UNIT 3 SOLVING RECURRENCES

Structure **Page Nos.**

3.0 INTRODUCTION

In the two previous units of this block, you have studied about setting up recurrences and how to solve them by the use of generating functions. In this unit we concentrate on other methods of finding solutions of recurrence equations.

To begin with, we shall develop the general theory for solving a linear homogeneous recurrence with constant coefficients. Following this, we shall discuss some general theory for solving a linear non-homogeneous recurrence whose non-homogeneous part is a polynomial or an exponential function. We shall conclude the unit by illustrating several techniques developed for solving recurrences which may otherwise be hard to solve by more standard methods. We shall also look at examples of real-life applications of the theory we discuss.

As you can see, this unit is closely linked with Unit 1. So, please glance over that unit again before going further.

Let us now clearly spell out the objectives of this unit.

3.1 OBJECTIVES

After going through this unit, you should be able to

- Find the characteristic polynomial, equation and roots of a linear, homogeneous recurrence relation with constant coefficients;
- Solve any linear, homogeneous recurrence relation with constant coefficients;
- Solve linear, non-homogeneous recurrences with constant coefficients when the non-homogenous part is either a polynomial or an exponential function;
- Solve recurrence relations by the method of inspection / telescopic sums/iteration/substitution, wherever applicable.

3.2 LINEAR HOMOGENEOUS RECURRENCES

You would recall from Unit 1 that the general form of a linear, non-homogeneous recurrence of order k is

$$u_n = f_1(n)\,u_{n-1} + f_2(u)\,u_{n-2} + \cdots + f_k(n)\,u_{n-k} + g(n), n \geq k,$$

where each f_j and g is a function of n. It is homogeneous if g is identically zero, and non-homogeneous otherwise.

Now, let us assume that g is non-zero. Then, associated with the non-homogeneous recurrence is the homogeneous recurrence

$$u_n = f_1(n)\,u_{n-1} + f_2(n)\,u_{n-2} + \cdots + f_k(u)\,u_{n-k}, n \geq k,$$

which we get by simply setting to zero the non-homogeneous part.

Let us concentrate on recurrences whose homogeneous parts are linear. You know that the most general linear homogeneous equation with constant coefficients is

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_k u_{n-k}, n \geq k, \qquad (1)$$

where the c_i are constants.

Related to this is an equation that we shall now define.

Definitions: The characteristic equation, or auxiliary equation, of the linear, homogeneous recurrence (1) is the equation

$$z^k - c_1 z^{k-1} c_2 z^{k-2} - \cdots - c_{k-1} - c_k = 0. \qquad (2)$$

The roots of the characteristic equation (2) are called the characteristic roots of (1).

The **multiplicity of a characteristic root** α of (1) is the greatest integer m such that $(z-\alpha)^m$ is a factor of the **characteristic polynomial** of (1) i.e., of $z^k - c_1 z^{k-1} - \cdots - c_{k.}$

Notice that the characteristic equation is simply obtained by setting the m^{th} term of the sequence $\{u_n\}$ equal to z^m in the recurrence, and simplifying.

For instance, the characteristic equation of the recurrence

$$u_{n+2} = 2u_n - u_{n-2},\ n \geq 2,$$

is

$$z^{n+2} = 2z^n - z^{n-2}, \text{i.e.,} z^4 = 2z^2 - 1.$$

Therefore, the characteristic roots of this recurrence are 1 and –1, both with multiplicity 2.

Now, given the characteristic roots of a recurrence, how do we solve it? As you know from Unit 2, solving a recurrence means finding a sequence $\{a_n\}$ that satisfies it, where a_n is a function of n. Often, if we can find such a sequence, then we shall (somewhat carelessly!) say a_n is a solution.

Now, to try and understand how to solve recurrences like (1), let us consider the recurrence

$$a_n = 16a_{n-2}$$

You can check that $a_n = A(4)^n + B(-4)^n$, satisfies the recurrence where A and B are constants. Observe that 4 and –4 are the roots of the characteristic equation, $z^2=16$. of the given recurrence. Both these roots have multiplicity 1.

Now let us consider the recurrence

$$a_{n+2} = 2a_{n+1} + 4a_n - 8a_{n-1}.$$

You can check that its characteristic polynomial is

$$z^3 - 2z^2 - 4z + 8, \text{i.e.,} (z-2)^2\,(z+2).$$

So, its characteristic roots are 2 (with multiplicity 2) and –2 (with multiplicity 1).

You can also check that the general solution of the given recurrence is

$$a_n = (A_0 + A_1 n)\,(2)^n + B_0(-2)^n, A_0, A_1, B_0 \in C.$$

We can write this as

$$a_n = A'_0, A'_1, B_0 \in C. A'_0\, C(n,0)2^n + A'_1\, C(n+1,0)2^n + B_0 C(n,0)(-2)^n$$

Have these examples given you an inkling of the general form of the solution of (1) in terms of its characteristic roots? Match your conclusions with the following theorem.

Theorem 1: A sequence $\{a_n\}$ satisfies the linear, homogeneous recurrence relation with constant coefficients.

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_k u_{n-k}, n \geq k,$$

if and only if each a_n is a sum of expressions of the form

$$b_0\, C(n,0)\alpha_i^n + b_1\, C(1+n,1)\alpha_i^n + \cdots + b\, m_{i-1}\, C(m_i - 1 + n, m_i - 1)\alpha_i^n$$

where $\alpha_1, \alpha_2, \ldots$ are the characteristic roots of multiplicity m_1, m_2, … , respectively, and the b_js are constants.

Proof: We recall from Theorem 1 in Unit 2 that the generating function, U(z), of the sequence $\{u_n\}$ is of the form p (z) / q (z), where p and q are polynomials with deg p < deg q, and q (z) = $1 - c_1 z - c_2 z^2 - \cdots - c_k z^k$.

Now, $z^k - c_1 z^{k-1} - c_2 z^{k-2} - \cdots - c_{k-1} z - c_k = \prod\limits_i (z - \alpha_i)^{m_i}$

$$\Leftrightarrow z^k \left[1 - c_1\left(\frac{1}{z}\right) - c_2\left(\frac{1}{z}\right)^2 - \cdots - c_k\left(\frac{1}{z}\right)^k\right] = z^k \prod_i \left(1 - \frac{\alpha_i}{z}\right)^{m_i}$$

$$\Leftrightarrow 1 - c_1 t - c_2 t^2 - \cdots - c_k t^k = \prod_i (1 - \alpha_i t)^{m_i} \text{ where we put } t = \frac{1}{z}.$$

$$\therefore U(z) = \frac{p(z)}{q(z)}, \text{ where } q(z) = \prod_i (1 - \alpha_i z)^{m_i} \text{ and deg } p < \deg q.$$

So, using partial fractions, we can express U(z) as a linear combination of terms of the form $(1 - \alpha_i z)^{-j-1}$, where $0 \leq j \leq m_i - 1$. Since the coefficient of z^n in the expansion of $(1 - \alpha_i z)^{-j-1}$ equals $C(-j-1, n)\alpha_i^n$, i.e., $C(j+n, j)\alpha_i^n$, the theorem follows.

For $j \geq 0, C(-j, n) = (-1)^n C(j+n-1, n)$

Note that each a_n in the theorem is actually a finite linear combination of terms of the form $n^j \alpha^n$, where α is a characteristic root of multiplicity m, and $0 \leq j \leq m-1$. This is because the binomial coefficients C (j+n, j) are themselves polynomials of degree j in the variable n. It is often easier to express the solution in this form, as for instance, **when the characteristic roots are all distinct, i.e., of multiplicity one**. In this situation, the form of the solution sequence is

$$u_n = \sum_{j=1}^{k} A_j \alpha_j^n, n \geq 0,$$

where the α_js are the characteristic roots and A_js are constants that are to be determined by the initial conditions.

Let us look at some examples of how Theorem 1 can be applied. While doing so, let us see how the solution depends on the initial conditions.

Example 1: Solve the recurrence $a_n = 4a_{n-2}$, where

a) $a_0 = 4, a_1 = 6$

b) $a_0 = 6, a_2 = 20$

c) $a_1 = 6, a_2 = 20$

Solution: The roots of the characteristic equation of the recurrence, $z^2 = 4$, are ± 2. Thus, by Theorem 1, the general solution is of the form $a_n = A(2)^n + B(-2)^n$, where A and B are arbitrary constants.

a) Now, if $a_0 = 4$ and $a_1 = 6$, then the general solution gives us

$$A+B = 4 \text{ and } 2A-2B = 6.$$

$$\therefore A = \frac{7}{2}, B = \frac{1}{2}.$$

So the solution is $a_n = 7(2)^{n-1} - (-2)^{n-1}$.

b) If $a_0 = 6$ and $a_2 = 20$, the general solution yields

$$A+B = 6 \text{ and } 4A + 4B = 20.$$

Since these equations are inconsistent, **there is no solution.**

c) If $a_1 = 6, a_2 = 20$, we get

$$2(A-B) = 6 \text{ and } 4(A+B) = 20.$$

So, A = 4, B = 1, and the solution is

$$a_n = 4(2)^n + (-2)^n.$$

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In the example above you have seen how important the initial conditions are. You have also seen that sometimes these conditions can be such that no solution is possible.

Now consider a second order linear homogeneous recurrence with constant coefficients that you solved in Unit 2 by making use of generating functions. This equation can also be solved by applying Theorem 1, as you will just see.

Example 2: Obtain the solution for the recurrence relation satisfied by the Fibonacci sequence (see Unit 1).

Solution: Recall that the Fibonacci sequence $\{f_n\}$ satisfies

$$f_n - f_{n-1} - f_{n-2} = 0 \text{ if } n \geq 3, \text{ and } f_1 = 1 = f_2. \quad (3)$$

The characteristic equation, $z^2 - z - 1 = 0$, has distinct roots $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Therefore, by Theorem 1, for some constants A and B,

$$f_n = A\alpha^n + B\beta^n, n \geq 1. \quad (4)$$

This is the **general solution** for the recurrence (3).

As you have seen in the previous example, the values of A and B depend on the initial conditions, i.e, the first two terms of the sequence.

Since $f_1 = 1, (4) \Rightarrow 1 = A\alpha + B\beta$

Since $f_2 = 1, (4) \Rightarrow 1 = A\alpha^2 + B\beta^2$.

Also, since α and β are roots of $z^2 - z - 1 = 0$,

$$\alpha^2 = \alpha + 1 \text{ and } \beta^2 = \beta + 1.$$

So, we get $1 = A\alpha^2 + B\beta^2 = A(\alpha+1) + B(\beta+1) = (A\alpha + B\beta) + (A+B) = 1 + (A+B)$.

Therefore, A+B = 0

Therefore, A $(\alpha - \beta) = 1$, and

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}, n \geq 1$$

* * *

Now consider an example in which no initial conditions are given.

Example 3: Solve the sixth order linear, homogeneous recurrence relation

$$u_n + u_{n-1} - 11u_{n-2} - 13u_{n-3} + 26u_{n-4} + 20u_{n-5} - 24u_{n-6} = 0.$$

Solution: The first step is to identify the characteristic roots together with their multiplicities. The characteristic equation is

$$z^6 + z^5 - 11z^4 - 13z^3 + 26z^2 + 20z - 24 = 0$$

$$\text{i.e.,} (z-1)^2 (z-3)(z+2)^3 = 0.$$

Since the root 1 is of multiplicity two, the root 3 of multiplicity one and the root (–2) of multiplicity three, by Theorem 1 we know that u_n is a linear combination of six terms

$$C(0+n,0).1^n, C(1+n,1).1^n, C(0+n,0)3^n, C(0+n,0)(-2)^n, C(1+n,1)(-2)^n \text{ and } C(2+n,2)(-2)^n,$$

$$\text{i.e., } u_n = a + b(1+n) + c.3^n + d(-2)^n + e(1+n)(-2)^n + f.\frac{(1+n)(2+n)}{2}.(-2)^n,$$

where a, …, f are constants which can be determined if any six consecutive terms (typically, the first six) of the sequence are known. Since no initial conditions are given, we can only simplify the expression to the form

$$u_n = A + Bn + C.3^n + (D + En + Fn^2)(-2)^n \text{ where A, …, F are constants.}$$

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So far we have solved linear recurrence relations by using Theorem 1. Now let us solve a non-linear recurrence relation, by reducing it to a linear relation.

Example 4: Solve the recurrence $a_{n+1}^2 = 5a_n^2$, where $a_n > 0$ and $a_0 = 2$. And find a_8.

Solution: The given recurrence is a quadratic relation. But, if we put

$b_n = a_n^2$, the relation becomes

$b_{n+1} = 5b_n$, $b_0 = 4$.

From Theorem 1, you know that its solution is

$b_n = A(5)^n$, A a constant

Now, $b_0 = 4 \Rightarrow A = 4$.

$\therefore b_n = 4(5)^n$.

Since a_n is the positive square root of b_n,

$a_n = 2(5)^{n/2}$ for $n \geq 0$.

$\therefore a_8 = 1250.$

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Why don't you try some exercises now?

E1) Find the general solution of the recurrence relation

$a_n = 3a_{n-1}$.

E2) Determine constants c_1 and c_2 such that the recursion

$$u_n + c_1 u_{n-1} + c_2 u_{n-2} = 0 \text{ has the characteristic roots } 1 \pm \sqrt{-1}.$$

E3) Find the solution of the following recurrence equation satisfied by P_n^2, the number of partitions of n into two parts in non-increasing order:

$$P_n^2 = P_{n-1}^2 + P_{n-2}^2 - P_{n-3}^2, n \geq 3, P_1^2 = 0, P_2^2 = 1, P_3^2 = 1$$

In this section we have seen some ways of solving linear homogeneous recurrences with constant coefficients. You have also seen how some non-linear recurrences can be reduced to such linear recurrences, and hence solved. Let us now see how to use what we have discussed here for solving non-homogeneous recurrences with constant coefficients.

3.3 LINEAR NON-HOMOGENEOUS RECURRENCES

In this section we shall look at some general theory pertaining to finding solutions of equations like $u_n = 3u_{n-2} + 3n^5 - 2^n$. More generally, we shall study equations of the form

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \cdots + c_k u_{n-k} + g(n), n \geq k. \quad (5)$$

Looking at (5), you may wonder if the solutions of (1) and (5) are linked. The following theorems tell us something about this.

Theorem 2: If $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are two sequences, each satisfying the non-homogeneous recurrence (5), then $\{d_n\}$, with $d_n = a_n - b_n$, $n \geq 0$, satisfies the associated homogeneous recurrence (1).

Proof: Since $\{a_n\}$ and $\{b_n\}$ satisfy (5), and $d_n = a_n - b_n$, for $n \geq 0$ we get

$$\begin{aligned} d_n &= a_n - b_n \\ &= \left[c_1 a_{n-1} + \cdots + c_k a_{n-k} + g(n)\right] - \left[c_1 b_{n-1} + \cdots + c_k b_{n-k} + g(n)\right] \\ &= c_1 d_n + \cdots + c_k d_{n-k} \end{aligned}$$

This shows that $\{d_n\}$ satisfies (1), i.e., we have proved the statement.

Now, can you see how we can use Theorem 2 along with Theorem 1 to find the general form of any solution of (5)? The following result explicitly answers this question.

A particular solution of (5) is any sequence $\{a_n\}$ that satisfies (5)

Theorem 3: Every solution of the recurrence (5) is of the form a_n+b_n, where a_n is any particular solution of (5) and b_n is any solution of its associated homogeneous recurrence (1).

Proof: Let a_n be any particular solution of (5). Now, Theorem 2 tells us that the difference of any two solutions of (5) is a solution of (1). So, every solution u_n of (5) satisfies $u_n - a_n = b_n$, where b_n satisfies (1). Therefore, $u_n = a_n + b_n$, where a_n is a particular solution of (5) and b_n is a solution of (1).

We have proved the two theorems above only for linear recurrence relations with constant coefficient. But they hold true in the general case also. This is what the following exercise is about.

E4) State and prove the analogues of Theorems 2 and 3 for general recurrences of the form

$u_n = f_1(n)u_{n-1}+f_2(n)u_{n-2}+\cdots+f_k(n)u_{n-k}+g(n)$,

where the f_i s and g are functions of n.

In view of the two theorems above, to solve (5) we must look for any one solution of (5) and the general form of the solution of (1). Let us consider an example.

Example 5: Find the complete solution of the recurrence

$$a_n=3a_{n-1}-4n,\ n \geq 1.$$

Solution: The required solution, as Theorem 3 says, is the sum of the general solution of $a_n=3a_{n-1}$ and any solution of the given recurrence.

From E1 you know that the general solution of $a_n=3a_{n-1}$ is $a_n=b.3^n$, where b is a constant.

Now, let us consider the non-homogeneous part too. We have $a_n=3a_{n-1}-4n$.

Let us see if a_n can be of the form An + B, A, B $\in$ C.

If it is, then

$$An + B = 3[A(n-1) + B] - 4n = n(3A - 4) \quad - 3A + 3B.$$

Comparing the coefficients of n, we get

$$A = 3A - 4 \text{ and } B = 3B - 3A,$$
i.e. A = 2 and B = 3.

So, the total solution of the recurrence will be

$$a_n=b.3^n + 2n + 3,\ b \in C.$$

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In the example above, we have obtained a particular solution by guess work. In many cases we need to use such an approach. Unlike the homogeneous case, there is no general method to obtain a particular solution for a non-homogeneous recurrence. But there are techniques available for certain recurrences, including the one given in Example 5. The following theorems tell us about two special cases.

Theorem 4: A particular solution of (5) with non-homogeneous part an^d, where a is a known constant and d $\in$ N, is of the form

i) $A_0 + A_1 n + \cdots + A_d n^d$, **if 1 is not a characteristic root** of (5);

ii) $A_0 n^m + A_1 n^{m+1} + \cdots + A_d n^{m+d}$, if 1 is a characteristic root of (5) with multiplicity m,

where $A_0, A_1, \ldots, A_d$ are constants.

Theorem 5: A particular solution of (5) with non-homogeneous part ar^n (where a is a known constant) is of the form

i) Ar^n, if r is not a characteristic root of (5);

ii) $An^m r^n$, if r is a characteristic root of (5) with multiplicity m, where A is a constant.

We will not prove these results here, but shall look at a few examples of their use.

Example 6: Find the solution to the recurrence namely, $L_n = L_{n-1} + n$, $n \geq 2$, with

$L_1 = 2$.

Solution: Observe that 1 is the only characteristic root of this recurrence. So, the general solution to the homogeneous part of this recurrence is simply $a.1^n = a$, where a is a constant.

Now, the non-homogeneous part of the recurrence is n. So, applying Theorem 4(ii) with m = 1 and d = 1, we see that a particular solution of this recurrence is of the form

$$A_0 n + A_1 n^2,\ A_0, A_1 \in C.$$

To find the values of A_0 and A_1, we set $L_n = A_0 n + A_1 n^2$ in the recurrence relation to get

$$A_0 n + A_1 n^2 = A_0(n - 1) + A_1(n - 1)^2 + n$$
$$= (-A_0 + A_1) + (A_0 - 2A_1 + 1)n + A_1 n^2$$

Comparing the constant terms and coefficients of n, we get

$$0 = -A_0 + A_1,\ A_0 = A_0 - 2A_1 + 1.$$

Therefore, $A_0 = A_1 = \frac{1}{2}$.

Now, taking the sum of both the solutions, we get

$$L_n = a + \frac{n(n+1)}{2}.$$

The initial condition $L_1 = 2$ tells us that a=1, so that

$$L_n = 1 + \frac{n(n+1)}{2},\ n \geq 1.$$

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Example 7: Rani takes a loan of R rupees which is to be paid back in T months. If I is the interest rate per month for the loan, what constant payment P must she make at the end of each period ?

Solution: Let a_n denote the amount Rani owes at the end of the nth month, i.e., after the nth payment. Then the problem can be written as

$$a_{n+1} = a_n + Ia_n - P,\ 0 \leq n \leq T - 1,\ a_0 = R,\ a_T = 0.$$

So, the homogeneous part contributes $b(1+I)^n$ to the solution, b being a constant.

Using Theorem 5(i), with r = 1, we see that the non-homogeneous part contributes A, a constant.

But then, putting $a_n = A$ in our recurrence relation, we get

$$A = A(1+1) - P \Rightarrow A = P/I.$$

Thus, $a_n = b(1+I)^n + P/I$.

Then, $a_0 = R \Rightarrow b + P/I = R \Rightarrow b = R - P/I$

Also, $a_T = 0 \Rightarrow b(1+I)^T + P/I = 0$

$$\therefore P = \frac{IR(1+I)^T}{[1-(1+I)^T]}$$

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Example 8: Solve the recurrence $u_n = au_{n-1}+c.a^n$, $n \geq 1$, where a and c are known constants.

Solution: Using Theorem 5, we get

$$u_n = Aa^n + Bna^n, \text{ A and B being constants.}$$
$$= a^n(A+Bn) \text{ for } n \geq 0$$

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Now, here are some simple exercises for you.

E5) Solve the recurrence $T_n = 2T_{n-1} + 1$, $n \geq 2$, with $T_1 = 1$(see Problem 2, Sec.7.2)

E6) The growth of population of a species of snails in a certain lake triples every year. Starting with 1000 such snails, and finding 1500 of them the following year, 200 are removed from the lake to increase them in other lakes. Similarly, at the end of every year 200 are removed. If a_n represents the snail population in the lake after n years, find and solve a recurrence relation for an, $n \geq 0$.

Now let us consider a result which tells us how to find a particular solution for recurrences with non-homogeneous parts which are linear combinations of n^d and r^n, r a constant.

Theorem 6 (Superposition Principle) : If $\{a_n\}$ is a solution of

$$u_n = c_1u_{n-1}+c_2u_{n-2}+\cdots+c_ku_{n-k}+g_1(n)$$

and $\{b_n\}$ is a solution of

$$u_n = c_1u_{n-1} + c_2u_{n-2}+\cdots+c_ku_{n-k}+g_2(n),$$

then, for constants A and B, Aa_n+Bb_n is a solution of

$$u_n = c_1u_{n-1}+\cdots+c_ku_{n-k}+Ag_1(n)+Bg_2(n).$$

Proof: For $n \geq k$, we have Aa_n+Bb_n

$$= A[c_1a_{n-1}+\cdots+c_ka_{n-k}+g_1(n)]+B[c_1 b_{n-1}+\cdots+c_kb_{n-k}+g_2(n)]$$
$$= c_1(Aa_{n-1}+Bb_{n-1})+\cdots+c_k(Aa_{n-k}+Bb_{n-k})+\{Ag_1(n)+Bg_2(n)\}$$

This means that Aa_n+Bb_n is a solution of (5) with $g(n) = Ag_1(n) + Bg_2(n)$.

In view of Theorem 6, we can combine the results of Theorems 4 and 5 to get solutions of non-homogeneous recurrences like the following one.

Example 9: Obtain the general solution of the recurrence

$$v_n - 7v_{n-1} + 12v_{n-2} = 5.2^n - 4.3^n, \ n \geq 2.$$

Solution: Since there are no initial conditions and the equation is of second order, we can only expect a general solution involving two constants.

To begin with, the homogeneous part $v_n - 7v_{n-1} + 12v_{n-2} = 0$ has the characteristic polynomial $z^2 - 7z + 12$, i.e., $(z-3)(z-4)$. Consequently, its general solution is of the form $a.3^n + b.4^n$, where $a, b \in C$.

Now, let's consider the non-homogeneous part. It consists of two terms, one of which is a power of one of the characteristic roots. By Theorems 5 and 6, we must set $v_n = c.2^n + dn.3^n$ in order to find a particular solution. When we do this, the recurrence relation gives us

$$2^{n-2}c(4-14+12) + 3^{n-2}d[9n-21(n-1)+12(n-2)] = 5.2^n-4.3^n.$$
$$\Rightarrow \quad 2^{n-1}(c-10) = 3^{n-1}(d-12)$$

Since this equality is true for every $n \geq 1$, we see that $2^{n-1}/(d-12)$ for every $n \geq 1$. This can only be true if $d-12=0$, i.e., $d=12$. This forces $c-10=0$ to be true, i.e., $c=10$.

Putting all this information together, we get

$$v_n = 10.2^n + (a+12n)3^n + b.4^n, \text{ where } a, b \in C.$$

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Let us go back to Theorem 6 for a moment. Will the superposition principle be true for linear homogeneous recurrences too ? Actually, it will, and we have been using this fact quite a lot. Try and pinpoint where we have first used it for such recurrences.

Here are some exercises now.

E7) If the recurrence $u_n+c_1u_{n-1}+c_2u_{n-2}=an+b$ has a general solution $u_n = A.2^n + B.5^n + 3n-5$, find a, b, c_1 and c_2.

E8) Solve the recurrence $v_n - 7v_{n-1}+16v_{n-2}-12v_{n-3} = 2^n + 3^n$, with the initial terms $v_0 = 1$, $v_1 = 0$, $v_2 = 1$.

So far we have seen how to solve (5) if g(n) is of the form an^d, ar^n or a linear combination of terms of these types. There is one more type of non-homogeneous part that we shall discuss now.

Theorem 7: A particular solution of (5) with non-homogeneous part an^dr^n, where a and r are known constants and $d \in N$, is of the form

i) $Ar^n(A_0 + A_1n+\cdots+A_dn^d)$, if neither r nor 1 are characteristic roots of (5);

ii) $An^mr^n(A_0+A_1n+\cdots+A_dn^d)$, if **either r or 1 (but not both)** is a characteristic root of (5) with multiplicity m;

iii) $An^{m_1+m_2}r^n\left(A_0+A_1n+\cdots+A_dn^d\right)$, **if r and 1 both are characteristic roots of (5) with multiplicities m_1 and m_2**, respectively,

where A, A_0, $A_1,\ldots,A_d$ are constants.

As before, we shall not prove this result, but shall show how it is applied.

Example 10: Find the general form of the solution to a linear homogeneous recurrence with constant coefficients for which the characteristic roots are 1 with multiplicity two, -1 with multiplicity three and 2 with multiplicity five. Further, assume that the non-homogeneous part is a linear combination of $n(-1)^n$, n^2, 2^n and 3^n plus a polynomial of degree three.

Solution: We wish to solve a recurrence which has 10 characteristic roots. So, it is of the form

$$u_n=c_1u_{n-1}+\cdots+c_{10}u_{n-10}+(a_0+a_1n+a_2n^2+a_3n^3)+bn(-1)^n+c.n^22^n+d.3^n,$$

where we know that the characteristic polynomial of the homogeneous part is $(z-1)^2(z+1)^3(z-2)^5$,

$$\text{i.e., } z^{10}-c_1z^9-\cdots-c_9z-c_{10} \equiv (z-1)^2(z+1)^3(z-2)^5$$

So, by Theorem 1, the form of the general solution to the homogeneous part will be

$$(A_0 + A_1n).1^n + (B_0+B_1n+B_2n^2)(-1)^n + (C_0+C_1n+\cdots+C_4n^4)2^n,. \qquad (6)$$

Where the As, Bs and Cs are constants.

Now, by Theorem 4, you know that the form of the particular solution corresponding to the third degree polynomial is

$n^2(D_0 + D_1n + D_2n^2 + D_3n^3)$, where the Ds are constants.

From Theorem 7 you know that the form of the solution corresponding to

$Bn(-1)^n$ is $n^5(-1)^n(E_0 + E_1n)$, and to $Cn^2 2^n$ is $n^7.2^n(F_0 + F_1n + F_2n^2)$, where the Es and Fs are constants.

From Theorem 5, you know that the part of the solution corresponding to $d.3^n$ is $G.(3)^n$, G being a constant.

Thus, the particular solution is of the form

$$n^2(D_0 + D_1n + D_2n^2 + D_3n^3) + n^5(-1)^n(E_0 + E_1n)$$
$$+ n^7(2^n)(F_0 + F_1n + F_2n^2) + G(3^n) \qquad (7)$$

Therefore, the complete solution is the sum of the expressions in (6) and (7).

* * *

Here's an exercise of the same type for you.

E9) Find the general form of the solution to a recurrence relation with constant coefficients for which the characteristic roots are 3 with multiplicity 1 and –2 with multiplicity 2. The relation also has a non-homogeneous part which is a linear combination of 2^n, $n(-1)^n$ and a polynomial of degree 2.

In this section we have considered some general methods for tackling special kinds of non-homogeneous recurrences. While studying them you would have noticed that the solution of the non-homogeneous part is dependent on whether a characteristic root of the recurrence occurs in this part.

Now that you have studied this section and Unit 2, can you solve all the problems given in Unit 1 ? What about the 'divide and conquer' problem ? To solve this problem and other recurrences with non-homogeneous parts different from the ones looked at in this section, we need to look at some other solution techniques. Let us do so now.

3.4 SOME OTHER METHODS

In the previous section we have seen how to deal with two kinds of non-homogeneous parts of linear recurrences. There are many other kinds of recurrences that we can solve by some special methods. We shall look at four of these methods in this section.

3.4.1 Method of Inspection

One simple way of solving a recurrence is to write down enough terms in the sequence until one feels comfortable in guessing the solution. However, unless the pattern of the sequence is fairly straightforward, it is not easy to make a good guess. Usually, if one has made a correct guess here, the principle of mathematical induction (see Unit 1 of MCS-013) can be used to prove the guess. Let us consider an example.

Example 11: Solve, by inspection, the recurrence relation $a_n = a_{n-1} + n!n$ if $n \geq 1$, and $a_0 = 0$.

Solution: If we compute the first five terms of this sequence, we get 0, 1, 5, 23 and 119. Can you see what the nth term might be ? Does adding one to each term in the sequence help ? Doing so would give us a sequence that you would recognize, i.e., $(n+1)!$. So our initial guess is $a_n = (n+1)! - 1$.

Having done the initial work of making a guess, let us attempt to prove it by using induction on n.

The base case is easy to check : $a_0 = (0+1)! - 1 = 0$

If we are to assume the result for n = k, for some $k \geq 0$, then

$$a_{k+1} = a_k + (k+1)!\,(k+1) = [(k+1)! - 1] + (k+1)!\,(k+1)$$
$$= (k+1)!\,(k+2) - 1 = (k+2)! - 1, \text{ as we hoped.}$$

This completes the proof by induction, and proves our guess.

* * *

Here's an exercise for you now.

E10) Use the method of inspection to solve the recurrence

$b_n = b_{n-1} + 4n^3 - 6n^2 + 4n - 1$ for $n \geq 1$ with $b_0 = 0$.

Let us now consider another method for solving recurrences.

3.4.2 Method of Telescoping Sums

This neat method is useful for solving recurrences of the form $u_n = u_{n-1} + g(n)$, particularly if $\sum_{n=1}^{k} g(n)$ is easy to find. More generally, it can be used to evaluate sums of series and products.

This method is based on the fact that the sum of the first N terms of a series whose nth term is of the form $a_n - a_{n-1}$ is simply

$$(a_1 - a_0) + (a_2 - a_1) + \cdots + (a_{N-1} - a_{N-2}) + (a_N - a_{N-1}) = a_N - a_0.$$

In much the same manner, the product of the first N terms of a series with nth term a_n/a_{n-1} is a_N/a_0 provided, of course, that none of the a_k's is zero.

Though this method appears easy, it is not often that this method can be applied, and often not easy to see how to use it even when it can be! Let us see a few instances of where it can be applied.

A sum of the form $\sum_i [f(i+1) - f(i)]$ is called **telescoping** in analogy with the thickness of a collapsed telescope, which is the difference between the outer radius of the outermost tube and the inner radius of the innermost tube

Example 12: Solve the linear recurrence
$a_n - a_{n-1} = f_{n+2} \cdot f_{n-1}$, $n \geq 1$,
where $a_0 = 2$ and f_i denotes the ith Fibonacci number.

Solution: From Example 2, you know that
$f_{n+2} \cdot f_{n-1} = (f_{n+1} + f_n)(f_{n+1} - f_n) = f_{n+1}^2 - f_n^2$

So, for n = 1,2,…, the recurrence gives us the following equations :

$a_1 - a_0 = f_2^2 - f_1^2$
$a_2 - a_1 = f_3^2 - f_2^2$
$a_3 - a_2 = f_4^2 - f_3^2$
$\vdots \qquad\qquad \vdots$
$a_n - a_{n-1} = f_{n+1}^2 - f_1^2$

On adding these equations, we find that
$a_n - a_0 = f_{n+1}^2 - f_1^2$

$\Leftrightarrow a_n = 2 + f_{n+1}^2 - 1 = f_{n+1}^2 + 1$

* * *

The next example should be familiar to you. Recall, from Sec.8.2, that σ_n^k denotes the sum of the kth powers of the first n positive integers.

Example 13: Compute σ_n^1, σ_n^2 and σ_n^3, using the method of telescoping sums.

Solution: To find σ_n^1, we sum both sides of the identity $(k+1)^2 - k^2 = 2k + 1$ from $k = 1$ to $k = n$. On doing so, we get

$$(n+1)^2 - 1 = \sum_{k=1}^{n}\{(k+1)^2 - k^2\} = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 2\sigma_n^1 + n.$$

$\therefore \sigma_n^1 = n(n+1)/2.$

Let us now find σ_n^2 and σ_n^3

Summing both sides of the identities $(k+1)^3 - k^3 = 3k^2 + 3k + 1$ and $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ from $k = 1$ to $k = n$, we obtain

$$(n+1)^3 - 1 = \sum_{k=1}^{n}\left\{(k+1)^3 - k^3\right\} = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 3\sigma_n^2 + 3\sigma_n^1 + n, \text{ and}$$

$$(n+1)^4 - 1 = \sum_{k=1}^{n}\left\{(k+1)^4 - k^4\right\} = 4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= 4\sigma_n^3 + 6\sigma_n^2 + \sigma_n^1 + n$$

From the first of these equations, and using the value of σ_n^1 from above,

$\sigma_n^2 = n(n+1)(2n+1)/6.$

Plugging in the values of σ_n^1 and σ_n^2 into the second equation, we now obtain

$\sigma_n^3 = \{n(n+1)/2\}^2$

While going through the example above, you may have felt that there is a much simpler method to compute σ_n^1. But the advantage of using telescoping sums is that it also works for computing σ_n^k for larger values of k, where the simpler method does not.

Now you can try and obtain the general formula for σ_n^k, $k \geq 1$.

E11) Find a recurrence relation satisfied by the sequence $\{\sigma_n^k\}_k$, and hence compute σ_n^4.

Let us now look at Problem 7 of Unit 1, namely, the number of derangements on k symbols, d_k.

Example 14: Solve the recurrence

$$d_k = kd_{k-1} + (-1)^k \text{ if } k \geq 2, \text{ with } d_1 = 0.$$

Solution: Looking at the recurrence, it doesn't seem to be in the form in which we can apply the method of telescoping sums to solve it. But we can alter it slightly to bring it into a suitable form. We simply divide each term by k! and get

$$\frac{d_k}{k!} - \frac{d_{k-1}}{(k-1)!} = \frac{(-1)^k}{k!}$$

Now we can apply the method since the terms are such that if we write down the equations from k = 2 to k = n, and add them, most of the terms will get cancelled. We will only be left with

$$\frac{d_n}{n!} - \frac{d_1}{1!} = \sum_{k=2}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Therefore,

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, n \geq 1.$$

In the next example we see how 'telescoping products' can be used for solving recurrences.

Example 15: Solve the recurrence $a_n = n^3 a_{n-1}$, $n \geq 1$, $a_0 = 2$.

Solution: Let us put k = 1,2,…,n, in the equation

$$\frac{a_k}{a_{k-1}} = k^3$$

We get

$$\frac{a_1}{a_0} = 1^3$$

$$\frac{a_2}{a_1} = 2^3$$

$$\vdots \qquad \vdots$$

$$\frac{a_n}{a_{n-1}} = n^3.$$

Multiplying these equations, we get

$$\frac{a_n}{a_0} = (n!)^{3.}$$

$$\Rightarrow a_n = 2(n!)^3$$

The technique in the example above can be used more generally for non-homogeneous recurrences of the type

$$a_n = f(n)a_{n-1} + g(n) \text{ , where } f(n) \neq 0 \text{ for all n.}$$

Let us consider an example of this.

Example 16: Solve the recurrence $u_n = \frac{1}{n} u_{n-1} + \frac{1}{n!}$, $n \geq 1$, $u_0 = 1$.

This method can be applied here since $\frac{1}{n} \neq 0 \; \forall \; n \geq 1$.

Solution: The homogeneous part of this recurrence is $\frac{a_n}{a_{n-1}} = \frac{1}{n}$. Using the method of telescoping products, we get

$$a_n = \frac{1}{n} \frac{1}{n-1} \ldots \frac{1}{1} = \frac{1}{n!}$$

Now, suppose that the solution of the given recurrence is of the form $u_n = a_n b_n$, where $b_0 = 1$. Then

$$a_nb_n = \frac{1}{n} a_{n-1}\, b_{n-1} + \frac{1}{n!}$$

$$= a_nb_{n-1} + \frac{1}{n!}, \text{ since } a_n = \frac{1}{n} a_{n-1}$$

$$\Rightarrow b_n = b_{n-1} + \frac{\frac{1}{n!}}{a_n} = b_{n-1} + 1 \text{ since } a_n = \frac{1}{n!}$$

Now, we can use the method of telescoping sums to solve the recurrence

$b_n = b_{n-1} + 1$, $b_0 = 1$. We get
$b_n = n + 1$

Therefore, $u_n = a_nb_n = \dfrac{n+1}{n!}$

* * *

Can you clearly spell out the steps we have gone through in Example 16? To obtain a solution of the recurrence $u_n = f(n)u_{n-1}+g(n)$, the steps are :

Step 1: See if $f(n) \geq 0\ \forall$ n. Only then can this method be applied.

Step 2: Find the solution $\{a_n\}$ for the homogeneous part of the recurrence.
So,
$a_n = f(n)a_{n-1}\ \forall\, n \geq 1$.

Step 3: Assume that the solution of the given recurrence is of the form

$u_n = a_nb_n.$
Then
$a_nb_n = f(n)\, a_{n-1}b_{n-1}+g(n)$
$= a_nb_{n-1}+g(n)$
Therefore, $b_n = b_{n-1}+g(n)/a_n$.
Here is where we use the fact $f(n) \neq 0\ \forall$ n. (How?)

Step 4: Solve the recurrence

$$b_n = b_{n-1} + \frac{g(n)}{a_n}$$

by whichever method you find suitable.

Step 5: Then the solution to be given recurrence is $u_n = a_nb_n$.

Here are some exercises now.

E12) Show that C(2n, n) is a solution of the recurrence
$$x_n = \frac{2(2n-1)}{n} x_{n-1},\ n \geq 1.$$

E13) Use the method of telescoping sums and products to solve the recurrence $a_n = n^3a_{n-1} + (n!)^2$ if $n \geq 1$ and $a_0 = 1$.

E14) Solve the recurrence $a_n = (n!n)a_{n-1}$, $n \geq 1$, $a_0 = 5$.

Let us now see how telescoping sums can be used efficiently to sum an infinite series. Although this is not an example involving recurrence relations, you would get some idea of how this method can be applied to different problems.

Example 17: Use the method of telescoping sums to sum the infinite series

$$\frac{3}{1.2.3}+\frac{5}{2.3.4}+\frac{7}{3.4.5}+\cdots+\frac{2n+1}{n(n+1)(n+2)}+\cdots$$

Solution: The central idea behind telescoping sums is the expression of the nth term as a difference of successive terms of a sequence. We would have been able to apply this had the nth term in this summation been a product of only two terms in the denominator. But don't worry! Let us try and extend the idea.

With three terms in the denominator, we first express the nth term as a partial fraction :

$$\frac{2n+1}{n(n+1)(n+2)}=\frac{1/2}{n}+\frac{1}{n+1}-\frac{3/2}{n+2}\ .$$

Now, if a_i denotes the ith term of the series, then

$$S_n = a_1+a_2+\cdots+a_n$$
$$=\left(\frac{1/2}{1}+\frac{1}{2}-\frac{3/2}{3}\right)+\left(\frac{1/2}{2}+\frac{1}{3}-\frac{3/2}{4}\right)+\left(\frac{1/2}{3}+\frac{1}{4}-\frac{3/2}{5}\right)$$
$$+\cdots+\left(\frac{1/2}{n}+\frac{1}{n+1}-\frac{3/2}{n+2}\right)$$

Since $\frac{-3/2}{n}+\frac{1}{n}+\frac{1/2}{n}=0$, cancelling groups of such terms, we get

$$S_n=\left(\frac{1/2}{1}+\frac{1}{2}\right)+\left(\frac{1/2}{2}\right)+\left(\frac{-3/2}{n+1}\right)+\left(\frac{1}{n+1}-\frac{3/2}{n+2}\right)$$
$$=\frac{5}{4}-\frac{1/2}{n+1}-\frac{3/2}{n+2}$$

Therefore, $S = \lim\limits_{n\to\infty} S_n = 5/4$.

Why don't you try some exercises now ?

E15) By methods of this sub-section, solve the recurrence $nx_n = (n-2)x_{n-1} + 1,\ n\geq 1$, where $x_0 = 0$.

E16) Using the method of telescoping sums, prove the following Fibonacci identities :

a) $\sum\limits_{k=1}^{n} f_k = f_{n+2}-1;$

b) $\sum\limits_{k=1}^{n} f_{2k-1} = f_{2n};$

c) $\sum\limits_{k=1}^{n} f_k^2 = f_n f_{n+1};$

d) $\sum\limits_{k=2}^{\infty} f_k/(f_{k-1}f_{k+1}) = 2;$

e) $\sum\limits_{k=2}^{\infty} \left(f_{k-1}f_{k+1}\right)^{-1} = 1$

And now we shall consider another very commonly used method for solving recurrences.

3.4.3 Method of Iteration

Iteration means 'to repeat'. In a sense, this is what we do in this method. More precisely, we successively express the nth term u_n in terms of some or all of the previous (n–1) terms $u_0, u_1, \ldots, u_{n-1}$, using the recurrence equation again and again. While doing so, we try and find an emerging pattern which can help us find u_n explicitly as a function of n.
To see how this method works, let us look at an example.

Example 18: Solve the recurrence relation given by

$u_n = 2u_{n-1} + 2^n - 1$, where $n \geq 1$ and $u_0 = 0$.

Solution: Replacing n by n–1, n–1 by n–2,… and so on in the recurrence equation, we get

$$\begin{aligned}
u_n &= 2u_{n-1} + 2^n - 1 \\
&= 2(2u_{n-2} + 2^{n-1} - 1) + 2^n - 1 \\
&= 2^2 u_{n-2} + 2.2^n - (1+2) \\
&= 2^2(2u_{n-3} + 2^{n-2} - 1) + 2.2^n - (1+2) \\
&= 2^3 u_n + 3.2^n - (1+2+2^2) \\
&\vdots \\
&= 2^n u_0 + n.2^n - (1+2+2^2+\cdots+2^{n-1}) \\
&= (n-1)2^n + 1, \text{ since } 1+2+2^2+\cdots+2^{n-1} = \frac{2^n - 1}{2-1}
\end{aligned}$$

In the example above, we began with the recurrence relation and reached an expression for the nth term in terms of n. In principle this method always works. But, it is not always easy to apply because the computation can sometimes get out of hand.

Why don't you try the following exercise now? You shouldn't have any difficulty in the computation. In fact, you may find it easier to solve by this method than by the method you used earlier.

E17) Solve the recurrence $u_n = \frac{1}{n} u_{n-1} + \frac{1}{n!}, n \geq 1$, with $u_0 = 1$ by the method of iteration.

Let us now consider an example which can be solved by iteration as well as by first solving the recurrence for a_k and then summing the series. Let us solve it by using the former method.

Example 19: Sum the first n terms of the series whose kth term, a_k, satisfies the recurrence $a_k = 3a_{k-1} + 1$, and whose initial term, is $a_1 = 2$.

Solution: From the recurrence, we find that

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= \sum_{k=1}^{n-1} a_k + (3a_{n-1} + 1) \\
&= \sum_{k=1}^{n-2} a_k + (1+3)(3a_{n-2} + 1) + 1 \\
&= \sum_{k=1}^{n-3} a_k + (1+3+3^2)(3a_{n-2} + 1) + \{1 + (1+3)\}
\end{aligned}$$

$$= \sum_{k=1}^{n-4} a_h + (1+3+3^2+3^3)(3a_{n-4}+1) + \{1+(1+3)+(1+3+3^2)\}$$

$$\vdots$$

$$= a_1 + (1+3+\cdots+3^{n-2})(3a_1+1)+\{1+(1+3)+\cdots+(1+3+\cdots+3^{n-3})\}$$

$$= 2(1+3+\cdots+3^{n-1}) + \frac{1}{2}\sum_{k=1}^{n-1}(3^k-1), \text{ since } a_1 = 2$$

$$\text{and } 1+3+\cdots+3^{k-1} = \frac{3^k-1}{3-1}$$

$$= \frac{5.3^n-3-2n}{4}$$

* * *

You may like to try and solve a similar problem now.

E18) Use the method of iteration to find the sum of the first n+2 terms of the series whose kth term, u_k, satisfies the recurrence $u_k = u_{k-1}+k$, and whose initial term is $u_1 = 1$.

Now let us discuss the fourth method of this section.

3.4.4 Method of Substitution

So far we have seen several techniques for solving a variety of recurrences, linear and non-linear. But there are some that defeat our entire arsenal. For instance, we are ill equipped to handle even some of the simplest non-linear recurrences and linear recurrences with non-constant coefficients. It is in some of these cases that we can call upon a substitution to extricate ourselves from this position.

The method of substitution is used to change the given recurrence to a form that can then be readily solved by one of the previously discussed techniques. As you might expect, the hard part is to figure out what the substitution should be. Let us see how this method works through examples related to the divide-and-conquer relations.

Example 20: Solve the recurrence relation

$a_n = a_{n/2} + 1$ for $n = 2^k$, $k \geq 1$, $a_1 = 0$.

Solution: Let us put $a_2k = u_k$. Then the given recurrence becomes

$u_k = u_{k-1} + 1$, $u_0 = 0$.

Now, we can apply the method of telescoping sums, to get the solution

$u_n = u_0+n = n$, i.e., $2^n = n$, i.e., $a_m = \log_2 m$ for $m \geq 1$.

Example 21: Solve the recurrence obtained by 'merge sort' in sec.1.4, namely,

$a_n = 2a_{n/2} + n–1$, $n = 2^k$, $k \geq 1$, $a_1 = 0$.

Solution: As in the previous example, we put $2^k = u_k$. Then the recurrence becomes

$u_k = 2u_{k-1} + 2^k – 1$, $u_0 = 0$.

Now, as in Example 18, we get

$$u_k = (k-1)2^k + 1$$

$$\text{i.e., } a_{2^k} = (k-1)2^k + 1$$

$$\text{i.e., } a_n = (\log_2 n - 1)n + 1.$$

Let us now look at another recurrence of the same type.

Example 22: Solve the recurrence

$$u_n = a u_{\frac{n}{b}} + n^c \quad \text{where } n = b^k$$

Solution: Let $v_k = u_{b^k}$ then,

$$v_k = a\, v_{k-1} + b^{kc} \qquad (8)$$

The solution to the homogeneous part is $v_k = A\, a^k$ where A is a constant.

Case 1: If $b^c = a$, then a particular solution to (8) is Bka^k since $v = b^c = a$ is a characteristic root of (8). Substituting this in (8), we get

$$Bka^k = a\left(B(k-1)a^{k-1}\right) + a^k$$

Putting k = 0, we get B = 1.
Therefore $v_k = Aa^k + ka^k$. Putting k = 0, we get

$A = u_1$, so, $v_k = u_1 a^k + ka^k$ or

$v_k = u_1 (b^c)^k + k(b^c)^k$

$b^k = n \therefore k = \log_b n$

$\therefore u_n = u_1 n^c + n^c \log_b n$

Case 1: If $b^c \neq a$, then the particular solution is $B\, b^{ck}$. Substituting in (8) we get

$$B\, b^{ck} = a\,(B\, b^{c\,(k-1)}) + b^{ck}$$

$$B = \frac{b^{ck}}{b^{ck} - ab^{c(k-1)}} = \frac{b^c}{\left(b^c - a\right)} \qquad (9)$$

We have $v_k = Aa^k + B$ bck
Putting k = 0 and using $u_1 = v_o$, we have

$u_1 = A + B$

$$A = u_1 + \frac{b^c}{(a - b^c)} \qquad (10)$$

$$\therefore \text{We have } a^k = \left(b^{\log_b a}\right)^k = \left(b^k\right)^{\log_b a}$$

$$= n^{\log_b a}$$

So, $u_n = A\, n^{\log_b a} + Bn^c$ when A and B are given by (9) and (10), respectively.

Here are some recurrences for you to solve now.

E19) Using an appropriate substitution, solve the recurrence

$$y_n = \frac{n-1}{n} y_{n-1} + \frac{1}{n}, n \geq 1, \text{ where } y_0 = 5.$$

E20) Solve the recurrence $t_n = 3t_{n/2} + n^2$, $t_1 = 2$, by substitution, and specify for which values of n the conversion is valid.

Let us look at another example, one which makes it a natural candidate for the substitution technique.

Example 23: Solve the second order, non-linear recurrence

$x_n = \left(2\sqrt{x_{n-1}} + 3\sqrt{x_{n-2}}\right)^2 \; n \geq 2,$ with the initial conditions $x_0 = 1$, $x_1 = 4$.

Solution: Looking at the recurrence, you probably feel that we have not developed the tools to solve an equation of this type. Let's see if we can transform this into a linear recurrence. Let us make the substitution $y_n = \sqrt{x_n}$, $n \geq 0$. (Note that this substitution is valid because each x_n is non-negative.)

The substitution does not quite make the recurrence linear, but at least it gets rid of the square root symbol – the problem now becomes

$$y_n^2 = (2y_{n-1} + 3y_{n-2})^2, n \geq 2, \text{ with } y_0 = 1, y_1 = 2.$$

Extracting square roots of each side, we now get

$$y_n = 2y_{n-1} + 3y_{n-2}, n \geq 2.$$

This is a second order linear recurrence with constant coefficients, and can be solved by standard methods discussed earlier. We leave it to you to verify that the solution is $y_n = A.3^n + B(-1)^n$, $n \geq 0$, for some constants A, B.

Using the initial conditions, we further get

$$A + B = 1 \text{ and } 3A - B = 2, \text{ so that } A = ¾ \text{ and } B = ¼.$$

$$\therefore x_n = y_n^2 = \frac{\{3^{n+1} + (-1)^n\}^2}{16}, n \geq 0.$$

As a final example, we look at another non-linear recurrence with an exponential-type relation between the terms.

Example 24: Solve the recurrence given by $x_n = x_{n-1}^7 / x_{n-2}^{12}$, together with the initial conditions $x_0 = 1$ and $x_1 = 2$.

Solution: Taking the logarithm to any convenient base (we are only going to be dealing with positive numbers in this sequence after all!) reduces the right side of this to a form we can quite easily handle.

$$\log_2 x_n = 7 \log_2 x_{n-1} - 12 \log_2 x_{n-2}$$

Now, let $y_n = \log_2 x_n$. The sequence $\{y_n\}$ satisfies the recurrence

$$y_n - 7y_{n-1} + 12y_{n-2} = 0.$$

So its characteristic roots are 3 and 4.

Therefore, $y_n = a.3^n + b.4^n$, $n \geq 0$, a, b $\in$ C.

Now, the initial conditions $x_0 = 1$ and $x_1 = 2$ yield $y_0 = 0$ and $y_1 = 1$.

Putting n = 0 and 1 in $y_n = a.3^n + b.4^n$ gives a = –1, b = 1.

Therefore, $y_n = 4^n - 3^n$

Thus, $x_n = 2^{y_n} = 2^{4^n - 3^n}, n \geq 0.$

Observe that the choice of base in the example above does not alter the final answer, as indeed it must not ! We choose 2 because x_0 and x_1 are both powers of two.

Now, for some exercises.

E21) Find the solution of $\sqrt{x_n} - 5\sqrt{x_{n-1}} + 6\sqrt{x_{n-2}} = 0$, $n \geq 2$, where $x_0 = 4$ and $x_1 = 25$.

E22) Solve the recurrence $x_n = 4n(n-1)x_{n-2} + \frac{5}{9}n!(3^n)$ for $n \geq 2$, if $x_0 = 1$ and $x_1 = -1$.

E23) Let $\{u_n\}$ satisfy the non-homogeneous recurrence

$$u_n = c_1u_{n-1} + c_2u_{n-2} + c_3u_{n-3} + c_4u_{n-4} + g(n),$$

such that the associated homogeneous recurrence has

$$(z-2)(z-3)(z-4)^2$$

as its characteristic polynomial, and $\{g(n)\}$ satisfies a fifth order linear homogeneous recurrence with constant coefficients whose characteristic polynomial is $(z-2)^2(z-3)(z-5)^2$. Determine u_n.

E24) Let $\{v_n\}$ satisfy the second order recurrence

$v_n + b_1v_{n-1} + b_2v_{n-2} = 5r^n$,
with b_1, b_2 and r as constants. Prove that the sequence also satisfies the third order homogeneous linear recurrence with constant coefficients having

$(z^2 + b_1z + b_2)(z-r)$ as its characteristic polynomial.

E25) Let $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ be two solutions of the recurrence
$u_n + a_1u_{n-1} + a_2u_{n-2} = 0$, where a_1 and a_2 are constants.

d) Show that $\{x_ny_n\}_{n\geq 0}$ satisfies a third order linear homogeneous recurrence with constant coefficients.

e) Show that $\{x_{2n}\}_{n\geq 0}$ satisfies a second order linear homogeneous recurrence with constant coefficients.

E26) Assume that for positive real numbers a, b and r, there exists $m \in N$ such that $(a+bn)r^n < n!$ for $n \geq m$. Using this, prove that there does not exist any second order homogeneous linear recurrence with constant coefficients satisfied by the sequence $\{n!\}$.

With this, we have come to the end of this unit and block on recurrences. Let us take a quick look at what we have covered in this unit.

3.5 SUMMARY

In this unit we have discussed the following points.

1) The solution of the linear homogeneous recurrence with constant coefficients,
$u_n = c_1u_{n-1} + c_2u_{n-2} + \cdots + c_ku_{n-k}$, $n \geq k$,
is

$$\sum_{j=1}^{m}\left[\sum_{i=0}^{t_j} b_{ij}C(i+n,i)\right]\alpha_j^n$$

where $\alpha_1, \alpha_2, ..., \alpha_m$ are the distinct characteristic roots of this recurrence with multiplicity $t_1,\ldots,t_m$, respectively.

2) The solution of a linear non-homogeneous recurrence is the sum of the general solution of its homogeneous part and a particular solution of the whole recurrence.

3) A particular solution of
$u_n = c_1u_{n-1} + c_2u_{n-2} + \cdots + c_ku_{n-k} + an^d$, $n \geq k$,

is of the form

$n^m(A_0 + A_1n + \cdots + A_dn^d)$

where $m \geq 0$ is the multiplicity of 1 as a characteristic root of the equation, and the A_is are constants.

4) A particular solution of
$u_n = c_1u_{n-1} + c_2u_{n-2} + \cdots + c_ku_{n-k} + ar^n$

is of the form

An^mr^n,

where $m \geq 0$ is the multiplicyt of r as a characteristic root of the equation and A is a constant.

5) A particular solution of
$u_n = c_1u_{n-1} + \cdots + c_ku_{n-k} + an^dr^n$

is of the form

$$n^{m_1+m_2} r^n \left(A_0 + A_1 n + \cdots + A_d n^d\right)$$

where $m_1 \geq 0$ and $m_2 \geq 0$ are the multiplicities of r and 1, respectively, as characteristic roots of the equation, and the A_is are constants.

6) The methods of inspection and telescoping sums for solving linear recurrences with constant coefficients.

7) The methods of iteration and substitution for solving linear recurrences with constant and non-constant coefficients.

3.6 SOLUTIONS/ANSWERS

E1) The characteristic equation is z = 3. So the characteristic root is 3, with multiplicity 1. Therefore, the solution is

$a_n = bC(0 + n,0).3^n$, $b \in C$,
i.e., $a_n = b3^n$

E2) The recursion $u_n + c_1u_{n-1} + c_2u_{n-2} = 0$ has the characteristic equation

$z^2 + c_1z + c_2 = 0$.
We also know that the roots of this equation are 1 + i and 1 – i. Therefore, you know that c_1 equals the negative of the sum of its characteristic roots, i.e., –2, and c_2 is the product of these roots, i.e., 2.

E3) The characteristic equation of the given recurrence is

$z^3 - z^2 - z + 1 = (z-1)^2 (z+1) = 0$.

So, $P_n^2 = (an + b) + c(-1)^n$, $n \geq 0$, for some constants a, b, c.

The initial conditions are a + b – c = 0, 2a + b + c = 1 and

3a + b – c = 1. Therefore,

$$P_n^2 = \frac{2n - 1 + (-1)^n}{4}, \; n \geq 0.$$

E4) **Statements :** 1) If $\{a_n\}$ and $b_n\}$ are two solution sequences of the non-homogeneous recurrence

$$u_n = f_1(n)u_{n-1} + f_2(n)u_{n-2} + \cdots + f_k(n)u_{n-k} + g(n), \tag{11}$$

then $\{c_n\}$ is a solution sequence of its associated homogeneous recurrence, where $c_n = a_n - b_n$.

2) Every solution of (11) is of the form $a_n + b_n$, where a_n is a particular solution of (8) and b_n is any solution of its associated homogeneous recurrence.

$$u_n = f_1(n)u_{n-1} + \cdots + f_k(n)u_{n-k} \tag{12}$$

The **proofs** are exactly on the lines of the proofs for the constant coefficients case.

E5) The characteristic root of the recurrence $T_n = 2T_{n-1}$ is z = 2. Therefore, the general solution of the homogeneous part is

$T_n = a.2^n$, $n \geq 1$.

The particular solution to the non-homogeneous part is $T_n = b$, by Theorem 5. Plugging in this value of T_n into the recurrence gives $b = -1$.

Adding the solutions of the homogeneous and non-homogenous parts, and using the initial condition $T_1 = 1$, we get

$T_n = 2^n - 1, n \geq 1.$

E6) Here $a_{n+2} - a_{n+1} = 3(a_{n+1} - a_n) - 200, n \geq 0,$

i.e., $a_{n+2} - 4a_{n+1} + 3a_n = -200.$
The solution corresponding to the homogeneous part is
$a.3^n + b(1)^n$, i.e., $a.3^n + b$, where $a, b \in C$.
Now, $-200 = (-200)(1)^n$, and 1 is a characteristic root.
So, by Theorem 5, a particular solution is An, A a constant.
Putting $a_n = An$ in the recurrence, we get
$A(n+2) - 4A(n+1)+3An = -200 \Rightarrow A = 100.$
$\therefore a_n = a.3^n + b + 100n.$
$a_n = 100(3)^n + 900 + 100n, n \geq 0.$

E7) The recurrence $u_n + c_1u_{n-1}+c_2u_{n-2} = 0$ has the characteristic equation

$z^2 + c_1z + c_2 = 0$. From the given solution we see that its roots are 2 and 5.
Therefore, $c_1 = -(2+5) = -7$ and $c_2 = 2 \times 5 = 10$.
Now, setting the given particular solution $u_n = 3n - 5$ in the given equation, we get
$(3n-5) - 7(3n-8)+10(3n-11) = an + b.$
Therefore, $a = 12$ and $b = -59$.

E8) The recurrence $v_n - 7v_{n-1} + 16v_{n-2}-12v_{n-3} = 0$ has the characteristic equation

$z^3 - 7z^2 + 16z - 12 = 0$, i.e., $(z-2)^2(z-3) = 0$.

So, $v_n = (an + b)2^n + c.3^n, n \geq 0$, for some a,b,c.

The particular solution is of the form $v_n = An^2\, 2^n + Bn3^n$.
Therefore, the recurrence reduces to
$A.2^{n-1}\{2n^2 - 7(n-1)^2 + 8\,(n-2)^2 - 3(n-3)^2\}$
$+ B.3^{n-2}\,\{9n - 21(n-1) + 16(n-2) - 4(n-3)\} = 2^n + 3^n.$
Solving this, we get $A = -1, B = 9$.
Therefore, $v_n = (-n^2 + an + b)2^n + (9n + c)3^n, n \geq 0.$
The initial conditions lead to the equations
$b + c = 1$, $2(a+b-1) + 3(c+9) = 0$ and $4(2a+b-4) + 9\,(c+18) = 1$.
Solving these equations we get $a = 7$, $b = 42$ and $c = -41$.
Therefore, $v_n = (-n^2 + 7n + 42)2^n + (9n - 41)3^n, n \geq 0.$

E9) We know that 3 and -2 are the only characteristic roots with multiplicity 1 and 2, respectively. Therefore, the solution corresponding to the homogeneous part is

$A.3^n + (Bn + C)(-2)^n.$
The non-homogeneous part of the recurrence is
$a.2^n + bn(-1)^n + (cn^2 + dn + e)$,a,…,e are constants.
Therefore, the solution corresponding to this part is
$D(2^n) + (-1)^n\,(E_0 + E_1n) + Fn^2 + Gn + E.$
The complete solution is the sum of the two solutions.

E10) Since the first few terms of the sequence are 0,1,16,81,…, it is reasonable to guess that $b_n = n^4$ for $n \geq 0$.

Let's check this guess by induction.

Now, this guess is correct for n = 0 and n = 1.
Let's assume that it is true for n–1. Now
$n^4 = (n-1)^4 + (4n^3 - 6n^2 + 4n - 1)$ for $n \geq 1$.
So, the principle of mathematical induction proves our guess.

E11) Summing both sides of the identity

$(j+1)^{k+1} - j^{k+1} = \sum_{r=0}^{k} C(k+1, r)\, j^r$ from j = 1 to j = n, we get the recurrence equation

$$(n+1)^{k+1} - 1 = \sum_{j=1}^{n} \sum_{r=0}^{k} C(k+1, r)\, j^r$$

$$= \sum_{r=0}^{n} \left\{ C(k+1, r) \sum_{j=1}^{n} j^r \right\}$$

$$= \sum_{r=0}^{k} \{C(k+1, r)\sigma_n^r\}$$

In particular, k = 4 gives

$$(n+1)^5 - 1 = \sum_{r=0}^{4} \{C(5, r)\sigma_n^r\} = \sigma_n^0 + 5\sigma_n^1 + 10\sigma_n^2 + 10\sigma_n^3 + 5\sigma_n^4$$

Therefore,

$$\sigma_n^4 = \frac{n^5 + 5n^4 + 10n^3 + 10n^2 + 4n}{5} - \frac{n^2(n+1)^2}{2}$$

$$= \frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$$

E12) **Method 1 :** Since $C(2n, n) = \frac{(2n)!}{(n!)^2} = \frac{2(2n-1)}{n} \frac{(2n-2)!}{[(n-1)!]^2}$, it follows that C(2n,n) is a solution of the given recurrence.

Method 2 : $x_n = x_0 \prod_{k=1}^{n} x_k / x_{k-1} = x_0 \prod_{k=1}^{n} 2(2k-1)/k$

$= x_0 2^n[1.3.5\ldots(2n-1)]/n! = x_0(2n)!/(n!)^2,\ n \geq 0.$

E13) Let us apply the technique of Example 16. We can do so, since

$n^3 \neq 0\ \forall\ n \neq 1.$
From Example 15 we know that the solution to the homogeneous part is
$u_n = u_0(n!)^3$.
Suppose the solution of the given recurrence is
$a_n = u_n v_n$, where $u_0 v_0 = 1$.
Then $u_n v_n = n^3 u_{n-1} v_{n-1} + (n!)^2$
$= u_n v_{n-1} + (n!)^2$

$$\Rightarrow v_n = v_{n-1} + \frac{1}{u_0} \cdot \frac{1}{n!}$$

Now applying telescoping sums, we get

$$v_n = v_0 + \frac{1}{u_0}\left(\sum_{k=1}^{n}\frac{1}{k!}\right)$$

Then the solution of the recurrence is

$$a_n = (n!)^3\left[1+\sum_{k=1}^{n}\frac{1}{k!}\right]$$

E14) $\frac{a_n}{a_{n-1}} = n!n \ \forall n \geq 1$

$$\therefore \frac{a_n}{a_0} = \prod_{k=1}^{n}\frac{a_k}{a_{k-1}} = \prod_{k=1}^{n} k!k = 1!2!\ldots(n-1)!(n!)^2$$

$\therefore a_n = 5\,[1!2!\ldots(n-1)(n!)^2]$

E15) Multiplying by n–1 reduces this recurrence to

$n(n-1)x_n - (n-1)(n-2)\,x_{n-1} = n-1,\ n \geq 1.$

Substituting $d_n = n(n-1)x_n$, we get

$d_n - d_{n-1} = n-1,\ d_0 = 0.$

$$\therefore\ d_n = \sum_{k=1}^{n}(k-1) = n(n-1)/2$$

$\therefore x_n = 1/2,\ n \geq 1.$

E16) For convenience, let us define $f_0 = f_2 - f_1 = 0$.

a) $$\sum_{k=1}^{n} f_k = \sum_{k=1}^{n}(f_{k+1} - f_{k-1}) = (f_{n+1} + f_n) = f_{n+2} - 1$$

b) $$\sum_{k=1}^{n} f_{2k-1} = \sum_{k=1}^{n}(f_{2k} - f_{2k-2}) = f_{2n} - f_0 = f_{2n}.$$

c) $$\sum_{k=1}^{n} f_k^2 = \sum_{k=1}^{n}(f_{k+1}f_k - f_k f_{k-1}) = f_{n+1}f_n - f_1 f_0 = f_{n+1}f_n.$$

d) $$\sum_{k=2}^{\infty}[f_k/(f_{k-1}f_{k+1})] = \sum_{k=2}^{\infty}(f_{k-1}^{-1} - f_{k+1}^{-1})$$

$$= \lim_{n\to\infty}\{(f_1^{-1} + f_2^{-1}) - (f_n^{-1} + f_{n+1}^{-1})\} = 2$$

e) $$\sum_{k=2}^{\infty}(f_{k-1}f_{k+1})^{-1} = \sum_{k=2}^{\infty}\{(f_{k-1}f_k)^{-1} - (f_k f_{k+1})^{-1}\}$$

$$= \lim_{n\to\infty}\{(f_1f_2)^{-1} - (f_nf_{n+1})^{-1}\} = 1.$$

E17) Repeatedly replacing n by n–1, we get

$$u_n = \frac{1}{n} u_{n-1} + \frac{1}{n!}$$

$$= \frac{1}{n}\left\{\frac{1}{n-1} u_{n-2} + \frac{1}{(n-1)!}\right\} + \frac{1}{n!} = \frac{1}{n(n-1)} u_{n-2} + \frac{2}{n!}$$

$$= \frac{1}{n(n-1)(n-2)} u_{n-3} + \frac{2}{n!}$$

$$\vdots$$

$$= \frac{1}{n!} u_0 + \frac{n}{n!} = \frac{n+1}{n!}, n \geq 0.$$

E18) By the iteration method, we have

$u_n = u_{n-1} + n$
$= u_{n-2} + (n-1) + n$
$\vdots$
$= u_1 + [n-(n-2)] + \ldots + (n-1) + n \ldots + n$

$$= \frac{n(n+1)}{2}$$

$$\therefore \quad \sum_{k=1}^{n+2} u_k = \frac{1}{2}\sum_{k=1}^{n+2} k(k+1)$$

$$= \frac{1}{2}\left(\sum_{1}^{n+2} k^2 + \sum_{1}^{n+2} k\right)$$

$$= \frac{1}{2}\left[\frac{(n+2)(n+3)(2n+5)}{6} + \frac{(n+2)(n+3)}{2}\right]$$

$$= \frac{(n+2)(n+3)(n+4)}{6}$$

E19) Rewriting the recurrence in the form $ny_n - (n-1)y_{n-1} = 1$, suggests the substitution $x_n = ny_n$, $n \geq 1$. The recurrence then reduces to the form

$x_n - x_{n-1} = 1$, and telescopes to

$$x_n - x_0 = \sum_{k=1}^{n} (x_k - x_{k-1}) = n$$

Therefore, $x_n = n$ and $y_n = 1$ for $n \geq 1$.

E20) n has to be of the form 2^k for the recurrence to be valid. Now, let us put

$t_{2^k} = u_k$ in the recurrence. Then it reduces to $u_n = 3u_{n-1} + 2^{2n}$, $n \geq 1$, $u_0 = 2$.
The solution of the homogeneous part is $u_n - A.3^n$.
The solution of the non-homogeneous part is $u_n = B.2^{2n} = B4^n$.
Putting this in the recurrence, we get B = 4.
$\therefore \quad u_n = A.3^n + 4^{n+1}$.
$\therefore$ Now, using the initial condition, we get A = –2.
$\therefore u_n = (-2)3^n + 4^{n+1.}$
$\therefore t_2^n = (-2)3^n + 2^{2(n+1)}$

E21) Let $y_n = \sqrt{x_n}$. Then, $y_n - 5y_{n-1} + 6y_{n-2} = 0$ has the characteristic roots 2 and 3.

So $y_n = a.2^n + b.3^n$, for some a,b.
Since $y_0 = 2$ and $y_1 = 5$, $a = 1 = b$.
Therefore, $x_n = y_n^2 = (2^n + 3^n)^2$ for $n \geq 0$.

E22) The term n! of the non-homogeneous part provides us with the hint that we should divide both sides by n!. If we do, we get

$y_n - 4y_{n-2} = \frac{5}{9} \times 3^n, n \geq 2$, with $y_0 = 1$, $y_1 = -1$, where $y_n = x_n/n!$.

Since the homogeneous part of this has the characteristic polynomial $z^2 - 4 = 0$, it follows that

$y_n = a.2^n + b(-2)^n + c.3^n$, $n \geq 0$, for some a,b.c.

Inserting this value into the recurrence gives c=1, while the initial conditions give rise to a+b+c=a+b+1=1 and

2a–2b+3c=2a–2b+3=–1, solving which we get a= –1, b=1.

Therefore, $x_n = \{2^n - (-2)^n + 3^n\}n!$, $n \geq 0$.

E23) Write u_n as a sum of its homogeneous solution, $u_n^{(h)}$, has particular solution, $u_n^{(p)}$. Then,

$u_n^{(h)} = a_1.2^n + a_2.3^n + (a_3 + a_4n)4^n, a_i \in C \ \forall$ i.

Since g(n) is of the form $(A+Bn)2^n + C.3^n + (D+En)5^n$, the form that the particular solution takes is

$u_n^{(p)} = [A_0n + (b_0 + b_1n)n]2^n + C_0n.3^n + D_0.5^n + (E_0 + E_1n)5^n$

$= (A_1n + B_1n^2)2^n + C_0n.3^n + D_0.5^n + (E_1n)5^n$, where the capital letters are constants.

Therefore, $u_n = u_n^{(h)} + u_n^{(p)}$.

E24) Let r_1,r_2 be the roots of the characteristic polynomial

$z^2 + b_1z+b_2 = 0$. Then

$$v_n = \begin{cases} a_1r_1^n + a_2r_2^n + cr^n & \text{if } r_1, r_2, r \text{ are all distinct,} \\ (a_1 + cn)r_1^n + a_2r_2^n & \text{if } r = r_1 \neq r_2, \\ (a + bn)r_1^n + cr^n & \text{if } r_1 = r_2 \neq r, \\ (a + bn + cn^2)r^n & \text{if } r_1 = r_2 = r. \end{cases}$$

In any of these cases, the characteristic polynomial of the linear homogeneous recurrence with constant coefficients satisfied by $\{v_n\}$ has roots r_1, r_2 and r, not necessarily distinct. In other words, this polynomial is

$(z-r_1)(z-r_2)(z-r) = (z^2 + b_1z + b_2)(z-r)$.

E25) Let $z^2 + a_1z+a_2 = (z-\alpha)(z-\beta)$

If $\alpha \neq \beta$, $x_n = A\alpha^n + B\beta^n$ and $y_n = C\alpha^n + D\beta^n$ for some constants A,B,C,D and all $n \geq 0$.

If $\alpha = \beta, x_n = (A + Bn)\alpha^n$ and $y_n = (C+Dn)\alpha^n$ for some constants A,B,C,D and all $n \geq 0$.

a) So, if the roots are distinct,

$x_ny_n = AC(\alpha^2)^n + (AD + BC)(\alpha\beta)^n + BD(\beta^2)^n$, $n \geq 0$.

So, $\{x_ny_n\}$ satisfies a third order linear homogeneous recurrence with constant coefficients and distinct characteristic roots α^2, $\alpha\beta$ and β^2.

More explicitly, the characteristic polynomial is

$(z-\alpha^2)(z-\alpha\beta)(z-\beta^2) = z^3 - (a_1^2 - a_2)z^2 + a_2(a_1^2 - a_2)z - a_2^3$, and the recurrence relation is

$v_n - (a_1^2 - a_2)v_{n-1} + a_2(a_1^2 - a_2)v_{n-2} - a_2^3v_{n-3} = 0$

If the roots are equal,

$x_ny_n = AC(\alpha^2)^n + (AD + BC)n(\alpha^2)^n + BDn^2(\alpha^2)^n, n \geq 0.$

So $\{x_n y_n\}$ again satisfies a third order linear homogeneous recurrence with constant coefficients and the characteristic root α^2 of multiplicity three.

More explicitly, the characteristic polynomial is
$(z-\alpha^2)^3 = z^3 - 3a_2 z^2 + 3a_2^2 z - a_2^3$,
and the recurrence relation is
$v_n - 3a_2 v_{n-1} + 3a_2^2 v_{n-2} - a_2^3 v_{n-3} = 0$.

b) In this case, if the roots are distinct,

$x_{2n} = A(\alpha^2)^n + B(\beta^2)^n, n \ge 0$, and $\{x_{2n}\}$ satisfies a second order linear homogeneous recurrence with constant coefficients and distinct characteristic roots α^2 and β^2.
More explicitly, the characteristic polynomial is

$(z-\alpha^2)(z-\beta^2) = z^2 - (a_1^2 - 2a_2)z + a_2^2$, and the recurrence relation is
$w_n - (a_1^2 - 2a_2)w_{n-1} + a_2^2 w_{n-2} = 0$

If the roots are equal,

$x_{2n} = (A+2Bn)(\alpha^2)^n$, $n \ge 0$,

and $\{x_{2n}\}$ again satisfies a second order linear homogeneous recurrence with constant coefficients and the characteristic root α^2 of multiplicity two.
More explicitly, the characteristic polynomial is

$(z-\alpha^2)^2 = z^2 - 2a_2 z + a_2^2$,

and the recurrence relation is
$w_n - 2a_2 w_{n-1} + a_2^2 w_{n-2} = 0$

E26) If the sequence $\{n!\}$ is to satisfy a second order homogeneous linear recurrence with constant coefficients, its nth term must be of the form $a_1 r_1^n + a_2 r_2^n$ for some a_1, a_2 provided $r_1 \neq r_2$, or of the form $(a+bn)r^n$ for some a,b.
if $r_1 \neq r_2$,

$\left|a_1 r_1^n + a_2 r_2^n\right| \le |a_1||r_1|^n + |a_2||r_2|^n$
$\le \left(|a_1| + |a_2|\right) r^n$, where $r = \max\left(|r_1|, |r_2|\right)$
So, in either case, $n! \le (A+Bn)\alpha^n$, $n \ge 0$ for some positive A, B and α, and this is a contradiction to our supposition.