LU DECOMPOSITION METHOD

The Gauss Elimination Method has the disadvantage that all right-hand sides (i.e. all the ***b*** *vectors of interest for a given problem) must be known in advance for the elimination step to proceed. The LU Decomposition Method outlined here has the property that the matrix modification (or decomposition) step can be performed independent of the right hand side vector. This feature is quite useful in practice - therefore, the LU Decomposition Method is usually the Direct Scheme of choice in most applications.*

To develop the basic method, let's break the coefficient matrix into a product of two matrices,

$$\boldsymbol{A} = \boldsymbol{L}\,\boldsymbol{U} \qquad (3.12)$$

where ***L*** *is a lower triangular matrix and* ***U*** *is an upper triangular matrix.*

Now, the original system of equations,

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b} \qquad (3.13)$$

becomes

$$\boldsymbol{L}\boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{b} \qquad (3.14)$$

This expression can be broken into two problems,

$$\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b} \qquad \text{and} \qquad \boldsymbol{U}\,\boldsymbol{x} = \boldsymbol{b} \qquad (3.15)$$

The rationale behind this approach is that the two systems given in eqn. (3.15) are both easy to solve; one by forward substitution and the other by back substitution. In particular, because ***L*** *is a lower diagonal matrix, the expression,* $\boldsymbol{Ly} = \boldsymbol{b}$ *, can be solved with a simple forward substitution step. Similarly, since* ***U*** *has upper triangular form,* $\boldsymbol{U\,x} = \boldsymbol{b}$ *can be evaluated with a simple back substitution algorithm.*

Thus the key to this method is the ability to find two matrices, ***L*** *and* ***U****, that satisfy eqn. (3.12). Doing this is referred to as the Decomposition Step and there are a variety of algorithms available. Three specific approaches are as follows:*

Doolittle Decomposition*:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{32} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad (3.16)$$

*Because of the specific structure of the matrices, a systematic set of formulae for the components of **L** and **U** results.*

***Crout Decomposition**:*

$$\begin{bmatrix} \ell_{11} & 0 & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & \ell_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad (3.17)$$

*The evaluation of the components of **L** and **U** is done in a similar fashion as above.*

***Cholesky Factorization**:*

For symmetric, positive definite matrices, where

$$\boldsymbol{A = A^T \;\; and \;\; x^T A\, x > 0 \;\; for\; x \neq 0} \qquad (3.18)$$

then,

$$\boldsymbol{U = L^T} \quad and \quad \boldsymbol{A = L\, L^T} \qquad (3.19)$$

*and a simple set of expressions for the elements of **L** can be obtained (as above).*

*Once the elements of **L and U** are available (usually stored in a single NxN matrix), the solution step for the unknown vector **x** is a simple process [as outlined above in eqn. (3.15)].*

A procedure for decomposing an $N \times N$ matrix **A** into a product of a ower triangular matrix L and an upper triangular matrix U,

$$L\,U = A.$$

Written explicitly for a 3×3 matrix the decomposition is

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} l_{11}\, u_{11} & l_{11}\, u_{12} & l_{11}\, u_{13} \\ l_{21}\, u_{11} & l_{21}\, u_{12} + l_{22}\, u_{22} & l_{21}\, u_{13} + l_{22}\, u_{23} \\ l_{31}\, u_{11} & l_{31}\, u_{12} + l_{32}\, u_{22} & l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33}\, u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

(3.20)

This gives three types of equations

$$i < j \quad l_{i1} u_{1j} + l_{i2} u_{2j} + \ldots + l_{ii} u_{ij} = a_{ij} \tag{3.21}$$
$$i = j \quad l_{i1} u_{1j} + l_{i2} u_{2j} + \ldots + l_{ii} u_{jj} = a_{ij} \tag{3.22}$$
$$i > j \quad l_{i1} u_{1j} + l_{i2} u_{2j} + \ldots + l_{ij} u_{jj} = a_{ij}. \tag{3.23}$$

This gives N^2 equations for $N^2 + N$ unknown (the decomposition is not unique), and can be solved using either using Doolittle or Crout's method.

Doolittle Method : Here $l_{ii} = 1$, $i = 1$ to N. In this case , equation (3.20) gives

$$u_{1j} = a_{1j} \qquad j = 1 \text{ to } N$$

$$l_{i1} = a_{i1} / a_{11}, \qquad i = 2 \text{ to } N$$

$$u_{2j} = a_{2j} - l_{21} \,.\, u_{1j}, \quad j=2 \text{ to } N$$

$$l_{i2} = (a_{i2} - l_{i1}\, u_{12}) / u_{22}, \; i = 3 \text{ to } N,$$

and so on

Crout's Method : Here $u_{ii} = 1$, $i = 1$ to N . In this case , we get

$$l_{i1} = a_{i1}, \qquad i=1 \text{ to } N$$

$$u_{1j} = a_{1j} / a_{11}, \qquad j= 2 \text{ to } N$$

$$l_{i2} = a_{i2} - l_{i1}\, u_{12}, \qquad i = 2 \text{ to } N$$

$$u_{2j} = (a_{2j} - l_{21}\, u_{1j}) / l_{22}, \; j= 3 \text{ to } N,$$

and so on

Example 7: *Given the following system of linear equations, determine the value of each of the variables using the LU decomposition method.*

$$\begin{aligned} 6x_1 - 2x_2 &= 14 \\ 9x_1 - x_2 + x_3 &= 21 \\ 3x_1 - 7x_2 + 5x_3 &= 9 \end{aligned} \tag{3.24}$$

Solution *:*

Upper Triangular	*Explanation of Step*	*Lower Triangular*
$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$	*<--- Beginning Matrix* *Matrix Storing Elementary Row Operations --->*	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & -1/3 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$	*In order to force a value of 1 at position (1,1), we must multiply row 1 by 1/6. Thus storing its reciprocal, 6, in position (1,1) in the lower matrix.*	$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix}$	*Introducing zeros to positions (2,1) and (3,1) require multiplications by -9 and -3 respectively. So we will store the opposite of these numbers in their respective locations.*	$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 8 & 5 \end{bmatrix}$	*On to the next position in the main diagonal, (2,2). To replace the value in this position with a 1, multiply row 2 by 1/2, thus storing a 2 (the reciprocal) in position (2,2) in the lower triangular matrix.*	$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$	*Replacing the position under the leading 1, position (3,2), with a zero can be done with a multiplication of -8. We will then store 8, the opposite of -8, in the lower matrix at that position.*	$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & -8 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$	*Only a multiplication of 1 is necessary to introduce a 1 to the next diagonal position. In fact nothing is being done to the upper triangular matrix, but we need the 1 in the lower matrix to show that.*	$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & -8 & 1 \end{bmatrix}$

If a matrix A can be decomposed into an LU representation, then A is equal to the product of the lower and upper triangular matrices. This can be shown with one matrix multiplication.

$$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & -8 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (3.25)$$

Solving Systems of Equations using the LU decomposition.

Systems of linear equations can be represented in a number of ways. In the Gauss-Jordan elimination method, the system was represented as an augmented matrix. In this method, we will represent the system as a matrix equation.

1. *Rewrite the system* $A\mathbf{x} = \mathbf{b}$ *using the LU representation for A. Making the system* $LU\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & -8 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \\ 9 \end{bmatrix}$$

2. *Define a new column matrix* ***y*** *so that U****x*** *=* ***y****.*

$$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

3. *Rewrite step one with the substitution from step two yielding L****y*** *=* ***b****.*

$$\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & -8 & 1 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 21 \\ 9 \end{bmatrix}$$

4. *Solve step three for* ***y*** *using forward substitution.*

$y_1 = 7/3$, $y_2 = 29/6$, $y_3 = 33/2$

5. *Using the results from step four, solve for* ***x*** *in step two using back substitution.*

$$\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

$x_1=43/36$, $x_2=-41/12$, $x_3=33/2$