Recall (from last time):

SA scheme: $x_{n+1} = x_n + a(n)\left[h(x_n) + M_{n+1}\right]$
$x_0 = \overline{x_0} \in \mathbb{R}^d.$

Assumptions:

(A1) h is Lipschitz

(A2) Step sizes satisfy the usual assumptions:

$$\sum_n a(n) = \infty, \quad \sum_n a^2(n) < \infty$$

(A3) $\{M_n\}$ is an $\{\mathcal{F}_n\}$-martingale difference sequence

$$\mathbb{E}\left[\|M_{n+1}\|^2 \mid \mathcal{F}_n\right] \leq K\left(1 + \|x_n\|^2\right) \text{ a.s.}, \; n \geq 0$$

for some $0 < K < \infty$.

Some consequences:

- (A1) $\Rightarrow$ h grows at most linearly
- (A1) and (A3) $\Rightarrow$ $\mathbb{E}[\|x_n\|^2]$, $\mathbb{E}[\|M_{n+1}\|^2]$ are finite for each $n \geq 0$.

(A4) The iterates $\{x_n\}$ remain bounded a.s., i.e.,

$$C_0 = \sup_n \|x_n\| < \infty \text{ a.s.}$$

(Usually the difficult assumption to check in applications).

• Setting up the basic convergence lemma:

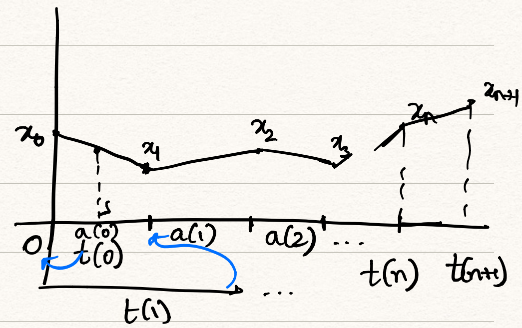
1) Observations: $\dot{x}(t) = h(x(t))$ is well-posed.

2) Interpolated trajectory:

$$t(0) = 0 \quad , t(n) = \sum_{m=0}^{n-1} a(m) \quad , n \geq 1 \quad , I_n = [t(n), t(n+1))$$

$\bar{x}(t)$, $t \geq 0$ is defined as:

$$\bar{x}(t(n)) = x_n, \quad t = t(n), \quad n \geq 0.$$



and $\bar{x}(t) = x_n + \dfrac{(t - t(n))}{(t(n+1) - t(n))} * (x_{n+1} - x_n)$, for $t \in I_n$.

[Linear interpolation]

3) $x^s(t)$, $t \geq s$, is the unique solution to the ode $\dot{x} = h(x)$, starting at s, with initial condition $x^s(s) = \bar{x}(s)$.

For each $s \geq 0$, we get a different trajectory
$\{x^s(\cdot), s \geq 0\}$ is a collection of trajectories

4) Likewise, $x_s(t)$, $t \leq s$ is the unique sol^n to the ode ending at s, with terminal condition $x_s(s) = \bar{x}(s)$.

In particular,

$$\dot{x}^s(t) = h(x^s(t)), \quad t \geq s$$
$$\dot{x}_s(t) = h(x_s(t)), \quad t \leq s.$$

with $x_s(s) = \bar{x}(s)$

Lemma [Basic Convergence Lemma] Fix T, $0 < T < \infty$. Assume (A1)-(A4).

Then,

(1) $$\lim_{s\to\infty} \sup_{t\in[s,s+T]} \|\bar{x}(t) - x^s(t)\| = 0 \text{ a.s.}$$

(2) $$\lim_{s\to\infty} \sup_{t\in[s-T,s]} \|\bar{x}(t) - x_s(t)\| = 0 \text{ a.s.}$$

Pf: We will focus first on s being one of the embedded time points $t(n)$.

Consider $[t(n), t(n)+T]$.

Take m : $t(n+m) \in [t(n), t(n)+T]$

& let $[t] = \max\{t(k) : t(k) \leq t\}$, i.e., the last iteration time point or embedded time point prior to t.

$m-1$

• $\bar{x}(t(n+m)) = \bar{x}(t(n)) + \sum_{k=0}^{\cdots} a(n+k)\left[h(\bar{x}(t(n+k))) + M_{n+k+1}\right]$

$$= \bar{x}(t(n)) + \sum_{k=0}^{m-1} a(k)\, h(\bar{x}(t(n+k))) + \underbrace{\delta_{n,n+m}}_{\text{noise.}},$$

where $\delta_{n,n+m} = \xi_{n+m} - \xi_n$, where

$$\xi_n = \sum_{m=0}^{n-1} a(m) M_{m+1}, \quad n \geq 0$$

✓ Yeah.

[Also a martingale, since a scaled sum of mart. diff.]

• $x^{t(n)}(t(n+m)) = \bar{x}(t(n)) + \int_{t(n)}^{t(n+m)} h(x^{t(n)}(t))\,dt$

Note

$$= \bar{x}(t(n)) + \sum_{k=0}^{m-1} \overbrace{a(n+k)\, h(x^{t(n)}(t(n+k)))}^{\text{sample \& hold } x^{t(n)}(t) \text{ at } x^{t(n)}(t(n+k))}$$

$$+ \underbrace{\boxed{\int_{t(n)}^{t(n+m)} \left[h(x^{t(n)}(t)) - h(x^{t(n)}([t]))\right] dt}}_{\text{time-discretization error.}}$$

• Error analysis:

a) $\xi_n = \sum_{m=0}^{n-1} a(m) M_{m+1}$ is an $\{\mathcal{F}_n\}$-martingale, square integrable (Cauchy-Schwartz)

We shall evaluate:

$$A_\infty = \sum_{n=0}^{\infty} \mathbb{E}\left[\|\xi_{n+1} - \xi_n\|^2 \middle| \mathcal{F}_n\right]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\left[a^2(n)\, \|M_{n+1}\|^2 \middle| \mathcal{F}_n\right]$$

$$\overset{(A3)}{\leq} \sum_{n=0}^{\infty} a^2(n)\left(1 + \|x_n\|^2\right) \quad a.s.$$

$$\overset{(A4)}{\leq} (1 + C_0) \sum_{n=0}^{\infty} a^2(n) \quad a.s.$$

$$< \infty \quad a.s.$$

Thus, by one of the martingale convergence theorems, ξ_n converges a.s.

Then the tail sums $\sup_{k \geq 0} \|\delta_{n, n+k}\| \xrightarrow[n\to\infty]{} 0$ a.s.

(Check that for a convergent series,

Need to check here (CB) $\left\{\begin{array}{l} \text{(i) Tail sum converges to } 0 \\ \text{(ii) sup over partial tail sums also conv to } 0\text{)} \end{array}\right.$

(b) Time-discretization error also goes to zero a.s.:

$\bullet \ \|h(x^s(t))\| \leq C_T$ a.s.

(i) $\|h(x)\| \leq \|h(0)\| + L\|x\|$

(ii) $t \in [s, s+T]$

$$x^s(t) = \bar{x}(s) + \int_s^t h(x^s(\tau))\, d\tau$$

$$\therefore \ \|x^s(t)\| \leq \|\bar{x}(s)\| + \int_s^t \left(\|h(0)\| + L\|x^s(\tau)\|\right) d\tau$$

$$\leq \left(C_0 + \|h(0)\|T\right) + L\int_s^t \|x^s(\tau)\|\, d\tau$$

$$\underset{\text{ineq}}{\overset{\text{(Gronwall)}}{\leq}} \left(C_0 + \|h(0)\|T\right) e^{LT}, \quad t \in [s, s+T].$$

(iii) $\|h(x^s(t))\| \leq \|h(0)\| + L\|x^s(t)\|$

$$\leq \|h(0)\| + L\left(C_0 + \|h(0)\|T\right) e^{LT} := C_T.$$

Now,

Fix K s.t. $0 \leq K \leq m-1$, $t \in [t(n+K), t(n+K+1)] = I_{n+K}$.

For this interval,

$$\| x^{t(n)}(t) - x^{t(n)}(t(n+k)) \|$$

$$= \left\| \int_{t(n+k)}^{t} h(x^{t(n)}(\tau))\, d\tau \right\| \quad \left(\text{Using fund. thm of calculus}\right)$$

$$\leq C_T \underbrace{a(n+k)}_{(t(n+k+1) - t(n+k))}$$

- Applying this to the time-discretization error:

$$\left\| \int_{t(n)}^{t(n+m)} h(x^{t(n)}(t)) - h(x^{t(n)}([t]))\, dt \right\|$$

$$\leq L \sum_{k=0}^{m-1} \int_{t(n+k)}^{t(n+k+1)} \| x^{t(n)}(\tau) - x^{t(n)}(t(n+k)) \|\, d\tau$$

$$\leq C_T L \sum_{k=0}^{m-1} a(n+k) \cdot a(n+k) \leq L C_T \sum_{k=0}^{\infty} a(n+k)^2$$

$$\longrightarrow 0 \quad \text{as } n \to \infty \text{ a.s.}$$

(c) Comparison at embedded time points, $m : t(n+m) \leq t(n) + T$.

So

$$\underbrace{\left\| \bar{x}(t(n+m)) - x^{t(n)}(t(n+m)) \right\|}_{z_m} \le \underbrace{\sup_k \|\delta_{n,n+k}\| + C_T L \sum_{k=0}^{\infty} a^2(n+k)}_{K_{T,n}}$$

$$+ L \sum_{k=0}^{m-1} a(n+k) \underbrace{\left\| \bar{x}(t(n+k)) - x^{t(n)}(t(n+k)) \right\|}_{z_k}$$

$$\le K_{T,n}\, e^{L(t(n+m)-t(n))}$$

[Discrete Gronwall]

$$\le K_{T,n}\, e^{LT}$$

$$\xrightarrow{n\to\infty} 0 \quad [\text{as } K_{T,n} \to 0]$$

(d) Comparison at other points. $t \in I_{n+k}$

$$\left\| x^{t(n)}(t) - \bar{x}(t) \right\| = \Big\| \lambda x^{t(n)}(t) - \lambda \bar{x}(t(n+k)) + (1-\lambda) x^{t(n)}(t) - (1-\lambda)\bar{x}(t(n+k+1)) \Big\|$$

$$= \Big\| \lambda x^{t(n)}(t(n+k)) + \lambda \int_{t(n+k)}^{t} h(x^{t(n)}(\tau))\,d\tau - \lambda \bar{x}(t(n+k)) + (1-\lambda) x^{t(n)}(t(n+k+1)) - (1-\lambda) \int_{t}^{t(n+k+1)} h(x^{t(n)}(\tau))\,d\tau$$

$$- (1-\lambda)\, \bar{x}(t(n+k+1)) \|$$

$$\leq \lambda K_{T,n} e^{LT} + (1-\lambda) K_{T,n} e^{LT} + (\lambda + (1-\lambda)) \int_{t(n+k)}^{t(n+k+1)} \| h(x^{t(n)}(\tau)) \| d\tau$$

$$\leq K_{T,n} e^{LT} + C_T a(n+k)$$

$$\leq K_{T,n} e^{LT} + C_T b_n \quad , \quad b_n = \sup_{k:\ t(n+k) \in [t(n),\ t(n)+T]} a(n+k)$$