# UNIT 3 SOLUTION OF LINEAR ALGEBRAIC EQUATIONS

**Structure** **Page Nos.**



## 3.0 INTRODUCTION

Systems of linear equations arise in many areas of study, both directly in modelling physical situations and indirectly in the numerical solution of other mathematical models. Linear algebraic equations occur in the linear optimization theory, least square fitting of data, numerical solution of ordinary and partial differential equations, statistical interference etc. Therefore, finding the numerical solution of a system of linear equations is an important area of study.

From study of algebra, you must be familiar with the following two common methods of solving a system of linear equations :

1) By the elimination of the variables by elementary row operations.
2) By the use of determinants, a method better known as *Cramer's rule*.

When smaller number of equations are involved, Cramer's rule appears to be better than elimination method. However, Cramer's rule is completely impractical when a large number of equations are to be solved because here n+1 determinants are to be computed for $n$ unknowns.

Numerical methods for solving linear algebraic systems can be divided into two methods, **direct** and **iterative.** Direct methods are those which, in the absense of round-off or other errors, yield exact solution in a finite number of arithmetic operations. Iterative methods, on the other hand, start with an initial guess and by applying a suitable procedure, give successively better approximations.

To understand, the numerical methods for solving linear systems of equations, it is necessary to have some knowledge of properties of the matrices. You might have studied matrices, determinants and their properties in your linear algebra course.

In this unit, we shall discuss two direct methods, namely, **Gauss elimination method** and **LU decomposition method**, and two iterative methods, *viz.*; **Jacobi method, Gauss – Seidel method** and **Successive over relaxation method**.These methods are frequently used to solve systems of linear equations.

## 3.1 OBJECTIVES

After studying this unit, you should be able to:

- state the difference between direct and iterative methods for solving a system of linear equations;
- learn how to solve a system of linear equations by Gauss elimination method;
- understand the effect of round off errors on the solution obtained by Gauss elimination method;
- learn how to modify Gauss elimination method to Gaussian elimination with partial pivoting to avoid pitfalls of the former method;
- learn LU decomposition method to solve a system of linear equations;
- learn how to find inverse of a square matrix numerically;
- learn how to obtain the solution of a sytem of linear equations by using an iterative method, and
- state whether an iterative method will converge or not.

## 3.2 GAUSS ELIMINATION METHOD

One of the most popular techniques for solving simultaneous linear equations is the Gaussian elimination method. Karl Friedrich Gauss, a great 19th century mathematician, suggested this elimination method as a part of his proof of a particular theorem. Computational scientists use this "proof" as a direct computational method. The approach is designed to solve a general set of *n* equations and *n* unknowns

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\
&\quad . \qquad\quad . \qquad\quad . \qquad\quad . \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n
\end{aligned} \tag{1}$$

In matrix form, we write Ax = b, where

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_n \end{bmatrix}$$

Gaussian elimination consists of two steps:

1) *Forward Elimination:* In this step, the elementary row operations are applied on the augmented matrix [A|b] to transform the coefficient matrix A into upper triangular form.

2) *Back Substitution:* In this step, starting from the last equation, each of the unknowns is found by back substitution.

**Forward Elimination of Unknowns**: In this first step the first unknown, $x_1$ is eliminated from all rows below the first row. The first equation is selected as the pivot equation to eliminate $x_1$. So, to eliminate $x_1$ in the second equation, one divides the first equation by $a_{11}$ (hence called the pivot element) and then multiply it by $a_{21}$. That is, same as multiplying the first equation by $a_{21}/a_{11}$ to give

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

Now, this equation is subtracted from the second equation to give

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n = b_2 - \frac{a_{21}}{a_{11}}b_1$$

or $\quad a'_{22}x_2 + \ldots + a'_{2n}x_n = b_2$

where $a'_{22} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}, \ldots,$

$$a'_{2n} = a_{2n} - \frac{a_{21}}{a_{11}}a_{1n},\ b'_2 = b_2 - \frac{a_{21}}{a_{11}}b_1 .$$

This procedure of eliminating $x_1$, is now repeated for the third equation to the $n^{th}$ equation to reduce the set of equations as

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
&a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b_2 \\
&a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n = b_3 \\
&\quad . \qquad . \qquad . \qquad . \\
&a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n = b'_n .
\end{aligned} \tag{2}$$

This completes the first step of forward elimination. Now, for the second step of forward elimination, we start with the second equation as the pivot equation and $a'_{22}$ as the pivot element. So, to eliminate $x_2$ in the third equation, one divides the second equation by $a'_{22}$ (the pivot element) and then multiply it by $a'_{32}$. That is, same as multiplying the second equation by $a'_{32}/a'_{22}$ and subtracting from the third equation. This makes the coefficient of $x_2$ zero in the third equation. The same procedure is now repeated for the fourth equation till the nth equation to give

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
&a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2 \\
&a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_3 \\
&\quad . \qquad . \\
&a''_{n3}x_3 + \ldots + a''_{nn}x_n = b''_n
\end{aligned} \tag{3}$$

The next steps of forward elimination are done by using the third equation as a pivot equation and so on. That is, there will be a total of (n–1) steps of forward elimination. At the end of (n–1) steps of forward elimination, we get the set of equations

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
&a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2
\end{aligned}$$

$$a''_{33}x_3 + \ldots + a''_n x_n = b''_3 \quad (4)$$

$$\vdots$$

$$a_{nn}^{(n-1)} x_n = b_n^{(n-1)}$$

**Back Substitution:** Now, the equations are solved starting from the last equation as it has only one unknown. We obtain

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Now, we solve the (n–1)th equation to give

$$x_{n-1} = \frac{1}{a_{n-1,n-2}^{(n-2)}} \left[ b_{n-1}^{(n-2)} - a_{n-1,n}^{(n-2)} x_n \right]$$

since $x_n$ is determined.

We repeat the procedure until $x_1$ is determined. The solution is given by

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

and $$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}}, \text{ for } i = n-1, n-2, \ldots, 1 \quad (5)$$

**Example 1:** Solve the following linear system of equations

$x_1 + x_2 + x_3 = 3,$
$4x_1 + 3x_2 + 4x_3 = 8,$
$9x_1 + 3x_2 + 4x_3 = 7$

using the Gauss elimination method.

**Solution:** In augmented form, we write the system as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 3 & 4 & 8 \\ 9 & 3 & 4 & 7 \end{array}\right]$$

Subtracting 4 times the first row from the second row gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 9 & 3 & 4 & 7 \end{array}\right]$$

Subtracting 9 times the first row from the third row, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & -5 & -20 \end{array}\right]$$

Subtracting 6 times the second row from the third row gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & -5 & 4 \end{array}\right]$$

Restoring the transformed matrix equation gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 4 \end{bmatrix},$$

Solving the last equation, we get $x_3 = \frac{-4}{5}$. Solving the second equation, we get $x_2 = 4$ and the first equation gives $x_1 = 3 - x_2 - x_3 = 3 - 4 + \frac{4}{5} = \frac{-1}{5}$.

**Example 2:** Use Gauss Elimination to solve

$10x_1 - 7x_2 = 7$

$-3x_1 + 2.099x_2 + 6x_3 = 3.901$

$5x_1 - x_2 + 5x_3 = 6$

correct to six places of significant digits.

**Solution :** In matrix form , we write

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Multiply the first row by 3/10 and add to the second equation, we get

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 6 \end{bmatrix}$$

Multiply the first row by 5/10 and subtract from the third equation, we get

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

This completes the first step of forward elimination.

Multiply the second equation by 2.5/(–0.005)= –2500 and subtract from the third equation, we obtain

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15005 \end{bmatrix}$$

We can now solve the above equations by back substitution. From the third equation, we get

$15005x_3 = 15005$, or $x_3 = 1$.

Substituting the value of $x_3$ in the second equation, we get

$-0.001x_2 + 6x_3 = 6.001$, or $-0.001\, x_2 = 6.001 - 6 = 0.001$, or $x_2 = -1$

Substituting the values of $x_3$ and $x_2$ in the first equation, we get

$10x_1 - 7x_2 = 7$ , or $10x_1 = 7 + 7x_2 = 0$, or $x_1 = 0$.

Hence, the solution is $\begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T$,

## 3.3 PITFALLS OF GAUSS ELIMINATION METHOD

There are two pitfalls in the Gauss elimination method.

*Division by zero*: It is possible that division by zero may occur during forward elimination steps. For example, for the set of equations

$$10x_2 - 7x_3 = 7$$
$$6x_1 + 2.099x_2 - 3x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

during the first forward elimination step, the coefficient of $x_1$ is zero and hence normalisation would require division by zero.

*Round-off error*: Gauss elimination method is prone to round-off errors. This is true, when there are large numbers of equations as errors propagate. Also, if there is subtraction of almost equal numbers, it may create large errors. We illustrate through the following examples.

**Example 3:** Solve the following linear equations

$$10^{-5}\, x + y = 1.0,$$
$$x + y = 2.0 \qquad (6)$$

correct to 4 places of accuracy.

**Solution:** For 4 places of accuracy the solution is , $x \approx y \approx 1.0$ .
Applying the Gauss elimination method, we get (by dividing with the pivotal element)

$$x + 10^5\, y = 10^5$$
$$(1 - 10^5)\, y = 2.0 - 10^5.$$

Now, $10^5 - 1$ when rounded to four places of accuracy, become is $10^5$. Similarly, $10^5 - 2$ when rounded to four places of accuracy becomes $10^5$.

Hence, from the second equation we get, $10^5\, y = 10^5$, or $y = 1.0$.

Substituting in the first equation, we get x = 0.0, which is not the solution.
Such errors can also arise when we perform computations with less number of digits. To avoid these computational disasters, we apply partial pivoting to gauss elimination.

## 3.4 GAUSS ELIMINATION METHOD WITH PARTIAL PIVOTING

We perform the following modification to the Gauss elimination method. At the beginning of each step of forward elimination, a row interchange is done, if necessary, based on the following criterion. If there are $n$ equations, then there are (n − 1) forward elimination steps. At the beginning of the k$^{th}$ step of forward elimination, we find the maximum of

$|a_{kk}|,\ |a_{k+1,k}|,\ \ldots\ldots\ldots\ldots,\ |a_{nk}|$

That is, maximum in magnitude of this elements on or below the diagonal element.

Then, if the maximum of these values is $|a_{pk}|$ in the p$^{th}$ row, $k \le p \le n$, then interchange rows p and k.. The other steps of forward elimination are the same as in Gauss elimination method. The back substitution steps remain exactly the same as in Gauss elimination method.

**Example 4:** Consider Example 3. We now apply partial pivoting on system (6).

**Solution:** We obtain the new system as

Since, $a_{11} < a_{21}$, we interchange the first and second rows (equations).

$$x + y = 2.0$$
$$10^{-5}x + y = 1.0$$

On elimination , we get second equation as y = 1.0 correct to 4 places. Substituting in the first equation, we get x = 1.0, which is the correct solution.

## 3.5 LU DECOMPOSITION METHOD

The Gauss elimination method has the disadvantage that the right-hand sides are modified (repeatedly) during the steps of elimination). The LU decomposition method has the property that the matrix modification (or decomposition) step can be performed independent of the right hand side vector. This feature is quite useful in practice. Therefore, the LU decomposition method is usually chosen for computations.

In this method, the coefficient matrix into a product of two matrices is written as

$$\mathbf{A = L\,U} \tag{7}$$

where **L** is a lower triangular matrix and **U** is an upper triangular matrix.

Now, the original system of equations, **A x = b** becomes

$$\mathbf{LU\,x = b} \tag{8}$$

Now, set **U x = y,** then, (8) becomes

$$\mathbf{Ly = b} \tag{9}$$

The rationale behind this approach is that the two systems given in (9) are both easy to solve. Since, **L** is a lower triangular matrix, the equations, **Ly = b** , can be solved for *y* using the forward substitution step. Since **U** is an upper triangular matrix, **U x = y** can be solved for *x* using the back substitution algorithm.

We define writing **A** as **LU** as the Decomposition Step. We discuss the following three approaches of Decomposition using $4 \times 4$ matrices.

**Doolittle Decomposition**

We choose $l_{ii} = 1$, i=1, 2, 3, and write

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{10}$$

Because of the specific structure of the matrices, we can derive a systematic set of formulae for the components of **L** and **U** .

**Crout Decomposition**:

We choose $u_{ii} = 1$ , I = 1, 2, 3, 4 and write

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{11}$$

The evaluation of the components of **L** and **U** is done in a similar fashion as above.

**Cholesky Factorization**:

If A is a symmetric and positive definite matrix, then we can write the decomposition as

Where L is the lower triangular matrix

$$L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \tag{12}$$

We now describe the rationale behind the choice of $l_{ii} = 1$ in (10) or $u_{ii} = 1$ in (11). Consider the decomposition of a $3 \times 3$ matrix as follows.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{13}$$

We note that L has 1+2+3=6 unknowns and U has 3+2+1=6 unknowns, that is, a total of 12 unknowns. Comparing the elements of the matrices on the left and right hand sides of (13), we get 9 equations to determine 12 unknowns . Hence, have 3 arbitrary parameters, the choice of which can be done in advance. Therefore, to make computations easy we choose $l_{ii} = 1$ in Doolittle method and $u_{ii} = 1$ in Crout's method.

In the general case of decomposition of an N $\times$N matrix, L has 1+2+3+….+N= $[N(N+1)/2]$ And U also has N(N+1)/2 unknowns, that is a total of $N^2 + N$ unknowns comparing the elements of A and the product LU, we obtain $N^2$ equations. Hence, we have N arbitrary parameters. Therefore, we choose either $l_{ii} = 1$ or $u_{ii} = 1$ ,i = 1,2,….n,

Now , let us give the solution for the Doolittle and Crout decomposition.

**Doolittle Method:** Here $l_{ii} = 1$, i = 1 to N. In this case, generalisation of (13) gives

$u_{1j} = a_{1j}$, j = 1 to N
$l_{i1} = a_{i1} / a_{11}$, i = 2 to N
$u_{2j} = a_{2j} - l_{21} . u_{1j}$ , j=2 to N
$l_{i2} = ( a_{i2} - l_{i1} u_{12} ) / u_{22}$, i = 3 to N, and so on

**Crout's Method:** Here $u_{ii} = 1$, i = 1 to N . In this case , we get

$l_{i1} = a_{i1}$ , i=1 to N
$u_{1j} = a_{1j} / a_{11}$ , j= 2 to N
$l_{i2} = a_{i2} - l_{i1} u_{12}$, i = 2 to N
$u_{2j} = ( a_{2j} - l_{21} u_{1j} ) / l_{22}$, j= 3 to N, and so on

**Example 5:** Given the following system of linear equations, determine the value of each of the variables using the LU decomposition method.

$6x_1 - 2x_2 = 14$
$9x_1 - x_2 + x_3 = 21$
$3x_1 - 7x_2 + 5x_3 = 9$

**Solution:** We write A = LU, with $u_{ii} = 1$ as

$$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & +7 & 5 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{33} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

We obtain $l_{11} = 6$, $l_{21} = 9$, $l_{31} = 3$, $l_{11}u_{12} = -2$, $u_{12} = -1/3$;

$l_{11} u_{13} = 0$, $u_{13} = 0$

$l_{21}u_{12} + l_{22} = -2$, $l_{22} = -1 + 3 = 2$; $l_{21}u_{13} + l_{22} u_{23} = 1$

$u_{23} = 1/2$, $l_{31}u_{12} + l_{32} = +7$, $l_{32} = +7 + 1 = 8$;

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 5$, $l_{33} = 5 - 4 = +1$.

Hence, $L = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & +1 \end{bmatrix}, U = \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

Solving $L_y = b$, $\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \\ 9 \end{bmatrix}$

We get, $y_1 = \frac{14}{6} = \frac{7}{3}$; $9\,y_1 + 2\,y_2 = 21$, $y_2 = \frac{1}{2}(21–21) = 0$

$3\,y_1 + 8\,y_2 + y_3 = 9$, $y_3 = 9 - 7 = 2$.

Solving $U_x = y$, $\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/3 \\ 0 \\ 2 \end{bmatrix}$

We get, $x_3 = 2$, $x_2 + \frac{1}{2}\,x_3 = 0$, $x_2 = -1$; $x_1 - \frac{1}{3}\,x_2 = \frac{7}{3}$, $x_1 = \frac{7}{3} - \frac{1}{3} = 2$.

The solution vector is [2 –1 2].

**Solving the system of Equations**

After decomposing A as A = LU, the next step is to computate the solution. We have,

LUX = b, set UX = y

Solve first Ly = b, by forward substitution. Then, solve Ux = y, by backward substitution to get the solution vector x.

## 3.6 ITERATIVE METHODS

Iterate means repeat. Hence, an iterative method repeats its process over and over, each time using the current approximation to produce a better approximation for the true solution, until the current approximation is sufficiently close to the true solution – or until you realize that the sequence of approximations resulting from these iterations is not converging to the true solution.

Given an initial guess or approximation $\boldsymbol{x}^{(0)}$ for the true solution $\boldsymbol{x}$, we use $\boldsymbol{x}^{(0)}$ to find a new approximation $\boldsymbol{x}^{(1)}$, then we use $\boldsymbol{x}^{(1)}$ to find the better approximation $\boldsymbol{x}^{(2)}$, and so on. We expect that $\boldsymbol{x}^{(k)} \rightarrow \boldsymbol{x}$ as $k \rightarrow \infty$; that is, our approximations should become closer to the true solution as we take more iterations of this process.

Since, we do not actually have the true solution $\boldsymbol{x}$, we cannot check to see how close our current approximation $\boldsymbol{x}^{(k)}$ is to $\boldsymbol{x}$. One common way to check the closeness of $\boldsymbol{x}^{(k)}$ to $\boldsymbol{x}$ is instead by checking how close $\boldsymbol{Ax}^{(k)}$ is to $\boldsymbol{Ax}$, that is, how close $\boldsymbol{Ax}^{(k)}$ is to $\boldsymbol{b}$.

Another way to check the accuracy of our current approximation is by looking at the magnitude of the difference in successive approximations, $|\, \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)} \,|$. We expect $\boldsymbol{x}^{(k)}$ to be close to $\mathbf{x}$ if $|\, \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)} \,|$ is small .

**The Jacobi Method**

This method is also called Gauss – Jacobi method. In Jacobi method, the first equation is used to solve for $x_1$, second equation is used to solve $x_2$ etc. That is,

$$x_1 = \frac{1}{a_{11}}\left[b_1 - (a_{12}x_2 + \ldots\ldots + a_{1n}x_n)\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - (a_{21}x_1 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n)\right]$$

If in the ***i***th equation

$$\sum_{j=1}^{n} a_{i,j}x_j = b_i \tag{15}$$

we solve for the value of $\boldsymbol{x_i}$, we obtain ,

$$x_i = (b_i - \sum_{j\neq 1}^{n} a_{i,j}x_j) / a_{i,j} \tag{16}$$

This suggests an iterative method defined by

$$x_i^{(k)} = (b_i - \sum_{j\neq i} a_{i,j}x_j^{(k-1)} / a_{i,j} \tag{17}$$

which is the Jacobi method. Note that the order in which the equations are solved is irrelevant, since the Jacobi method treats them independently. For this reason, the Jacobi method is also known as the *method of simultaneous displacements*, since the updates could in principle be done simultaneously.

Jacobi method can be written in matrix notation.

Let A be written as A = L + D + U, where L is strictly lower triangular part, D the diagonal part and U is strictly upper triangular part.

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \ddots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, D = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$$

Therefore, we have

(L + D + U) X = b, or DX = – (L + U) x + b

Since, $a_{ii} \neq 0$, $D^{-1}$ exists and is equal to

$D^{-1}$ = diag ( $1/a_{11}$, $1/a_{12}$, ……$1/a_{nn}$).

Inverting D, we write the intersection as

$$X^{(k+1)} = -D^{-1}(L + U)X^{(k)} + D^{-1}b \tag{18}$$

$$= M_J x^{(k)} + C \tag{19}$$

where $M_J - D^{-1}(L + U)$ and $C = D^{-1}b$.

The matrix $M_J$ is called the iteration matrix. Covergence of the method depends on the properties of the matrix $M_J$.

**Diagonally dominant**: A matrix A is said to be diagonally dominant if

$$| a_{ii} | \geq \sum_{j=1, i \neq j}^{n} | a_{ij} | \qquad (20)$$

with inequality satisfied for atleast one row.

**Convergence**: (i) The Jacobi method converges when the matrix **A** is diagonally dominant. However, this is a sufficient condition and necessary condition.

(ii) The Jacobi method converges if Spectral radius $(M_J) < 1$. Where Spectiral radius of a matrix = $\max_i = |\lambda_i|$ and $x_i$ are eigenralues of $M_J$.This is a necessary and sufficient condition. If no intial approximation is known, we may assume $X^{(0)} = 0$.

**Exercise 1**: Are the following matrices diagonally dominant?

$$\mathbf{A} = \begin{bmatrix} 2 & -5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

**Solution:** In A, all the three rows violate the condition (20). Hence, A is not diagonally dominant.

In B, in the row $| -129 | = 129 < 96 + 34 = 130$. Therefore, B is also not diagonally dominant.

**Example 2:** Solve the following system, of equations
$x + y - z = 0$, $-x + 3y = 2$, $x - 2z = -3$ by Jacobi Method, both directly and in matrix form. Assume the initial solution vector as $[0.8 \quad 0.8 \quad 2.1]^T$.

**Solution :** We write the Jacobi method as
$x^{(k+1)} = -y^{(k)} + z^{(k)}$, $y^{(k+1)} = \frac{1}{3}(2 + x^{(k)})$, $z^{(k+1)} = \frac{1}{2}(3 + x^{(k)})$
with $x^{(0)} = 0.8$, $y^{(0)} = 0.8$, $z^{(0)} = 2.1$, we get the following approximations.

$x^{(1)} = 1.3$, $y^{(1)} = 0.9333$, $z^{(1)} = 1.9$;
$x^{(2)} = 0.9667$, $y^{(2)} = 1.1$, $z^{(2)} = 2.5$;
$x^{(3)} = 1.0500$, $y^{(3)} = 0.9889$, $z^{(3)} = 1.98335$;
$x^{(4)} = 0.99445$, $y^{(4)} = 1.01667$, $z^{(4)} = 2.025$;
$x^{(5)} = 1.00833$, $y^{(5)} = 0.99815$, $z^{(5)} = 1.997225$;
$x^{(6)} = 0.988895$, $y^{(6)} = 1.00278$, $z^{(6)} = 2.004165$;
$x^{(7)} = 1.001385$, $y^{(7)} = 0.99630$, $z^{(7)} = 1.99445$;
$x^{(8)} = 0.99815$, $y^{(8)} = 1.00046$, $z^{(8)} = 2.00069$;
$x^{(9)} = 1.00023$, $y^{(9)} = 0.99938$, $z^{(9)} = 1.999075$.

At this stage, we have,

$| x^{(9)} - x^{(8)} | = 0.002$, $| y^{(9)} - y^{(8)} | = 0.0019$, $| z^{(9)} - z^{(8)} | = 0.0016$.

Therefore, the $9^{th}$ interaction is correct to two decimal places.

Let us represent the matrix $A$ in the form

$$A = L + D + U = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have,

$$Mj = -D^{-1}(L+U) = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1/3 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

$$c = D^{-1}b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 2/3 \\ 3/2 \end{bmatrix}$$

Therefore, Jacobi method gives,

$$X^{(k+1)} = \begin{bmatrix} 0 & -1 & 1 \\ 1/3 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} X^{(k)} + \begin{bmatrix} 0 \\ 2/3 \\ 3/2 \end{bmatrix}$$

The intial approximation is given as $X^{(0)} = [0.8 \quad 0.8 \quad 2.1]^T$

Then, we have

$$X^{(1)} = \begin{bmatrix} 0 & -1 & 1 \\ 1/3 & 0 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.8 \\ 2.1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2/3 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0.9333 \\ 1.9 \end{bmatrix}$$

which is same as $X^{(1)}$ obtained earlier.

Since, the two procedures (direct and in matrix form) are identical, we get the same approximations $x^{(2)}$, ….. $x^{(9)}$. The exact solution is $x = [1 \quad 1 \quad 2]^T$.

Note that the coefficient matrix A is not diagonal dominant. But, we have obtained the solution correct to two decimal places in 9 interactions. This shows that the requirement of A being diagonal dominant is a sufficient condition.

**Example 3:** Solve by Jacobi's method the following system of linear equations.

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ x_1 + 2x_2 - x_3 &= 6 \\ x_1 - x_2 + 2x_3 &= -3. \end{aligned}$$

**Solution:** This system can be written as

$$\begin{aligned} x_1 &= 0.5\, x_2 - 0.5\, x_3 - 0.5 \\ x_2 &= -\,0.5\, x_1 + 0.5\, x_3 + 3.0 \\ x_3 &= -\,0.5\, x_1 + 0.5\, x_2 - 1.5 \end{aligned}$$

So the Jacobi iteration is

$$\begin{bmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{bmatrix} = \begin{bmatrix} 0.0 & 0.5 & -0.5 \\ -0.5 & 0.0 & 0.5 \\ -0.5 & 0.5 & 0.0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} + \begin{bmatrix} -0.5 \\ 3.0 \\ -1.5 \end{bmatrix}$$

Since, no intial approximation is given, we start with $x^{(0)} = (0, 0, 0)^T$. We get the following approximations.

$\mathbf{X}^{(1)} = [-0.5000 \quad 3.0000 \quad -1.5000]^T$
$\mathbf{X}^{(2)} = [1.7500 \quad 2.5000 \quad 0.2500]^T$
$\mathbf{X}^{(3)} = [0.6250 \quad 2.2500 \quad -1.1250]^T$
$\mathbf{X}^{(4)} = [1.1875 \quad 2.1250 \quad -0.6875]^T$
$\mathbf{X}^{(5)} = [0.9063 \quad 2.0625 \quad -1.0313]^T$
$\mathbf{X}^{(6)} = [1.0469 \quad 2.0313 \quad -0.9219]^T$
$\mathbf{X}^{(7)} = [0.9766 \quad 2.0156 \quad -1.0078]^T$
$\mathbf{X}^{(8)} = [1.0117 \quad 2.0078 \quad -0.9805]^T$
$\mathbf{X}^{(9)} = [0.9941 \quad 2.0039 \quad -1.0020]^T$
$\mathbf{X}^{(10)} = [1.0029 \quad 2.0020 \quad -0.9951]^T$
$\mathbf{X}^{(11)} = [0.9985 \quad 2.0010 \quad -1.0005]^T$
$\mathbf{X}^{(12)} = [1.0007 \quad 2.0005 \quad -0.9988]^T$
$\mathbf{X}^{(13)} = [0.9996 \quad 2.0002 \quad -1.0001]^T$
$\mathbf{X}^{(14)} = [1.0002 \quad 2.0001 \quad -0.9997]^T$

After 14 iteractions, the errors in the solutions are

$| x_1^{(14)} - x_1^{(13)} | = 0.0006$, $| x_2^{(14)} - x_2^{(13)} | = 0.0001$, $| x_3^{(14)} - x_3^{(13)} | = 0.0004$.

The solutions $x^{(14)}$ are therefore almost correct to 3 decimal places.

## The Gauss-Seidel Method

We observe from Examples 2 and 3 that even for a 3 *x* 3 system, the number of iterations taken by the Jacobi method (to achieve 2 or 3 decimal accuracy) is large. For large systems, the number of iterations required may run into thousands. Hence, the Jacobi method is slow. We also observe that when the variable $x^i$ is being iterated in say the k-th iteration, the variables, $x_1, \ldots. x_{i-1}$ have already been updated in the k-th iteration. However, these values are not being used to compute $x_i^{(k)}$. This is the disadvantage of the Jacobi method. If we use all the current available values, we call it the Gauss-Seidel method.

Therefore, Gauss – seidel method is defined by

$$x_i^{(k)} = (bi - \sum_{j<i}^{n} a_{i,j} x_j^{(k)} - \sum_{j>i}^{n} a_{i,j} x_j^{(k-1)}) / a_{i,\,i} \qquad (21)$$

Two important facts about the Gauss-Seidel method should be noted. First, the computations in (21) are serial. Since, each component of the new iterate depends upon all previously computed components, the updates cannot be done simultaneously as in the Jacobi method. Second, the new iterate depends upon the order in which the equations are being used. The Gauss-Seidel method is sometimes called the *method of successive displacements* to indicate the dependence of the iterates on the ordering. If this ordering is changed, the *components* of the new iterate (and not just their order) will also change.

To derive the matrix formulation, we write,

AX = (L + D + U) X = b     or     (L + D) X = – UX + b.

The Gauss-Seidel method can be expressed as

$$X^{(k+1)} = -(L+D)^{-1} U X^{(k)} + (L+D)^{-1} b$$
$$= M_G X^{(k)} + C \qquad (22)$$

where $M_G = -(L+D)^{-1} U$ is in iteration matrix and $C = (L+D)^{-1}$ b.

Again, convergence depends on the properties of $M_G$ if Spectral radius($M_G$) < 1, the iteraction converges always for any intial solution vector. Further, it is known that Gauss-Seidel method converges atleast two times faster than for Jacobi method.

**Example 4:** Solve the system in Example 2, by the Gauss-Seidel method. Write its matrix form.

**Solution:** Gauss-Seidel method for solving the system in Example 2 is given by

$$x^{(k+1)} = -y^{(k)} + z^{(k)},\ y^{(k+1)} = \frac{1}{3}(2 + x^{(k)}),\ z^{(k+1)} = \frac{1}{2}(3 + x^{(k)})$$

with $x^{(0)} = 0.8$, $y^{(0)} = 0.8$, $z^{(0)} = 2.1$, we obtain the following results.

$x^{(1)} = 1.3$, $y^{(1)} = 1.1$, $z^{(1)} = 2.15$;
$x^{(2)} = 1.05$, $y^{(2)} = 1.01667$, $z^{(2)} = 2.025$;
$x^{(3)} = 1.00833$, $y^{(3)} = 1.00278$, $z^{(3)} = 2.004165$;
$x^{(4)} = 1.001385$, $y^{(4)} = 1.00046$, $z^{(4)} = 2.00069$;
$x^{(5)} = 1.00023$, $y^{(5)} = 1.000077$, $z^{(5)} = 2.000115$;

The errors after the $5^{th}$ iteractions are

$| x^{(5)} - x^{(4)} | = 0.0012$, $| y^{(5)} - y^{(4)} | = 0.00038$, $| z^{(5)} - z^{(4)} | = 0.00057$.

In 5 iteractions, we have for two place accuracy, while 9 iteractions we required in the Jacobi method.

The matrix function can be written as

$$X^{(k+1)} = -\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X^{(k)} + \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$= +\frac{1}{6}\begin{bmatrix} -6 & 0 & 0 \\ -2 & -2 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X^{(k)} = \frac{1}{6}\begin{bmatrix} -6 & 0 & 0 \\ -2 & -2 & 0 \\ -3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$= +\frac{1}{6}\begin{bmatrix} 0 & -6 & 6 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{bmatrix} X^{(k)} - \frac{1}{6}\begin{bmatrix} 0 \\ -4 \\ -9 \end{bmatrix}$$

Starting with $X^{(0)} = [\ 0.8 \quad 0.8 \quad 2.1]^T$, we get the same iterated values as above.

**Example 5:** Solve the system given in Example 3 by Gauss-Seidel method.

**Solution:** For Gauss- Seidel iterations, the system in Example 3 can be written as

$$x_1^{(k+1)} = 0.5\, x_2^{(k)} - 0.5\, x_3^{(k)} - 0.5$$
$$x_2^{(k+1)} = -0.5\, x_1^{(k+1)} + 0.5\, x_3^{(k)} + 3.0$$
$$x_3^{(k+1)} = -0.5\, x_1^{(k+1)} + 0.5\, x_2^{(k+1)} + 1.5$$

Start with (0, 0, 0), we get the following values

$\mathbf{X}^{(1)} = [-0.5000 \quad 3.2500 \quad 0.3750]^T$
$\mathbf{X}^{(2)} = [0.9375 \quad 2.7188 \quad -0.6094]^T$
$\mathbf{X}^{(3)} = [1.1641 \quad 2.1133 \quad -1.0254]^T$
$\mathbf{X}^{(4)} = [1.0693 \quad 1.9526 \quad -1.0583]^T$
$\mathbf{X}^{(5)} = [1.0055 \quad 1.9681 \quad -1.0187]^T$
$\mathbf{X}^{(6)} = [0.9934 \quad 1.9939 \quad -0.9997]^T$
$\mathbf{X}^{(7)} = [0.9968 \quad 2.0017 \quad -0.9976]^T$
$\mathbf{X}^{(8)} = [0.9996 \quad 2.0014 \quad -0.9991]^T$
$\mathbf{X}^{(9)} = [1.0003 \quad 2.0003 \quad -1.0000]^T$
$\mathbf{X}^{(10)} = [1.0001 \quad 1.9999 \quad -1.0001]^T$

After 10 iterations, the errors in solutions are
$| x_1^{(10)} - x_1^{(9)} | = 0.0002$, $| x_2^{(10)} - x_1^{(9)} | = 0.0.0004$, $| x_3^{(10)} - x_3^{(9)} | = 0.0001$.

The solutions are correct to 3 decimal places.

## 3.7 SUMMARY

In this unit, we have discussed direct and interative methods for solving a system of linear equations. Under these categories of methods, used to solve a system of linear equations, we have discussed Gauss elimination method and LU decomposition method. We have also discussed the method of finding inverse of a square matrix. Further, under the category of iterative methods for root determination we have discussed Jacobi and Gauss Scidel method.

## 3.8 EXERCISES

E1. Solve the following systems using the Gauss elimination method.

(a) $3x_1 + 2x_2 + 3x_3 = 5,$
$x_1 + 4x_2 + 2x_3 = 4,$
$2x_1 + 4x_2 + 8x_3 = 8,$

(b) $3x_1 + x_2 + x_3 = 1.8,$
$2x_1 + 4x_2 + x_3 = 2.7,$
$x_1 + 3x_2 + 5x_3 = 4.0,$

(c) $x_1 - x_2 + x_3 = 0,$
$2x_1 + 3x_2 + x_3 - 2x_4 = -7$
$3x_1 + x_2 - x_3 + 4x_4 = 12,$
$3x_2 - 5x_3 + x_4 = 9$

(d) $3x_1 + x_2 = 5,$
$x_1 + 3x_2 + 6x_3 = 6$
$4x_2 + x_3 + 3x_4 = 7$
$x_3 + 5x_4 = 8,$

E2. Solve the following systems using the LU decomposition method.

(a) $3x + y + z = 3,$
$x + 4y + 2z = 0,$
$2x + y + 5z = 4,$

(b) $2x + y + z = 5,$
$x + 3y + 2z = 4,$
$-x + y + 6z = 4,$

(c) $4x + y + 2z = 3.6,$
$x + 3y + z = 2.5,$
$2x + y + 2z = 4.0,$

(d) $3x + y = -2,$
$x + 3y - z = 0,$
$-y + 7z = 13.$

E3. For problems in 1(a), (b); 2(a), (b), (c), (d), obtain the solution to 3 decimals using the Jacobi and Gauss Seidel methods. Write the matrix formulations also. Assume the initial solution vectors respectively as

(i) $[0.8, 0.6, 0.5]^T$,
(ii) $[0.3, 0.3, 0.6]^T$,
(iii) $[0.9, -0.6, 0.6]^T$,
(iv) $[1.9, 0.2, 0.9]^T$,
(v) $[0.2, 0.5, 1.1]^T$,
(vi) $[-1.1, 0.9, 2.1]^T$.

## 3.9 SOLUTIONS TO ANSWERS

1. (a) 1, ½, ½ . (b) 0.3, 0.4, 0.5.
   (c) 1, −1, −2, 2. (d) 3/2, ½, ½, 3/2.

2. (a) 1, −1/2, 1/2. (b) 2, 0, 1.
   (c) 0.3, 0.4, 1. (d) −1, 1, 2, (You can also try $LL^T$ decomposition)

3. Refer to Page 49 and 52.