Recall:

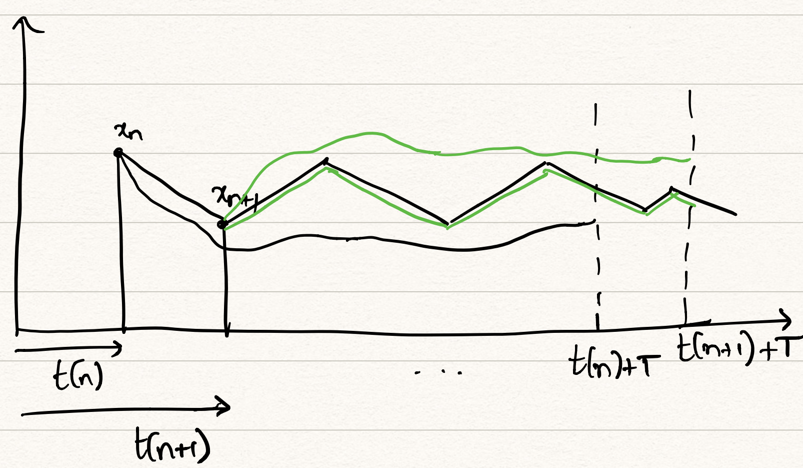
Lemma (Basic Convergence Lemma): Fix $T>0$.

$$\lim_{s\to\infty} \sup_{t\in[s,s+T]} \|x^s(t) - \bar{x}(t)\| = 0$$

$$\lim_{s\to\infty} \sup_{t\in[s-T,s]} \|x_s(t) - \bar{x}(t)\| = 0$$

Need to show ↑ for s being between $t(n)$ & $t(n+1)$, for some n.

If x_n and x_{n+1} are close, then so are the solutions to the ode $x^s(s)$ with initial conditions x_n and x_{n+1}, resp.



(Shift both green lines back to $t(n)$ & compare).

Now, $\|x_{n+1} - x_n\| \leq a(n)\|h(x_n)\| + a(n)\|M_{n+1}\|$

$$\overset{a.s.}{\leq} \underbrace{a(n)\left[\|h(0)\| + LC_0\right]}_{\xrightarrow{n\to\infty} 0} + \underbrace{a(n)\|M_{n+1}\|}_{\longrightarrow 0}$$

(Since $a(n) \xrightarrow{n\to\infty} 0$)   as $\xi_n = \sum_{n=0}^{\infty} M_n$ is convergent.

So the ode solutions are close to each other.

For $t > s$, $\|x^s(t) - x^{t(n)}(t)\|$

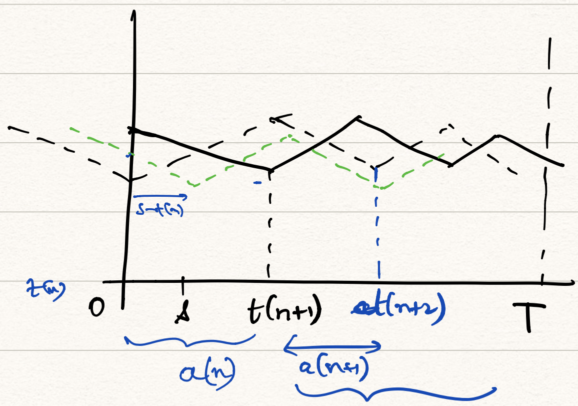
$= \|\bar{x}(s) - \bar{x}(t(n)) + \int_s^t h(x(\tau))d\tau + \int_t^{t(n)} h(x(\tau))d\tau\|$

$\overset{n\to\infty}{\leq} 0 + \int_s^{t(n)} \|h(x(\tau))\| d\tau \overset{n\to\infty}{\leq} 0 + 0$ ✓ (Since $a(n) \to 0$ & $\|x_n\|$ is bdd). a.s.

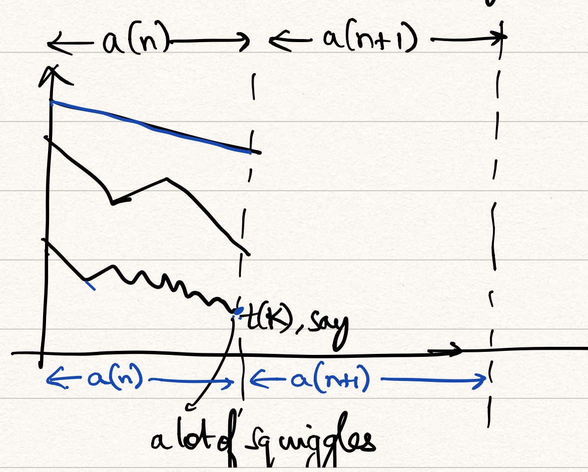
Besides, the ode's solution starting at s is also close to the two ode solutions starting at x_{n+1}, x_n.

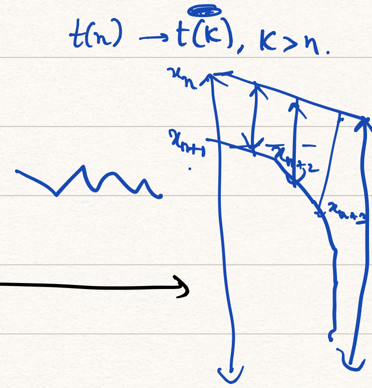
Verified.

By Δ-ineq, it suffices to check that the 3 segments of the same interpolated trajectory, $\bar{x}(\cdot)$ are close to one another.



The issue is when the following happens: (after shifting)





So the max-difference between the shifted interpolated trajectories goes to zero.

$$\text{Max diff.} \leq \max\Big\{ \Big(\overbrace{\|x_{n+1}-x_n\| + \|x_{n+2}-x_{n+1}\| + \dots + \|x_{k+1}-x_k\|}^{\text{Just for the first interval },\, a(n).} \Big), \dots \Big\}$$

$$(\|h(0)\| + LC_0) + \|M_{n+1}\|$$

$$\leq \max\Big\{ \underbrace{\Big(a(n)K + \underbrace{\big(a(n+1)K + \dots + a(k)K\big)}_{\leq (a(n)+a(k+1))K} \Big)}_{\xrightarrow{n\to\infty} 0}, \dots \Big\}$$

So

& hence the max diff $\xrightarrow{n\to\infty} 0$.

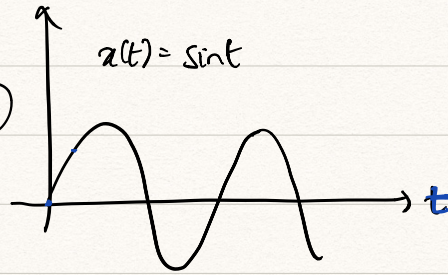
Preliminaries (Contd...): Asymptotic behaviour of dynamical systems.

- System: $\dot{x}(t) = h(x(t)), \; t \geq 0$
  $x(0) = a \in \mathbb{R}^d$

- **Defn:** ω-limit set: Given a trajectory passing through a at time 0, its ω-limit set is defined as

$$\Omega_a = \{ b \in \mathbb{R}^d : \exists \{t_k\}, t_k \uparrow \infty, x(t_k) \to b, x(0) = a \}$$

eg.

① 

$\Omega_0 = [-1, 1]$

② $x(t) = e^{-t}$

$\Omega_1 = \{0\}$

Likewise, α-limit set $(t_k \searrow -\infty)$:

Notation: A_a

$\mathbb{R}^1$ $(a,b) \to [a,b]$

Lemma: $\Omega_a = \bigcap_{t>0} \overline{\{x(s) : s > t\}}$ (as usual, $\overline{}$ denotes closure)

Proof: Dir^n ① Take $b \in \bigcap_{t>0} \overline{\{x(s) : s > t\}}$

(as an eg, for $x(t) = e^{-t}$, We are essentially taking $\bigcap_{t>0} \overline{[0, e^{-t}]} = \{0\}$ (closure))

Take $t_i > 0$.

Hence $b \in \overline{\{x(s) : s > t_i\}}$.

$\boxed{\text{So there exists } t_{i+1} > t_i + 1 : \|x(t_{i+1}) - b\| < \frac{1}{i+1}}$ ✓ Note style of presentation

We have $t_k \to \infty$, $x(t_k) \to b$ as $k \to \infty$.

This, by defn of ω-limit set, says that $b \in \Omega_a$.

Dir ②:

- Take $b \in \Omega_a$. $\exists \{t_k\}$, $t_k \uparrow \infty$: $x(t_k) \to b$ as $k \to \infty$.

$$\Rightarrow b \in \overline{\{x(s) : s > t_k\}} \text{ , for every } t_k \text{ in the seq. } \{t_k\}.$$

Note that: Validity for $s > t_k$ implies validity for $s > t$, for every $t \leq t_k$,

and since $t_k \to \infty$, we have shown $b \in \bigcap_{t>0} \overline{\{x(s) : s > t\}}$.

$$\text{Hence, } \Omega_a \subset \bigcap_{t>0} \overline{\{x(s) : s > t\}}. \quad \blacksquare$$

Observations and examples:

(i) Ω_a depends on a.

(ii) $\dot{x}(t) = H x(t)$,

$$\Rightarrow x(t) = e^{Ht} x(0), \text{ where } e^{Ht} = \sum_{m=0}^{\infty} \frac{H^m t^m}{m!}$$

Let's say H has eigenvalues $\lambda_1, \dots, \lambda_d$. We focus on $d=2$.

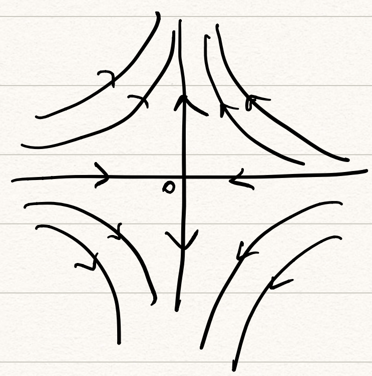
1. $Re(\lambda_i) < 0$, $i=1,2$. $\Omega_a = \{0\}$.

2. $Re(\lambda_i) = 0$, $i=1,2$. $\Omega_a = \{\underline{b} : \|\underline{b}\| = \|\underline{a}\|\}$

(Ω_a depends on a).

3. $Re(\lambda_i) > 0$, $i=1,2$. $\Omega_a = \phi$.

4. $d>2$, $Re(\lambda_i) < 0$, $Re(\lambda_j) > 0$



$\boxed{\Omega_a \text{ can be } \{0\} \text{ or } \phi.}$