Recall:

(f) A compact invariant set M is internally chain transitive (ict)

if $\forall x, y \in M$, $\forall \epsilon > 0$, $\forall T > 0$, there exist $n+1$ points in M

$$x = x_0, x_1, x_2, \dots, x_{n-1}, x_n = y$$

such that a trajectory initiated at x_i meets the nbd of x_{i+1} at time $t_i \geq T$, $i = 0, 1, \dots, n-1$.

Example:
$$\dot{x}(t) = \begin{cases} x(1-x^2) & , |x| < \sqrt{2} \\ \text{linear} & , |x| > \sqrt{2} \end{cases}$$

$M_1 = [-1, 1]$ was invariant, but is not ict. However

$M = \{-1\}$ or $M = \{0\}$ or $M = \{+1\}$ are ict.

Thm: $x_n \longrightarrow A$ a.s., where A is a (possibly) sample path dependent connected ict set for the dynamics.

Def: (We will restrict attention to compact metric spaces)
($E = \{x \in \mathbb{R}^d : \|x\| \leq C_0\}$).

A set $M \subset E$ is connected if it cannot be written as

(disjoint) → This is sufficient

$M \subset U \cup V$, $U, V \to$ open subsets of E",

$M \cap U \neq \phi$, $M \cap V \neq \phi$.

Remark: Path connected $\Rightarrow$ Connected
$\not\Leftarrow$

(M is path connected if we can find a continuous path between any two points in M.)

$p : [0,1] \to M$,
$p(0) = x$, $p(1) = y$, $p \to$ continuous.

Proof of Theorem:

(a) Identification of A:

By (A4), $\sup_n \|x_n\| = C_0 < \infty$ a.s.

By BCL, for any fixed T, convergence to the ode holds as the starting location for comparison $s \to \infty$.

Fix an ω such that both of these hold:

$$A := \bigcap_{t>0} \overline{\{\bar{x}(s) : s \geq t\}}$$, the ω-limit set of $\bar{x}(\cdot)$.

(b) A is non-empty, compact, connected

We Know: $\bar{x}(\cdot)$ is continuous, bounded.

Hence $\overline{\{\bar{x}(s): s \geq t\}}$ is a nested, decreasing family (indexed by t) of nonempty, compact, connected sets.

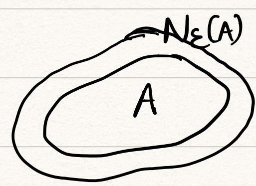
($\{\bar{x}(s): s \geq t\}$ is path connected, but its closure may not be so. But closure of a connected set is connected) Obviously

Hence, $\bigcap_{t>0} \overline{\{\bar{x}(s): s \geq t\}}$ is non-empty, compact, connected.

(To be proved later).

(c) Convergence to A:

Fix $\varepsilon > 0$.



$$N_\varepsilon(A)^c \cap A = \phi$$

$$N_\varepsilon(A)^c \cap \bigcap_{t>0} \overline{\{\bar{x}(s) : s \geq t\}} = \phi$$

$$\Rightarrow \bigcap_{t>0} \underbrace{N_\varepsilon(A)^c \cap \overline{\{\bar{x}(s): s \geq t\}}} = \phi$$

compact, nested decreasing

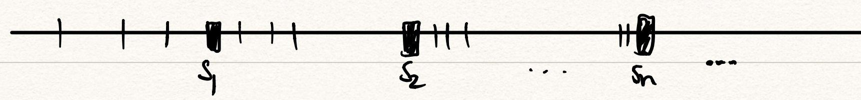
$$\Rightarrow \quad N_\varepsilon(A)^c \cap \{\bar{x}(s) : s \geq t_0\} = \phi, \text{ for some } t_0.$$

$$\Rightarrow \{\bar{x}(s) : s \geq t_0\} \subset N_\varepsilon(A)$$

Since ε was arbitrary, $\bar{x}(t) \xrightarrow{t\to\infty} A$, and so $x_n \longrightarrow A$.

(d) Any $x \in A$ is approachable via the embedded time points.

Take $x \in A$. Then $\exists \{s_n\} \nearrow \infty$, $\bar{x}(s_n) \longrightarrow x$.



(look at immediate prev. time instant).

$$\max_{s \in [t(k), t(k+1)]} \|\bar{x}(s) - \bar{x}(t(k))\| \leq \|x_{k+1} - x_k\|$$

$$\leq a(k)\|h(x_k)\| + a(k)\|M_{k+1}\|$$

$$\leq a(k)\left(\|h(0)\| + LC_0\right) + a(k)\|M_{k+1}\|$$

$$\xrightarrow{k\to\infty} 0$$

Each s_n lies in $[t(m(n)), t(m(n)+1))$. So,

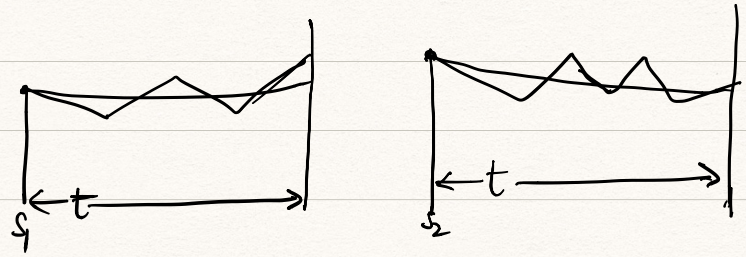
$$\|x-\bar{x}(t(m(n)))\| \le \|x-\bar{x}(s_n)\| + \|\bar{x}(s_n)-\bar{x}(t(m(n)))\|$$

$$\xrightarrow{n\to\infty} 0$$

(e) A is invariant to the dynamics → ode dynamics.

We will show positive invariance. Let $x \in A$.

By (d), $\exists \{s_n\} \uparrow \infty$, $\boxed{\text{embedded time points}}$, $\bar{x}(s_n) \longrightarrow x$. Let $t>0$. To show $\phi_t(x) \in A$.



The ode solution $x^{s_n}(s_n+t) = \phi_t(x^{s_n}(s_n)) \xrightarrow[n\to\infty]{} \phi_t(x)$.

By (c), $\bar{x}(s_n+t) \xrightarrow[n\to\infty]{} A$. ⊛ ⊛

By BCL, the error between $\bar{x}(s_n+t)$ and $x^{s_n}(s_n+t) = \phi_t(\bar{x}(s_n))$ goes to 0 as $n\to\infty$. Then, $\phi_t(x) \in A$, thereby showing +ve invariance. ⊛

(f) A is ict for the dynamics. (cb) [Not clearly written]

Given : $\tilde{x}_0, \tilde{x}_1 \in A$, $\varepsilon > 0$, $T > 0$.

Pick $\delta < \frac{\epsilon}{3}$ s.t. $\|x - y\| < \delta \Rightarrow \sup_{t \in [0, 2T]} \|\phi_t(x) - \phi_t(y)\| < \frac{\epsilon}{3}$

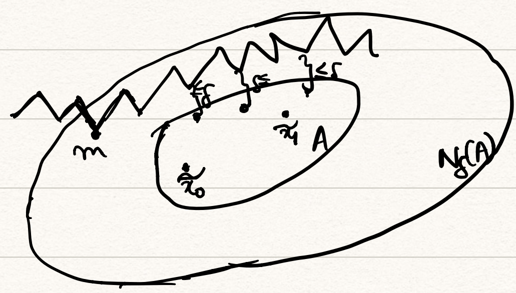
(Uniform continuity for the fixed $2T$)

(Follows from $\phi_t(\cdot)$ being Lipschitz)

Pick $m > 1$ sufficiently large so that

$s \geq t(m) \Rightarrow$ 1) $\bar{x}(s + \cdot) \in N_\delta(A)$ [Possible, since $\bar{x}(s_n + \cdot) \xrightarrow[n\to\infty]{} A$]

2) $\sup_{t \in [s, s+2T]} \|\bar{x}(t) - x^s(t)\| < \delta$



$\exists n_0 > m$ s.t. $\|\bar{x}(t(n_0)) - \tilde{x}_0\| < \delta$

$\exists n_1 > n_0$ s.t. $\|\bar{x}(t(n_1)) - \tilde{x}_1\| < \delta$, $t(n_1) - t(n_0) \geq T$.

- $KT \leq t(n_1) - t(n_0) < KT + T$
- $s(0) = t(n_0)$, $s(i) = s(0) + iT$, $i = 1, 2, \ldots, K-1$

  $s(K) = t(n_1)$

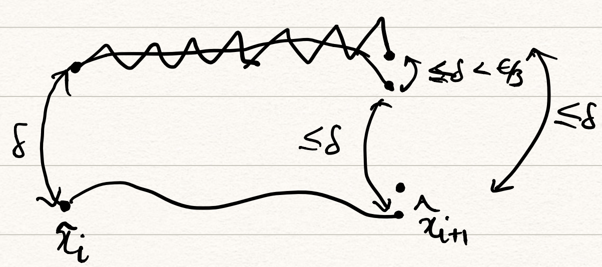
- Pick $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{K-1}$ in A and in the δ-nbd of $\bar{x}(s(i))$.

- Set $\tilde{x}_0 = \hat{x}_0, \hat{x}_1, \hat{x}_2, \dots, \hat{x}_{k-1}, \hat{x}_k = \tilde{x}_1$.

<u>Claim</u>: $(s(i), \hat{x}_i)_{i=0}^{k}$ satisfy the conditions for ict.

- All time segments are of duration T except the last one which has duration between T and $2T$.

- An ode started at $\hat{x}_i$ and traversed for the chosen duration is within ϵ of $\hat{x}_{i+1}$.



By Δ-inequality,

$$\left\|\phi_{s(i+1)-s(i)}(\hat{x}_i) - \hat{x}_{i+1}\right\| \le \overbrace{\left\|\phi_{s(i+1)-s(i)}(\hat{x}_i) - \phi_{s(i+1)-s(i)}\left(x^{s(i)}(s(i+1))\right)\right\|}^{\text{Unif. cont.} \ \le \epsilon/3}$$

$$+ \underbrace{\left\|\phi_{s(i+1)-s(i)}\left(x^{s(i)}(s(i+1))\right) - \bar{x}(s(i+1))\right\|}_{\le \delta, \text{ By BCL}}$$

$$+ \underbrace{\left\|\bar{x}(s(i+1)) - \hat{x}_{i+1}\right\|}_{\le \delta, \text{ by choice.}}$$