$x^s(s) = \bar{x}(s) \in \overline{H^M}.$$

T units of time later, $x^s(s+T) \in H^\eta$, i.e., $V(x^s(s+T)) < \eta$.

By BCL: $\|\bar{x}(s+T) - x^s(s+T)\| < \delta$

By uniform continuity of V,

$$|V(\bar{x}(s+T)) - V(x^s(s+T))| < \eta$$

$$\Rightarrow V(\bar{x}(s+T)) < V(x^s(s+T)) + \eta < 2\eta.$$

Pick $t_1 = s_0 + T$. This suffices to make $V(\bar{x}(t)) < 2\eta$. Since $\eta \to$ arbitrary, $\bar{x}(t) \longrightarrow H$, and we know from earlier that it goes to a connected ict set. $\blacksquare$

Extensions:

1. Suppose $\{\sup_n \|x_n\| < \infty\}$ does not hold a.s.

S'pose this holds w.p. $p > 0$.

Our argument was a pathwise argument. On all such ω where this holds, we still have

$$\sum a(n)\, \mathbb{E}[\|M_{n+1}\|^2 \mid \mathcal{F}_n] < \infty.$$

So the theorem continues to hold for all such ω.

2. Random $a(n)$, but (A2) holds wp 1.

$$\mathcal{F}_n := \sigma\{x_m, M_m, a(m), m \le n\}.$$

Take this $\mathcal{F}_n$ in (A3) $\left(E[\|M_{n+1}\|^2 | \mathcal{F}_n] \le K(1+\|x_n\|).\right)$

Then the theorem continues to hold.

3. Additional random or deterministic noise

$$x_{n+1} = x_n + a(n)\left[h(x_n) + M_{n+1} + \varepsilon(n)\right], \; n \ge 0.$$

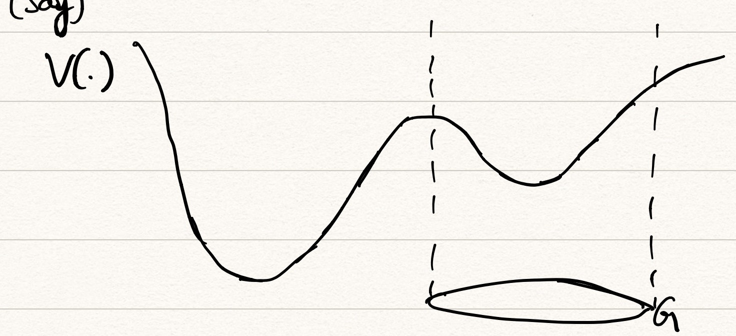
eg. $\underbrace{h_n(x_n) - h(x_n)}_{\varepsilon(n)} + h(x_n)$, $h_n \longrightarrow h$ uniformly over compact sets.

BCL and all others continue to hold if $\{\varepsilon(n)\}$ is bdd and $o(1)$ a.s. We will have an additional error which goes to 0 as $n \to \infty$.

Chapter 3: Lock-in Probability:

What is the chance of being locked into an equilibrium when we are close to it?

- Assume (A1) – (A3) hold. [Relaxing (A4)].
- For simplicity, $a(n) = \frac{1}{n+1}$ $\left(\begin{array}{l} a(n)\text{s are decreasing} \\ \sup_n a(n) \leq 1 \end{array} \right)$.
- Local Liapunov function



$G \subset \mathbb{R}^d$, open. $V: G \longrightarrow [0,\infty)$, $V \in C^1$, lower level sets are compact.

$H = \{x \in G : V(x) = 0\}$, compact.

$\langle \nabla V(x), h(x) \rangle \leq 0$, $\forall x \in G$ with equality iff $x \in H$.

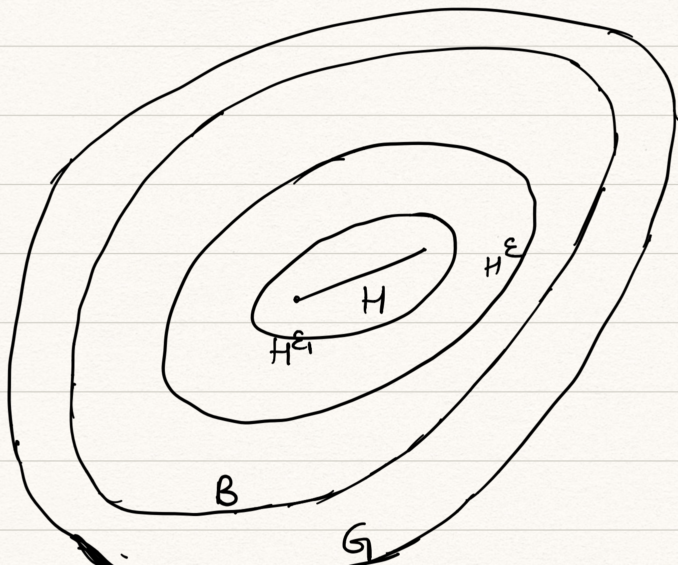
Observation: H is asymptotically stable. (Is Liapunov stable & is an attractor). ≈OK (Intuitively true).

• Let us say we have an open subset $B \subset G$.

Assume $H \subset B \subset \overline{B} \subset G$.

• If $\{x_n\}$ remain bounded in B a.s., $x_n \xrightarrow{n\to\infty}$ connected ict subset of H.

✓ ((A4) holds)



(ch.) Not clear

Fix $0<\varepsilon_1<\varepsilon$ s.t., $H^{\varepsilon_1} \subset \underbrace{H^{\varepsilon} \subset B}$

this is possible since H is compact.

Fix $\delta>0$ s.t. $N_\delta(H^{\varepsilon_1}) \subset H^{\varepsilon} \subset N_\delta(H^{\varepsilon}) \subset B$.

✓ Yes, possible to find such a $\delta>0$.

Starting from any location in $\overline{B}$, the ode would have entered H^{ε_1}, after

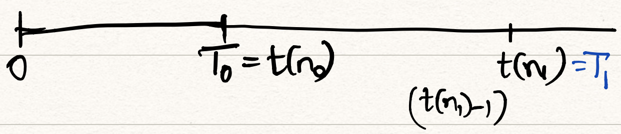
$$T = \max_{x\in\overline{B}} V(x) - \varepsilon_1$$

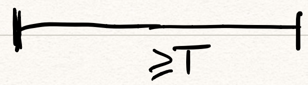
(something like M

$\overline{\min_{x\in \bar{B}\setminus H^{\varepsilon}} |\langle \nabla V(x), h(x)\rangle|}$ ($\overline{\Delta}$ /

- n_0 : some time when $x_{n_0} \in B$.

$n_m = \min\{n : t(n) \geq t(n_{m-1}) + T\}$ This is fixed by considerations above

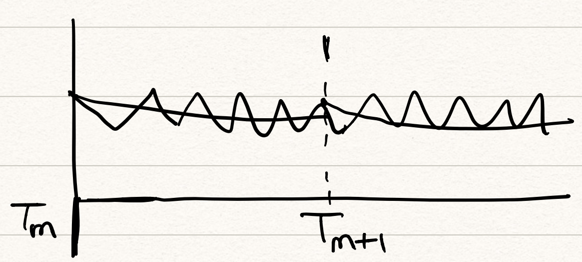




$T_m = t(n_m)$

$I_m = [T_m, T_{m+1}]$, each of size $\geq T$

- $\delta_m := \sup_{t \in I_m} \| \bar{x}(t) - x^{T_m}(t) \|$



<u>Lemma</u>: Suppose $x_{n_0} \in B$ and suppose $\delta_m < \delta$, $\forall m \geq 0$.

Then (1) $\bar{x}(t) \in N_r(H^{\varepsilon})$, $\forall t \geq T_1$

(2) $\bar{x}(t) \longrightarrow H$ as $t \to \infty$.

→ since the x_ns are now bounded, after $t = T_1$.

<u>Proof</u> : of (1) only. (2) follows from Liapunov fn result.