UNIT 2 NUMERICAL INTEGRATION

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2.0 INTRODUCTION

In Unit 1, we developed methods of differentiation to obtain the derivative of a function f(x), when its values are not known explicitly, but are given in the form of a table. In this unit, we shall derive numerical methods for evaluating the definite integrals of such functions f(x). You may recall that in calculus, the definite integral of f(x) over the interval [a, b] is defined as

$$\int_a^b f(x)\,dx = \lim_{h \to 0} R[h]$$

where R[h] is the left-end Riemann sum for n subintervals of length $h = \frac{(b-a)}{n}$ and is given by

$$R[h] = \sum_{k-0}^{n-1} h\, f(x_k)$$

for the nodal points $x_0, x_1, \ldots, x_n$, where $x_k = x_0 + kh$ and $x_0 = a$, $x_n = b$.

The need for deriving accurate numerical methods for evaluating the definite integral arises mainly, when the integrand is either

i) a complicated function such as $f(x) = e^{-x^2}$ or $f(x) = \frac{\sin(x)}{x}$ etc. which have no anti-derivatives expressible in terms of elementary functions, or

ii) when the integrand is given in the form of tables.

Many scientific experiments lead to a table of values and we may not only require an approximation to the function f(x) but also may require approximate representations of the integral of the function. Also, for functions the integrals of which can be calculated analytically, analytical evaluation of the integral may lead to transcendental, logarithmic or circular functions. The evaluation of these functions for a given value of x may not be accurate. This motivates us to study numerical integration methods which can be easily implemented on calculators.

In this unit we shall develop numerical integration methods where in the integral is approximated by a linear combination of the values of the integrand i.e.

$$\int_a^b f(x)\,dx = \beta_0 f(x_0) + \beta_1 f(x_1) + \ldots\ldots\ldots\ldots + \beta_n f(x_n) \qquad (1)$$

where $x_0, x_1, \ldots\ldots\ldots x_n$ are the points which divide the interval [a, b] into n sub-intervals and $\beta_0, \beta_1, \ldots\ldots\ldots \beta_n$ are the weights to be determined. We shall discuss in this unit, a few techniques to determine the unknowns in Eqn. (1).

2.1 OBJECTIVES

After going through this unit you should be able to:

- state the basic idea involved in numerical integration methods for evaluating the definite integral of a given function;
- use numerical integration method to find the definite integral of a function f(x) whose values are given in the form of a table;
- use trapezoidal and Simpson's rules of integration to integrate such functions and find the errors in these formulas.

2.2 METHODS BASED ON INTERPOLATION

In Block 2, you have studied interpolation formulas, which fits the given data (x_k, f_k), k = 0, 1, 2, …………….., n. We shall now see how these interpolation formulas can be used to develop numerical integration methods for evaluating the definite integral of a function which is given in a tabular form. The problem of numerical integration is to approximate the definite integral as a linear combination of the values of f(x) in the form

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{n} \beta_k f_k \tag{2}$$

where the n + 1 distinct points x_k, k = 0, 1, 2, ………., n are called the nodes or abscissas which divide the interval [a, b] into n sub-intervals with $(x_0 < x_1, < x_2, \ldots\ldots < x_{n-1} < x_n)$ and β_k, k = 0, 1, ….n, are called the weights of the integration rule or quadrature formula. We shall denote the exact value of the definite integral by I and denote the rule of integration by

$$I_h[f] = \sum_{k=0}^{n} \beta_k f_k \tag{3}$$

The error of approximating the integral I by $I_h[f]$ is given by

$$E_h[f] = \int_a^b f(x)dx - \sum_{k=0}^{n} \beta_k f_k \tag{4}$$

The order of the integration method (3) is defined as follows:

Definition : An integration method of the form (3) is said to be of order p if it produces exact results for all polynomials of degree less than or equal to p.

In Eqn. (3) we have 2n+2 unknowns viz., n + 1 nodes x_k and the n + 1 weights, β_k and the method can be made exact for polynomials of degree ≤ 2n + 1. Thus, the method of the form (3) can be of maximum order 2n + 1. But, if some of the values are prescribed in advance, then the order will be reduced. If all the n + 1 nodes are prescribed, then we have to determine only n + 1 weights and the corresponding method will be of maximum order n.

We first derive the numerical method based on Lagrange's interpolation.

2.2.1 Methods Using Lagrange's Interpolation

Suppose we are given the n + 1 abscissas x_k's and the corresponding values f_k's are known that the unique Lagrange's interpolating polynomial $P_{nn}(x)$ of degree $\le$ n, satisfying the interpolatory conditions $P_n(x_k) = f(x_k)$, k = 0, 1, 2, ….. n, is given by

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} L_k(x) f_k \qquad (5)$$

with the error of interpolation

$$E_{n+1}[P_n(x)] = \frac{\pi(x)}{(n+1)!} f^{n+1}(\alpha) \text{ , with } x_0 < \alpha < x_n \qquad (6)$$

where $L_k(x) = \dfrac{(x-x_0)(x-x_1)....(x-x_{k-1})(x-x_{k+1})....(x-x_n)}{(x_k-x_0)(x_k-x_1)......(x_k-x_{k-1})(x_k-x_{k+1})....(x_k-x_n)}$

and $\pi(x) = (x-x_0)(x-x_1)...(x-x_n)$.

We replace the function f(x) in the definite integral (2) by the Lagrange interpolating polynomial $P_n(x)$ given by Eqn. (5) and obtain

$$I_n[f] = \int_a^b P_n(x)dx = \sum_{k=0}^{n} \int_a^b L_k(x) f_k dx = \sum_{k-0}^{n} \beta_k f_k \qquad (7)$$

where $\beta_k = \int_a^b L_k(x)\,dx.$ (8)

The error in the integration rule is

$$E_n[f] = \int_a^b E_{n+1}[P_n(x)]\,dx = \int_a^b \frac{\pi(x)}{(n+1)!} f^{n+1}(\alpha)dx \qquad (9)$$

We have

$$|E_n[f]| \le \frac{M_{n+1}}{(n+1)!} \int_a^b |\pi(x)|dx \qquad (10)$$

where $M_{n+1} = \max_{x_0 < x < x_n} |f^{n+1}(x)|$

Let us consider now the case when the nodes x_k 's are equispaced with $x_0 = a$, $x_n = b$ with the length of each subinterval as $h = \dfrac{b-a}{n}$. The numerical integration methods given by (7) are then known as **Newton-Cotes formulas** and the weights β_k's given by (8) are known as **Cotes numbers.** Any point x ε [a, b] can be written as $x = x_0 + sh$.

With this substitution, we have

$\pi(x) = h^{n+1} s(s-1)(s-2)(s-3)……(s-n)$

$$L_k(x) = \frac{(-1)^{n-k} s(s-1)....(s-k+1)(s-k-1).....(s-n)}{k!(n-k)!} \qquad (11)$$

Using $x = x_0 + sh$ and changing the variable of integration from x to s, we obtain

$$\beta_k = \frac{(-1)^{n-k}}{k!(n-k)!} h \int_0^n s(s-1)(s-2).........(s-k+1)(s-k-1)......(s-n)\, ds \qquad (12)$$

and $$|E_n[f]| \le \frac{h^{n+2} M_{n+1}}{(n+1)!} \int_0^n s(s-1)(s-2).....(s-n) ds \qquad (13)$$

We now derive some of the Newton Cotes formulas viz. trapezoidal rule and Simpson's rule by using first and second degree Lagrange polynomials with equally spaced nodes.

Trapezoidal Rule

When $n = 1$, we have $x_0 = a$, $x_1 = b$ and $h = b - a$. Using Eqn. (12) the Cotes numbers can be found as

$$\beta_0 = -h \int_0^1 (s-1) ds = \frac{h}{2};$$

and $$\beta_1 = h \int_0^1 s\, ds = \frac{h}{2}.$$

Substituting the values of β_0 and β_1 in Eqn. (7), we get

$$I_T[f] = \frac{h}{2}[f_0 + f_1] \qquad (14)$$

The error of integration is

$$|E_T[f]| \le \frac{h^3}{2} M_2 \int_0^1 |s(s-1)| ds = \left| -\frac{h^3}{12} M_2 \right| = \frac{h^3}{12} M_2$$

Where $M_2 = \max\limits_{x_0 < x < x_1} |f''(x)|$

Thus, by trapezoidal rule, $\int_a^b f(x)\, dx$ is given by

$$\boxed{I[f] = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12} M_2}$$

The reason for calling this formula the trapezoidal rule is that geometrically when f(x) is a function with positive values then $h(f_0 + f_1)/2$ is the area of the trapezium when height $h = b - a$ and parallel sides as f_0 and f_1. This is an approximation to the area under the curve $y = f(x)$ above the x-axis bounded by the ordinates $x = x_1$ (see Fig. 1.)

Since the error given by Eqn. (15) contains the second derivative, trapezoidal rule integrates exactly polynomials of degree $\le$ 1.

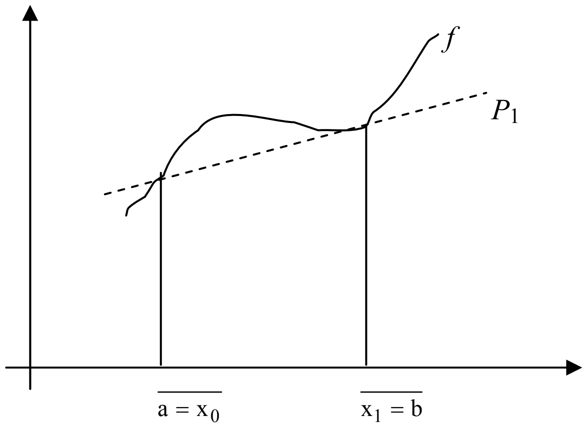

Fig. 1

Let us now consider an example.

Example 1: Find the approximate value of

$$I = \int_0^1 \frac{dx}{1+x}$$

using trapezoidal rule and obtain a bound for the error. The exact value of $I = \ln_2 = 0.693147$ correct to six decimal places.

Solution: Here $x_0 = 0$, $x_1=1$, and $h = 1 - 0 = 1$. Using Eqn. (14), we get

$$I_T\ [f] = \frac{1}{2}\left(1+\frac{1}{2}\right) = 0.75$$

Actual error = 0.75 – 0.693147 = 0.056853.

The error in the trapezoidal rule is given by

$$|\,E_T\,|[\,f\,] \le \frac{1}{12}\max\left|\frac{2}{(1+x)^3}\right| = \frac{1}{6} = 0.166667$$

Thus, the error bound retain is much greater than the actual error.

We now derive the Simpson's rule.

Simpson's Rule

For n = 2, we have $h = \frac{b-a}{2}$, $x_0 = a$, $x_1 = \frac{a+b}{2}$ and $x_2 = b$.

From (12), we find the Cotes numbers as

$$\beta_0 = \frac{h}{2}\int_0^2 (s-1)(s-2)\,ds = \frac{h}{3}$$

$$\beta_1 = h\int_0^2 s(s-2)\,ds = \frac{4h}{3}, \beta_2 = \frac{h}{2}\int_0^2 s(s-1)\,ds = \frac{h}{3}.$$

Eqn. (7) in this case reduces to

$$I_S[f] = \frac{h}{3}[f_0 + 4f_1 + f_2] \quad (17)$$

Eqn. (17) is the Simpson's rule for approximating $I = \int_a^b f(x)\,dx$

The magnitude of the error of integration is

$$|E_2[f]| \le \frac{h^4 M_3}{3!}\int_0^2 |s(s-1)(s-2)|\,ds$$

$$= \frac{h^4 M_3}{3!}\left[\int_0^1 s(s-1)(s-2)\,ds + \int_1^2 s(s-1)(s-2)\,ds\right]$$

$$= \frac{h^4 M_3}{3!}\left[\left(\frac{s^4}{4} - s^3 + s^2\right)_0^1 + \left(\frac{s^4}{4} - s^3 + s^2\right)_1^2\right]$$

$$= \frac{h^4 M_3}{3!}\left[\frac{1}{4} - \frac{1}{4}\right] = 0$$

This indicates that Simpson's rule integrates polynomials of degree 3 **also** exactly. Hence, we have to write the error expression (13) with n = 3. We find

$$|E_S[f]| \le \frac{h^5 M_4}{24}\int_0^2 |s(s-1)(s-2)(s-3)|\,ds$$

$$= \frac{h^5 M_4}{24}\left[\int_0^1 s(s-1)(s-2)(s-3)\,ds + \int_1^2 s(s-1)(s-2)(s-3)\,ds\right] \quad (18)$$

$$= \frac{-h^5 M_4}{90}$$

where $M_4 = \max_{x_0 < x < x_2} \left| f^{IV}(X) \right|$

Since the error in Simpson's rule contains the fourth derivative, Simpson's rule integrates exactly all polynomials of degree ≤ 3.

Thus, by Simpson's rule, $\int_a^b f(x)\,dx$ is given by

$$\boxed{I_S[f] = \frac{h}{3}[f_0 + 4f_1 + f_2] - \frac{h^5}{90}M_4}$$

Geometrically, $\frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$ represents the area bounded by the quadratic curve passing through (x_0, f_0), (x_1, f_1) and (x_2, f_2) above the x-axis and lying between the ordinates $x = x_0$, $x = x_2$ (see figure. 2).

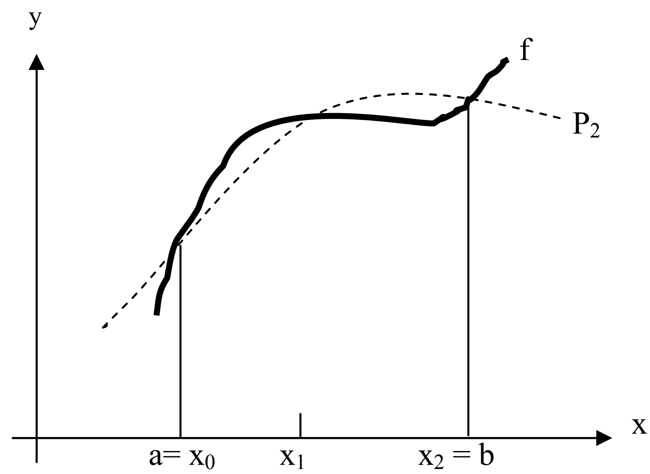

Fig. 2

In case we are given only one tabulated value in the interval [a, b], then h = b – a, and the interpolating polynomial of degree zero is $P_0(x) = f_k$. In this case, we obtain the rectangular integration rule given by

$$\boxed{I_R\left[f\right] = \int_a^b f_k \, dx \approx hf_k} \qquad (19)$$

The error in the integration rule is obtained from Eqn. (13) as

$$E_R\left|f\right| \le \frac{h^2 M_1}{2} \qquad (20)$$

where $M_1 = \max_{a < x < b} |f'(x)|$

If the given tabulated value in the interval [a, b] is the value at the mid-point then we have $x_k = \frac{(a+b)}{2}$, and $f_k = f_{k+\frac{1}{2}}$. In this case h = b – a and we obtain the integration rule as

$$\boxed{{}_M\ f = \int_a^b {}_{k+\frac{}{2}} \, dx \qquad {}_{k+\frac{1}{2}}} \qquad (21)$$

Rule (21) is called the mid-point rule. The error in the rule calculated from (13) is

$$E_M\left[f\right] = \frac{h^2}{2}\int_{-1/2}^{1/2} s \, ds = 0$$

This shows that mid-point rule integrates polynomials of degree one exactly. Hence the error for the mid-point rule is given by

$$E_M[f] \le \frac{h^3 M_2}{2}\int_{-1/2}^{1/2} s(s-1)\,ds = \frac{h^3 M_2}{24} \qquad (22)$$

where $M_2 = \max_{a < x < b} |f''(x)|$ and $h = b - a$

We now illustrate these methods through an example.

Example 2 : Evaluate $\int_0^1 e^{-x^2}\,dx$, using

a) Rectangular rule b) mid-point rule c) trapezoidal rule and d) Simpson's rule. If the exact value of the integral is 0.74682 correct to 5 decimal places, find the error in these rules.

Solution: The values of the function f(x) = e^{-x^2} at $x = 0,\ 0.5$ and 1 are

f(0) = 1, f(0.5) = 0.7788, f(1) = 0.36788

Taking h = 1 and using

a) $I_R[f] = hf_0,$ we get $I_R[f] = 1.$

b) $I_M[f] = hf_{1/2},$ we get $I_M[f] = 0.7788$

c) $I_T[f] = \frac{h}{2}[f_0 + f_1]$ we get $I_T[f] = \frac{1}{2}(1 + 0.36788) = 0.68394.$

Taking h = 0.5 and using Simpson's rule, we get

d) $I_S[f] = \frac{h}{3}[f_0 + 4f_1 + f_3]$

$= \frac{h}{3}[f(0) + 4f(0.5) + f(1)]$

= 0.74718

Exact value of the integral is 0.74682.

The errors in these rules are given by

$E_R[f] = -0.25318, E_M[f] = -0.03198$

$E_T[f] = 0.06288, E_s[t] = -0.00036.$

You may now try the following exercise:

Ex.1) Use the trapezoidal and Simpson's rule to approximate the following integrals. Compare the approximations to the actual value and find a bound for the error in each case.

a) $\int_1^2 \ln x\,dx$

b) $\int_{0}^{0.1} x^{1/3}\,dx$

c) $\int_{0}^{\pi/4} \tan x\,dx$

We now derive integration methods using Newton's forward interpolation formula.

2.2.2 Methods Using Newton's Forward Interpolation

Let the data be given at equi-spaced nodal points $x_k = x_0 + sh, s = 0, 1, 2, ..., n,$ Where $x_0 = a$ and $x_n = x_0 + nh = b$.

The step length is given by $h = \dfrac{b-a}{n}$.

The Newton's forward finite difference interpolation formula interpolating this data is given by

$$f(x) \approx P_n(x) = f_0 + s\Delta f_0 + s(s-1)\frac{\Delta^2 f_0}{2} + ... + \frac{s\,(s-1)(s-2)...(s-n+1)\,\Delta^n f_0}{n!} \tag{23}$$

with the error of interpolation

$$E_n[f] = \frac{h^{n+1}\,s(s-1)(s-2)...(s-n)}{(n+1)!} f^{n+1}(\alpha)$$

Integrating both sides of Eqn. (23) w.r.t. x between the limits a and b, we can approximate the definite integral I by the numerical integration rule

$$I_h[f] = \int_a^b P_n(x)\,dx = h\int_0^1 \left[f_0 + s\;\Delta f_{0+}\;\frac{s\,(s-1)}{2}\Delta^2 f_0 + ...\right] ds \tag{24}$$

The error of interpolation of (24) is given by

$$|E_h(f)| \le \frac{h^{n+2} M_{n+1}}{(n+1)!}\int_0^1 s(s-1)(s-2)...(s-1)\,ds$$

We can obtain the trapezoidal rule (14) from (24) by using linear interpolation i.e., f(x) $\approx P_1(x) = f_0 + s\,\Delta f_0$. We then have

$$I_T(x) = h\int_0^1 [f_0 + s\Delta f_0]\,ds$$

$$= h\left[s f_0 + \frac{s^2}{2}\Delta f_0\right]_0^1$$

$$= h\left[f_0 + \frac{\Delta f_0}{2}\right] = \frac{h}{2}[f_0 + f_1]$$

with the error of integration given by (15).

Similarly Simpson's rule (16) can be obtained from (24) by using quadratic interpolation i.e., $f(x) \approx P_2(x)$.

Taking $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h = b$, we have

$$I_s[f] = \int_a^b f(x)\,dx \approx h\int_0^2 \left[f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 \right] ds$$

$$= h\left[2f_0 + 2\Delta f_0 + \frac{\Delta^2 f_0}{3} \right]$$

$$= \frac{h}{3}[f_n + 4f_1 + f_2].$$

The error of interpolation is given by Eqn. (18).

Example 3: Find the approximate value of $I = \int_0^1 \frac{dx}{1+x}$ using

Simpson's rule. Obtain the error bound and compare it with the actual error. Also compare the result obtained here with the one obtained in Example 1.

Solution: Here $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$ and $h = \frac{1}{2}$.

Using Simpson's rule we have

$$I_s[f] = \frac{h}{3}[f(0) + 4f(0.5) + f(1)] = \frac{1}{6}\left[1 + \frac{8}{3} + 0.5\right] = 0.694445$$

Exact value of I = ln 2 = 0.693147.

Actual error = 0.001297. The bound for the error is given by

$$|E_s[f]| \le \frac{h^5}{90} M_4 = 0.00833, \text{ where } M_4 = \max\left|\frac{24}{(1+x)^5}\right| = 24$$

Here too the actual error is less than the given bound.

Also actual error obtained here is much less than that obtained in Example 1. You may now try the following exercise.

Ex. 2) Find an approximation to $\int_{1.1}^{1.5} e^x\,dx$, using

a) the trapezoidal rule with h = 0.4
b) Simpson's rule with h = 0.2

The Newton-Cotes formulas as derived above are generally unsuitable for use over large integration intervals. Consider for instance, an approximation to

$\int_0^4 e^x$ dx, using Simpson's rule with h = 2. Here

$$\int_0^4 e^x dx \approx \frac{2}{3}\left(e^0 + 4e^2 + e^4\right) = 56.76958.$$

Since the exact value in this case $e^4 - e^0 = 53.59815$, the error is -3.17143. This error is much larger than what we would generally regard as acceptable. However, large error is to be expected as the step length $h = 2.0$ is too large to make the error expression meaningful. In such cases, we would be required to use higher order formulas. An alternate approach to obtain more accurate results while using lower order methods is the use of composite integration methods, which we shall discuss in the next section.

2.3 COMPOSITE INTEGRATION

In composite integration we divide the given interval [a, b] into a number of subintervals and evaluate the integral in each of the subintervals using one of the integration rules. We shall construct composite rules of integration for trapezoidal and Simpson's methods and find the corresponding errors of integration when these composite rules are used.

Composite Trapezoidal Rule

We divide the interval [a, b] into N subintervals of length $h = \frac{(b-a)}{N}$. We denote the subintervals as

$(x_{k-1}, x_k), k = 1, 2, .., N$ where $x_0 = a, x_N = b$. Then

$$I = \int_a^b f(x)\,dx = \sum_{k=1}^{N} \int_{x_{k-1}}^{x_k} f(x)\,dx \tag{25}$$

Evaluating each of the integrals on the right hand side by trapezoidal rule, we have

$$\boxed{I_T[f] = \sum_{k=1}^{N} \frac{h}{2}\left[f_{k-1} + f_k\right] = \frac{h}{2}\left[f_0 + f_N + 2\left(f_1 + f_2 + \ldots.. + f_{N-1}\right)\right]} \tag{26}$$

`

$$E_T[f] = -\frac{h^3}{12}\left[\sum_{i=1}^{N} f''(\alpha_i)\right], x_{k-1} < \alpha_i < x_k, \text{ for } k = 1...., N.$$

Now since f is a continuous function on the interval [a, b] we have as a consequence of Intermediate-value theorem

$$\sum_{i=1}^{N} f''(\alpha_i) = f''(\xi)\sum_{i=1}^{N} 1, \text{ where } a < \xi < b.$$

$$\therefore E_T[f] = \frac{-h^3}{12} f''(\xi)\,N, a < \xi < b.$$

$$= \frac{-Nh}{12} h^3 f''(\xi)$$

$$= \frac{-(b-a)h^{2.}}{12} f''(\xi).$$

If $M_2 = \max\limits_{a < \xi < b} \left| f''(\xi) \right|$. Then

$$\left| E_T[f] \right| \le \frac{(b-a)\,h^2}{12} M_2 \qquad (27)$$

The error is of order h^2 and it decreases as h decreases

Composite trapezoidal rule integrates exactly polynomials of degree $\le$ 1. We can try to remember the formula (26) as

$$I_T[f] = \left(\frac{h}{2}\right)\left[\text{first ordinate} + \text{last ordinate} + 2\,(\text{sum of the remaining ordinates})\right].$$

Composite Simpson's Rule

In using Simpson's rule of integration (17), we need three abscissas. Hence, we divide the interval (a, b) into an even number of subintervals of equal length giving an odd number of abscissas in the form $a = x_0 < x_1 < x_2 < \ldots\ldots < x_{2n} = b$ with $h = \frac{b-a}{2N}$ and $x_k = x_0 + kh$, $k = 0, 1, 2, \ldots, 2N$. We then write

$$I = \int_a^b f(x)\,dx = \sum_{k=1}^{N} \int_{x_{2k-2}}^{x_{2k}} f(x)\,dx \qquad (28)$$

Evaluating each of the integrals on the right hand side of Eqn. (28) by the Simpson's rule, we have

$$) \qquad (29)$$

The formula (29) is known as **the composite Simpson's rule of numerical integration.** The error in (29) is obtained from (18) by adding up the errors. Thus we get

$$E_s[f] = -\frac{h^5}{90}\left[\sum_{k=1}^{N} f^{IV}(\alpha)\right], x_{2k-2} < \alpha_k < x_{2k}$$

$$= -\frac{h^5}{90} f^{iv}(\xi) \sum_{i=1}^{N} 1, \quad a < \xi < b$$

$$= -\frac{Nh^5}{90} f^{IV}(\xi)$$

$$= -\frac{(b-a)\,h^4}{180} f^{IV}(\xi).$$

If $M_4 = \max\limits_{a \le \xi \le b} \left| f^{IV}(\xi) \right|$ write using $h = \frac{(b-a)}{2N}$

$$\left| E_s[f] \right| \le \frac{(b-a)}{180} h^4 M_4 \qquad (30)$$

The error is of order h^4 and it approaches zero very fast as $h \to 0$. The rule integrates exactly polynomials of degree $\le$ 3. We can remember the composite Simpson's rule as

$$I_S[f] = \left(\frac{h}{3}\right)[\text{first ordinate} + \text{last ordinate} + 2\,(\text{sum of even ordinates}) + 4\,(\text{sum of the remaining odd ordinates})]$$

We now illustrate composite trapezoidal and Simpson's rule through examples.

Example 4: Evaluate $\int_0^1 \frac{dx}{1+x}$ using

(a) Composite trapezoidal rule (b) composite Simpson's rule with 2, 4 and 8 subintervals.

Solution: We give in Table 1 the values of f (x) with $h = \frac{1}{8}$ from x = 0 to x = 1.

Table 1

x	:	0	1/8	2/8	3/8	4/8	5/8	6/8	7/8	1
f(x)	:	1 f_0	8/9 f_1	8/10 f_2	8/11 f_3	8/12 f_4	8/13 f_5	8/14 f_6	8/15 f_7	8/16 f_8

If N = 2 then h = 0.5 and the ordinates f_0, f_4 and f_8 are to be used

We get

$$I_T[f] = \frac{1}{4}[f_0 + 2f_4 + f_8] = \frac{17}{24} = 0.708333$$
$$I_s[f] = \frac{1}{6}[f_0 + 4f_1 + f_8] = \frac{25}{36} = 0.694444$$

If N = 4 then h = 0.25 and the ordinates f_0, f_2, f_4, f_6, f_8 are to be used.

We have

$$I_T[f] = \frac{1}{8}[f_0 + f_8 + 2(f_2 + f_4 + f_6)] = 0.697024$$
$$I_T[f] = \frac{1}{12}[f_0 + f_8 + 4(f_2 + f_6) + 2\ f_4] = 0.693254$$

If N = 8 then h = 1/8 and all the ordinates in Table 1 are to be used.

We obtain

$$I_T[f] = \frac{1}{16}[f_0 + f_8 + 2(f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7)] = 0.694122$$

$$I_s[f] = \frac{1}{24}[f_0 + f_8 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6)] = 0.693147$$

The exact value of the given integral correct to six decimal places is ln2 = 0.693147.

We now give the actual errors in Table 2 below.

Table 2

N	E_T (f)	E_s (f)
2	0.015186	0.001297
4	0.003877	0.000107
8	0.000975	0.000008

Note that as h decreases, the errors in both trapezoidal and Simpson's rule also decreases. Let us consider another example.

Example 5: Find the minimum number of intervals required to evaluate $\int_0^1 \frac{dx}{1+x}$ with an accuracy 10^{-6}, by using the Simpson rule.

Solution: In example 4 you may observe from Table 2 that N $\approx$ 8 gives (1.E–06) accuracy. We shall now determine N from the theoretical error bound for Simpson's rule which gives 1.E – 06 accuracy. Now

$$|E_s[f]| \le \frac{(b-a)^5 M_4}{2880N^4}$$

where

$$M_4 = \max_{0<x<1} |f^{IV}(x)|$$

$$= \max_{0<x<1} \left| \frac{24}{(1+x)^5} \right| = 24$$

To obtain the required accuracy we should therefore have

$$\frac{24}{2880N^4} \le 10^{-6}, \text{or } N^4 \ge \frac{24 * 10^6}{2880} = 8333.3333$$

$$\therefore N \ge 9.5$$

We find that we cannot take N = 9 since to make use of Simpson's rule we should have even number of intervals. We therefore conclude that N = 10 should be the minimum number of subintervals to obtain the accuracy 1.E – 06 (i.e., 10^{-6})

Ex.3) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by subdividing the interval (0, 1) into 6 equal parts and using.

(a) Trapezoidal rule (b) Simpson's rule. Hence find the value of π and actual errors.

Ex. 4) A function f(x) is given by the table

Ex.5) The speedometer reading of a car moving on a straight road is given by the following table Estimate the distance traveled by the car in 12 minutes using (a) Trapezoidal rule (b) Simpson's rule.

Ex.6) Evaluate $\int_{0.2}^{0.4} (\sin x - \ln x + e^{x})\, dx$

using (a) Trapezoidal rule (b)

Simpson's rule taking h = 0.1. Find the actual errors.

Ex.7) Determine N so that composite trapezoidal rule gives the value of $\int_0^1 e^{-x^2}\, dx$

correct upto 3 digits after the decimal point, assuming that e^{-x^2} can be calculated accurately.

2.4 SUMMARY

In this unit, we learnt the following :

1) If a function f(x) is not known explicitly but a table of values of x is given or when it has no anti-derivative expressible in terms of elementary functions then its integral cannot be obtained by calculus methods. In such cases numerical integration methods are used to find the definite integral of f(x) using the given data.

2) The basic idea of numerical integration methods is to approximate the definite integral as a linear combination of the values of f(x) in the form

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{n} \beta_k f(x_k) \qquad \text{(See Eqn. (21))}$$

where the (n+1) distinct nodes $x_k, k = 0,1,\ldots, n$, with $x_0 < x_1 < x_2 < \ldots < x_n$ divide the interval (a, b) into n subintervals and $\beta_k, k = 0,1,\ldots,$ n are the weights of the integration rule. The error of the integration methods is then given by

$$\left|E_h[f]\right| = \int_a^b f(x)\,dx - \sum_{k=0}^{n} \beta_k f(x_k) \qquad \text{(see Eqn. (4))}$$

3) For equispaced nodes, the integration formulas derived by using Lagrange interpolating polynomials $P_n(x)$ of degree $\leq$ n, satisfying the interpolatory conditions $P_n(x_k) = f(x_k), k = 0,1,\ldots,$ n are known as Newton-Cotes formulas. Corresponding to n = 1 and n = 2, Newton-Cotes formulas viz., trapezoidal rule and Simpson's rule are obtained.

4) For large integration intervals, the Newton-Cotes formulas are generally unsuitable for they give large errors. Composite integration methods can be used in such cases by dividing the interval into a large number of subintervals and evaluating the integral in each of the subintervals using one of the integration rules.

2.5 SOLTIONS/ANSWERS

E1) a) $I_T[f] = \frac{h}{2}[f_0 + f_1] = 0.346574$

$$I_s[f] = \frac{h}{3}[f_0 + 4f_1 + f_2]$$

$$= \frac{0.5}{3}[4\ln 1.5 + \ln 2] = 0.385835$$

Exact value of I = 0.386294

Actual error in $I_T[f] = 0.03972$

Actual error in $I_S[f] = 0.0000459$

Also

$$|E_T[f]| \le -\frac{h^3}{12} \max_{1<x<2} \left|\frac{1}{x^2}\right| = -\frac{1}{12} = -0.083334$$

$$|E_s[f]| \le -\frac{h^5}{90} \max_{1<x<2} \left|\frac{6}{x^4}\right| = -0.002083$$

b) $I_T[f] = 0.023208, \; |E_T[f]| = \text{none}.$

$I_S[f] = 0.032296, \; |E_S[f]| = \text{none}.$

Exact value = 0.034812.

c) $I_T[f] = 0.39270, \; |E_T[f]| = 0.161.$

$I_S[f] = 0.34778, \; |E_S[f]| = 0.00831.$

Exact value = 0.34657

E2) $I_T[f] = 1.49718$

$I_S[f] = 1.47754$

E3) With h = 1/6, the values of $f(x) = \frac{1}{1+x^2}$

From x = 0 to 1 are

x	:	0	1/6	2/6	3/6	4/6	5/6	1
f(x)	:	1	0.972973	0.9	0.8	0.692308	0.590167	0.5

Now

$$I_T[f] = \frac{h}{2}[f_0 + f_6 + 2(f_1 + f_2 + f_3 + f_4 + f_5)]$$

$= 0.784241$

$$I_S[f] = \frac{h}{3}[f_0 + f_6 + 4\,(f_1 + f_3 + f_5) + 2(f_2 + f_4)]$$

$= 0.785398$

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}.$$ Exact $\pi = 3.141593$

Value of π from $I_T[f] = 4x0.784241 = 3.136964$

Error in calculating π by $I_T[f]$ is $E_T[f] = 0.004629$

Value of π from $I_s[f] = 4\,x\,0.785398 = 3.141592$

Error in π by $I_s[f]$ is $E_s[f] = 1.0 \times 10^{-6}$

E4) $I_T[f] = \left(\frac{h}{2}\right)[f_0 + f_4 + 2(f_1 + f_2 + f_3)]$

$= (1/4)[1 + 25 + 2(2.875 + 7 + 14.125)] = 18.5$

$I_s[f] = \left(\frac{h}{3}\right)[f_0 + f_4 + 2f_2 + 4(f_1 + f_3)]$

$= (1/6)[1 + 25 + 2\,x7 + (2.875 + 14.125)] = 18$

E5) Let $v_0 = 0,\ v_1 = 15, v_2 = 25, v_3 = 40, v_4 = 45, v_5 = 20, v_6 = 0.$ Then

$$I = \int_0^{12} v\,dt,$$

$I_T[v] = \left(\frac{h}{2}\right)[v_0 + v_6 + 2(v_1 + v_2 + v_3 + v_4 + v_5)] = 290$

$I_s[v] = \frac{880}{30} = 293.33.$

E6) The values of f(x) = sin x = 1n x + e^x are

$f(0.2) = 3.02951, f(0.3) = 2.849352, f(0.4) = 2.797534$

$$I_T[f] = \left(\frac{0.1}{2}\right)[f(0.2) + 2f(0.3) + f(0.4)] = 0.57629$$

$$I_S[f] = \left(\frac{0.1}{3}\right)[f(0.2) + 4f(0.3) + f(0.4)] = 0.574148$$

Exact value = 0.574056

$E_T = 2.234 \times 10^{-3}$

$E_s = 9.2 \times 10^{-5}$

E7) Error in composite trapezoidal rule

$$E_T[f] = -\frac{(b-a)^3}{12N^2} M_2\,,\ M_2 = \max_{0<x<1} |f''(x)|.$$

Thus

$$|E_T[f]| \le \frac{1}{12N^2} \max_{0 \le x \le 1} |f''(x)|$$

$$f(x) = e^{-x^2}, f''(x) = e^{-x^2}(4x^2 - 2)$$

$$f'''(x) = e^{-x^2}\, 4x(3 - 2x^2) = 0 \text{ when or } x = 0, x = \sqrt{1.5}$$

$$\max |[f''(0), f''(1)]| = \max [2, 2e^{-1}] = 2$$

For getting the correct value upto 3 digits, we must have

$$\frac{2}{12N^2} < 10^{-03} \text{ or } N^2 > \frac{10^3}{6} = \frac{10^4}{60}$$

or

$$N > \frac{100}{\sqrt{(60)}} \approx 12.9.$$

The integer value of N =13.