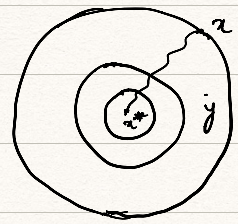
Suppose $\dot{x}(t) = h(x(t))$, $t \geq 0$, Lipschitz. x^* is the globally asymp. stable eqbm.

$\{x^*\}$ is the only ict set.

Let A be an ict set. Let $x, y \in A$.



Can never reach y from x, unless $y = x^*$, for large T & small ϵ.

Hence $A = \{x^*\}$.

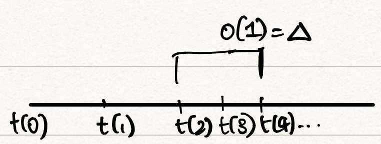
Averaging the natural time scale :

$$x_{n+1} = x_n + a(n)\left[h(x_n, Y_n) + M_{n+1}\right]$$

Y_{n+1}, given the past, $\sim p(dy_{n+1} \mid y_n, x_n)$ [simpler settings : $p(dy_{n+1} \mid y_n)$, if density exists, $= p(y_{n+1} \mid y_n)\, dy_{n+1}$; $p(dy_{n+1})$, iid]

$a(n) \longrightarrow 0$, $\bar{x}(t)$ slowing down (in discrete time-steps, $x_{n+1} - x_n$ is getting smaller)

y_n continues in the natural time scale ($n \to n+1$ & so on).
(In steps of 1).



Not clear

$\bar{x}(t+\Delta) - \bar{x}(t)$. In the Δ-time interval, we can see many y_n's influencing dynamics of x_n. But x is quasi-static.

We anticipate that ergodic thm will apply and the average of the y_n's will determine the dynamics.

Fix x. Consider the x-parameterised MC $\{Y_n\}_{n\geq 1}$ $\left(p(dy_{n+1}|y_n, x)\right)$.

Assume that it is ergodic & the unique invariant $\nu_x(dy)$ is known.
So y is likely to have equilibriated.

In small time Δ, $h(x(t), y)$ would have seen values with density $\nu_x(dy)$.

One anticipates $\dot{x}(t) = \int_Y h(x(t), y)\, \nu_x(dy) =: \bar{h}(x(t))$, $t \geq 0$. $\|$

Theorem (without proof):
Assume that for each x, the x-parameterized MC is ergodic & has ν_x as its unique invar. measure.

Assume $\bar{h}$ is Lipschitz. Assume, (A2) and (A3) with $\mathcal{F}_n = \sigma\{x_m, M_m, Y_m, m \leq n\}$ and (A4). Then $x_n \xrightarrow[n\to\infty]{} A$, some connected ict set for the dynamics
OK $\dot{x}(t) = \bar{h}(x(t))$.

6. Asynchronous Schemes:

$$X_n = \begin{pmatrix} X_n(1) \\ X_n(2) \\ \vdots \\ X_n(d) \end{pmatrix}$$

Often in practice, different components updated by different agents in different locations.

(There may also be delays, but we don't deal with this, for now).

$$x_{n+1}(i) = x_n(i) + a(\nu(i,n))\,\mathbb{1}\{i \in Y_n\}\left[h_i(x_n) + M_{n+1}(i)\right]$$

i^{th} component

(i) $Y_n \subset \{1,2,\ldots,d\}$, subset of components updating at global clock n.

(ii) $\nu(i,n) = \sum_{m=0}^{n} \mathbb{1}\{i \in Y_m\}$ = # of updates of the i^{th} agent.

(A5) Updates are done comparably often

$$\liminf_{n\to\infty} \frac{\nu(i,n)}{n} > 0 \quad \text{a.s.}$$

[So we don't have slow-fast timescales]

(A4) $\sup_n \|x_n\| < \infty$ a.s.

(A3) $\mathcal{F}_n = \sigma(x_m, M_m, Y_m, m \le n)$, $n \ge 0$.

$$E[M_{n+1}(i) \mid \mathcal{F}_n] = 0$$

$$E[\|M_{n+1}(i)\|^2 \mid \mathcal{F}_n] \le K\left(1 + \sup_{m\le n} \|x_m\|^2\right)$$

[To take into account delayed updates]

(A2) Write $\bar{a}(n) = \max_{i \in Y_n} a(\nu(i,n))$.

Under original (A2), $\sum_n \bar{a}(n) \ge \sum_n a(\nu(i,n))\,\mathbb{1}\{i \in Y_n\}$ for a fixed i

$$= \sum_m a(m) = \infty \quad \text{(Thanks to comparably often assumption).}$$

IIIly. $\sum \bar{a}(n)^2 \le d \sum a^2(m) < \infty$ $\rightarrow$ Easy.

$\left(\max_i \leq \mathrm{Sum}\right)$

Then interpolate as usual with $t(n)$'s coming from $\bar{a}(n)$.

$$x_{n+1}(i) = x_n(i) + \bar{a}(n)\left[\underbrace{\frac{a(\nu(i,n))}{\bar{a}(n)}\,\mathbb{1}\{i \in Y_n\}}_{q(i;n)\,\in\,[0,1]}\left[h_i(x_n) + M_{n+1}(i)\right]\right]$$

entire guy.

$\longrightarrow$ goes to limiting value.

$$\dot{x}(t) = \Lambda(t)\, h(x(t)), \quad t \geq 0.$$