UNIT 2 GENERATING FUNCTIONS

Structure **Page Nos.**

2.0 INTRODUCTION

In your earlier Mathematics courses, you may come across power series expansions of functions like e^x, sin x etc. There, we have to worry about **convergence** questions, i.e for which values of x does the expansion represent the function. Here, we will discuss power series expansions from a different point of view. We will not be interested in convergence questions because we will never substitute numerical values for x; rather we will be interested in the combinatorial properties of the power series. Sequences of numbers that have combinatorial significance appear as the coefficients of power series. We call the power series where coefficients are the terms of a sequence as the **generating function** for the sequence. For example, we will see later in this Unit that the coefficients of the power series of $\frac{z}{(1-z-z^2)}$ are the Fibonacci numbers. Thus,

$\frac{z}{(1-z-z^2)}$ is the generating function for the Fibonacci numbers.

In sec.2.2, we shall explain the concept and some elementary uses of generating functions. In Sec.2.3, we shall introduce you to a particular type of generating functions that are used to solve arrangement problems in combinatorics.

In sec.2.4, we shall explore the power of the generating functions as a tool when, for example, it is used to derive some combinatorial identities, solve some combinatorial problems involving general integer equations, find the number of partitions and solve certain recurrence relations.

2.1 OBJECTIVES

After going through this unit, you should be able to

- define and construct generating functions for sequences arising in various types of combinatorial problems;
- use generating functions to find the number of integer solutions to linear equations.
- find the generating function associated with a sequence in closed form in some simple cases
- find the exponential generating function associated with a sequence in closed form in some simple cases;
- solve recurrence relations using generating functions; and
- use generating functions to prove identities involving combinatorial coefficients.

2.2 GENERATING FUNCTIONS

Often, we can relate the solutions of a combinatorial problem to the coefficients of a power series. In the next example we will see how to relate the number of integer solutions to certain linear equations to the coefficients of a power series.

Example 1: Determine the number of integer solutions to linear equation

$$X_1 + X_2 = 3, \text{with } 0 \le X_1 \le 1 \text{ and } 0 \le X_2 \le 2.$$

Solution: By explicit enumeration, the possible values are given below.

X_1	X_2	Sum
0	0	0
0	1	1
0	2	2
1	0	1
1	1	2
1	2	3

Thus, there are two ways to obtain a sum of 1 (also 2) and one way to obtain the sum 3.

Now consider the following product of polynomials:

$$(z^0 + z^1)(z^0 + z^1 + z^2),$$

Where the exponents of symbol z in the first factor corresponded to the possible values of X_1 and in the second factor to the possible values of X_2. On expanding this product, we get

$$\begin{aligned}\left(z^0 + z^1\right)\left(z^0 + z^1 + z^2\right) &= \left(z^0 z^0 + z^0 z^1 + z^0 z^2 + z^1 z^0 + z^1 z^1 + z^1 z^2\right)\\ &= 1 + 2z + 2z^2 + z^3.\end{aligned}$$

Adding the exponents of the symbol z after multiplication corresponds to considering the sum of the values of X_1 and X_2.

We note that the coefficient of z^r , $1 \le r \le 3$, in this expression gives the number of integer solutions to $X_1 + X_2 = r$, with $0 \le X_1 \le 1$ and $0 \le X_2 \le 2$. In particular, because the coefficient of z^3 in the above expression is 1, and there is only one pair of values viz. (1,2), which satisfy the given linear equation.

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Suppose we intend to find non-negative integer solutions to the linear equation

$$X_1 + X_2 + X_3 = 10, \text{with } 0 \le X_1 \le 4, X_2 > 0, \text{and } X_3 \ge 0.$$

Then, by arguments given in the example above, we take the product of the following three polynomials.

$$\left(1 + z + z^2 + z^3 + z^4\right)\left(z + z^2 +\right)\left(1 + z + z^2 + \cdots\right)$$

In the above product, both second and third factors are infinite because there is no upper bound on X_1 and X_2. Also, second factor does not contain the constant term owing to the fact that $X_2 > 0$. Then, as before, the coefficient of z^{10} in the above expression will give us a solution to the linear equation given above.

For finding the coefficients of a power series, we often use the following **results.**

Result 1: (Binomial Theorem)

Let n > 0 then

a) $(1+z)^n = \sum_{r=0}^{\infty} C(n,r)\, z^r$

b) $(1+z)^{-n} = \sum_{r=0}^{\infty} C(n-1+r,r)(-1)^r z^r$

c) $(1-z)^{-n} = (1+z+z^2+\cdots)^n = 1+\sum_{r=1}^{\infty} C(n-1+r,r) z^r.$

Result 2: $\frac{1-z^n}{1-z} = 1+z+z^2+\cdots+z^{n-1}, z \neq 1.$

Next, we illustrate the technique of identifying the power series associated with a combinatorial problem with the help of following example.

Example 2: Find a power series associated with the problem where we have to find the number of ways to select a **dozen** pieces of fruit from 5 Apples, 10 Bananas and 15 Coconuts.

Solution: To begin with, let us use the letters A, B and C for Apples, Bananas and Coconuts, respectively. So, if we select k Apples, ℓ Bananas and m Coconuts, then we must have k+ ℓ +m =12, with the restriction that $0 \le k \le 5, 0 \le \ell \le 10$ and $0 \le m \le 15$. Let us see what we could do to set up the problem using the symbols A, B and C.

Here we may denote k Apples by A^k, ℓ Bananas by B^{ℓ}, m Coconuts by C^m. Then we have picked the correct number of pieces provided the degree (i.e. the sum k+ ℓ +m) of the term $A^k B^{\ell} C^m$ equals 12. Thus, to find the required number of ways of selecting a dozen pieces of fruit, you simply have to find the number of terms in the expansion.

$$(A^0 + A^1 + \cdots + A^5)(B^0 + B^1 + \cdots + B^{10})(C^0 + C^1 + \cdots + C^{15}) \qquad (1)$$

whose degree equals 12. This will be the sum of the coefficients of all the terms $A^k B^{\ell} C^m$ in (1) such that k+ ℓ +m =12 i.e. of $A^0B^0C^{12}$, $A^0B^{11}C^{12}$, etc.

At this point it is important to observe that any selection of fruits with the given restriction on the numbers k, ℓ and m correspondes to precisely one term in this product. For instance, if you pick 3 Apples, 4 Bananas and 5 Coconuts, the corresponding term in the product (1) is $A^3B^4C^5$. Conversely, the term AB^2C^9 represents the choice of 1 Apple, 2 Bananas and 9 Coconuts. Thus product (1) when expanded as $\sum_{i,j,k} a_{ijk} A^i B^j C^k$, gives the required (finite) power series for the given problem.

* * *

Now, since our real interest is in the degree of $A^k B^{\ell} C^m$ (i.e. in the sum k+ ℓ +m), we may as well replace each of these symbols in (1) by a common symbol, say z. Then, as before, we are led to determine the coefficient of z^{12} in the following product of polynomials.

$$(1+z+\cdots+z^5)(1+z+\cdots+z^{10})(1+z+\cdots+z^{15}).$$

Now we don't need to look into the possible ways in which A^kB^ℓ and C^m add up to 12 fruits.

Next, let us ask a similar question for the problem given in the following example.

Example 3: How can a power series be associated with the problem in which we have to find the number of selections of fruits if we have Rs.50 with us and it is given that an Apple costs Rs. 5, a Banana Rs.2 and a Coconut Rs.3

Solution: Since here we don't have any restriction on the number of pieces of fruit, the required power series (in terms of money) is of the form

$$(A^0+A^5+A^{10}+\cdots)(B^0+B^2+B^4+\cdots)(C^0+C^3+C^6+\cdots),$$

which is the product of three polynomials (infinite because there is no restriction on the number of pieces of fruit). Because an Apple costs Rs.5, so, purchase of k Apples would mean that we have to spend Rs.5k. Similarly, purchase of ℓ Bananas and m Coconuts will amount to spending Rs. (2ℓ+3m). Thus purchase of (k+ℓ+m) fruits correspond to the term $A^{5k}B^{2\ell}C^{3m}$ in the above product of three polynomials. Also because we have Rs.50 only, we must have 5k+2ℓ+3m = 50. On the other hand, each term $A^{5k}B^{2\ell}C^{3m}$ (with 5k+2ℓ+3m = 50) in the above series gives a choice for purchasing k Apples ℓ Bananas and m Coconuts.

Thus, in view of given cost of the Apple, Banana and Coconut, power of symbols A, B and C in the first, second and third polynomials are multiples of 5, 2 and 3, respectively. As before, in this expression we seek the number of terms with degree 50. However, by our discussion following Example 2, if we replace each of these symbols by a common symbol z (say) then the required number is given by the coefficient of z^{50} in the expression.

$$(1+z^5+z^{10}+\cdots)(1+z^2+z^4+\cdots)(1+z^3+z^6+\cdots). \qquad (*)$$

Hence, this product on expansion gives the power series associated with the above problem.

In above example, if we impose some restrictions on our selection of the fruits, then there will be a corresponding change in the associated power series (*).
This is what we want you to see in the following exercise.

E1) Find the power series associated with the problem given in Example 3,
a) when all our selections are required to have 1 Apple at least;
b) when each selection has to have at least one fruit of each type.

You have seen above how to associate a power series with a combinatorial problem, such that, the solution of the problem is given by certain coefficients of that series. Certain series can be written in a functional form which we call as **closed form**. For example, it follows from Binomial theorem (see Result 1, a) given above) that $(1-z)^{-1}$ is the closed form (or a functional form) of the power series $\sum_{r=0}^{\infty} z^r$

To distinguish them from exponential generating functions (which we will define in the next section), they are sometimes called **ordinary generating functions.**

Definition: The generating function A (z) (say) for the sequence of real (or complex) numbers, $\{a_0, a_1, \ldots, a_n, \ldots\}$ is given by the powers series

$$A(z)=\sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \ldots + a_n z^n + \ldots$$

Thus, the (n+1)th term a_n of the sequence $\{a_n\}$, $n \geq 0$ is simply the coefficient of z^n in A (z). As said before, the generating function thus serves the purpose of identifying the different terms of a sequence by different powers of the symbol z.

Let G (z) be the generating function of the **geometric progression** $\{ar^n\}$ $n \geq 0$, i.e.

$$G(z) = a + (ar)z + (ar^2)z^2 + \cdots$$

Then,

$$G(z) - a = rz\left[a + (ar)z + \left(ar^2\right)z^2 + \right]$$
$$= rz(G(z))$$

which gives, on simplification, G (z) = a/ $(1 - rz)$

Why don't you try an exercise now?

E2) Verify that

a) The generating function for the finite geometric progression $\{a, ar, ar^2, \ldots, ar^{k-1}\}$ is $a(1 - r^k z^k)/(1 - rz)$.

b) The generating function for the sequence of Binomial coefficients $\{C(k,0), C(k,1)a, C(k,2)a^2, \ldots\}$ is $(1 + az)^k$.

c) the generating function for the sequence of Binomial coefficients $\{C(k-1,0), C(k,1)a, C(k+1,2)a^2, \ldots\}$ is $(1 - az)^{-k}$.

Note that the generating function for a finite sequence is the generating function for a corresponding infinite sequence which can be obtained by setting to zero every term not previously defined. Thus for a finite polynomial $a_0 + a_1 z + a_2 z^2$ we write

$$a_0 + a_1 z + a_2 z^2 + 0.z^3 + 0.z^4 + \cdots$$

Now let us see how the technique of associating a series with a sequence is helpful in solving a combinatorial problem. We try to understand this with the help of following example.

Example 4: Determine the number of subsets of a set of n elements, $n \geq 0$.

Solution: Let s_n denote the number of subsets that a set of n elements can have. In the previous unit, you have seen that the recurrence relation satisfied by the sequence $\{s_n\}$ is given by

$s_n = 2s_{n-1}$ if $n \geq 1$ and $s_0 = 1$. (see Example 7, Unit 1)

Let S (z) stand for the generating function of the sequence $\{s_n\}_{n \geq 0}$. So, we can write

$$S(z) = \sum_{n=0}^{\infty} s_n z^n = 1 + \sum_{n=1}^{\infty} s_n z^n$$
$$= 1 + 2\sum_{n=0}^{\infty} s_{n-1} z^n \text{ (by definition of } S_n, n \geq 1)$$
$$= 1 + 2z\sum_{n=0}^{\infty} s_n z^n = 1 + 2z S(z),$$

$$\text{i.e. } S(z) = 1 + 2z S(z).$$

Solving last equation for S (z), we get

Two symbolic series $\sum a_n z^n$ and $\sum b_n z^n$ are **equal** iff $\mathbf{a_n} = \mathbf{b_n}$, $\forall$n.

$$S(z) = \frac{1}{1 - 2z} = \sum_{n=0}^{\infty} 2^n z^n. \quad \text{(by Binomial theorem)}$$

Finally, comparing the coefficients of z^n on both sides of above equations, we get $s_n = 2^n$, $n \geq 0$. Thus, the number of subsets of a set of n element is 2^n, $\forall$n.

As you have seen in above example, some (algebraic) operations are needed at the middle stage of the process while writing the general term of a sequence explicitly. These operations on generating functions, which we are defining below, have a crucial role to play in solving combinatorial problems.

Apart from the usual operations of addition, subtraction, multiplication and division of series, we may need to integrate or differentiate a power series. It is important to observe that, while performing last two operations, our aim is to associate with the object $\frac{d}{dz}\left(\sum a_n z^n\right)$ $\left(\text{and} \int\left(\sum a_n z^n\right)dz\right)$ a new power series as given in the right hand side of O_3 (and O_4, respectively).

O_1 (Sum and Difference) $$\sum a_n z^n \pm \sum b_n z^n = \sum (a_n \pm b_n) z^n$$

O_2. (Multiplication) $$\left(\sum a_n z^n\right)\left(\sum b_n z^n\right) = \sum\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n;$$

O_3. (Differentiation) $$\frac{d}{dz}\left(\sum a_n z^n\right) = \sum (n+1)\, a_{n+1} z^n;$$

O_4. (Integration) $$\int\left(\sum a_n z^n\right)dz = \sum \frac{a_n}{n+1} z^{n+1}.$$

O_5. (Division) $$\left(\sum a_n z^n\right) \Big/ \left(\sum b_n z^n\right) = \sum c_n z^n$$

$$\Leftrightarrow \left(\sum b_n z^n\right)\left(\sum c_n z^n\right) = \sum a_n z^n, \text{i.e.}, a_n = \sum_{k=0}^{n} b_k c_{n-k}.$$

The quotient of two power series defined in O_5 above is via the product in the usual manner. In fact, there is no really convenient expression for the quotient.

Next, let us now look at some general results which provide connection between the generating functions of various sequences, terms of which are related in some manner to each other. These results are particularly useful when we know the generating functions of some of these, and want to find the same for others.

If $\{a_n\}$ and $\{b_n\}$ are two sequences, the sequence $\{c_n\}$, where $c_n = \sum_{0}^{n} a_k b_{n-k}$, is called the **convolution** of the sequences. If A(z) and B(z) are the generating functions of the sequences $\{a_n\}$ and $\{b_n\}$, respectively, according to O_2, the generating function of the convolution of $\{a_n\}$ and $\{b_n\}$ is A(z)B(z).
Here is an exercise involving convolution of sequences.

E3) Prove the Binomial identity $\sum_{j=0}^{k} C(m, j)C(n, k-j) = C(m+n, k)$, using Lemma 1. Hence deduce the Binomial identity $\sum_{j=0}^{k} C(k, j)^2 = C(2k, k)$

We next prove another useful lemma of similar nature.

Lemma 1: Suppose that the sequence $\{a_n\}$ $n \geq 0$, has the generating function

A (z). Then, generating function B(z) (say) for the sequence $\{b_n\} n \geq 0$, where $b_n = a_n - a_{n-1}$ for $n \geq 1$, and $b_0 = a_0$, is given by

$$B(z) = (1-z)A(z).$$

Proof: By definition, the generating function for the sequence $\{b_n\}$ is

$$\begin{aligned} B(z) &= \sum_{n=0}^{\infty} b_n z^n \\ &= b_0 + \sum_{n=1}^{\infty} b_n z^n \\ &= a_0 + \sum_{n=1}^{\infty} a_n z^n - z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} \text{ (using definition of } b_n) \\ &= a_0 + [A(z) - a_0] - zA(z) \\ &= (1-z)A(z) \end{aligned}$$

This completes the proof of the lemma.

Try the following exercise now.

E4) a) Use Lemma 1 to find the generating function A(z) (say) for the sequence in arithmetic progression $\{a, a+d, a+2d, \ldots\}$.

b) Suppose that A(z) is the generating function for the sequence $\{a_n\}, n \geq 0$, Show that the generating function S(z) (say) for the sequence $\{s_n\}$ of its partial sums viz. $s_n = \sum_{k=0}^{n} a_k, (n \geq 0)$ is given by $S(z) = \frac{A(z)}{1-z}$.

c) Use (b) to find the generating function for the sequence $\{1, 3, 6, \ldots\}$.

We next look at a problem which you might have solved earlier by different methods. Using generating functions, we shall give you alternative methods of solving them. This is an example involving the sum of k-th power of the first n natural numbers which we denote by σ_n^k

i.e $\sigma_n^k = 1^k + 2^k + \ldots + n^k = \sum_{i=1}^{n} i^k, \; k \geq 1.$

You already know how a formula for $\sigma_n^k \, (1 \leq k \leq 3)$ can be verified by induction (see Unit 2 of MCS-013). Let us see how generating function technique makes this task easier. You will see this in operation for the evaluation of $\sigma_n^2 = \sum_{j=1}^{n} j^2$ in the following example.

Example 5: Differentiating the Binomial function $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$, we get

$$\sum_{j=1}^{\infty} jz^{j-1} = (1-z)^{-2} \qquad \text{(see } O_3)$$

Multiplying this by z on both sides, we get

$$\sum_{j=1}^{\infty} jz^j = z(1-z)^{-2}$$

Repeating this process of first differentiating and then multiplying by z, we get

$$A(z) = \sum_{j=1}^{\infty} j^2 z^j = z(1+z)(1-z)^{-3},$$

where we write A (z) for the generating function of the sequence $\{j^2\}_{j \geq 1.}$

Then

$$\sum_{n=1}^{\infty} \sigma_k^2 z^n = \sum_{k=1}^{k}\left(\sum_{j=1}^{\infty} j^2\right) z^n$$
$$= \frac{A(z)}{(1-z)} \text{ (by E4, b)}$$
$$= z(1+z)(1-z)^{-4}$$

Therefore, σ_n^2 is the coefficient of z^n in the series which can be obtained by expanding the function $z(1+z)(1-z)^{-4}$. However, because

$$z(1+z)(1-z)^{-4} = z(1-z)^{-4} + z^2(1-z)^{-4}$$

this is the same as looking for the sum of coefficients of z^{n-1} and z^{n-2} in the expanded form of the Binomial function $(1-z)^{-4}$. Thus, in view of Binomial identity

C (n, k) = C $(n-k)$ we have

$$\sigma_n^2 = C(n+2,3) + C(n+1,3) = n(n+1)(2n+1)/6.$$

Try the following exercise now.

E5) Find the sum σ_n^1 of the first n natural numbers using generating functions.

So far, you learnt how to identify generating functions and use them to solve some simple combinatorial problems. However, there are several combinatorial problems which are hard to crack by using these functions. This is particularly true of problems that involve arrangements (in which **order** plays a crucial role) and distributions of distinct objects (see Block 2 of MCS-013 for more details). In the next section we introduce you to a slightly different kind of generating function which will prove useful for solving these type of problems.

2.3 EXPONENTIAL GENERATING FUNCTIONS

In this section, we shall study a modified form of the series we discussed in the last section. To understand the difference, let us consider the problem of finding the number of three-letter **words** i.e., a string of three letters which can be formed from a two-alphabet set {a, b} (say), with the restriction that not all letters in these **words** are identical.

Thus, we may use either two a's and one b or two b's and one a to form all the three letter **words** out of the two-element set {a, b}. Each of these two possibilities (by our discussion of permutations of objects, not necessarily distinct, in Block 2) give 3!/2!1!=3 distinct **words** viz. aab, aba, baa in the first case, and bba, bab, abb in the second, for a total of six **words.**

An ordered pair (x,y) of positive integers is a solution to the linear equation m+n =3, iff x+y=3.

Now, could we say that the number of distinct possibilities in the problem above is merely the number of positive integer solutions to the linear equation m+ n =3, if we think of this as using m a's and n b's, where m, n $\geq$ 1? This would have been so if we had not been interested in the position of a and b, in which case aab and aba would mean the same to us. But this is not the case. We are considering the number of three-letter **words** i.e., different strings of three letters. So, the position of the letters is important. Consequently, we would like each integer solution to contribute not 1 but 3 (so total is 3!) to the total number of **words.**

Now, as we wish to count the number of three letter **words**, we should look for the coefficient of z^3 in a series that counts $(m+n)!/m!n!$ each time $z^m z^n = z^3$ appears in that. So, we try the product.

$$\left(\frac{z}{1!}+\frac{z^2}{2!}\right)\left(\frac{z}{1!}+\frac{z^2}{2!}\right)=\frac{z^2}{1!1!}+\frac{z^3}{1!2!}+\frac{z^3}{2!1!}+\frac{z^4}{2!2!}.$$

For r = 1, 2, 3, 4, the coefficient of z^r in this is term of the form $1/m!n!$, where $m+n = r$, $m, n \geq 1$. We need to multiply this by $(m+n)!$ in order to get the answer we are looking for. Since the coefficient of z^3 in the above expansion is 1, we end up multiplying this by 3! to get a right answer to above problem.
An exponential generating function is precisely the power series of this type. A formal definition is given below.

Definition: The **exponential generating function** $A_{exp}(z)$ (say) for the sequence of real or complex numbers $\{a_0, a_1, \ldots, a_n, \ldots,\}$ is given by the power series.

$$A_{exp}(z)=\sum_{k=0}^{\infty}\frac{a_k}{k!}z^k = a_0 + \frac{a_1}{1!}z+\cdots+\frac{a_n}{n!}z^n+\cdots$$

As you can see, the nth term a_n of the given sequence is no longer the coefficient of z^n in $A_{exp}(z)$, rather it is n! times that coefficient.

For example, the exponential generating function for the constant sequence {1, 1, 1, …} is given by

$$\exp(z)=e^z=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=1+z+\frac{z^2}{2!}+\cdots$$

Does it remind you of some function? Of course, it resembles exponential function with which you are familiar but here z is just a symbol and not a variable. It is this resemblance from where these type of generating functions have derived their name.

Try the following exercise now.

E6) Find the exponential generating function of the sequence $\{P(n,k)\}_{n=1}^{k}$ for a fixed $n \in \mathbf{N}$ where P (n, k) denotes the number of k-permutations of n objects.

As before, let us try to identify the exponential generating functions associated with the combinatorial problem given in the following example.

Example 6: Show that the exponential generating function associated with the problem of finding the number of ways to choose some subset of m objects and distribute them into n boxes in such a way that the order within the same box is important, is given by $e^z(1-z)^{-n}$.

Solution: First of all, there are $n(n+1)\ldots(n+k-1)$ ways to arrange the objects into n boxes. Let us see why this is so.
Let us first look at an example. Suppose we want to arrange 4 objects, numbered 1, 2, 3, 4 in 3 different boxes labelled a, b, c. Let us first consider the objects to be indistinguishable. Then, the number of ways of distributing 4 indistinguishable objects in 3 distinguishable boxes is C (4+3 −1,4) = C(6,4) = C(6,2) = 15 (See page 68

of Block 2, MCS-013). Let us fix one such distribution, say 2 in the first box, 2 in the second box and none in the third box. One such distribution with objects considered distinct is given in Fig. 1. Let us apply any permutation of 1, 2, 3, 4, say 1 to 2, 2 to 3 and 3 to 1. Then, we will get the arrangement given in Fig. 2. There are 4! such permutations. So, corresponding to each arrangement with objects considered indistinguishable, there are 4! arrangement with objects considered distinguishable. So, the number of arrangements of 4 objects in 3 boxes, when the order inside the boxes matters, is $4! \times C(4+3-1,4) = 24 \times 15 = 360.$

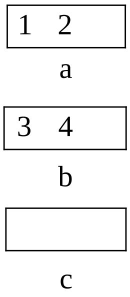

Fig. 1

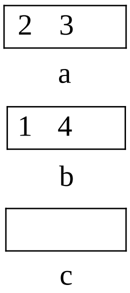

Fig. 2

Let us now look at the general situation. Let us first assume that the objects are identical and only the number of objects in each box matters. The number of ways of distributing k indistinguishable objects in n distinguishable boxes is $C(n+k-1,k)$. Let us fix any one arrangement and apply all possible permutations of r objects. We will then get r! different arrangements when the order within the boxes are taken into account. So, there are $r! \times C(n+k-1,k) \ = n(n+1)\ldots(n+k-1)$ arrangements. There are C(m,k) ways of choosing k out of m objects. Thus, the total number of ways to choose some subset of m objects and distribute the objects into n boxes in such a way that the order in the same box is matters, are

$$C(m,0) + \sum_{k=1}^{m} n(n+1)\ldots(n+k-1)C(m,k)$$

$$= m!\left[\frac{1}{m!} + \sum_{k=1}^{m} \frac{1}{(m-k)!k!} \times n(n+1)\ldots(n+k-1)\right]$$

Here, we may take n to be fixed, and consider this a sequence in m alone. Therefore, the corresponding exponential generating function for this sequence is

$$\sum_{m=0}^{\infty}\left[\frac{1}{m!} + \sum_{k=1}^{m} \frac{1}{(m-k)!k!} \times n(n+1)\ldots(n+k-1)\right] z^m,$$

which, in turn, is a product of the series

$$\left(\sum_{m=0}^{\infty} \frac{1}{m!} z^m\right) \text{and} \left(1 + \sum_{m=1}^{\infty} \frac{n(n+1)\ldots(n+m-1)}{m!} z^m\right) . (\text{see } O_2)$$

Now the first series equals e^z (by definition), while the second equals $(1-z)^{-n}$, by Binomial theorem. Hence, we have obtained the associated exponential generating function, as claimed.

Let us work out few examples to get a feeling about some elementary uses of the exponential generating functions in solving combinatorial problems.

Example 7: Find the number of bijections on a set of n elements, $n \geq 1$.

Solution: Let b_n denote the number of bijections on a set of n elements, $n \geq 1$. Recall from the previous unit (Example 6) that the recurrence relation satisfied by the sequence $\{b_n\}$ is given by

$$b_n = nb_{n-1} \text{if } n \geq 2 \text{ and } b_1 = 1.$$

Since we do not know b_0, we will ignore this term. The exponential generating function B (z) (say) of the sequence $\{b_n\}$ is given by

$$B(z)=\frac{b_1}{1!}z+\frac{b_2}{2!}z^2+\frac{b_3}{3!}z^3+\cdots+\frac{b_r}{r!}z^r+\cdots$$

Then

$$\begin{aligned}B(z)&=\sum_{n=1}^{\infty}\frac{b_n}{n!}z^n\\ &=z+\sum_{n=2}^{\infty}\frac{nb_{n-1}}{n!}z^n \quad (\text{by definition of } b_n, n\geq 2)\\ &=z+z\sum_{n=1}^{\infty}\frac{b_n}{n!}z^n=z+z.B(z).\end{aligned}$$

Solving for B (z), we get

$$B(z)=z/(1-z)=\sum_{n=1}^{\infty}z^n. \qquad \text{(by Binomial theorem)}$$

So, by comparing coefficients of z^n, we get from the last equality $b_n = n$ for $n\geq 1$

At times, the exponential generating functions are also useful in calculating the sum of an infinite series. Let us see an example of this.

Example 8: Find the sum of the series

$$\sum_{k=0}^{\infty}\frac{(k+1)^2}{k!}=\frac{1^2}{0!}+\frac{2^2}{1!}+\cdots+\frac{(n+1)^2}{n!}+\cdots$$

using exponential generating functions.

Solution: Multiplying by z on the both side of $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, we get

$$ze^z=\sum_{n=0}^{\infty}\frac{z^{n+1}}{n!}$$

This equation when differentiated once, gives

$$(1+z)e^z=\sum_{n=0}^{\infty}\frac{(n+1)z^n}{n!}. \qquad (\text{See} O_3)$$

Since we have already got one n+1 term in the numerator we are probably on the right track. We repeat the first two steps viz. multiply each side of the last equation by z and then differentiate, we get.

$$(1+3z+z^2)\,e^z=\sum_{n=0}^{\infty}\frac{(n+1)^2z^n}{n!}.$$

The rest of the job is easy. Put $z=1$ in the last equation to get 5e = $\sum_{n=0}^{\infty}(n+1)^2/n!$.

Therefore, the required sum of the given series is 5e.

Why don't you try an exercise now?

E7) Using exponential generating functions, find the number d_n of derangements of n objects. (see Unit 1 and Unit 3, Block 2 of MCS-013 for more details on derangements.)

In the previous two sections, you have seen some elementary use of two type of generating functions. In the next section, we shall give some more applications of generating functions.

2.4 APPLICATIONS

In this section we will see some applications of generating functions. We will see how to derive some combinatorial identities using generating functions. After that we will see how to find the number of integer solutions of linear equations using generating functions. We will also see applications of generating functions to partitions and for solving recurrences.

So let us start by applying generating functions to solve some simple combinatorial identities, particularly those that involve Binomial coefficients.

2.4.1 Combinatorial Identities

By Binomial theorem:

$$(1+z)^n = \sum_{k=0}^{n} C(n,k)z^k, \tag{2}$$

we know that $(1+z)^n$ is the generating function of the finite sequence $\{C(n,k)\}_{k=0}^{n}$. We shall use this to derive some combinatorial identities given in the following two examples.

Example 9: Prove the Binomial identity

$$C(n,1)+3C(n,3)+5C(n,5)+\cdots = n2^{n-2} = 2C(n,2)+4C(n,4)+6C(n,6)+\cdots$$

Solution: Differentiating both sides of (2) with respect to z, we get

$$n(1+z)^{n-1} = \sum_{k=0}^{n} kC(n,k)z^{k-1}.$$

Now setting $z=1$ and $z=-1$ in the resulting expression, we get

$$\sum_{k=1}^{n} kC(n,k) = n2^{n-1}, \text{and} \tag{3}$$

$$\sum_{k=1}^{n} (-1)^{k-1} kC(n,k) = 0, \text{ respectively.} \tag{4}$$

Shifting negative terms to the r.h.s. in (4), we have

$$C(n, 1) + 3C(n, 3) + 5C(n, 5) + \cdots = 2C(n, 2) + 4C(n,4) + 6C(n, 6) + \cdots$$

Now, on adding terms 2C(n,2), 4C(n,4), 6C(n, 6) ··· so on, to both sides of above identity, we get

$$\sum_{n=1}^{\infty} kC(n,k) = 2[2C(n,2) + 4C(n,4) + 6C(n,6) + \cdots]. \quad (5)$$

From this, using (3) it follows that r.h.s of (5) equals $\frac{n2^{n-1}}{2} = n2^{n-2}$. With this we have established the Binomial identity stated above.

Our next application concerns k-permutations of a set of n elements. By E12, of Unit 7, you know that the number of k-permutations of n distinct objects, P (n, k), satisfies the recurrence relation

$$P(n,k) = P(n-1, k) + kP(n-1, k-1), n, k \geq 1. \quad (6)$$

Example 10: For fixed n, find an explicit formula for P (n, k) by making use of its exponential generating function, $P_{exp}(z;n)$ (say) as defined below.

$$\mathbf{P_{exp}(z;n) = \sum_{k=0}^{\infty} (P(n,k)/k!)\, z^k.}$$

Solution: Using (6) and the definition of P_{exp} (z; n) we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{P(n,k)}{k!} z^k &= \sum_{k=1}^{\infty} \frac{P(n-1,k)}{k!} z^k + \sum_{k=1}^{\infty} \frac{P(n-1,k-1)}{k!} z^k \\
\text{i.e.} \sum_{k=1}^{\infty} \frac{P(n,k}{k!} z^k &= \sum_{k=1}^{\infty} \frac{P(n-1,k)}{k!} z^k + z\sum_{k=1}^{\infty} \frac{P(n-1,k-1)}{(k-1)!} z^{k-1} \\
\Rightarrow P_{exp}(z;n) - P(n,0) &= \left[P_{exp}(z;n-1) - P(n-1,0)\right] + zP_{exp}(z;n-1) \\
\Rightarrow P_{exp}(z;n) &= (1+z)P_{exo}(z;n-1)\,(\text{as}\,P(n,0) = P(n-1,0)) \\
\Rightarrow P_{exp}(z;n) &= (1+z)^n P_{exp}(z;0) = (1+z)^n. \ (\text{by iteration})
\end{aligned}$$

Since the coefficient of z^k in $(1+z)^n$ is C (n, k) (by Binomial theorem), it follows by comparing coefficients, that

$$\frac{P(n,k)}{k!} = C(n,k) \Rightarrow P(n,k) = k!C(n,k) = \frac{n!}{(n-k)!}.$$

Of course, if k > n, C (n, k) = 0, and hence P (n, k) = 0 then. So, we have obtained P(n, k), explicitly.

Try the following exercise now.

E8) Evaluate, using generating function technique, the sum $\sum_{k=1}^{n} k3^k C(n,k)$.

We next consider the application of generating functions to general integer equations.

2.4.2 Linear Equations

Generating functions are also particularly handy when one is looking for non-negative integer solutions to linear equations of the type $a_1 + a_2 + \cdots + a_k = n$. You may recall that we showed earlier (see Theorem 5 of Unit 2, Block 2 of MCS-013) that this equals $C(n+k-1, k-1)$ by elementary counting techniques. If, on the other hand, each a_j is a positive integer, then the number of such solutions equals $C(n-1, k-1)$.

Generating functions often provide a simpler way to solve such equations. This is illustrated in the following example.

Example 11: Find the number of integer solutions of the linear equation.

$$a_1 + a_2 + \cdots + a_k = n,$$

using generating function techniques when a) $a_i \geq 0$ b) $a_i \geq 1$.

Solution: a) The required number is the coefficient of z^n in the following product of polynomials (see discussion following Example 1)

$$(1+z+z^2+\cdots)\cdots(1+z+z^2+\cdots) \qquad \text{(k times)}$$

Each term of this product equals $(1-z)^{-1}$ (by Binomial theorem) and the coefficient of z^n in $(1-z)^{-k}$ is

$$C\,(n+k-1, n) = C\,(n+k-1, k-1);$$

b) If each $a_j \geq 1$ instead, we seek the coefficient of z^n in the expansion

$$(z+z^2+z^3+\cdots) \ldots (z+z^2+z^3+\cdots). \qquad \text{(k times)}$$

Each term of this product equals $z\,(1-z)^{-1}$ (by Binomial theorem) and the coefficient of z^n in $z^k\,(1-z)^{-k}$ is the coefficient of z^{n-k} in $(1-z)^{-k}$. This equals

$C(n-k) +k-1, n-k) = C\,(n-1, k-1)$.

Of course, this means that there is no solution if $n < k$, as should be the case.

* * *

If, in the example above, we require that one or more of the solutions, a_j are bounded at both ends, and if we allow a_j to be negative, then the number of solutions, even for $k = 2$ or 3 becomes a tedious computation. The method of generating functions is just what you could use for such problems. We illustrate this in the following example.

Example 12: Find the number of integer solutions to $a_1 + a_2 + a_3 = n$, where $-1 \leq a_1 \leq 1$, $1 \leq a_2 \leq 3$ and $a_3 \geq 3$.

Solution: Let us bring this into the situation of Example 11. For this, we put $b_1 = a_1+1$ and $b_3 = a_3 - 3$. Then our problem is same as looking for the number of integer solutions to

$$b_1 + b_2 + b_3 = n-2, \text{ where } 0 \leq b_1 \leq 2, 1 \leq b_2 \leq 3 \text{ and } b_3 \geq 0.$$

Now in view of these bounds on b_i's, it follows that associated generating function is given by

$$(1+z+z^2)(z+z^2+z^3)(1+z+z^2+\cdots)=\frac{1-z^3}{1-z}\times\frac{z(1-z^3)}{1-z}\times\frac{1}{1-z},$$

by using Binomial theorem and Result 2. As before, we want the coefficient of z^{n-2} in this expansion, which is same as the coefficient of z^{n-3} in

$$(1-z^3)^2(1-z)^{-3}=(1-z)^{-3}-2z^3(1-z)^{-3}+z^6(1-z)^{-3}.$$

Let us assume that C(n,k) = 0 for k > n.

$$(1-z)^{-3}=\sum_{k=0}^{\infty}C(3+k-1,2)z^k$$

$$z^3(1-z)^{-3}=\sum_{k=0}^{\infty}C(3+k-1,2)z^{k+3}$$

$$z^6(1-z)^{-3}=\sum_{k=0}^{\infty}C(3+k-1,2)z^{k+6}$$

So, the coefficient of z^{n-3} in $(1-z)^{-3}(1-z^3)^2$ is

$$C(3+n-3-1,2)-2C(3+n-6-1,2)+C(3+n-9-1,2)$$
$$=C(n-1,2)-2C(n-4,2)+C(n-7,2)$$

Since we have assumed that C(n,k) = 0 if $n<k$ all the terms are non-zero only if $n-7\ge 2$ or $n\ge 9$. If this is the case,

$$C(n-1,2)-2C(n-4,2)+C(n-7,2)$$
$$=\frac{(n-1)(n-2)}{2}-2\frac{(n-4)(n-5)}{2}+\frac{(n-7)(n-8)}{2}$$
$$=\frac{n^2-3n+2-(2n^2-18n+40)+(n^2-15n+56)}{2}=9.$$

If $n-4\ge 2$ and $n-7\le 1$, $6\le n\le 8$. In this case, the answer is

$$\frac{(n-1)(n-2)}{2}-2\frac{(n-4)(n-5)}{2}=\frac{n^2-3n+2-2n^2+18n-40}{2}$$
$$=\frac{-n^2+15n-38}{2}$$

For n = 6, 7, 8, this quantity is 8, 9, 9, respectively.

If $n-4\le 1$ and $n-1\ge 2$, $3\le n\le 5$ and the value is $\frac{(n-1)(n-2)}{2}$. If $n-1<2$ or $n<3$, all the terms are 0, i.e. there are no solutions.

The technique adopted in the example given above is no different if we have more than three summands or if the bounds we had are more general. In principle, therefore, we are in a position to find the number of integer solutions to

$$a_1+a_2+\cdots+a_k=n,\text{with}\, m_j\le a_j\le M_j, m_j, M_j\in Z. \quad (1\le j\le k)$$

Why don't you check your understanding of Example 12 by attempting the following exercise?

E9) How many integer solutions are there to $a_1+a_2+a_3+a_4+a_5 = 28$ with $a_k > k$ for each k, $1\le k\le 5$?

Another illustration of the use of generating functions is in the mathematical theory of partitions – historically one of the first problems studied with generating functions. We shall talk about this next.

2.4.3 Partitions

We shall only see one aspect of partitions namely their connection with generating functions. You already had some exposure to them in your earlier Mathematics course. Here we will go a little deeper. For this, we should first define the sequence of partitions, P_n.

Definition: The nth term of the sequence $\{P_n\}$, $n \geq 1$, counts the number of ways in which n can be expressed as a sum of positive integers such that the order of the summands (parts) is not important. We define $P_0 = 1$.

For Example, $P_4 = 5$ since 4 = 3+1=2+2=2+1+1=1+1+1+1. So, partitioning n is the same as distributing n non-distinct objects into n non-distinct boxes, with the empty box allowed (e.g. 4=3+1+0+0). In terms of linear equations discussed above, P_n is the number of non-negative integer solutions to the integer equation.

$$X_1 + X_2 + \cdots + X_k + \cdots = n, X_i = ia_i\ (\forall i),$$

where a_k denotes the number of k's in the partition. Note here, that the number of X_i's is not bounded. It can grow very large according to the size of n.

Let us look at the form that the generating function, P (z) of the sequence $\{P_n\}_{n\geq 0}$ must take.

Note that, in the above linear equation, for each integer $k \geq 1$, we may use none, one or more k's according to the value of $a_k \geq 0$. There is no other restriction on a_k's. Therefore, for each term $X_i = i.a_i$ ($a_i \geq 0$) the corresponding term in the associated generating function is simply $(1 + z^k + z^{2k} + \cdots)$.

$$P(z) = \prod_{k=1}^{\infty} (1 + z^k + z^{2k} + \cdots) = \prod_{k=1}^{\infty} \frac{1}{1 - z^k}$$

Generating functions of related sequences are not any harder to determine. They play a significant role in proving identities involving partitions. We illustrate this with the help of the following example.

Example13: Show that every nonnegative integer can be written as a unique sum of distinct powers of 2.

Solution: The generating function for the sequence $\{a_n\}$, where a_n denotes the number of ways n can be written as sum of distinct power of 2, is

$$(1+z)\,(1+z^2)\,(1+z^4)\,(1+z^8)\ldots$$

Now we have

$$\begin{aligned}
&(1-z)\,(1+z)\,(1+z^2)\,(1-z^4)\,(1+z^8)\cdots \\
&= (1-z^2)\,(1+z^2)\,(1+z^4)\,(1+z^8)\cdots \\
&= (1-z^4)\,(1+z^4)\,(1+z^8)\cdots \\
&= (1-z^{2n})\,(1+z^{2n})\cdots \\
&= 1 \quad (\text{assuming } |z| < 1.)
\end{aligned}$$

Thus, in view of operation O_5 and Binomial theorem, it follows that

$$(1+z)(1+z^2)(1+z^8)\cdots = \frac{1}{1-z} = 1 + z + z^2 + \cdots$$

From this, by comparing coefficients, we conclude that the coefficient of z^n in the 1.h.s of the equation is 1. Hence, the number a_n of partitions of n into distinct parts of size 1, 2, 4, 8, 16, …, so on, is 1. In other words, every non-negative integer can be **uniquely** expressed as the sum of distinct powers of 2.

Why don't you try the following exercise now?

E10) Show that the generating function for the sequence of the number of partitions of n with:

a) Parts each of which is at most m is $\prod_{k=1}^{m}(1-z^k)^{-1}$;

b) Unequal parts is $\prod_{k=1}^{\infty}(1+z^k)$;

c) Parts each of which is odd is $\prod_{k=1}^{\infty}(1-z^{2k-1})^{-1}$

E11) Find the generating function for the sequence of the number of partitions of n

i) into primes;
ii) into distinct primes.

Next, we shall discuss one of the most important uses of generating functions, viz., its utility as a tool to solve the recurrence relations.

2.4.4 Recurrence Relations

In unit 1, you have learnt how to set up recurrences for a combinatorial problem. Though we had not talked about how to solve them, we gave you some solutions, which you verified.

For solving a recurrence, we need to know the terms of a sequence explicitly. In other words, for a sequence $\{a_n\}$ that satisfies a given recurrence, we shall use its generating function A (z) (say) to find an explicit formula for a_n in terms of n.

Let us look at an example to see how we can solve recurrences using generating functions.

Example 14: Solve the recurrence $L_n = L_{n-1} + n$ for $n \geq 2$, and $L_1 = 2$.

Solution: The recurrence relation satisfied by the sequence $\{L_n\}$ is $L_n = L_{n-1} + n$ for $n \geq 2$, and $L_1 = 2$. If the same recurrence were to hold for $n \geq 1$ instead, then L_0 must equal 1.

Starting the sequence at L_0, the generating function L (z) (say) of the sequence $\{L_n\}_{n \geq 0}$ is given by

$$L(z) = \sum_{n=0}^{\infty} L_n z^n .$$

Now, by using the recurrence relation, we get

$$\begin{aligned} L(z) &= 1 + \sum_{n=1}^{\infty}(L_{n-1} + n)z^n \\ &= 1 + z\sum_{n=1}^{\infty} L_{n-1}z^{n-1} + z\sum_{n=1}^{\infty} nz^{n-1} \\ &= 1 + z\sum_{n=0}^{\infty} L_n z^n + z\sum_{n=1}^{\infty} nz^{n-1} \\ &= 1 + z.L(z) + \frac{z}{(1-z)^2}. \end{aligned}$$

Solving for L (z) in the last equation, we get

$$L(z) = \frac{1}{1-z} + \frac{z}{(1-z)^3}.$$

So, using Binomial theorem, we get

$$L(z) = \sum_{n=0}^{\infty}\left\{1 + \frac{1}{2}n(n+1)\right\}z^n.$$

Finally, equating coefficients of z^n on both sides of the last equation, we get

$$L_n = \frac{1}{2}n(n+1) + 1, n \geq 1.$$

We next consider the sequence of Fibonacci numbers $\{f_n\}$ which satisfy the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \text{ if } n \geq 3, \text{ and } f_1 = 1 = f_2.$$

Example 15: Find the generating function associated with the sequence of Fibonacci sequence $\{ f_n \} n \geq 1$. Then deduce a formula for f_n , $n \geq 1$.

Solution: We write F (z) for the associated generating function. Then by definition, we have

$$\begin{aligned} F(z) &= \sum_{n=1}^{\infty} f_n z^n \\ &= z + z^2 + \sum_{n=3}^{\infty}(f_{n-1} + f_{n-2})z^n \\ &= z + z^2 + z\sum_{n=2}^{\infty} f_n z^n + z^2\sum_{n=1}^{\infty} f_n z^n \\ &= z + z^2 + z[F(z) - z] + z^2.F(z). \end{aligned}$$

Then $(1-z-z^2)$ F (z) = z. Therefore, with $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$,

$$\beta = (1-\sqrt{5})/2,$$

$$\begin{aligned} F(z) &= \frac{z}{(1-\alpha z)(1-\beta z)} \qquad \text{(by solving equation } z^2+z-1=0) \\ &= \frac{1}{\alpha-\beta}\left(\frac{1}{1-\alpha z}-\frac{1}{1-\beta z}\right) \\ &= \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}(\alpha^n-\beta^n)z^n. \text{ (by binomial theorem)} \end{aligned}$$

Comparing coefficients of z^n now gives $f_n = (\alpha^n - \beta^n)/(\alpha-\beta)$, for all $n \geq 1$.
Try the following exercise now.

E12) Solve the recurrence relation $T_n = 2T_{n-1}+1$ if $n \geq 2$ and $T_1 = 1$, using generating functions technique. (See Tower of Hanoi problem in Unit 1).

If you have understood the steps that we followed in solving the recurrence relation involving Fibonacci sequence in previous example, then it should not be difficult for you to understand the proof of the following general result.

Theorem 1: The generating function, denoted by U(z), for a general linear, homogeneous recurrence relation with constant coefficients, of order k,

$u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_ku_{n-k}, n \geq k$, with $u_0 = c_0, \ldots, u_{k-1} = c_{k-1}$ satisfies the equation

$$\left(1 - a_1z - a_2z^2 - \cdots - a_kz^k\right)U(z) = c_0 + \sum_{n=1}^{k-1}(c_n - a_1c_{n-1} - \cdots - a_nc_0)z^n.$$

Proof: We have, by definition,

$$\begin{aligned} U(z) &= \sum_{n=0}^{\infty} u_n z^n \\ &= (u_0 + u_1z + \ldots + u_{k-1}z^{k-1}) + \sum_{n=k}^{\infty} u_n z^n \end{aligned}$$

$$\begin{aligned} \sum_{n=k}^{\infty} u_n z^n &= \sum_{n=k}^{\infty}\left(a_1u_{n-1} + a_2u_{n-2} + \cdots + a_ku_{n-k}\right)Z^n \\ &= a_1z\sum_{n=k}^{\infty} u_{n-1}z^{n-1} + a_2z^2\sum_{n=k}^{\infty} u_{n-2}z^{n-2} + \cdots + a_kz^k\sum_{n=k}^{\infty} u_n z^{n-k} \\ &= a_1z\sum_{n=k-1}^{\infty} u_n z^n + a_2z^2\sum_{n=k-2}^{\infty} u_n z^n + \cdots + a_kz^k\sum_{n=0}^{\infty} u_n z^n \\ &= a_1z\left(U(z) - \left(u_0 + u_1z + u_2z^2 + \cdots + u_{k-2}z^{k-2}\right)\right) \\ &\quad + a_2z^2\left(U(z) - \left(u_0 + u_1z + u_2z^2 + \cdots + u_{k-3}z^{k-3}\right)\right) \\ &\quad + \cdots + a_{k-1}z^{k-1}\left(U(z) - u_0\right) + a_kz^kU(z) \end{aligned}$$

Using $u_i = c_i$ for $0 \leq i \leq k$,

$$
\begin{aligned}
\sum_{n=k}^{\infty} u_n z^n &= \left(a_1 z + a_2 z^2 + \cdots + a_k z^k\right)U(z) - a_1 z\left(c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-2} z^{k-2}\right) \\
&\qquad - a_2 z^2\left(c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-2} z^{k-2}\right) \\
&\qquad \cdots \\
&\qquad - a_{k-2} z^{k-2}\left(c_0 + c_1 z\right) - a_{k-1} c_0 z^{k-1} \\
&= \left(a_1 z + a_2 z^2 + \cdots + a_k z^k\right)U(z) - \left(a_1 c_0 z + (a_1 c_1 + a_2 c_0) z^2\right. \\
&\quad + \cdots + \left(a_1 c_{k-3} + a_2 c_{k-4} + \cdots + a_{k-3} c_1\right) z^{k-2} \\
&\qquad \left. + \left(a_1 c_{k-2} + a_2 c_{k-3} + \cdots + c_0 a_{k-1}\right) z^{k-1}\right) \\
\therefore U(z) &= c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-1} z^{k-1} - \sum_{n=k}^{\infty} u_n z^n \\
&= \left(a_1 z + a_2 z^2 + \cdots + a_k\right)\left(U(z)\right) + c_0 + \left(c_1 - a_1 c_0\right) z + \left(c_2 - a_1 c_1 - a_2 c_0\right) z^2 \\
&\quad + \cdots + \left(c_{k-1} - a_1 c_{k-2} - a_2 c_{k-3} - \cdots - c_0 a_{k-1}\right) z^k
\end{aligned}
$$

$$
\left(1 - \left(a_1 z + a_2 z^2 + \cdots + a_k z^k\right)\right)U(z) = c_0 + \sum_{i=1}^{k-1} \left(c_i - c_{i-1} a_1 - c_{i-2} a_2 + \cdots + c_0 a_i\right) z^i
$$

$$
\therefore U(z) = \frac{Q_k(z)}{P_{k-1}(z)}
$$

where

$$
Q_k(z) = 1 - \left(a_1 z + a_2 z^2 + \cdots + a_k z^k\right)
$$

and $P_{k-1}(z) = c_0 + \sum_{n=0}^{k-1} (c_n - a_1 c_{n-1} - \cdots - a_n c_0) z^n$

This completes the proof of the theorem.

A first conclusion that you can easily deduce from the theorem above, is given in the following result.

Corollary 1: The generating function of linear, homogeneous recurrence relations with constant coefficients given in Theorem 1 is a rational function, p(z) / q (z), with the numerator, p(z), a polynomial of degree at most one less than the order of the recurrence.

Also observe that 1+q(z) is equal to the polynomial obtained from the R. H. S. of the given recurrence relation given in Theorem 1 by replacing u_{n-i} with z^i $(1 \le i \le k)$. While applying this corollary, you need to pay careful attention to the form of q (z). You should not try to memorize p (z) at all. After all, once you know q (z), p (z) can be obtained by multiplying q (z) by the generating series $\sum_{n=0}^{\infty} u_n z^n$.

Let us employ Theorem 1 and Corollary 1 to solve the following recurrence relation.

Example 16: Solve the third-order recurrence

$$
u_n - 9u_{n-1} + 26u_{n-2} - 24u_{n-3} = 0, n \ge 3,
$$

with the initial conditions $u_0 = 6$, $u_1 = 17$ and $u_2 = 53$.

Solution: We denote by U (z) the generating function for the sequence $\{u_n\}$. Then, by Theorem 1, we know that $(1 - 9z + 26z^2 - 24z^3)\,U(z) = P(z)$ is a polynomial of degree 4 in z.. Now, a little more calculations will lead you to conclude that

$$
(1 - 9z + 26z^2 - 24^3)\,U(z) = (1 - 2z)(1 - 3z)(1 - 4z)\,U(z)
$$

$$= \sum_{n=0}^{\infty} u_n z^n - 9\sum_{n=0}^{\infty} u_n z^{n+1} + 26\sum_{n=0}^{\infty} u_n z^{n+2} - 24\sum_{n=0}^{\infty} u_n z^{n+3}$$

$$= u_0 + (u_1 - 9u_0)z + (u_2 - 9u_1 + 26u_0)z^2$$

$$+ \sum_{n=3}^{\infty} (u_n - 9u_{n-1} + 26u_{n-2} - 24u_{n-3})z^n$$

$= 6 - 37z + 56z^2$, by using the given recurrence relation and substituting the values $u_0 = 6,\ u_1 = 17$ and $u_2 = 53$.

Therefore,

$$U(z) = (6 - 37z + 56z^2) / (1-2z)(1-3z)(1-4z).$$

Decomposing the R.H.S. into partial fractions, we then get

$$U(z) = 3(1-2z)^{-1} + (1-3z)^{-1} + 2(1-4z)^{-1}.$$

Expanding the R.H.S. using Binomial theorem and comparing the co-efficient of z^n both sides, we get

$$u_n = 3.2^n + 3^n + 2.4^n,\ n \geq 0.$$

Try the following exercise now.

E13) Determine the generating function for the sequence $\{t_n\}_{n=0}^{\infty}$ given by the recurrence relation $(n \geq 3)$

$$t_n = \begin{cases} t_{n-3} & \text{if } n \text{ is even} \\ t_{n-3} + \dfrac{n + (-1)^{(n+1)/2}}{4} & \text{if } n \text{ is odd} \end{cases}$$

You may take $t_0 = t_1 = t_2 = 0$.

In yet another situation, let us next consider the case of nonhomogenous recurrences viz. when the nonhomogenous term (s) are either of the types $r^n\ (r \in C)$ or $n^k\ (k \in N \cup \{0\})$. Below we consider the case when it is of the form r^n. The method of generating functions, and in particular Theorem 1, can still be of use to good effect as the following example shows.

Example 17: Solve the third-order nonhomogeneous linear recurrence with constant coefficients viz. $u_n - 3u_{n-2} - 2u_{n-3} = an + b.2^n$ in terms of the initial conditions u_0, u_1 and u_2.

Solution: Write U (z) for the generating function of the sequence $\{u_n\}_{n\geq 0}$, then

$$(1 - 3z^2 - 2z^3)U(z) = (1+z)^2(1-2z)U(z)$$

$$= \sum_{n=0}^{\infty} u_n z^n - 3\sum_{n=0}^{\infty} u_n z^{n+2} - 2\sum_{n=0}^{\infty} u_n z^{n+3}$$

$$= u_0 + u_1 z + (u_2 - 3u_0) z^2 + \sum_{n=3}^{\infty} (u_n - 3u_{n-2} - 2u_{n-3}) z^n$$
$$= u_0 + u_1 z + (u_2 - 3u_0) z^2 + az \sum_{n=3}^{\infty} n z^{n-1} + b \sum_{n=3}^{\infty} (2z)^n$$
$$= (u_0 - b) + (u_1 - a - 2b) z + (u_2 - 3u_0 - 2a - 4b) z^2$$
$$- \frac{a}{(1-z)^2} - \frac{a}{1-z} + \frac{b}{1-2z}.$$

The rest of the calculation is tedious, but routine. We employ partial fractions, to get U (z) in the form

$A(1-z)^{-1} + B(1-z)^{-2} + C(1+z)^{-1} + D(1+z)^{-2} + E(1-2z)^{-1} + F(1-2z)^{-2}$ for some choice of A, … , F. In terms of these constants.

$$u_n = A + B(n+1) + C(-1)^n + D(-1)^n (n+1) + E.2^n + F.2^n (n+1), n \geq 0.$$

Try the following exercise now.

E14) Use Theorem 1 to solve the recurrence
$a_n - 3a_{n-1} - 10a_{n-2} = 28 \times 5^n$ for $n \geq 2$, with $a_0 = 25$ and $a_1 = 120$.

It is sometimes possible to solve even non-linear recurrences with the help of generating functions. We illustrate this by solving a recurrence about which you have read before in Unit 1.

Example 18: Solve the recurrence relation for Catalan numbers given by

$a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1}, n \geq 2,$ with $a_n \geq 0 \, (\forall n)$ and $a_1 = 1.$

Solution: In order to extend the validity of the given recurrence to $n \geq 1$. we define $a_0 = 0$. If we denote its generating function by A (z), we get

$$\sum_{n=2}^{\infty} a_n z^n = \sum_{n=2}^{\infty} (a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1}) z^n$$
$$\Rightarrow A(z) - a_1 z - a_0 = \{A(z)\}^2 - (a_1 a_0 + a_0 a_1) z - a_0{}^2 \quad (\text{by } O_2)$$
$$\Rightarrow \{A(z)\}^2 - A(z) + z = 0$$
$$A(z) = \frac{1 \pm \sqrt{1-4z}}{2}.$$

Now, using Binomial Theorem, the coefficient of z^n in $(1-4z)^{1/2}$ is equal to

$$\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!}(-4)^n,$$

which you can easily simplify to $-\frac{2}{n} C(2n-2, n-1)$.

We choose the solution $A(z) = (1-\sqrt{1-4z})/2$, so that the terms a are non negative for $n \geq 1$, we thus have

$$a_n = \frac{1}{n} C(2n-2, n-1) = \frac{(2n-2)!}{(n-1)!n!}.$$

Why don't you try an exercise now?

E15) Using Theorem 1, find the n^{th} term, L_n of the Lucas sequence given by $L_n = L_{n-1} + L_{n-2}, n \geq 3, \text{with } L_1 = 1, L_2 = 3.$

So far we have discussed the use of generating functions in various areas. Regarding linear recurrence relations, we have seen how useful they are for finding solutions of such equations. There are several others methods for solving equations of this kind. We shall discuss them in the next unit. For now let us summarise what we have covered in this limit.

2.5 SUMMARY

In this unit we have seen how to:

1) construct generating functions for sequences arising from combinatorial problems.
2) find the number of integer solutions to linear equations.
3) find the generating functions associated with sequences in closed form in certain simple cases.
4) find the exponential generating functions associated with sequences in closed form in certain special cases.
5) solve recurrence relations in certain special cases.
6) use generating functions to prove identities involving combinatorial co-efficients.

2.6 SOLUTIONS / ANSWERS

Note: In all the following solutions, we will skip some steps and you are encouraged to work out the individual steps to ensure understanding of the computational procedure. In most cases, second block of MCS-013 will be helpful.

E1) a) The associated power series is
$(z^5 + z^{10} + \cdots)(1 + z^2 + z^4 + \cdots)(1 + z^3 + z^6 + \cdots).$
Here the first polynomial does not contain the constant term because of the given condition.

b) Since each k, ℓ and m are positive by given condition, and so, for a choice of (k+ ℓ +m) fruits (with 5k+2 ℓ +3m=50) , the associated power series is $(z^5 + z^{10} + \cdots)(z^2 + z^4 + \cdots)(z^3 + z^6 + \cdots)$.

E2) a) The generating function for the finite geometric progression is
$$\sum_{n=0}^{k-1} ar^n z^n = a\sum_{n=0}^{k-1}(rz)^n = a(1-r^k z^k)/(1-rz),$$ by result **R2**.

b) Replacing z by az in the Binomial theorem, it follows that $(1+az)^k$ is the generating function for the sequence $\{C(k,n)a^n\}_{n=0}^{\infty}$, if k is negative. This gives solution of (b).

c) Replacing z by az in Binomial theorem result IC, we get the result.

E3) For positive m and n, since $(1+z)^m$ is the generating function for the sequence $\{C(m,k)\}_{k=0}^{\infty}$, and $(1+z)^n$ is the generating function for $\{C(n,k)\}_{k=0}^{\infty}$, the function $(1+z)^m (1+z)^n$ is the generating function for the sequence with kth term $\sum_{j=0}^{k} C(m,j)C(n,k-j)$. However, $(1+z)^{m+n}$ is the generating function for

$\{C(m+n,k)\}_{k=0}^{\infty}$. Hence the first identity.The second identity follows from the first by taking m = n = k and using the identity C (n, k) = C (n, n-k).

E4) a) Write $a_n = a+nd,\ n \geq 0$. Then $a_n - a_{n-1} = d,\ \forall n \geq 1$. and $a_0 = a$. Let $\{b_n\}$ denote the sequence, where $b_0 = a$ and $b_n = d, \forall\, n \geq 1$.
By definition, $B(z) = a+dz+dz^2+ \ldots = a+zd\,[1+z+z^2+\ldots] = a+dz\,(1-z)^{-1}$, which is the generating function for the sequence $\{b_n\}_{n\geq 1.}$ Thus, by Lemma 2, B (z) = (1–z) A (z)

$$\Rightarrow \quad A(z) = a(1-z)^{-1} + z\,d(1-z)^{-2} = \{a+(d-a)z\}(1-z)^{-2}.$$

b) Since $a_n = s_n - s_{n-1}$, for $n \geq 1$, and $a_0 = s_0$, so, we have
$(1-z)S(z) = A(z).$ (by Lemma1)
Finally, proof is complete by using the definition O_5 of quotients of series.

c) The n th term of the given sequence is the n th partial sum of the sequence {1, 2, 3, …,} whose generating function A (z) (say) is $(1-z)^{-2}$ by (a). Hence, by (b), the generating function for the sequence {1, 3, 6,…} equals $(1-z)^{-3}$.

E5) Differentiating the Binomial function $(1-z)^{-1} = \sum_{j=0}^{\infty} z^j$, we get

$$\sum_{j=0}^{\infty} jz^{j-1} = (1-z)^{-2} \qquad (\text{see } O_3).$$

On multiplying this by z both sides, we get

$$A(z) = \sum_{j=1}^{\infty} jz^j = z(1-z)^{-2},$$

where we write A (z) for the generating function of the sequence $\{j\}_{j\geq 1}$. Then

$$\begin{aligned}\sum_{k=1}^{\infty} \sigma_k^1 z^k &= \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} j\right) z^k \\ &= \frac{A(z)}{(1-z)} \qquad (\text{by E4 b}) \\ &= z(1-z)^{-3}.\end{aligned}$$

Therefore, σ_n^1 is the coefficient of z_n in the series which can be obtained by expanding the function $z\,(1-z)^{-3}$. However, this is the same as looking for the coefficients of z^{n-1}. In the expanded form of the Binomial function $(1-z)^{-3}$. Thus, in view of Binomial identity C (n, k) = C (n, n– k), we have

$$\sigma_n^1 = C(n+1, n-1) = C(n+1, 2) = n(n+1)/2.$$

E6) By definition, exponential generating function of the sequence $\{P(n,k)\}_{k=1}^{n}$ is

$$\sum_{k=0}^{\infty} \frac{P(n,k)}{k!} z^k = \sum_{k=0}^{\infty} C(n,k) z^k = (1+z)^n.$$

E7) A first-order recurrence equation that the sequence $\{d_n\}$ satisfies is given by
$d_n = nd_{n-1} + (-1)^n, n \geq 2$, with $d_1 = 0, d_2 = 1$. (see Problem 7 of Unit 1).
In order that the recurrence also holds for n =1, we define d_0=1.Then, with

$D_{exp}(z) = \sum_{n=0}^{\infty}(d_n / n!)z^n$, we have

$$\sum_{n=1}^{\infty}\frac{d_n}{n!}z^n = \sum_{n=1}^{\infty}\frac{nd_{n-1}}{n!}z^n + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n$$

$$\Rightarrow D_{exp}(z) - d_0 = zD_{exp}(z) + (e^{-z} - 1)$$

$$\Rightarrow D_{exp}(z) = \frac{e^{-z}}{1-z}.$$

Now the coefficient of z^n in the expansion of e^{-z} equals $\frac{(-1)^n}{n!}$ and so, the coefficient of z^n in the expansion of $D_{exp}(z)$ is $\sum_{k=0}^{n}(-1)^k / k!$ (see E3 (b)). It then follows that $d_n = n! \sum_{k=0}^{n}(-1)^k / k! \ \ \forall n,$ by comparing coefficients of z^n.

E8) Differentiating and then multiplying by z on both sides of the identity $\sum_{k=0}^{n} C(n,k)z^k = (1+z)^n$, We get $\sum_{k=1}^{n} kC(n,k)z^k = nz(1+z)^{n-1}$.
Putting z = 3 yields
$\sum_{k=1}^{n} k3^k C(n,k) = 3 \times 4^{n-1}n.$

E9) Since the required generating function is

$$(z^2 + z^3 + z^4 + \cdots)(z^3 + z^4 + z^5 + \cdots)(z^4 + z^5 + z^6 + \cdots)$$
$$\times (z^5 + z^6 + z^7 + \cdots)(z^6 + z^7 + z^8 + \cdots).$$

$= z^{20}(1 + z + z^2 + \cdots)^5$, the number of integer solutions is the coefficient of z^8 in $(1-z)^{-5}$, which is C(12,4)=495.

E10) a) The contribution to the generating function from a part k is $(1 + z^k + z^{2k} + \ldots)$. Since $1 \le k \le m$, the required generating function is $\prod_{k=1}^{m}\left(1 + z^k + z^{2k} + \cdots\right) = \prod_{k=1}^{m}\left(1 - z^k\right)^{-1}$.

b) If we use unequal parts, no part k may be repeated. The corresponding term in the generating function is $(1+z^k)$, so that k may be used at most once. Therefore, the generating function is $\prod_{k=1}^{\infty}(1 + z^k)$.

c) The contribution from the odd part, 2k–1, is $(1 + z^{2k-1} + z^{2(2k-1)} + \cdots)$. Thus, the required generating function is $\prod_{k=1}^{\infty}(1 + z^{2k-1} + z^{2(2k-1)} + \cdots) = \prod_{k=1}^{\infty}(1 - z^{2k-1})^{-1}$.

E11) i) By above discussion, the required generating function is

$(1 + z^{p_1} + z^{2p_1} + \cdots)(1 + z^{p_2} + z^{2p_2} + \cdots)$ where p_1, p_2, … are the prime numbers.

ii) Similarly, here generating function will be

$\left(1 + z^{p_1}\right)\left(1 + z^{p_1}\right)\cdots$

E12) Defining $T_0 = 0$, so that the recurrence is valid for $n \ge 1$, and writing T (z) for the generating function of $\{T_n\}_{n=0}^{\infty}$, we have

$$T(z)=\sum_{n=0}^{\infty} T_n z^n = T_0 + 2\sum_{n=1}^{\infty} T_{n-1} z^n + 2\sum_{n=1}^{\infty} T_{n-1} z^n + \sum_{n=1}^{\infty} z^n$$

$$= 2z.T(z) + z/(1-z)$$

Therefore, $T(z)=z/(1-z)(1-2z)=(1-2z)^{-1}-(1-z)^{-1}$ and hence $T_n = 2^n - 1$, $n \ge 0$, by comparing coefficients after applying Binomial theorem on r.h.s. of the last equality.

E13) Let $T(z) = \sum_{n=0}^{\infty} t_n z^n$. Then,

$$T(z) = (t_0 + t_1 z + t_2 z^2) + \sum_{n=3}^{\infty} t_{n-3} z^n + \sum_{n=1}^{\infty} \frac{2n+1+(-1)^{n+1}}{4} z^{2n+1}$$

$$= z^3.T(z) + \frac{z}{4}\sum_{n=1}^{\infty}(2n+1)z^2 + \frac{z^3}{4}\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

$$\Rightarrow (1-z^3)T(z) = \frac{z}{4}\frac{d}{dz}\sum_{n=1}^{\infty} z^{2n+1} + \frac{z^3}{4(1-z^2)}$$

$$= \frac{z}{4}\frac{d}{dz}\left(z\sum_{n=1}^{\infty} z^{2n}\right) + \frac{2^3}{4(1-z^{2)}}$$

$$= \frac{z}{4}\frac{d}{dz}\left(\frac{z}{(1-z^2)}\right) + \frac{z^3}{4(1-z^2)}$$

$$= \frac{3z^3}{4(1-z^3)^2} + \frac{z^3}{4(1-z^2)}$$

$$\Rightarrow T(z) = \frac{z^3(4-3z^2-2z^3+z^6)}{4(1-z^3)^3(1-z^2)}.$$

E14) Let $A(z)=\sum_{n=0}^{\infty} a_n z^n$. Then,

$$(1-3z-10z^2)A(z) = a_0 + (a_1 - 3a_0)z + \sum_{n=2}^{\infty}(a_n - 3a_{n-1} - 10a_{n-2})z^n$$

$$= 25+45z+28\sum_{n=2}^{\infty}(5z)^n = 25+45z+28\left\{\frac{1}{1-5z}-(1+5z)\right\}$$

$=(25-80z+475z^2)/(1-5z)$.

Using partial fractions, we get

$$A(z)=(25-80z+475z^2)/(1+2z)(1-5z)^2 =$$

$15(1+2z)^{-1} - 10(1-5z)^{-1} + 20(1-5z)^{-2}$. Equating coefficients of z^n, we get

$$a_n = 15(-2)^n - 10.5^n + 20(n+1)5^n = 15(-2)^n + (10+20n)5^n, n \ge 0.$$

E15) We set $L_0 = L_2 - L_1 = 2$, so that the recurrence is valid for $n \ge 2$. By Theorem 1, $(1-z-z^2)\,L(z) = L_0 + (L_1 - L_0)z = 2 - z$. Therefore,

$L(z)=(1-\alpha z)^{-1}+(1+\beta z)^{-1}$, where $\alpha+\beta=1=-\alpha\beta$.

Comparing the coefficients of z^n, we get $L_n = \alpha^n + \beta^n, n \ge 0$.