METHOD OF SUCCESSIVE ITERATION

The first step in this method is to write the equation in the form

$$x = g(x) \qquad (14)$$

For example, consider the equation $x^2 - 4x + 2 = 0$ *. We can write it as*

$$x = \sqrt{4x - 2} \qquad (15)$$

$$x = (x^2 + 2)/4 \qquad (16)$$

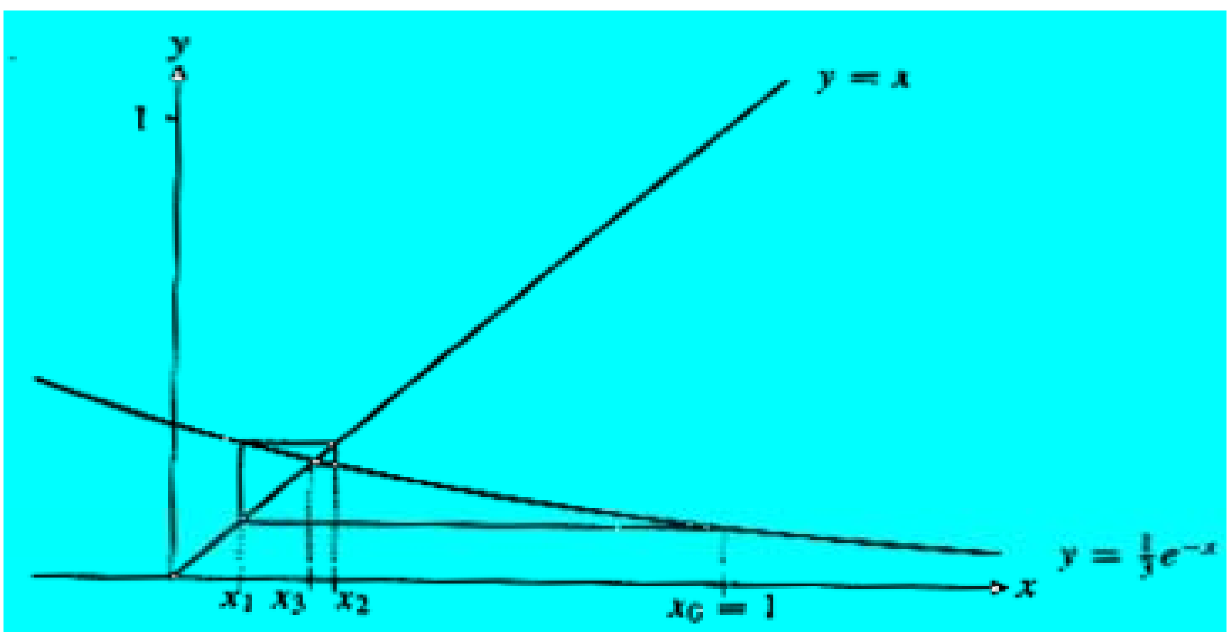
$$x = \frac{2}{4 - x} \qquad (17)$$

Thus, we can choose form (1) in several ways. Since $f(x) = 0$ *is the same as* $x = g(x)$ *, finding a root of* $f(x) = 0$ *is the same as finding a root of* $x = g(x)$*, i.e. finding a fixed point* α *of* $g(x)$ *such that* $\alpha = g(\alpha)$ *. The function* $g(x)$ *is called an* **iterative function** *for solving* $f(x) = 0$ *.*

If an initial approximation x_0 *to a root* α *is provided , a sequence* x_1 *,* x_2 *,….. may be defined by the iteration scheme*

$$x_{n+1} = g(x_n) \qquad (18)$$

with the hope that the sequence will converge to α *. The successive iterations are interpreted graphically , shown in the following figure*



Convergence will certainly occur if , for some constant M such that $0< M <1$, the inequality

$$| g(x) - g(\alpha) | \leq M | x - \alpha | \quad (19)$$

holds true whenever $|x- \alpha | \leq | x_0 - \alpha |$. For, if (6) holds, we find that

$$| x_{n+1} - \alpha | = | g(x_n) - \alpha | = | g(x_n) - g(\alpha) | \leq M | x_n - \alpha | \quad (20)$$

Proceeding further,

$$| x_{n+1} - \alpha | \leq M | x_n - \alpha | \leq M^2 | x_{n-1} - \alpha | \leq M^3 | x_{n-2} - \alpha | \quad (21)$$

Continuing in this manner, we conclude that

$$| x_n - \alpha | \leq M^n | x_0 - \alpha | \quad (22)$$

Thus, $\lim x_n = \alpha$ *, as* $\lim M^n = 0$.

Condition (6) is clearly satisfied if function g(x) possesses a derivative g'(x) such that $| g'(x) | < 1$ for $| x-\alpha | < | x_0 - \alpha|$.

If x_n is closed to α , then we have

$$\begin{aligned} | x_{n+1} - \alpha | &= | g(x_n) - g(\alpha) | \\ &\leq g'(\xi) | x_n - \alpha | \end{aligned} \quad (23)$$

for some ξ between x_0 and α .

Therefore, condition for convergence is $| g'(\xi) | < 1$.

Example 9 :

Lets consider $f(x) = x^3 + x - 2$, which we can see has a single root at x=1. There are several ways $f(x)$=0 can be written in the desired form, x=$g(x)$.

The simplest is

$$x_{n+1} = x_n + f(x_n) = x_n^3 + 2x_n - 2$$

In this case, $g'(x) = 3x^2 + 2$, and the convergence condition is

$$1 > |g'(x)| = 3x^2 + 2, \qquad -1 > 3x^2$$

Since this is never true, this doesn't converge to the root.

An alternate rearrangement is

$$x_{n+1} = 2 - x_n^3$$

This converges when

$$1 > |g'(x)| = |-3x^2|, \qquad x^2 < \frac{1}{3}, \qquad |x| < \frac{1}{\sqrt{3}}$$

Since this range does not include the root, this method won't converge either.

Another obvious rearrangement is

$$x_{n+1} = \sqrt[3]{2 - x_n}$$

In this case the convergence condition becomes

$$\frac{1}{3}\left|(2 - x_n)^{-\frac{2}{3}}\right| < 1, \qquad (2 - x_n)^{-2} < 3^3, \qquad |x_n - 2| > \sqrt{27}$$

Again, this region excludes the root.

Another possibility is obtained by dividing by x^2+1

$$x_{n+1} = \frac{2}{x_n^2 + 1}$$

In this case the convergence condition becomes

$$\frac{4|x|}{(1 + x^2)^2} < 1, \qquad 4|x| < (1 + x^2)^2$$

Consideration of this ineqaulity shows it is satisified if $x>1$, so if we start with such an x, this will converge to the root.

Clearly, finding a method of this type which converges is not always straightforwards