Recall:

Defn: Stopping time: A $\mathbb{Z}_+ \cup \{\infty\}$-valued RV $T$ is a stopping time wrt $\{\mathcal{F}_n\}$ if for each $n \in \mathbb{Z}_+ \cup \{\infty\}$, the event $\{T \leq n\}$ is in $\mathcal{F}_n$, where $\mathcal{F}_\infty = \bigvee_{n=1}^{\infty} \mathcal{F}_n$.

Defn: $\mathcal{F}_T = \{A \in \mathcal{F} \mid A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{Z}_+ \cup \{\infty\}\}$.

Thm: Optional stopping theorem:

Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$-submartingale. Let $T$ be a stopping time. Then $\{X_{T\wedge n}\}_{n\geq 0}$ is an $\{\mathcal{F}_n\}$-submartingale.

Pf: $H_n = \mathbb{1}\{T \geq n\} \in \mathcal{F}_{n-1}$

Def $(H \cdot X)_n = \sum_{k=1}^{n} H_k (X_k - X_{k-1}) = X_{T\wedge n}$ (Verifiable)

Claim: $(H \cdot X)_n$ is an $\{\mathcal{F}_n\}$-submartingale

Pf: $\mathbb{E}[(H \cdot X)_{n+1} \mid \mathcal{F}_n] = (H \cdot X)_n + \underbrace{\mathbb{E}[H_{n+1}(X_{n+1} - X_n) \mid \mathcal{F}_n]}$

$= H_{n+1} \underbrace{\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n]}_{\geq 0}$

$\geq (H \cdot X)_n.$

Thm: (Optional Sampling Theorem)

Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$- submartingale. Let $T_n, n \geq 0$ be a sequence of bounded stopping times such that $T_0 \leq T_1 \leq \dots$ a.s. Then $\{X_{T_n}\}$ is an $\{\mathcal{F}_{T_n}\}$- submartingale.

Pf: • $X_{T_n}$ is adapted to $\mathcal{F}_{T_n}$.

Let $T_n \leq N_n$, $n \geq 0$ ($N_n$ is the bound on $T_n$)

$$\{X_{T_n} \in B\} = \bigcup_{k=0}^{N_n} \{T_n = k\} \cap \{X_k \in B\}$$

Note that: $\{T_n = k\} \cap \{X_k \in B\}$

$$= \left(\{T_n \leq k\} \setminus \{T_n < k\}\right) \cap \{X_k \in B\}$$

$$\in \mathcal{F}_k \quad , \forall k.$$

So $\{X_{T_n} \in B\} \in \mathcal{F}_{T_n}$ (Recall the $\mathcal{F}_T$ defn from earlier)

• $|X_{T_n}| \leq |X_0| + |X_1| + \dots + |X_{N_n}|$ & since each is integrable

• It suffices to show for stopping times $S, T$ with $S \leq T$ a.s.

Need to show: $\mathbb{E}[X_T \mid \mathcal{F}_S] \geq X_S$ a.s.

common UB on $T, S$.

Now, $$X_T - X_S = \sum_{k=1}^{N} \sum_{j=0}^{k-1} \mathbb{1}\{T=k\} \, \mathbb{1}\{S=j\} \, (X_k - X_j)$$

$$= \sum_{k=1}^{N} \mathbb{1}\{T=k\} \, \mathbb{1}\{S \le k-1\} \, X_k$$

$$- \sum_{j=0}^{N-1} \mathbb{1}\{T>j\} \, \mathbb{1}\{S=j\} \, X_j$$

$$= \sum_{k=1}^{N} \mathbb{1}\{T=k\} \, \mathbb{1}\{S \le k-1\} \, X_k$$

$$- \sum_{k=1}^{N} \mathbb{1}\{T \geq k\} \, \mathbb{1}\{S = k-1\} \, X_{k-1} \qquad (*)$$

To this, add & subtract (add to 2nd term)

$$\sum_{k=1}^{N} \mathbb{1}\{S<k-1\} \, \mathbb{1}\{T \geq k\} \, X_{k-1}$$

↓ to 1st term in $(*)$.

$$= \sum_{k=2}^{N} \mathbb{1}\{S<k-1\} \, \mathbb{1}\{T \geq k\} \, X_{k-1}$$

[Since $k=1$ leads to an empty sum]

$$= \sum_{k'=1}^{N} \mathbb{1}\{S<k'\} \, \mathbb{1}\{T \geq k'+1\} \, X_{k'}$$

So we get

$$X_T - X_S = \sum_{k=1}^{N} \mathbb{1}\{S<k\} \, \mathbb{1}\{T \geq k\} \, (X_k - X_{k-1})$$

Take conditional expectation, conditioned on $\mathcal{F}_S$.

$$\mathbb{E}[X_T - X_S \mid \mathcal{F}_S] = \mathbb{E}\left[\sum_{k=1}^{N} \mathbb{1}\{S<k\}\mathbb{1}\{T \geq k\}(X_k - X_{k-1}) \middle| \mathcal{F}_S\right]$$

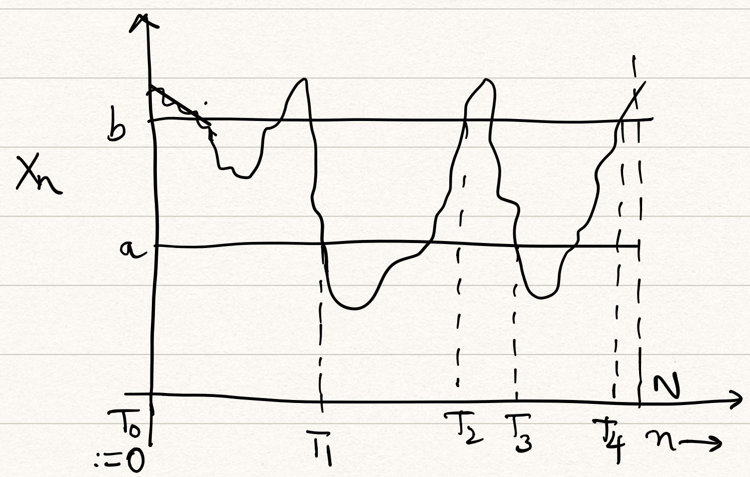
$\left(\mathbb{1}\{S \leq k-1\} \cap \{X_{k-1} \in B\} \text{ is m'able}\right)$

$$= \sum_{k=1}^{N} \mathbb{1}\{S \leq k-1\}\mathbb{1}\{T > k-1\}\underbrace{\left(\mathbb{E}[X_k \mid \mathcal{F}_{k-1}] - X_{k-1}\right)}_{\geq 0}$$

$\blacksquare$

Doob's Up-crossing Inequality:

Setting: Let $\{X_n\}$ be an $\{\mathcal{F}_n\}$-sub-martingale. Let $(a,b)$, $a<b$, be a given interval. Let $N \geq 1$ be fixed, finite.




$T_1 = \min\{0 \leq n \leq N \mid X_n \leq a\}$, if empty, $T_1 = N$

$T_2 = \min\{T_1 \leq n \leq N \mid X_n \geq b\}$ —— " $T_2 = N$

$\vdots$

$T_{2m-1} = \min\{T_{2m-2} \leq n \leq N \mid X_n \leq a\}$ —"—

$T_{2m} = \min\{T_{2m-1} \leq n \leq N \mid X_n \geq b\}$ —"— .

$\beta_N(a,b)$ = # of upcrossings of interval $(a,b)$ upto time $N$.

<u>Thm</u>: (Doob's Upcrossing Inequality)

$$\mathbb{E}[\beta_N(a,b)] \leq \frac{\mathbb{E}[(X_N-a)^+]}{b-a}$$

<u>Pf</u>: Take $a=0$, $X_n \geq 0 \ \forall n$, $X_0 = 0$.

$$X_N = X_N - X_0 = \sum_{m\geq 0}(X_{T_{2m+1}} - X_{T_{2m}}) + \underbrace{\sum_{m\geq 0}(X_{T_{2m}} - X_{T_{2m-1}})}_{\geq b\,\beta_N(0,b)}$$

$$\geq \sum_{m\geq 0}\left(X_{T_{2m+1}} - X_{T_{2m}}\right) + b\,\beta(0,b)$$

$$\therefore \mathbb{E}[X_N] \geq \mathbb{E}\Big[\sum_{m\geq 0}\overbrace{\big[\mathbb{E}[X_{T_{2m+1}} \mid \mathcal{F}_{T_{2m}}] - X_{T_{2m}}\big]}^{\geq 0}\Big] + b\beta(0,b)$$

$$\geq b\beta(0,b)$$

Ex: Work out general case, with $\phi(X) = (X-a)^+$

(convex, inc)

$\Rightarrow \{\phi(X_n)\}$ is sub-mart.