UNIT 2 DEDUCTIVE INFERENCE RULES AND METHODS

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2.0 INTRODUCTION

In order to establish validity/invalidity of a conclusion C, in an argument, from a given set of facts/axioms $A_1, A_2, \ldots, A_n$; so far, we only know that *either* a truth table should be constructed for the formula P: $A_1 \wedge A_2 \wedge \ldots \wedge A_n \rightarrow C$, *or* this formula should be converted to CNF or DNF through substitutions of equivalent formulas and simplifications. There are other alternative methods also. However, the problem with these methods is that as the number *n* of axioms becomes larger, the formula becomes complex (imagine n = 50) and the number of involved variables, say *k*, also, generally, increases. With number of variables *k* involved in the argument, the size of Truth-table becomes 2^k. For large *k*, the number of rows, i.e. 2^k becomes, almost unmanageable. Therefore, we need to search for alternative methods which instead of processing the whole of the argument as a single formula, process each of the *individual* formulas $A_1, A_2, \ldots$, and C of the argument and their derivatives by applying some rules which preserve validity.

In **Section 3.2**, we introduce eight inference rules for drawing valid conclusions in PL. Next, in **Section 3.3**, we introduce four quantification rules, so that all the twelve inference rules are used to validate conclusions in FOPL. The methods of drawing valid conclusions, discussed so far, are cases of an *approach* of drawing valid conclusions, called **natural deduction approach** of making inferences in which the reasoning system initiates reasoning process from the axioms, uses inferencing rules and, if the conclusion can be validly drawn, then ultimately reaches the intended conclusion.

On the other hand, there is another *approach* called **Refutation approach** of drawing valid conclusions. According to this approach, negation of the intended conclusion is taken as an additional axiom. If the conclusion can be validly drawn from the axioms, then through application of inference rules, a contradiction is encountered, i.e., two formulas which are mutual negations, are encountered during the process of making inference.

Resolution method is a single rule refutation method. Resolution method and its applications for PL are discussed in **Section 3.4**. Resolution Method and its applications for FOPL are discussed in **Section 3.5**.

2.1 OBJECTIVES

After going through this unit, you should be able to:

- enumerate basic inference rules of PL and also be able to apply these in solving problems requiring PL reasoning;
- enumerate four basic quantification rules and be able to apply these rules alongwith basic rules of PL to solve problems involving FOPL reasoning;
- explain Resolution method for PL and apply it in solving problems requiring PL reasoning, and
- explain Resolution method for FOPL to solve problems involving FOPL reasoning.

2.2 BASIC INFERENCE RULES AND APPLICATIONS IN PL

In this section, we study a method which uses a number of *rules of inference* for drawing valid conclusions, and later we study *Resolution Method* for establishing validity of arguments.

We introduce eight rules of inference. Each of these rules has a specific name. In order to familiarize ourselves with

(i) what a rules of inference is ,
(ii) how a rule is represented, and
(iii) how a rule of inference helps us in solving problems.

We discuss in some detail, one of the rules known as Modus Pones.

Rule 1 Modus Ponens (M. P.)

Notations for M. P.: $\dfrac{P \to Q, P}{Q}$

(The comma is read as 'and'. The rule may also be written as

$\dfrac{P, P \to Q}{Q}$ *, i.e,. we may assume commutativity of comma)*

The rule states that if formulas P and $P \to Q$ *(of either propositional logic or predicate logic) are True then we can assume the Truth of Q.*
The assumption is based on the fact that through truth-table method or otherwise we can show that if P and $P \to Q$, each is assigned truth value T then Q must have truth value T.

Consider the Table

P	**Q**	$P \to Q$
T	*T*	*T*
T	F	F
F	T	T
F	F	T

From the above table, we can see that P and $P \rightarrow Q$ both are True only in the first row and in the first row Q, the formula which is inferred, is also True.

The same is the reason for allowing use of other rules of inference in deducing new facts.

Rule 2 Module Tollens (M. T.)

$$\frac{P \rightarrow Q,\ \sim Q}{\sim P}$$

The rule states ***if*** $P \rightarrow Q$ *is True, but Q , the consequent of* $P \rightarrow Q$ *is False then the antecedent P of* $P \rightarrow Q$ *is also False.*

The validity of the rule may again be established through truth-table as follows:

P	**Q**	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	*F*	*T*

In the above table $P \rightarrow Q$ *is T and Q is False simultaneously* only in the last row and in this row P, the formula which is inferred, is False.

Note: The validity of the rest of the rules will not be established. However, it is desirable that the students verify the validity of the other inference rules also through Truth-Table or otherwise.

Rule 3 Hypothetical Syllogism (H.P.)

$$\frac{P \rightarrow Q,\ Q \rightarrow R}{P \rightarrow R}$$

The rule states that if we assume that both the formulas $P \rightarrow Q$ *and* $Q \rightarrow R$ *are True then we may assume* $P \rightarrow R$ *is also True.*

Rule 4 Simplification (Simp.)

(i) $\dfrac{P \wedge Q}{P}$ *and* $(ii)\dfrac{P \wedge Q}{Q}$

The rule says that if $P \wedge Q$ *is True then P can be assumed to be True (and similarly Q may be assumed to be True.)*

Some of us may be surprised at the mention of the rule, thinking that if $P \wedge Q$ *is True then P must be True. The symbol* $\wedge$ *is generally read as 'and'. But the significance of the rule is that '* $\wedge$ *' is merely a symbol and its meaning in the sense of 'and' comes only through this rule of inference.*

Rule 5 Conjunction (Conj.)

$$\frac{P,\ Q}{P \wedge Q}$$

The rule states if formulas P and Q are simultaneously True then the formula $P \wedge Q$ can be assumed to be True.

Rule 6 Disjunctive Syllogism (D.S.)

(i) $\dfrac{P \vee Q,\ \sim P}{Q}$ and (ii) $\dfrac{P \vee Q, \sim Q}{P}$

The two rules above state that if it is given that (a) $P \vee Q$ is true and (b) one of P or Q is False, then other must be True

Rule 7 Addition (Add.)

(i) $\dfrac{P}{P \vee Q}$ and (ii) $\dfrac{Q}{P \vee Q}$

The rules state that if one of P and Q is assumed to be True, then we can assume $P \vee Q$ to be True.

Rule 8 Dilemma (Dil.)

$$\frac{P \to Q,\ R \to S, P \vee R}{Q \vee S}$$

The rule states that if both the formulas $P \to Q$ and $R \to S$ are assumed to be True and if $P \vee R$, i.e. disjunction of the antecedents is assumed to be 'True', then assume Truth of $Q \vee S$, which is disjunction of consequents.

We demonstrate how the above-mentioned rules of inference can be used in solving problems.

Example: Symbolize and construct a proof for the following valid argument using rules of inference:

(i) If you smoke or drink too much then you do not sleep well, and if you do not sleep well or do not eat well then you feel rotten, (ii) If you feel rotten, you do not exercise well and do not study enough, (iii) You do smoke too much, *therefore,* (iv) You do not study enough.

Solution: *Let us symbolize the statements in the argument as follows:*

S: You smoke too much

D: You drink too much

W: You sleep well

E: You eat well

R: You feel rotten

X : You exercise well

T: You study enough

Then the three given statements marked as (i), (ii) and (iii) are symbolized as follows:
(i) $((S \vee D) \rightarrow \sim W) \wedge ((\sim W \vee \sim E) \rightarrow R)$

(ii) $R \rightarrow (\sim X \wedge \sim T)$

(iii) S

(iv) ~T .(*To show*)

Through simplification of (i), i.e., by using $\frac{P \wedge Q}{P}$, *we get*

(v) $S \vee D \rightarrow \sim W$

Using Add on (iii) $\left(\text{i.e. by using } \frac{S}{S \vee D}\text{ , we get}\right)$

(vi) $S \vee D$

and using syllogism on (v) and (vi) we get

(vii) ~ W

Again through simplification of (i), we get

(viii) $(\sim W \vee \sim E) \rightarrow R$

and by addition of (vii), we get

(ix) $\sim W \vee \sim E$

From (viii) and (ix) using M.P., we get

(x) R

Again using M.P. with (ii) & (x), we get

(xi) $\sim X \wedge \sim T$

Again using simplification of (xi) we get the required formula

(xi) ~ T

Example: Symbolize and construct a proof for the following valid argument: (i) If the Bible is literally true then the Earth was created in six days, (ii) If the Earth was created in six days then carbon dating techniques are useless and scientists are frauds, (iii) Scientists are not frauds, (iv) The Bible is literally true, *therefore,* (iv) God does not exist.

Solution: Let us symbolize as follows:

B: Bible is literally true
E: The Earth was created in six days
C: Carbon dating techniques are useless
S: Scientists are frauds
G: God exists

Therefore the statements in the given arguments are symbolically represented as :

(i) $B \to E$
(ii) $E \to C \wedge S$
(iii) ~ S
(iv) B
(v) ~ G (*to show*)

Using M.P. on (i) and (iv), we get
(vi) E
Using M.P. on (ii) & (vi) we get
(vii) $C \wedge S$
Using Simp on (vii), we get
(viii) S

Using Addition on (viii), we get
(ix) $S \vee \sim G$
Using (D.S.) on (iii) & (ix) we get
(x) ~ G
The last statement is what is to be proved.

Remarks: In the above deduction, (iii) and (viii) are contradicting each other. In general, if in the process of derivation, we encounter two statement (like S and ~S) which contradict each other, then we can **deduce any statement** even if the statement can never be True in any sense. Thus, if both S and ~ S have already occurred in the process of derivation, then we can assume the truth of any statement. For example, we can assume the truth of the statement: '*Moon is made of green cheese'*

The **technique, to prove any non-sense** statement say NON-SENSE in a situation where already two mutually contradicting statements say S and ~ S have already been encountered, is the same as is used in deriving (ix) by applying Addition Rule to (viii).

Thus, once we encounter S, where ~S has already occurred, use Addition rule to get S $\vee$ NON-SENSE from S. Then use D.S. on $S \vee NON\text{-}SENSE$ and *~S*, we get NON-SENSE.

Ex.1 Given the following three statements:

(i) *Matter always existed*
(ii) *If there is God, then God created the universe.*
(iii) *If God created the universe, then matter did not always exist.*

Show the truth of the statement: (iv) *There is no God.*

Ex.2 Using propositional logic, show that, if the following statements are assumed to be true:

(i) *There is a moral law.*
(ii) *If there is a moral law, then someone gave it.*
(iii) *If someone gave the moral law, then there is God.*

then the following statement is also true:

(iv) *There is GOD*

2.3 BASIC INFERENCING RULES AND APPLICATIONS IN FOPL

In the previous unit, we discussed eight inferencing rules of Propositional Logic (PL) and further discussed applications of these rules in exhibiting validity/invalidity of arguments in **PL.** In this section, the earlier eight rules are extended to include four more rules involving quantifiers for inferencing. Each of the new rules, is called a **Quantifier Rule**. The extended set of 12 rules is then used for validating arguments in First Order Predicate Logic (**FOPL)**.

Before introducing and discussing the Quantifier rules, we briefly discuss why, at all, these rules are required. For this purpose, let us recall the argument discussed earlier, which Propositional Logic could not handle:

(i) Every man is mortal.
(ii) Raman is a man.
(iii) Raman is mortal.

The equivalent symbolic form of the argument is given by:

(i') $(\forall x)$ (Man (x) $\rightarrow$ Mortal (x)
(ii') Man (Raman)
(iii') Mortal (Raman)

If, instead of (i') we were given
(iv) Man (Raman) $\rightarrow$ Mortal (Raman) ,
(which is a formula of Propositional Logic also)
then using Modus Ponens on (ii') & (iv) in *Propositional Logic*, we would have obtained (iii') *Mortal (Raman).*
However, from (i') & (ii') we cannot derive in Propositional Logic (iii'). This suggests that there should be mechanisms for dropping and introducing quantifier appropriately, i.e., in such a manner that *validity* of arguments is not violated. Without discussing the validity-preserving characteristics, we introduce the four Quantifier rules.

(i) Universal Instantiation Rule (U.I.):

$$\frac{(\forall x)\, p(x)}{p(a)}$$

Where is an a arbitrary constant.

*The rule states if **(∀x) p(x)** is True, then we can assume **P(a)** as True for any constant **a** (where a constant **a** is like Raman). It can be easily seen that the rule associates a formula **P(a) of Propositional Logic** to a formula **(∀x) p(x) of FOPL**. The significance of the rule lies in the fact that once we obtain a formula like **P(a)**, then the reasoning process of Propositional Logic may be used. The rule may be used , whenever, its application seems to be appropriate.*

(ii) Universal Generalisation Rule (U.G.)

$$\frac{P(a),\ for\ all\ a}{(\forall x)\, p(x)}$$

The rule says that if it is known that for all constants a, the statement P(a) is True, then we can, instead, use the formula $(\forall x)\, p(x)$.

The rule associates with a set of formulas ***P(a) for all*** **a of Propositional Logic**, a formula $(\forall x)\, p(x)$ *of FOPL.*

Before using the rule, we must ensure that P(a) is True for all a,
Otherwise it may lead to wrong conclusions.

(iii) Existential Instantiation Rule (E. I.)

$$\frac{(\exists x)\, P(x)}{P(a)} \qquad (E.I.)$$

The rule says if the Truth of $(\exists x)\ P(x)$ *is known then we can assume the Truth of P(a) for* ***some fixed*** *a . The rule, again, associates a formula P(a) of Propositional Logic to a formula* $(\forall x)\, p(x)$ *of FOPL.*

An inappropriate application of this rule may lead to *wrong* conclusions. The source of possible errors lies in the fact that the choice 'a' in the rule *is not arbitrary* and can not be known at the time of deducing P(a) from $(\exists x)\ P(x)$.

If during the process of deduction some other $(\exists y)\ Q(y)$ or $(\exists x)\,(\,R(x))$ or even another *(∃x)P(x)* is encountered, then each time a new constant say *b, c* etc. should be chosen to infer Q (b) from $(\exists y)\ Q(y)$ *or R(c)* from $(\exists x)\,(\,R(x))$ *or P(d)* from $(\exists x)\ P(x)$.

(iv) Existential Generalization Rule (E.G)

$$\frac{P(a)}{(\exists x) P(x)} \qquad \text{(E.G)}$$

The rule states that if P(a), a formula of Propositional Logic is True, then the Truth of $(\exists x)\ P(x)$, a formula of FOPL , may be assumed to be True.
The Universal Generalisation (U.G) and Existential Instantiation rules should be applied with utmost care, however, other two rules may be applied, whenever, it appears to be appropriate.

Next, The purpose of the two rules, viz.,
(i) Universal Instantiation Rule (U. I.)
(iii) Existantial Rule (E. I.)
is to associate formulas of Propositional Logic (PL) to formulas of FOPL in a manner, the validity of arguments due to these associations, is not disturbed. Once, we get formulas of PL, then any of the eight rules of inference of PL may be used to validate conclusions and solve problems requiring logical reasoning for their solutions.

The purpose of the other Quantification rules viz. for generalisation, i.e.,

(ii) $\dfrac{P(a),\ \text{for all a}}{(\forall x)\ P(x)}$

(iv) $\dfrac{P(a)}{(\exists x)\, P(x)}$

is that the conclusion to be drawn in FOPL is not generally a formula of PL but a formula of FOPL. However, while making inference, we may be first associating formulas of PL with formulas of FOPL and then use inference rules of PL to conclude formulas in PL. But the conclusion to be made in the problem may correspond to a formula of FOPL. These two generalisation rules help us in associating formulas of FOPL with formulas of PL.

Example: Tell, supported with reasons, which one of the following is a correct inference and which one is not a correct inference.

(i) *To conclude* $F(a) \wedge G(a) \rightarrow H(a) \wedge I(a)$
from $(\forall x)\big(F(x) \wedge G(x)\big) \rightarrow H(x) \wedge I(x)$
using Universal Instantiation (U.I.)

The above inference or conclusion is *incorrect* in view of the fact that the scope of universal quantification is only the formula: $F(x) \wedge G(x)$ and not the whole of the formula.

The occurrences of *x* in $H(x) \wedge I(x)$ are free occurrences. Thus, one of the correct inferences would have been:

$$F(a) \wedge G(a) \rightarrow H(x) \wedge I(x)$$

(ii) *To conclude* $F(a) \wedge G(a) \rightarrow H(a) \wedge I(a)$ *from*
$(\forall x)\ (F(x) \wedge G(x) \rightarrow H(x) \vee I(x))$ *using U.I.*
The conclusion is correct in view of the argument given in (i) above.

(iii) *To conclude* ~ *F(a) for an arbitrary a, from* ~ $(\forall x)$ *F(x) using U.I.*

The conclusion is incorrect, because actually
$\sim (\forall x)\ F(x) = (\exists x) \sim F(x)$

Thus, the inference is not a case of U.I., but of Existential Instantiation (E.I.)

Further, as per restrictions, we can not say for which a, ~ *F(x)* is True. Of course, ~ *F(x)* is true for some constant, but not necessarily for a pre-assigned constant *a*.

(iv) *to conclude* $\big((F(b) \wedge G(b) \rightarrow H(c)\big)$
from $(\exists x)\big((F(b) \wedge G(x)\big) \rightarrow H(c)$

Using E.I. is *not* correct

The reason being that the constant to be substituted for *x* cannot be assumed to be the same constant *b*, being given in advance, as an argument of *F*. However,

to conclude $\big((F(b) \wedge G(a) \rightarrow H(c)\big)$
from $(\exists x)\big((F(b) \wedge G(x)\big) \rightarrow H(c)\big)$ is correct.

Ex. 3: Tell for each of the following along with appropriate reasoning, whether it is a case of correct/incorrect reasoning.

(i) To conclude

$F(a) \wedge G(a)$ by applying E.I. to
$(\exists x)\ F(x) \wedge \exists\,(x)\ G\,(x)$

(ii) To conclude $F(a) \vee \left(G(a) \wedge H(a)\right)$ from
$(\exists x)\, F(x) \vee \left(G(x) \wedge H(x)\right)$

(iii) To conclude $(\exists x)\ (\sim F(x) \rightarrow \sim G\,(x))$ from
$\sim F\,(a) \rightarrow \sim G(a)$

(iv) To conclude $\sim\left((\exists x)(F(x) \wedge G(x)\right)$ from $\sim\left(F(a) \wedge G(a)\right)$

Step for using Predicate Calculus as a Language for Representing Knowledge & for Reasoning:

Step 1: Conceptualisation: First of all, all the relevant entities and the relations that exist between these entities are explicitly enumerated. Some of the implicit facts like, '*a person dead once is dead for ever*' have to be explicated.

Step 2: *Nomenclature* & Translation: Giving appropriate names to objects and relations. And then translating the given sentences given in English to formulas in FOPL. Appropriate names are essential in order to guide a reasoning system based on FOPL. It is well-established that no reasoning system is complete. In other words, a reasoning system may need help in arriving at desired conclusion.

Step 3: Finding appropriate sequence of reasoning steps, involving selection of appropriate rule and appropriate FOPL formulas to which the selected rule is to be applied, to reach the conclusion.

Applications of the 12 inferrencing rules (8 of Propositional Logic and 4 involving Quantifiers.)

Example: Symbolize the following and then construct a proof for the argument:

(i) Anyone who repairs his own car is highly skilled and saves a lot of money on repairs
(ii) Some people who repair their own cars have menial jobs. Therefore,
(iii) Some people with menial jobs are highly skilled.

Solution: Let us use the notation:

P(x)	:	x is a person
S(x)	:	x saves money on repairs
M(x)	:	x has a menial job
R(x)	:	x repairs his own car
H(x)	:	x is highly skilled.

Therefore, (i), (ii) and (iii) can be symbolized as:

(i) $(\forall x)\ (R(x) \rightarrow (H(x) \wedge S(x)))$
(ii) $\exists(x)\ (R(x) \wedge M(x))$
(iii) $(\exists x)\ (M(x) \wedge H(x))$ *(to be concluded)*

From (ii) using Existential Instantiation (E.I), we get, for some fixed a

(iv) $R(a) \wedge M(a)$

Then by simplification rule of Propositional Logic, we get

(v) $R(a)$

From (i), using Universal Instantiation (U.I.), we get

(vi) $R(a) \rightarrow H(a) \wedge S(a)$

Using modus ponens w.r.t. (v) and (vi) we get

(vii) $H(a) \wedge S(a)$

By specialisation of (vii) we get

(viii) $H(a)$

By specialisation of (iv) we get

(ix) $M(a)$

By conjunctions of (viii) & (ix) we get

$M(a) \wedge H(a)$

By Existential Generalisation, we get

$(\exists x)\ (M(x) \wedge H(x))$

Hence, (iii) is concluded.

Example:

(i) Some juveniles who commit minor offences are thrown into prison, and any juvenile thrown into prison is exposed to all sorts of hardened criminals.
(ii) A juvenile who is exposed to all sorts of hardened criminals will become bitter and learn more techniques for committing crimes.
(iii) Any individual who learns more techniques for committing crimes is a menace to society, if he is bitter.
(iv) Therefore, some juveniles who commit minor offences will be menaces to the society.

Example: Let us symbolize the statement in the given argument as follows:

(i)	J(x)	:	x is juvenile.
(ii)	C(x)	:	x commits minor offences.
(iii)	P(x)	:	x is thrown into prison.
(iv)	E(x)	:	x is exposed to hardened criminals.
(v)	B(x)	:	x becomes bitter.
(vi)	T(x)	:	x learns more techniques for committing crimes.
(vii)	M(x)	:	x is a menace to society.

The statements of the argument may be translated as:

(i) $(\exists x)\ (J(x) \wedge C(x) \wedge P(x)) \wedge ((\forall y)\ (J(y) \rightarrow E(y))$

(ii) $(\forall x)\ (J(x) \wedge E(x) \rightarrow B(x) \wedge T(x))$

(iii) $(\forall x)\ (T(x) \wedge B(x) \rightarrow M(x))$

Therefore,

(iv) $(\exists x)\ (J(x) \wedge C(x) \wedge M(x))$

By simplification (i) becomes

(v) $(\exists x)\ (J(x) \wedge C(x) \wedge P(x))$ and

(vi) $(\forall y)\ (J(y) \rightarrow E(y))$

From (v) through Existential Instantiation, for some fixed *b*, we get

(vii) $J(b) \wedge C(b) \wedge P(b)$

Through simplification (vii) becomes

(viii) $J(b)$

(ix) $C(b)$ and
(x) $P(b)$

Using Universal Instantiation, on (vi), we get
(xi) $J(b) \rightarrow E(b)$
Using Modus Ponens in (vii) and (xi) we get
(xii) $E(b)$
Using conjunction for (viii) & (xii) we get
(xiii) $J(b) \wedge E(b)$
Using Universal Instantiation on (ii) we get
(xiv) $J(b) \wedge E(b) \rightarrow B(b) \wedge T(b)$
Using Modus Ponens for (xiii) & (xiv), we get
(xv) $T(b) \wedge B(b)$
Using Universal Instantiation for (iii) we get
(xvi) $T(b) \wedge B(b) \rightarrow M(b)$
Using Modus Ponens with (xv) and (xvi) we get
(xvii) $M(b)$
Using conjunction for (viii), (ix) and (xvii) we get
(xviii) $J(b) \wedge C(b) \wedge M(b)$
From (xviii), through Existential Generalization we get the required (iv), i.e.
$(\exists x)\ (J(x) \wedge C(x) \wedge M(x))$

Remark: It may be noted the occurrence of quantifiers is not, in general, commutative i.e.,
$(Q_1x)\ (Q_2x) \neq (Q_2x)\ (Q_1x)$
For example
$(\forall x)\ (\exists y)\ F(x,y) \neq (\exists y)\ (\forall x)\ F(x,y)$ (A)

The occurrence of $(\exists y)$ on L.H.S depends on *x* i.e., occurrence of *y* on L.H.S is a function of *x*. However, the occurrence of $(\exists y)$ on R.H.S is independent of *x*, hence, occurrence of *y* on R.H.S is not a function of *x*.

For example, if we take F(x,y) to mean:
y and x are integers such that y>x,

then, *L.H.S of (A)* above states: *For each x there is a y such that y>x.*
The statement is true in the domain of real numbers.

On the other hand, *R.H.S of (A)* above states that: There is an integer *y* which is greater than *x*, for all *x*.

This statement is not true in the domain of real numbers.

Ex. 4 We are given the statements:

(i) *No feeling of pain is publically observable*
(ii) *All chemical processes are publically observable*

We are to prove that

(iii) *No feeling of pain is a chemical process.*

2.4 RESOLUTION METHOD IN PL

Basically, there are two different **approaches** for proving a theorem or for making a valid deduction from a given set of axioms:

i) natural deduction
ii) refutation method

In the **natural deduction** approach, one starts with a the set of axioms, uses some rules of inference and arrives at a conclusion. This approach closely resembles of the intuitive reasoning of human beings.

On the other hand, in a **refutation method**, one *starts with the negation of the conclusion to be drawn* and derives a contradiction or FALSE. Because of having assumed the conclusion as false, we derive a contradiction; **therefore, the assumption that the conclusion is wrong, itself is wrong**. Hence, the argument of resolution method leads to the validity of the conclusion.

So far, we have discussed methods, of solving problems requiring reasoning of propositional logic, that were based on

i) Truth-table construction
ii) Use of inference rules,

and follow, directly or indirectly, **natural deduction approach.**

In this section, we discuss another method, viz., **Resolution Method** suggested by **Robinson** in 1965 which is based on **refutation approach**. The method is important in view of the fact that the Robinson's method has been a basis for some automated theorem provers. Even, the logic programming language PROLOG (*subject matter of Unit 2, Block 3*) is based on Resolution Method. The resolution method, as mentioned above, is a **refutation method**.

In this section, we discuss how the resolution method is applied in solving problems using only Propositional Logic (PL). The general resolution method for FOPL is discussed in the next section.

The resolution method in PL is applied only after converting the given statements or wffs into clausal forms. A **clasual form** of a wff is obtained by *first* converting the wff into its equivalent Conjuctive Normal Form (CNF). We already know that a *clause* is a formula (only) of the form:
$A_1 \vee A_2 \vee \ldots\ldots\vee A_n$,
where A_i is either an atomic formula or negation of an atomic formula.

The resolution method is a generalization of the Modus Ponens, i.e., of

$\dfrac{P, P \to Q}{Q}$ when written in the equivalent form $\dfrac{P, \sim P \vee Q}{Q}$
(*replacing* $P \to Q$ *by* $\sim P \vee Q$*).*

This simple special case, of **general resolution principle** to be discussed soon, states that if the two formulas P and $\sim P \vee Q$ are given to be True, then we can assume Q to be True.

The validity of (general) resolution method can be established by constructing truth-table.

In order to discuss the resolution method, first we discuss some of its applications.

Example: Let $C_1 : Q \vee R$ and $C_2: \sim Q \vee S$ be two given clauses, so that, one of the literals i.e., Q occurs in one of the clauses (in this case C_1) and its negation $(\sim Q)$ occurs in the other clause C_2. Then application of resolution method in this case tells us to take disjunction of the remaining parts of the given clause C_1 and C_2, i.e., to take $C_3 : R \vee S$ as **deduction** from C_1 and C_2. Then C_3 is called a **resolvent** of C_1 and C_2. The two literals Q and $(\sim Q)$ which occur in two different clauses are called **complementary literals**.

In order to illustrate resolution method, we consider another example.

Example: Let us be given the clauses $C_1: \sim S \vee \sim Q \vee R$ and $C_2; \sim P \vee Q$.

In this case, complementary pair of literals viz. Q and $\sim Q$ occur in the two clause C_1 and C_2.
Hence, the resolution method states:

Conclude $C_3: \sim S \vee R \vee (\sim P)$

Example: Let us be given the clauses $C_1: \sim Q \vee R$ and $C_2: \sim Q \vee S$
Then, in this case, the clauses do not have any complementary pair of literals and hence, resolution method cannot be applied.

Example: Consider a set of three clauses
$C_1 : R$
$C_2: \sim R \vee S$
$C_3: \sim S$

Then, from C_1 and C_2 we conclude, through resolution:
C_4: S
From C_3 and C_4, we conclude,
C_5 : FALSE
However, a resolvent FALSE can be deduced only from an ***unstatisfiable set of clauses.*** Hence, the set of clauses C_1, C_2 and C_3 is an unsatisfiable set of clauses.

Example: Consider the set of clauses
$C_1: R \vee S$
$C_2: \sim R \vee S$
$C_3: R \vee \sim S$
$C_4: \sim R \vee \sim S$
Then, from clauses C_1 and C_2 we get the resolvent
$C_5 : S \vee S = S$
From C_3 and C_4 we get the resolvent
C_6 : ~ S
From C_5 and C_6 we get the resolvent
C_7 : FALSE

Thus, again the set of clauses C_1, C_2, C_3 and C_4 is unsatisfiable.

Note: We could have obtained the resolvent FALSE from only two clauses, viz., C_2 *and* C_3. *Thus, out of the given four clauses, even set of only two clauses viz,* C_2 *and* C_3 *is unsatisfiable. Also, a superset of any unsatisfiable set is unsatisfiable.*

Example: Show that the set of clauses:
$C_1: R \vee S$
$C_2: \sim S \vee W$
$C_3: \sim R \vee S$
$C_4: \sim W$ is unsatisfiable.

From clauses C_1 and C_3 we get the resolvent
C_7 : S
From the clauses C_7 and C_2 we get the resolvent
C_8 : W
From the clauses C_8 and C_4 we get
C_9 : FALSE
Hence, the given set of clauses is unsatisfiable.

Problem Solving using resolution method. We have mentioned earlier, the resolution method is a refutation method. Therefore, proof technique in solving problems will be as follows:

After symbolizing the problem under consideration, add the negation of the wff which represents conclusion, as an additional premise. From this enhanced set of premises/axioms, derive FALSE or contradiction. If we are able to conclude FALSE, then the conclusion, that was required to be drawn, is valid and problem is solved. ***However****, through all efforts, if we are* ***not able to*** *derive FALSE, then we* ***cannot say*** *whether the conclusion is valid or invalid. Hence, the* ***problem*** *with given axioms and the conclusion* ***is not solvable.***

Let us now apply Resolution Method for the problems considered earlier.

Example: Suppose the stock prices go down if the interest rate goes up. Suppose also that the most people are unhappy when stock prices go down. Assume that the interest rate goes up. Show that we can conclude that most people are unhappy.

To show the above conclusion, let us denote the statements as follows:

A : Interest rate goes up,
S : Stock prices go down
U : Most people are unhappy

The problem has the following four statements

1) If the interest rate goes up, stock prices go down.
2) If stock prices go down, most people are unhappy.
3) The interest rate goes up.
4) Most people are unhappy. (to conclude)

These statements are first symbolized as wffs of PL as follows:

(1′) $A \rightarrow S$
(2′) $S \rightarrow U$
(3′) A
(4′) U. *(to conclude)*

Converting to clausal form, we get
(i) $\sim A \vee S$
(ii) $\sim S \vee U$
(iii) A
(iv) U *(to be concluded)*

As per resolution method, assume (iv) as false, i.e., *assume ~ U as initially given statement, i.e., an axiom.*

Thus, the set of axioms in clasual form is:
(i) ~ A $\vee$ S
(ii) ~ S $\vee$ U
(iii) A
(iv) ~ U

Then from (i) and (iii), through resolution, we get the clause
(v) S.
From (ii) and (iv), through resolution, we get the clause
(vi) ~ S
From (vi) and (v), through resolution we get,
(viii) FALSE

Hence, the conclusion, i.e.,
(iv) *U : Most people are unhappy*
is valid.

We might have observed from the above solution using resolution method, that clausal conversion is a major time-consuming step after translation to wffs. Generally, once the clausal form is obtained, proof, at least, by a human being can be easily visualised.

Ex. 5: Given that if the Parliament refuses to enact new laws, then the strike will not be over unless it lasts more than one year and the president of the firm resigns, will the strike not be over if the Parliament refuses to act and the strike just starts?

2.5 RESOLUTION METHOD IN FOPL

In the beginning of the previous section, we mentioned that resolution method for FOPL requires discussion of a number of complex new concepts. Also, in Block 2, we discussed (Skolem) Standard Form and also discussed how to obtain Standard Form for a given formula of FOPL. In this section, we introduce two new, and again complex, concepts, viz., *substitution and unification.*

The complexity of the resolution method for FOPL mainly results from the fact that a clause in FOPL is generally of the form : P(x) $\vee$ Q (f(x), x, y) $\vee$….., in which the variables x, y, z, may assume any one of the values of their domain.

Thus, the atomic formula ($\forall$x) P(x), *which after dropping of universal quantifier, is written as just P(x)* stands for $P(a_1) \wedge P(a_2) \ldots \wedge P(a_n)$ where the set $\{a_1\ a_2\ldots, a_n\}$ is assumed here to be domain (x).

Similarly, ($\exists$x) P(x) stands for $(P(a_1) \vee P(a_2) \vee \ldots. \vee P(a_n))$
However, in order to resolve two clauses – one containing say P(x) and the other containing ~ P(y) where *x* and *y* are universal quantifiers, possibly having some restrictions, we have to know which values of x and y satisfy both the clauses. For this purpose we need the concepts of **substitution** and **unification** as defined and discussed in the rest of the section.

Instead of giving formal definitions of substitution, unification, unifier, most general unifier and resolvent, resolution of clauses in FOPL, we illustrate the concepts through examples and minimal definitions, if required

Example: Let us consider our old problem:

To conclude
(i) Raman is mortal

From the following two statements:

(ii) Every man is mortal and
(iii) Raman is a man

Using the notations

MAN (x) : x is a man
MORTAL (x) : x is mortal,

the problem can be formulated in symbolic logic as: Conclude

MORTAL (Raman)

from

(ii) $((\forall x)$ (MAN(x) $\rightarrow$ MORTAL (x))
(iii) MAN (Raman).

As resolution is a refutation method, assume
(i) ~ MORTAL (Raman)

After Skelomization and dropping $(\forall x)$, (ii) in standard form becomes

(i) ~ MAN (x) $\vee$ MORTAL (x)
(ii) MAN (Raman)

In the above *x* varies over the set of human beings including Raman. Hence, one special instance of (iv) becomes

(vi) ~ MAN (Raman) $\vee$ MORTAL (Raman)

At the stage, we may observe that

(a) *MAN(Raman)* and *MORTAL(Raman)* do not contain any variables, and, hence, their truth or falsity can be determined directly. Hence, each of like a formula of PL. In term of formula which does not contain any variable is called **ground term** or **ground formula**.

(b) Treating *MAN (Raman)* as formula of PL and using resolution method on (v) and (vi), we conclude

(vii) MORTAL (Raman),

Resolving (i) and (vii), we get **False**. Hence, the solution.

Unification: In the process of solution of the problem discussed above, we tried to make the two expression *MAN(x)* and *MAN(Raman)* identical. Attempt to make identical two or more expressions is called *unification.*

In order to unify *MAN (x)* and *MAN (Raman)* identical, we found that because one of the possible values of *x* is *Raman* also. And, hence, we replaced *x* by one of its possible values : *Raman*.

This replacement of a variable like *x*, by a term (*which may be another variable also*) which is one of the possible values of *x*, is called **substitution**. The substitution, in this case is denoted formally as {Raman/x}

Substitution, in general, **notationally** is of the form $\{t_1 / x_1, t_2 / x_2 \ldots t_m / x_m\}$ where $x_1, x_2 \ldots, x_m$ are variables and $t_2, t_2 \ldots t_m$ are terms and t_i replaces the variable x_i in some expression.

Example: (i) Assume Lord Krishna is loved by everyone who loves someone (ii) Also assume that no one loves nobody. Deduce Lord Krishna is loved by everyone.

Solution: Let us use the symbols

Love (x, y): *x loves y (or y is loved by x)*

LK : *Lord Krishna*

Then the given problem is formalized as :

(i) $(\forall x)\ ((\exists y)\ \text{Love}\ (x, y) \rightarrow \text{Love}\ (x, LK))$

(ii) $\sim (\exists x)\ ((\forall y) \sim \text{Love}\ (x, y))$

To show : $(\forall x)\ (\text{Love}\ (x, LK))$

As resolution is a refutation method, assume negation of the last statement as an axiom.

(iii) $\sim (\forall x)\ \text{Love}\ (x, LK)$

The formula in (i) above is reduced in standard form as follows:

$(\forall x)\ (\sim (\exists y)\ \text{Love}\ (x, y) \vee \text{Love}\ (x, LK))$

$= (\forall x)\ ((\forall y) \sim \text{Love}\ (x, y) \vee \text{Love}\ (x, LK))$

$= (\forall x)\ (\forall y)\ (\sim \text{Love}\ (x, y) \vee \text{Love L}\ (x, LK))$

(∵ (∀y) does not occurs in Love (x, LK))

After dropping universal quantifications, we get

(iv) $\sim \text{Love}\ (x, y) \vee \text{Love}\ (x, LK)$

Formula (ii) can be reduced to standard form as follows:

(ii) $= (\forall x)\ (\exists y)\ \text{Love}\ (x, y)$

y is replaced through skolomization by f(x)

so that we get

$(\forall x)\ \text{Love}\ (x, f(x))$

Dropping the universal quantification

(v) Love (x, f(x))

The formula in (iii) can be brought in standard form as follows:

(iii) = (∃x) (~ Love (x, LK))

As existential quantifier *x* is not preceded by any universal quantification, therefore, *x* may be substituted by a constant *a* , i.e., we use the substitution {a/x} in (iii) to get the standard form:

(vi) ~ Love (a, LK).

Thus, to solve the problem, we have the following standard form formulas for resolution:

(iv) ~ Love (x, y) ∨ Love (x, LK)
(v) Love (x, f(x))
(vi) ~ Love (a, LK).

Two possibilities of resolution exist for two pairs of formulas viz.

one possibility: resolving (v) and (vi).
second possibility : resolving (iv) and (vi).

The possibilities exist because for each possibility pair, the predicate *Love* occurs in complemented form in the respective pair.

Next we attempt to resolve (v) and (vi)

For this purpose we attempt to make the two formulas *Love(x, f(x))* and *Love (a, LK)* identical, through unification involving substitutions. We start from the left, matching the two formulas, term by term. First place where matching may fail is when 'x' occurs in one formula and 'a' occurs in the other formula. **As, one of these happens to be a variable**, hence, the substitution {a/x} can be used to unify the portions so far.

Next, possible disagreement through term-by-term matching is obtained when we get the two disagreeing terms from two formulas as f(x) and LK. **As none of f(x) and LK is a variable** *(note f(x) involves a variable but is itself not a variable),* hence, no unification and, hence, no resolution of (v) and (vi) is possible.

Next, we attempt unification of **(vi) Love (a, LK)** *with* **Love (x, LK)** of (iv).

Then first term-by-term possible disagreement occurs when the corresponding terms are 'a' and 'x' respectively. As one of these is a variable, hence, the substitution{a/x} unifies the parts of the formulas so far. Next, the two occurrences of LK, one each in the two formulas, match. Hence, the whole of each of the two formulas can be unified through the substitution {a/x}. Though the unification has been *attempted* in corresponding smaller parts, substitution has to be carried **in the whole of the formula**, in this case in whole of (iv). Thus, after substitution, (iv) becomes
(viii) ~ Love (a, y) ∨ Love (a, L K)
resolving (viii) with (vi) we get

(ix) ~ Love (a, y)

In order to resolve (v) and (ix), we attempt to unify **Love (*x*, f(x))** of (v) with **Love (a, y)** of (ix). The term-by-term matching leads to possible disagreement of *a* of (ix) with *x* of (v). As, one of these is a variable, hence, the substitution {a/x} will unify the portions considered so far. Next, possible disagreement may occur with f (x) of (v) and y of (ix). As one of these are a variable viz. y, therefore, we can unify the two terms through the substitution { f(x)/y}. Thus, the complete substitution *{a/x, f (x)/y}* is required to match the formulas.
Making the substitutions, we get
(v) becomes Love (a, f(x))
and (ix) becomes ~ Love (a, f (x))
Resolving these formulas we get **False.** Hence, the proof.

Ex. 6: Unify, if possible, the following three formulas:

(i) Q (u, f (y, z)),
(ii) Q (u, a)
(iii) Q (u, g (h (k (u))))

Ex. 7: Determine whether the following formulas are unifiable or not:

(i) Q (f (a), g(x))
(ii) Q (x, y)

Example: Find resolvents, if possible for the following pairs of clauses:

(i) ~ Q (x, z, x) ∨ Q (w, z, w) and
(ii) Q (w, h (v, v), w)

Solution: As two literals with predicate Q occur and are mutually negated in (i) and (ii),therefore, there is possibility of resolution of ~ *Q (x, z, x)* from (i) with *Q (w, h (v, v), w)* of (ii). We attempt to unify Q (x, z, x) and Q (w, h (v, v), w), if possible, by finding an appropriate substitution. First terms x and w of the two are variables, hence, unifiable with either of the substitutions {x/w} or {w/x}. Let us take {w/x}. Next pair of terms from the two formulas, viz, z and h(v, v) are also unifiable, because, one of the terms is a variable, and the required substitution for unification is { h (v, v)/z}.

Next pair of terms at corresponding positions is again {w, x} for which, we have determined the substitution {w/x}. Thus, the substitution {w/x, h(v, v)/z} unfies the two formulas. Using the substitutions, (i) and (ii) become resp. as

(iii) ~ Q (w, h (v, v), w) ∨ Q (w, h (v, v), w)
(iv) Q (w, h (v, v), w)

Resolving, we get
Q (w, h (v, v), w),
which is the required resolvent.

2.6 SUMMARY

In this unit, eight basic rules of inference for PL and four rules involving quantifiers for inferencing in FOPL, are introduced respectively in Section 3.2 and Section 3.3, and then these rules are used in solving problems. Further, a new method of drawing inference called Resolution method based on refutation approach, is discussed in the next two Sections. In Section 3.4, Resolution method for PL is introduced and applied in solving problems involving PL reasoning. In Section 3.5, Resolution method for FOPL is introduced and used for solving problems involving FOPL reasoning.

Problems with FOPL as a system of knowledge representation and reasoning:

FOPL is not capable of easily representing some kinds of information including information pieces involving.

(i) Properties of relations. For example, the mathematical statement:

Any relation which is symmetric and transitive may not be reflexive is not expressible in FOPL. A relation in FOPL can only be constant, and not a variable. Only in second and higher order logics, the relations may be variable. This type of logics are not within the scope of the course.

(ii) linguistic variable like hot, tall, sweat.
For example: *It is very cold today,*
can not be appropriately expressed in FOPL.

(iii) different belief systems.
For example, *I know that he thinks India will win the match, but I think India will lose,* also, cannot be appropriately expressed in FOPL.

2.7 SOLUTIONS/ANSWERS

Ex.1: Assuming the statements (i), (ii) and (iii) given above as True we are required to Show the truth of (iv)
The **first** step is to mark the logical operators, if any, in the statements of the argument/problem under consideration.

In the above-mentioned problem, statement (i) does not contain any logical operator. Each of the statements (ii) and (iii) contains the logical operator 'If….then….'

The next step is to use symbols, P, Q, R, for atomic formulas occurring in the problem. The symbols are generally mnemonic, i.e., names used to help memory.

Let us denote the atomic statements in the argument given above as follows:

M: Matter always existed,
TG: There is God,
GU: God created the universe.

Then **the given** *statements* in English, become respectively the following *formulas* of PL:

(i) M
(ii) TG $\rightarrow$ GU
(iii) GU $\rightarrow$ ~ M
(iv) ~ TG (*To show*)

Applying transposition to (iii) we get

(v) M $\rightarrow$ ~ GU

using (i) and (v) and applying Modus Ponens, we get

(vi) ~ GU

Again, applying transposition to (ii) we get

(vii) $\sim GU \rightarrow \sim TG$

Applying Modus Ponens to (vi) and (vii) we get

(vii) $\sim TG$

The formula (viii) is the same as formula (iv) which was required to be proved.

Ex.2 In order to translate in PL, let us use the symbols:

ML: there is a moral law,
SG: someone gave it, (the word 'it' stand for moral law)
TG: There is God.

Using these symbols, the **Statement** (i) to (iv) become the **formula** (i) to (iv) of PL as given below:

(i) *ML*
(ii) $ML \rightarrow SG$
(iii) $SG \rightarrow TG$ *and*
(iv) *TG*

Applying Modus Ponens to formulae (i) and (ii) we get the formula

(v) *SG*

Applying Modus Ponens to (v) and (iii), we get

(vi) *TG*

But formula (vi) is the same as (iv), which is required to be established. Hence the proof.

Ex. 3: (i) Concluding $F(a) \wedge G(a)$ from $(\exists x)F(x) \wedge (\exists x)G(x)$ is *incorrect*, because, as mentioned earlier also, the given Quantified Formula may be equivalently written as $(\exists x)F(x) \wedge (\exists y)G(y)$. And in the case of each existential quantification, we can not assign an already-used constant. Therefore, a correct conclusion may be of the form
$F(a) \wedge G(b)$

(ii) The conclusion of $F(a) \vee \big(G(a) \wedge H(a)\big)$
from $(\exists x)F(x) \vee \big(G(x) \wedge H(x)\big)$
is again *incorrect*, in view of the fact that scope of existential variable in the formula, is only *F(x)* and not the whole formula. Hence, the last two occurrences of *x* are free. Therefore, a correct conclusion can be $F(a) \vee (G(x) \wedge H(x))$

(iii) The conclusion is correct

(iv) The conclusion is *incorrect*, because , from the given fact $\sim \big(F(a) \wedge G(a)\big)$,

we may conclude $((\exists x)(\sim(F(x) \wedge G(x))$
which is equivalent to $\sim (\forall x)\big(F(x) \wedge G(x)\big)$
and not to $\sim (\exists x)(F(x) \wedge G(x))$

Ex. 4: *For translating the given statements (i), (ii) & (iii), let us use the notation:*

F(x): x is an instance of feeling of pain
O(x): x is an entity that is publically observable
C(x): x is a chemical process.

Then, the statement (i), (ii) and (iii) can be equivalently expressed as formulas of FOPL

(i) $(\forall x)\ (F(x) \rightarrow \sim O(x))$
(ii) $(\forall x)\ (C(x) \rightarrow O(x))$
To prove

(iii) $(\forall x)\ (F(x) \rightarrow \sim C(x))$

From (i) using generalized instantiation, we get
(iv) $F(a) \rightarrow \sim O(a)$, for any arbitrary a

Similarly, from (ii), using generalized instantiation, we get

(v) $C(b) \rightarrow O(b)$, *for arbitrary b*

From (iv) using transposition rule, we get

(vi) $O(a) \rightarrow \sim F(a)$, *for arbitrary a*

As b is arbitrary in (v), therefore we can rewrite (v) as

(vii) $C(a) \rightarrow O(a)$, *for arbitrary a*

From (vii) and (vi) and using chain rule, we get

(viii) $C(a) \rightarrow \sim F(a)$, *for any arbitrary a*

But as a is arbitrary in (viii), by generalized quantification, we get

(ix) $(\forall x)\ (C(x) \rightarrow \sim F(x))$

But (ix) is the same as (iii), which was required to be proved.

Ex. 5: Let us symbolize the statements in the problem given above as follows:
A: The Parliament refuses to act.
B: The strike is over.
R: The president of the firm resigns.
S: The strike lasts more than one year.

Then the facts and the question to be answered can be symbolized as:

E1: $(A \rightarrow (\sim B \vee (R \wedge S)))$ represents the statement: *If the congress refuses to enact new laws, then the strike will not be over unless it lasts more than one year and the president of the firm resigns.*
(Note: P unless Q = $P \vee Q$)
E2 : A represents the statement: *The congress refuses to act, and*

E3: ~ S represent the statement: *The strike just starts.*
E4: ~ B (*to be concluded*)

As we are going to use resolution method, we use **E_5 the negation of E_4** as an axiom in addition to E_1, E_2 and E_3.
E_5: B
As a first step, we convert E_1, E_2, E_3 and E_5 into clausal forms as follows:
E_1: $\sim A \vee (\sim B \vee (R \wedge S)))$
Using associativity of $\vee$, we get
$= (\sim A \vee \sim B) \vee (R \wedge S)$
Using distributivity of $\vee$ over $\wedge$, we get
$= (\sim A \vee \sim B \vee R) \wedge (\sim A \vee \sim B \vee S)$

replacing E_1 by two clauses
E_{11}: $(\sim A \vee \sim B \vee R)$ and
E_{12}: $(\sim A \vee \sim B \vee S)$

E_2, E_3 and E_5 are already in clausal form

We get the axioms, including the negation of the conclusion, in the clausal form as
E_{11}: $(\sim A \vee \sim B \vee R)$
E_{12}: $(\sim A \vee \sim B \vee S)$
E_2: A
E_3: $\sim S$
E_5: $\sim (\sim B) = B$
By resolving E_2 with E_{12}, we get the resolvent
E_6: $\sim B \vee S$
By resolving E_5 with E_6, we get the resolvent as
E_7 : S
By resolving E_7 with E_3, we get the resolvent as
E_8 : FALSE

Hence, the conclusion ~ *B: The strike will not be over,* is valid.

Ex. 6: First, we attempt to unify (i) and (ii)
As the predicate is Q in each of the given terms, therefore, we should attempt matching terms. The first terms match, as each is u. Next second terms are 'a' and f (y, z), none of which is a variable. Hence, (i) and (ii) are not unifiable.

In the similar manner (i) and (iii) are not unifiable as the second terms f (y, z) and g (h (k (u))) are such that none is a variable.

Ex. 7: The predicate symbols (*each being* Q) match. Hence, we may proceed. Next, the first two terms viz. f (a) and x, are not identical. However, as one of these terms is a variable viz. 'x', hence, the corresponding terms are unifiable with substitution {f (a)/x}.
Next, the two terms g (x) and y, one from each of the formula at corresponding positions, are again unifiable by the substitution { g(x)/y}.
Hence, the required substitutions { f (a)/x, g (f (a))/y} using the substitution {f (a)/x} in g (x)/y to get the substitution {g (f (a))/y}.

Therefore the two formulas are unifiable and after unification the formulas become Q (f(a), g (f (a)))

2.8 FURTHER READINGS

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