UNIT 2 THE PROPOSITIONAL LOGIC

Structure **Page Nos.**

2.0 INTRODUCTION

Symbolic logic may be thought of as a *formal language* for representing facts about objects and relationships between objects of a problem domain **alongwith** a precise *inferencing mechanism* for reasoning and deduction. An **inferencing mechanism** derives the knowledge, which is not explicitly/directly available in the knowledge base, but can be logically inferred from what is given in the knowledge base.

The reason why the subject-matter of the study is called **Symbolic Logic** is that **symbols** are used to denote facts about objects of the domain and relationships between these objects. Then the **symbolic representations** *and not the original facts and relationships* are manipulated in order to make conclusions or to solve problems.

Also, we mentioned that a Symbolic Logic, apart from having other characteristics, is a formal language. As a formal language, there must be clearly stated unambiguous rules for defining various constituents or constructs, viz. alphabet set, words, phrases, sentences etc. of the language and also for associating meaning to each of these constituents.

The study of **Symbolic Logic** is significant, specially, for academic pursuits, in view of the fact that it is not only **descriptive** (*i.e., it tells how the human beings reason*) but it is also **normative** (*i.e., it tells how the human beings should reason*).

In this unit, we shall first study the simplest form of symbolic logic, viz, the *Propositional Logic* (PL). In the next unit, we consider a more general form of logic called the *First-Order Predicate Logic (FOPL)*. Subsequently, we shall consider other symbolic systems including Fuzzy systems and some Non-monotonic systems.

In the *propositional logic*, we are interested in *declarative* sentences, i.e., sentences that can be either true or false, but not both. Any such declarative sentence is called **a proposition or a statement**. For example

(i) The proposition: "*The sun rises in the west*," is False,
(ii) The proposition: "*Sugar is sweet*," is True, and

(iii) The truth of the proposition: “*Ram has a Ph. D degree*.” depends upon whether Ram is actually a Ph. D or not.
Though, at present, it may not be known whether the statement is True or False, yet it is sure that the sentence is either True or False and not both True and False simultaneously.

For a given declarative sentence, its being ‘True’ or ‘False’ is called its *Truth-value*. Thus, truth-value of (i) above is False and that of (ii) is True.

On the other hand, none of the following sentences can be assigned a truth-value, and hence none of these, is a statement or a proposition:

(i) Who was the first Prime Minister of India? *(Interrogative sentence)*
(ii) Please, give me that book. *(Imperative sentence)*
(iii) Ram must exercise regularly. *(Imperative, rather Deontic)*
(iv) Hurrah! We have won the trophy. *(Exclamatory sentence)*

In propositional logic, as mentioned earlier also, symbols are used to denote propositions. For instance, we may denote the propositions discussed above as follows:

P : The sun rises in the west,
Q : Sugar is sweet,
R : Ram has a Ph.D. degree.

The symbols, such as P, Q, and R, that are used to denote propositions, are called **atomic formulas, or atoms.** As discussed earlier, in this case, the truth-value of P is False, the truth-value of Q is True and the truth-value of R, though not known yet, is exactly one of ‘True’ or ‘False’, depending on whether Ram is actually a Ph. D or not.

At this stage, it may be noted that once symbols are used in place of given statements in, say, English, then the propositional system, and, in general, a symbolic system is aware **only** of symbolic representations, and the associated truth values. The system operate only on these representations. And, except for possible final translation, is **not aware** of the original statements, generally given in some natural language, say, English.

We can build, from atoms, *more complex propositions*, sometimes called ***compound propositions***, by using logical connectives.

Examples of such propositions are:

(i) *Sun rises in the east* ***and*** *the sky is clear*, and
(ii) ***If*** *it is hot* ***then*** *it shall rain.*

The logical connectives in the above two propositions are “*and*” and “*if…then*”. In the propositional logic, **five logical operators or connectives**, viz., ~ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (*if… then*), and $\leftrightarrow$ (*if and only if*), are used. These five logical connectives can be used to build compound propositions from given atomic formulas. More generally, they can be used to construct more complicated compound propositions from compound propositions by applying the connectives repeatedly. For example, **if** each of the letters P, Q, C is used as a symbol for the corresponding statement, as follows:

P: The wind speed is high.
Q: Temperature is low.
C: One feels comfortable.

then the sentence:
"*If the wind speed is high and the temperature is low, then one does not feel comfortable*"
may be represented by the formula $((P \wedge Q) \rightarrow (\sim C))$. Thus, a compound proposition can express a complex idea. In the propositional logic, an expression that represents a proposition, such as P, or a compound proposition, such as $((P \wedge Q) \rightarrow (\sim C))$, is called a *well-formed formula*.

2.1 OBJECTIVES

After going through this unit, you should be able to:

- tell about what is Logic, Symbolic Logic, and Propositional Logic (PL); further, about why we study each of these; and about some detailed subject matter of each of these;
- tell the difference between a Proposition/Statement, which forms the basis of PL, and a sentence in a natural language;
- explain the difference between a logical operator and a non-logical operator; any symbolic logic uses only logical operators;
- explain the concept of arguments in a logical system and further should be able to explain mutual differences between a (i) valid argument (ii) sound argument (iii) invalid argument, and (iv) unsound argument;
- differentiate between an expression that is a well-formed formula **wff** of PL and an expression which is not a wff.;
- find the truth-value, or meaning, of a wff of PL and should be able to explain how the truth value of a wff is obtained from the truth values of atomic wffs.
- explain the difference between various types of wffs, viz, valid wff; consistent wff, invalid wff and inconsistent wff;
- explain about the various tools, like truth table, logical deduction and reduction to normal forms that are used to establish validity/invalidity of arguments, and further should be able to use these tools for the purpose, and
- use the tools and techniques of PL in solving problems that can be solved within a PL system.

2.2 LOGICAL STUDY OF VALID AND SOUND ARGUMENTS

Logic is the analysis and appraisal of arguments.

An **argument** is a set of statements consisting of a finite number of premises, i.e., assumed statements and a conclusion.

Valid Argument: A valid argument is one in which it would be contradictory for the premises to be true but the conclusion false.

In logical studies we are interested in valid arguments.

Example of Valid Argument

(i) If you overslept, you will be late
(ii) You are not late.
$\therefore$ you did not oversleep.

Example of Invalid argument

(i) If you overslept, you will be late
(ii) You did not oversleep
$\therefore$ you are not late

(This argument is invalid, because despite not having overslept, one may be late because of some other engagements or lazyness.)

Another Invalid Argument

(i) If we are close to the top of Mt. Everest then we have magnificent view.
(ii) We are having a magnificent view.
Therefore,
(iii) We are the near the top of Mt. Everest.

(This argument is invalid, because, we may have a magnificent view even if we are not close to the top of Mt. Everest. The two given statements do not falsify this claim)

How to establish logical validity/invalidity of an argument

We have already discussed invalidity of some arguments, but invalidity above was based on our *intuition*. However, intuition may also lead us to incorrect conclusion. To be sure about the validity of our argument, we need some formal method. In Section 1.5, we discuss how a Truth table *(a formal tool*) can be used to establish the validity/invalidity of an argument.

Sound Argument

We may note that, in the case of a *valid* argument, it is **not required** that the premises/axioms or assumed statements must be True. The *assumptions may not be True*, and still the *argument* may be *valid*. For example, **the following argument is *valid*, but its premises and conclusion both are *false*:**

Premise 1: If moon is made of green cheese
Then 2 + 2 = 5
Premise 2: Moon is made of green cheese
(False premise)
From Premise 1 and Premise 2, by applying Modus Ponens, we conclude through **valid** argument that 2 + 2 = 5 *(which is False).*

However, in order to solve problems of everyday life, we need generally to restrict to *only true premises* and *valid arguments*. Then such an argument is called **sound argument.**

Sound Argument: is an argument that is valid and has true premises.

(i) If you are reading this, then
you are not illiterate
(ii) You are reading this (*true premise*)
You are not illiterate (*sound conclusion*)

Example of valid but not sound argument with correct conclusion.

(i) If moon is made of green cheese then 2 + 2 = 4
(ii) Moon is made of green cheese (*False premise*)
To conclude 2 + 2 = 4 (*correct*) makes the argument a Valid Argument

Example of Invalid Argument

I (i) If you overslept, you are late.
(ii) you are late.
Therefore, you overslept.
II (i) If you are in Delhi, you are in India.
You are in India.
Therefore, you are in Delhi (*invalid argument, though conclusion may be True*)

2.3 NON-LOGICAL OPERATORS

One of reason why special **symbols:**

$\wedge \quad \vee \quad \sim \quad \leftrightarrow \quad \rightarrow$

are used in symbolic logic in stead of the corresponding natural languages **words:** and, or, not, if…. Then, if and only if, is that the **words** may have different meaning in different contexts. For example, the use of the word **and** in one sentences has different connotation or meaning from the use in others in the following:

(i) Ram **and** Mohan are good hockey players.

(the statement can be equivalently broken into two statements:
(i) Ram is a good hockey player (ii) Mohan is a good hockey player)

(ii) Ram **and** Mohan are good friends.

*(though the word **and** joins two words Ram & Mohan, but can not be equivalently broken into two statements viz. (i) Ram is a friend (ii) Mohan is a friend)*

(iii) Mohan drove a car to reach home, met an accident **and** got slightly injured.

(Here, the use of the word 'and' is not in a logical sense, but, it is in temporal sense of 'and then' because statement (iii) has different sense from the statement given in (iv) below)

(iv) Mohan met an accident, got slightly injured **and** drove a car to reach home.

Thus from the above statements, it can be seen that the natural language word **and** may have many senses, both logical and non-logical. Similarly, the words **since**, **hence** and **because** are frequently used in arguments to establish some facts. But as shown from the following two arguments, their use in logical arguments is **risky** in the sense that some of the arguments involving any of these words may lead to **incorrect** conclusions:

Argument (1): Using the word *because*, we get correct conclusion from True statements.

Let

P: Dr. Man Mohan Singh was Prime Minister of
India in the year 2006 (*True statement*)

Q: Congress party and its allies commanded majority in Indian Parliament in the year 2006 *(True statement)*

Then the following statement:
P because Q *(True statement/conclusion)*

Thus, by using the connective *because* we get a correct/True conclusion from two True statements viz. P and Q.

Argument (2)

In the following using the word, because**, we get incorrect/false conclusion from True statements**

Let

P: Dr. Man Mohan Singh was Prime Minister of
India in the year 2006 (*True statement*)

R: Chirapoonji, a town in north-east India, received maximum average rainfall in the world during 1901-2000. *(True statement)*

However to say
P because R, i.e., to say
Dr. Man Mohan Singe was Prime Minster of India in 2006, because Chirapoonji, a town in north-east India, received maximum average rainfall in the world during 1901-2000.
is at least **incorrect**, if not ludicrous.

Thus from two True statements, P and R and by using connective 'because', in this case, the conclusion is *incorrect*.

Thus, by using connective **because**, in one argument we get a correct conclusion from two True statements and, on the other hand, we get an incorrect conclusion from True statements.

2.4 SYNTAX OF PROPOSITIONAL LOGIC

A Well-formed formula, or *wff* or *formula* in short, in the propositional logic is defined recursively as follows:

1. An atom is a wff.
2. If A is a wff, then (~A) is a wff.
3. If A and B are wffs, then each of $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$ is a wff.
4. Any wff is obtained only by applying the above rules.

From the above recursive definition of a wff it is not difficult to see that expression: $((P \rightarrow (Q \wedge (\sim R))) \leftrightarrow S)$ is a wff; because , to begin with, each of P, Q , (~ R) and S, by definitions is a wff. Then, by recursive application, the expression: $(Q \wedge (\sim R))$ is a wff. Again, by another recursive application, the expression: $(P \rightarrow (Q \wedge (\sim R)))$ is a wff. And, finally the expression given initially is a wff.

Further, it is easy to see that according to the recursive definition of a wff, each of the expressions: $(P \rightarrow (Q \wedge))$ and $(P (Q \wedge R))$ is **not** a wff.

Some pairs of parentheses may be dropped, for simplification. For example, $A \vee B$ and $A \rightarrow B$ respectively may be used in stead of the given wffs $(A \vee B)$ and $(A \rightarrow B)$, respectively. We can omit the use of parentheses by assigning *priorities in increasing order* to the connectives as follows:

$$\leftrightarrow, \quad \rightarrow, \quad \vee, \quad \vee, \quad \sim.$$

Thus, '↔' has least priority and '~' has highest priority. Further, if in an expression, there are no parentheses and two connectives between three atomic formulas are used, then the operator with higher priority will be applied first and the other operator will be applied later.

For example: Let us be given the wff $P \rightarrow Q \wedge \sim R$ without parenthesis. Then among the operators appearing in wff, the operator '~' has highest priority. Therefore, ~ R is replaced by (~R). The equivalent expression becomes P → Q ∧ (~ R). Next, out of the two operators viz '→' and '∧', the operators '∧' has higher priority. Therefore, by applying parentheses appropriately, the new expression becomes P → (Q ∧ (~ R)). Finally, only one operator is left. Hence the *fully parenthesized expression* becomes (P → (Q ∧ (~ R)))

2.5 SEMANTICS/MEANING IN PROPOSITIONAL LOGIC

Next, we define the rules of finding the truth value or meaning of a wff, when truth values of the atoms appearing in the wff are known or given.

1. The wff ~ A is *True* when A is *False*, and ~ A is *False* when A is *true*. The wff ~ A is called the ***negation*** of A.
2. The wff (A ∧ B) is True if A and B are both True; otherwise, the wff A ∧ B is False. The wff (A ∧ B) is called the **conjunction** of A and B.
3. The wff (A ∨ B) is true if at least one of A and B is True; otherwise, (A ∨ B) is False. (A ∨ B) is called the **disjunction** of A and B.
4. The wff (A → B) is False if A is True and B is False; otherwise, (A → B) is True. The wff (A → B) is read as "If A, then B," or "**A implies B**." The symbol '→' is called **implication**.
5. The wff (A ↔ B) is True whenever A and B have the same truth values; otherwise (A ↔ B) is False. The wff (A ↔ B) is read as "**A if and only if B**."

Table 1.5

	A	B	~A	(A ∧ B)	(A ∨ B)	(A → B)	(A ↔ B)
(i)	T	T	F	T	T	T	T
(ii)	T	F	F	F	T	F	F
(iii)	F	T	T	F	T	T	F
(iv)	F	F	T	F	F	T	T

The above relations can be summarized by Table 1.5 given below.

The table may be read as follows:
Let the symbol T stand for True and the symbol F stand for False. Then, *Row (i)* is interpreted as: **if** we assign T (i.e. True) to A and T to B **then** the truth values of (~ A), (A ∧ B), (A ∨ B), (A → B) and (A ↔ B) are respectively F, T, T, T, T.
Further row (iii), for example, is interpreted as:, if we assign truth-value F (False) to A and T (True) to B then truth values of (~ A), (A ∧ B), (A ∨ B), (A → B) and (A ↔ B) are respectively T, F, T, T, F.

This table, shall be used to evaluate the truth values of a wff in terms of the truth values of the atoms occurring in the formula.

Now, we discuss the issue, raised in Section 1.2, of how to check validity/invalidity of an argument through formal means.

Validity through Truth-Table.

(i) If I overslept, then I am late, i.e., symbolically

$S \rightarrow L$

(ii) I am not late, i.e., symbolically

$\sim L$

To conclude

(iii) I did not oversleep, i.e., symbolically

$\sim S$

To establish the validity/Invalidity of the argument, consider the Truth-Table

S	L	$S \rightarrow L$	$\sim L$	$\sim S$
F	F	T	T	T
F	T	T	F	T
T	F	F	T	F
T	T	T	F	F

There is only one row, viz., first row, in which both the premises viz. $S \rightarrow L$ and $\sim L$ are True. But in this case the conclusion represented by $\sim S$ is also True. Hence, the conclusion is valid.

Invalidity through Truth-Table

(i) If I overslept, then I am late

$S \rightarrow L$

(ii) I did not oversleep, i.e.,

$\sim S$

To conclude

(iii) I would not be late, i.e.,

$\sim L$ (*invalid conclusion*)

S	L	$(S \rightarrow L)$	$\sim S$	$\sim L$
F	F	T	T	T
F	T	T	T	F
T	F	F	F	T
T	T	T	F	F

The invalidity of the argument is established, because, for validity last column must contain True in those rows for which all axioms/premises are True. But in the second row both $S \rightarrow L$ and $\sim S$ are True but $\sim L$ is False

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Ex. 1 Express the following statements in Propositional Logic.

a) If he campaigns hard, he will be elected.
b) If the humidity is high, it will rain either today or tomorrow.
c) Cancer will not be cured unless its cause is determined and a new drug for cancer is found.
d) It requires courage and skills to climb a mountain.

Ex. 2: Let

P : He needs a doctor,
Q : He needs a lawyer,
R : He has an accident,
S : He is sick,
U : He is injured.

State the following formulas in English.

a) $(S \rightarrow P) \wedge (R \rightarrow Q)$ b) $P \rightarrow (S \vee U)$
c) $(P \wedge Q) \rightarrow R$ d) $(P \wedge Q) \leftrightarrow (S \wedge U)$

2.6 INTERPRETATIONS OF FORMULAS

In order to find the truth value of a given formula G, the truth values for the *atoms* of the formula are either given or assumed. The set of initially given/assumed values of all the atomic formulas occurring in a formula say G, is called an **interpretation** of the formula G. Suppose that A and B are two atoms and that the truth values of A and B are T and F respectively. Then, according to third row of Table 1.5, when A is F and B is T we find that the truth values of $(\sim A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$ are T, F, T, T and F, respectively. By developing a Truth-table of a(ny) formula, its truth value can be evaluated in terms of its interpretation, i.e., in terms of the truth values associated with the constituent atoms.

Example

Consider the formula

$$G : ((A \wedge B) \rightarrow (R \leftrightarrow (\sim S))).$$

(Please note that the string, in this case G, before the symbol ':', is the name of the formula which is the name of the string of symbols after ':'. Thus, G is the name of the formula $((A \wedge B) \rightarrow (R \leftrightarrow (\sim S)))$.

The atoms in this formula are A, B, R and S. *Suppose the truth values of A, B, R, and S are given as T, F, T and T, respectively.* **Then *(in the following and elsewhere also, if there is no possibility of confusion, we use T for 'True' and F for 'False'.)***

- $(A \wedge B)$ is F since B is F;
- $(\sim S)$ is F since S is T;
- $(R \leftrightarrow (\sim S))$ is F since R is T and $(\sim S)$ is F; and hence,
- $(A \wedge B) \rightarrow (R \leftrightarrow (\sim S))$ is T since $(A \wedge B)$ is F (*and $(R \leftrightarrow (\sim S))$ is F, which does not matter)*.

Note: In view of the fact that when $(A \wedge B)$ is F, the truth-value of

$(A \wedge B) \rightarrow$ Any Formula

must be T and, hence, we need not compute the value of $(R \leftrightarrow (\sim S))$.
Therefore, the formula G is T if A, B, R, and S are assigned truth values T, F, T and T, respectively.

The assignment of the truth values T, F, T, T to A, B, R, S, respectively, is called an **interpretation of the formula G**. Since, each one of A, B, R, and S can be assigned one of the two values, viz., either T or F, there are $2^4 = 16$ possible interpretations of the formula G. In Table 1.6, we give the truth values of the formula G under all these 16 interpretations.

The above procedure may be repeated to find truth value of any formula from any interpretation, i.e., from any assignment to the atomic formulas occurring in the given formula.

Table 1.6 Truth Table of (A $\wedge$ B $\rightarrow$ (R $\leftrightarrow$ (~ S)

A	B	R	S	~S	(A $\wedge$ B)	(R $\leftrightarrow$ (~S))	(A $\wedge$ B) $\rightarrow$ (R $\leftrightarrow$ (~ S)
T	T	T	T	F	T	F	F
T	T	T	F	T	T	T	T
T	T	F	T	F	T	T	T
T	T	F	F	T	T	F	F
T	F	T	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	T	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	F	F	F	T
F	T	T	F	T	F	T	T
F	T	F	T	F	F	T	T
F	T	F	F	T	F	F	T
F	F	T	T	F	F	F	T
F	F	T	F	T	F	T	T
F	F	F	T	F	F	T	T
F	F	F	F	T	F	F	T

A table, such as given above, that displays the truth values of a formula G for *all possible assignments of truth values to atoms occurring in G is called* **a Truth table of G**.

NOTATION: If $A_1,\ldots.A_n$ are all the atoms in a formula, it may be more convenient to represent an *interpretation* by a set $(m_1,\ldots.m_n)$, where m_i is either A_i or $\sim A_i$. m_i is written as A_i if T is assigned to A_i. But m_i is written as $\sim A_i$ if F is assigned to A_i.

For example, the set {A, ~B, ~R,S} represents an interpretation of a formula in which A, B, R, and S are the only atoms and which are, respectively, assigned T, F, F, and T. We will use the notation throughout.

Ex. 3: Construct a truth table for the formula.
P: (~ A $\vee$ B) $\wedge$ (~ (A $\wedge$ ~ B))

2.7 VALIDITY AND INCONSISTENCY OF PROPOSITIONS

It may noted that in Section 1.2, we discussed the concept of valid **Argument**. Here, we study **formulas** or propositions. Next, we shall consider wff that are **true under all** possible interpretations and wff that are **false under all** possible interpretations.

Example

Let us consider the wff

G : (((A $\rightarrow$ B) $\wedge$ A) $\rightarrow$ B).

The formula G has 2^2 = 4 possible interpretations in view of the fact it has two atoms viz A and B. It can be easily seen from the following table that the wff G is True under all its interpretations. **Such as a wff which is True under all interpretation is called a valid formula (or a tautology).**

Truth Table of $(((A \rightarrow B) \wedge A) \rightarrow B)$

A	B	$(A \rightarrow B)$	$(A \rightarrow B) \wedge A$	$((A \rightarrow B) \wedge A) \rightarrow B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Consider another formula

$$G : ((A \rightarrow B) \wedge (A \wedge \sim B))$$

The truth table of the formula G given below shows that G is False under all its interpretations. Such a formula which is False under all interpretations is called an **inconsistent formula (or a contradiction)**.

Truth Table of $(A \rightarrow B) \wedge (A \wedge \sim B)$

A	B	~B	$(A \rightarrow B)$	$(A \wedge \sim B)$	$((A \rightarrow B) \wedge (A \wedge \sim B)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	F	F
F	F	T	T	F	F

Next, we formally define the concepts discussed above.

Definition: A formula is said to be **valid** if and only if it is **true under all** its interpretations. A formula is said to be **invalid** if and only if it is *not true under at least one* interpretation. A valid formula is also called a **Tautology.** A formula is **invalid** if there is **at least one** interpretation for which the formula has a truth value False.

Definition: A formula is said to be **inconsistent (or unsatisfiable)** if and only if it is **False under all** its interpretations. A formula is said to be **consistent or satisfiable** if and only if it is not inconsistent. In other words, a formula is *consistent if there is at least one interpretation* for which the formula has a truth value true.

From the definitions given above, it is easily seen that

(i) A formula is *valid* if and only if its negation is *inconsistent*.
(ii) A formula is *invalid if* and only if there is *at least one* interpretation under which the formula is *false*.
(iii) A formula is *consistent* if and only if there is *at least one* interpretation under which the formula is *true*.
(iv) If a formula is *valid*, then it is *consistent*, but not *vice versa*. *(example given below)*
(v) If a formula is *inconsistent*, then it is *invalid*, but not *vice versa*. *(example given below)*

Definition: If a formula P is *True under an interpretation I*, then we say that **I satisfied P**, or P is satisfied by I. If a formula *P is False under an interpretation I,* then we say that **I falsifies** P or P **is falsified by I**.

As for an example, the formula $(A \wedge (\sim B))$ is satisfied by the interpretation {A, ~ B} i.e, by taking A as T and B as F, but is **falsified** by the interpretation {A, B} i.e., when A is taken as T and B is taken as T. *An interpretation I that satisfies a formula P, is called a* ***model*** *of the formula F*.

Examples:

(i) A Valid Formula:

(a) Even *True* is a wff which is always True and, hence, True is a valid formula.

(b) G_1: A $\vee$ (~A) is True for all its interpretations. As G_1 has only one atom viz. A, terefore, it has only two interpretations. Let *one interpretation of G_1 be : A is True*. But then G_1 assumes the value (True $\vee$ (~ True)) = True. The *other interpretation of G_1 is : A is False*. Then G_1 assumes the value (False $\vee$ ~ False) = True.

(ii) Consistent (*True for at least one interpretation*) but not valid Formula (*i.e. is invalid, i.e., False for at least one interpretation*):

(a) The simplest example of such a formula is the formula *G_2: A*. Then, for the assignment A as True, G_2 is True. Therefore G_2 is consistent. On the other hand, the interpretation of G_2 with A as False, makes G_2 false. Therefore, G_2: A is not valid.

(b) Both G_3 : A $\vee$ B and G_4 : A $\wedge$ B are consistent but not valid. Both G_3 and G_4 are True under the assignment A as True and B as True. On the other hand, both are False under the interpretation A as False and B as False.

(iii) Invalid (*False for at least one interpretation*) but not inconsistent (*not False for all interpretations*): Any one of the examples in (ii) above

(iv) Inconsistent formula (*i.e., which is false for all interpretations*)

(a) Even 'False' is a wff; which is always False, and hence is inconsistent.

(b) G_5 : A $\wedge$ (~A) is False, for all interpretations of G_5. Actually, there are only two interpretations of G_5. One is : A is True. The other is : A is False. In both cases G_5 is False.

It will be shown later that the proof of the validity or inconsistency of a formula is a very important problem. In the propositional logic, since the number of interpretations of a formula is finite, one can *always decide* whether or not a formula in the propositional logic is valid (inconsistent) by exhaustively examining all of its possible interpretations.

Ex. 4: For each of the following formulas, determine whether it is valid, inconsistent, consistent or some combination of these.

(i) E: ~ (~A) $\rightarrow$ B
(ii) G: (A $\rightarrow$ B) $\rightarrow$ (~ B $\rightarrow$ ~ A)
(iii) H: (A $\vee$ ~ A) $\rightarrow$ (A $\wedge$ B) $\wedge$ (~ A)
(iv) J: (A $\wedge$ B) $\wedge$ (~ A) $\rightarrow$ (B $\vee$ ~ B)

2.8 EQUIVALENT FORMS IN THE PROPOSITIONAL LOGIC (PL)

Definition: Logically Equivalent Formulas: Two formulas G_1 and G_2 are said to be (logically) *equivalent* if for each interpretation i.e., truth assignment to all the atoms that occur in either G_1 or G_2; the truth values of G_1 and G_2 are identical. In other words, for each interpretation, G_1 is True if and only if G_2 is True. And, for each interpretation, G_1 is False if and only if G_2 is False.

As will be clear later, it is often necessary to transform a formula from one form to another, especially to a *normal form*. This is accomplished by replacing a formula in the given formula by a formula *equivalent* to it and repeating this process until the desired form is obtained.

Example

We can verify that the formula **E: $\sim (A \to B)$** is equivalent the formula **G: to $A \wedge \sim B$** by examining the following truth table. The corresponding values in the last two columns are identical.

Table Joint Truth table of $\sim (A \to B)$ and $(A \wedge \sim B)$

A	B	$\sim B$	$(A \to B)$	$\sim(A \to B)$	$A \wedge \sim B$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Solutions of problems using symbolic logic can be simplified, if we can simplify involved formulas by some equivalent simpler formulas given in table below. These equivalences can be verified by using truth tables.

Table of Equivalences of PL

(1.1)	$E \leftrightarrow G = (E \to G) \wedge (G \to E)$	
(1.2)	$E \to G = \sim E \vee G$	
(1.3)(a)	$E \vee G = G \vee E$;	(b) $E \wedge G = G \wedge E$
(1.4)(a)	$(E \vee G) \vee H = E \vee (G \vee H)$;	(b) $(E \wedge G) \wedge H = E \wedge (G \wedge H)$
(1.5)(a)	$E \vee (G \wedge H) = (E \vee G) \wedge (E \vee H)$;	(b) $E \wedge (G \vee H) = (E \wedge G) \vee (E \wedge H)$
(1.6)(a)	$E \vee$ False = E;	(b) $E \wedge$ True = E
(1.7)(a)	$E \vee$ True = True	(b) $E \wedge$ False = False
(1.8)(a)	$E \vee \sim E$ = True;	(b) $E \wedge\ E = E$
(1.9)	$\sim (\sim E) = E$	
(1.10)(a)	$\sim (E \vee G) = \sim E \wedge \sim G$;	(b) $\sim (E \wedge G) = \sim E \vee \sim G$

In the table given above, True denotes the fact that the wff is True under all interpretations and False denotes the wff that is False under all interpretations.

Laws (1.3a), (1.3b) are often, called **commutative laws**; (1.4a), (1.4b) **associative laws;** (1.5a), (1.5b), **distributive laws:** and (1.10a), (1.10b), **De Morgan's laws**.

2.9 NORMAL FORMS

Some Definitions: A **clause** is a disjunction of literals. For example, $(E \vee \sim F \vee \sim G)$ is a clause. But $(E \vee \sim F \wedge \sim G)$ is not a clause. A **literal** is either an atom, say A, or its negation, say $\sim A$.

Definition: A formula E is said to be in a **Conjunctive Normal Form (CNF)** if and only if E has the form $E : E_1 \wedge \ldots . \wedge E_n$, $n \geq 1$, where each of $E_1, \ldots, E_n$ is a **disjunction** of literals.

Definition: A formula E is said to be in **Disjunctive Normal Form (DNF)** if and only if E has the form E: $E_1 \vee E_2 \vee \ldots . E_n$, where each E_i is a **conjunction** of literals.

Examples: Let A, B and C be atoms. Then F: $(\sim A \wedge B) \vee (A \wedge \sim B \wedge \sim C)$ is a formula in a disjunctive normal form.

Example: Again G: $(\sim A \vee B) \wedge (A \vee \sim B \vee \sim C)$ is a formula in Conjunctive Normal Form, because it is a conjunction of the two disjunctions of literals viz of $(\sim A \vee B)$ and $(A \vee \sim B \vee \sim C)$

Example: Each of the following is neither in CNF nor in DNF

(i) $(\sim A \vee B) \vee (A \wedge \sim B \vee C)$
(ii) $(A \rightarrow B) \wedge (\sim B \wedge \sim A)$

Using table of equivalent formulas given above, any valid Propositional Logic formula can be transformed into CNF as well as DNF.

The steps for conversion to DNF are as follows

Step 1: Use the equivalences to remove the logical operators '$\leftrightarrow$' and '$\rightarrow$':

(i) $E \leftrightarrow G = (E \rightarrow g) \wedge (G \rightarrow E)$
(ii) $E \rightarrow G = \sim E \vee G$

Step 2 Remove ~'s, if occur consecutively more than once, using

(iii) $\sim (\sim E) = E$

(iv) Use De Morgan's laws to take '~' nearest to atoms

(v) $\sim(E \vee G) = \sim E \wedge \sim G$
(vi) $\sim(E \wedge G) = \sim E \vee \sim G$

Step 3 Use the distributive laws repeatedly

(vii) $E \vee (G \wedge H) = (E \vee G) \wedge (E \vee H)$
(viii) $E \wedge (G \vee H) = (E \wedge G) \vee (E \wedge H)$

Example

Obtain a disjunctive normal form for the formula $\sim (A \rightarrow (\sim B \wedge C))$.

Consider $A \rightarrow (\sim B \wedge C) = \sim A \vee (\sim B \wedge C)$ *(Using $(E \rightarrow F) = (\sim E \vee F)$)*

Hence, $\sim (A \rightarrow (\sim B \wedge C)) = \sim (\sim A \vee (\sim B \wedge C))$

$= \sim (\sim A) \wedge (\sim (\sim B \wedge C))$ *(Using $\sim (E \vee F) = \sim E \wedge \sim F$)*

$= A \wedge (B \vee (\sim C))$ *(Using $\sim (\sim E) = E$ and $\sim (E \wedge F) = \sim E \vee \sim F$*

$= (A \wedge B) \vee (A \wedge (\sim C))$ *(Using $E \wedge (F \vee G) = (E \wedge F) \vee (E \wedge G)$)*

However, if we are to obtain CNF of $\sim A (\rightarrow (\sim B \wedge C))$, in the last but one step, we obtain
$\sim (A \rightarrow (\sim B \wedge C)) = A \wedge (B \vee \sim C)$, which is in CNF, because, each of A and $(B \vee \sim C)$ is a disjunct.

Example: Obtain conjunctive Normal Form (CNF) for the formula: $D \rightarrow (A \rightarrow (B \wedge C))$

Consider
$D \to (A \to (B \wedge C))$ *(using $E \to F = \sim E \vee F$ for the inner implication)*
$= D \to (\sim A \vee (B \wedge C))$ *(using $E \to F = \sim E \vee F$ for the outer implication)*
$= \sim D \vee (\sim A \vee (B \wedge C))$
$= (\sim D \vee \sim A) \vee (B \wedge C)$ *(using Associative law for disjunction)*
$= ((\sim D \vee \sim A \vee B) \wedge (\sim D \vee \sim A \vee C)$

The last line denotes the conjunctive Normal Form of $D \to (A \to (B \wedge C))$ (using distributivity of $\vee$ over $\wedge$)

Note: *If we stop at the last but one stop, then we obtain $(\sim D \vee \sim A) \vee (B \wedge C) = \sim D \vee \sim A \vee (B \wedge C)$ is a* ***Disjunctive Normal Form*** *for the given formula: $D \to (A \to (B \wedge C))$*

Ex. 5: Transform the following into disjunctive normal forms.

(i) $\sim (A \vee \sim B) \wedge (S \to T)$ (ii) $(A \to B) \to R$

Ex. 6: Transform the following into conjunctive normal forms.
(i) $(A \to B) \to R$

(ii) $(\sim A \wedge B) \vee (A \wedge \sim B)$

Ex. 7: Verify each of the following pairs of equivalent formulas by transforming formulas on both sides of the sign = into the same normal form.

(i) $(A \to B) \to (A \wedge B) = (\sim A \to B) \wedge (B \to A)$
(ii) $A \wedge B \wedge (\sim A \vee \sim B) = \sim A \wedge \sim B \wedge (A \vee B)$

2.10 LOGICAL DEDUCTION

Definition: A formula G is said to be a logical **consequence** of given formulas $E_1,\ldots, E_n$ **(or G is logical derivation of $E_1,\ldots.E_2$)** if and only for any interpretation I in which $E_1 \wedge E_2 \wedge \ldots \wedge E_n$ is true, for the interpretation I, G is also true. The proposition $E_1, E_2\ldots,E_n$ are called *axioms/premises* of G.

Next, we state without proof two very useful theorems for establishing logical derivations:

Theorem 1: Given formulas $E_1,\ldots, E_n$ and a formula G, G is a logical derivation of $E_1,\ldots.,E_n$ if and only if the formula $((E_1 \wedge \ldots.\wedge E_n) \to G)$ is valid, i.e., True for all interpretations of the formula.

Theorem 2: Given formulas $E_1,\ldots,E_n$ and a formula G, G is a logical consequence or derivation of $E_1,\ldots E_n$ if only if the formula $(E_1 \wedge \ldots.. \wedge E_n \wedge \sim G)$ is inconsistent, i.e., False for all interpretations of the formula.

The above two theorems are very useful. They show that proving a particular formula as a logical consequence of a finite set of formulas is equivalent to proving that a *certain single* but related formula is valid or inconsistent.

Note: Significance of the above two theorems lies in the fact that logical consequence relates **two** formulas, where as validity/inconsistency is only about **one** formula. Also, there are a number of well-known methods, including truth-table method, for establishing inconsistency/validity of a formula. Thus, formula G logically follows from a given set of formulas, we check validity of single formula. And, for checking validity of a single formula, we already have some methods including Truth-table method.

Definition: If the formula G is a logical consequence of the formula $E_1, \ldots E_n$, then the single formula $((E_1 \wedge \ldots \wedge E_n) \rightarrow G)$ is called a **theorem,** and G is also called the **conclusion** of the theorem.

There are at least three alternative methods of establishing formula G as a logical consequence of given formulas $E_1, E_2, \ldots E_n$.

According to **one of these methods**, through truth table or otherwise, it should be established that for any interpretation for which each of $E_1, \ldots E_n$, is true then for that interpretation G must be true.

According to **second method**, using Theorem 1, we should show that the formula:

$$(E_1 \wedge E_2 \wedge \ldots \wedge E_n) \rightarrow G$$

is valid, i.e., True for each of its interpretations. Again validity can be shown either through a truth table or otherwise.

The **last of the three methods** uses Theorem 2. According to this method, in order to show, G as a logical consequence of $E_1, E_2, \ldots E_n$, it should be established that the formula $(E_1 \wedge E_2 \wedge \ldots \wedge E_n \wedge \sim G)$ is inconsistent, i.e., is False under all its interpretations. Next, we apply these methods through an example.

Example: We are given the formulas

$E_1 : (A \rightarrow B)$, $E_2 : \sim B$, $G : \sim A$

We are required to show that G is a logical consequence of E_1 and E_2.

Method 1: From the following Table, it is clear that whenever E_1: $A \rightarrow B$ and E_2: $\sim B$ both are simultaneously True, (*which is true only in the last row of the table*) then G: ~ A is also True. Hence, the proof.

A	B	$A \rightarrow B$	~ B	~ A
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Method 2: We prove the result by showing the validity of $E_1 \wedge E_2 \rightarrow G$, i.e., of $((A \rightarrow B) \wedge \sim B) \rightarrow \sim A$ by transforming it into a conjunctive normal form.

$$(A \rightarrow B) \wedge \sim B) \rightarrow \sim A = \sim ((A \rightarrow B) \wedge \sim B) \vee \sim A \quad \text{(using } E \rightarrow F = (\sim E \vee F))$$
$$= \sim ((\sim A \vee B) \wedge \sim B) \vee \sim A$$

$$= \sim ((\sim A \wedge \sim B) \vee (B \wedge \sim B)) \vee \sim A$$
$$= \sim ((\sim A \wedge \sim B) \vee \text{False}) \vee \sim A$$
$$= \sim ((\sim A \wedge \sim B)) \vee \sim A \quad \textit{(using De Morgan's Laws)}$$
$$= (A \vee B) \vee \sim A =$$
$$= (B \vee A) \vee \sim A$$
$$= B \vee (A \vee \sim A)$$
$$= B \vee \text{True}$$

= True (*always*)

Thus, $((A \rightarrow B) \wedge B) \rightarrow \sim A$ is valid.

Ex. 8: Using Truth Table show that G is a logical consequence of E_1 and E_2 where $E_1 : (A \rightarrow B)$, $E_2 : \sim B$, $G : \sim A$, by establishing validity of the formula $(E_1 \wedge E_2 \rightarrow G)$.

Ex. 9: Use (i) the truth table technique (ii) reduction to DNF/CNF to show that $(A \rightarrow B) \wedge \sim B \wedge A$ is inconsistent which, in turn proves that $\sim A$ is a logical consequence of $(A \rightarrow B)$ and $\sim B$.

2.11 APPLICATIONS

Next, we discuss some of the applications of Propositional Logic.

Example

Suppose the stock prices go down if the interest rate goes up. Suppose also that most people are unhappy when stock prices go down. Assume that the interest rate goes up. Show that we can conclude that most people are unhappy.

To show the above conclusion, let us denote the statements are as follows:

A : Interest rate goes up,
S : Stock prices go down
U : Most people are unhappy

The problem has the following four statements:

1) If the interest rate goes up, stock prices go down.
2) If stock prices go down, most people are unhappy.
3) The interest rate goes up.
4) Most people are unhappy. (*to conclude*)

The above-mentioned statements are symbolised as,

(1′) $A \rightarrow S$
(2′) $S \rightarrow U$
(3′) A
(4′) U. (*to conclude*)

In order to establish the conclusion, we should show that (4′) is logical consequence of (1′) , (2′) and (3′). For this purpose, we show that (4′) is true whenever (1′) $\wedge$ (2′) $\wedge$ (3′) is true.

We transform $((A \rightarrow S) \wedge (S \rightarrow U) \wedge A)$ (*representing (1′) $\wedge$ (2′) $\wedge$ (3′)*) into a normal form:

$((A \rightarrow S) \wedge (S \rightarrow U) \wedge A) = ((\sim A \vee S) \wedge (\sim S \vee U) \wedge A)$ *(by using $E \rightarrow F = \sim E \vee F$)*

$= (A \wedge (\sim A \vee S) \wedge (\sim S \vee U))$ *(by using $E \wedge F = F \wedge E$, (to bring the last clause A in the beginning)*

$= (((A \wedge \sim A) \vee (A \wedge S)) \wedge (\sim S \vee U))$ *(by using associative laws and then using distributivity of 'A ∧' over the next disjunct (~ A ∨ S))*

$= ((\text{False} \vee (A \wedge S)) \wedge (\sim S \wedge U))$ *(using False ∨ E = E)*

$= (A \wedge S) \wedge (\sim S \vee U)$

$= (A \wedge S \wedge \sim S) \vee (A \wedge S \wedge U)$

$= (A \wedge \text{False}) \vee (A \wedge S \wedge U)$ *(using A ∧ False = False)*

$= \text{False} \vee (A \wedge S \wedge U)$

$= A \wedge S \wedge U$

Therefore, if $((A\rightarrow S) \wedge (S\rightarrow U) \wedge A)$ is true, then $(A \wedge S \wedge U)$ is true. Since $(A \wedge S \wedge U)$ is true then each of A, S, and U is true, we conclude that U is true. Hence, U is a logical consequence of 1), 2) and 3) given above.

Ex. 10: Given that if the Parliament refuses to enact new laws, then the strike will not be over unless it lasts more than one year and the president of the firm resigns, will the strike not be over if the Parliament refuses to act and the strike just starts?

2.12 SUMMARY

In this unit, to begin with, we discuss what is Symbolic Logic **and** why it is it is important to study it. The subject matter of symbolic logic consists of **arguments**, where an argument consists of a number of statements — one of which is called the **conclusion** and is supposed to be logically drawn from the others. Each one of the other is called a **premise,** To be more specific, the subject of Symbolic Logic is the study of how to develop tools and technique to draw correct conclusions from a given set of premisses or to verify whether a conclusion is correct or not. A conclusion is correct in the sense: Whenever all the premisses are True then conclusion is necessarily True. An argument with correct conclusion is called a **valid** argument. Next, a **sound** argument is defined as a valid argument in which premises also have to be True.
(in some world).

In this unit, we study only a specific branch of symbolic logic, viz. Propositional Logic (PL).

Next, we discuss how a **statement**, also called a well-formed formula **(wff)** and also a **Proposition**, which is the basic unit of an argument in PL, is appropriately **denoted** and how it is **interpreted**, i.e., how a wff is given meaning. The meaning of a wff in PL is only in terms of True or False. The wffs are classified as *valid, invalid, consistent and inconsistent.*
Then tools and techniques in the form of Truth-table, logical deduction, normal forms etc are discussed to test these properties of wffs and also to test validity of arguments. Finally a number of applications of these concepts, tools and techniques of PL are used to solve problems that involve logical reasoning of PL systems.

2.13 SOLUTIONS/ANSWERS

Ex. 1

(a) Let H: He campaigns hard ; E: He will be elected
Then the statement becomes the formula:

$H \rightarrow E$

(b) Let H: The Humidity is high, RTY: It will rain today
RTW: It will rain tomorrow.
Then

$H \rightarrow RTY \vee RTW$

(c) Let C: Cancer will be cured
D: Cancer's cause will be determined
F: A new drug for cancer will be found
Then the statement becomes the formula:
$(\sim C) \vee (D \wedge F)$. This formula may also be written as:

$C \rightarrow D \wedge F$

(d) Let C: One has courage
S: One has skill
M: One climbs mountain
Then the statement becomes the formula:

$M \rightarrow C \wedge S$

Ex 2: (a) If he is sick then he needs a doctor, but, if he has an accident then he needs a lawyer
(b) If One requires a doctor then one must be either sick or injured.
(c) If he needs both a doctor and a lawyer then he has an accident.
(d) One requires a doctor and also a lawyer if and only if one is sick and also injured.

Ex. 3:
(i) Truth table of the formula: P: $(\sim A \vee B) \wedge (\sim (A \wedge \sim B))$ is as given below.

A	B	$\sim A$	$\sim B$	$\sim A \vee B$	$A \wedge \sim B$	$\sim (A \wedge \sim B)$	P
T	T	F	F	T	F	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

Ex. 4:

(i) Consistent but not valid, because, for For B as T and A as F, the formula is T. But, for A as T and B as F the formula is F.

(ii) It can be easily that $\sim B \rightarrow \sim A$ has same truth-value as $(A \rightarrow B)$ for any interpretation. Therefore, in stead of the given formula, we may consider the formula
$(A \rightarrow B) \rightarrow (A \rightarrow B)$
which can be further written as $P \rightarrow P$, writing $(A \rightarrow B)$ as P. Even $P \rightarrow$ can be written as $P \vee P \equiv P \equiv (A \rightarrow B)$, The last formula is F when *F* and A is T. The formula is T when A is F and B is T. Hence, the formula is neither valid nor inconsistent.
Therefore, the formula is consistent but not valid

(iii) For all truth assignments to A and B, L. H.S. of the formula is always T and R. H.S. is always F. Hence the formula is inconsistent, i.e., always F

(iv) The L. H. S. of the given formula is F under all interpretations. Hence, the formula is T under all interpretation. Therefore, the formula is valid.

Ex. 5: (i) *Removing '$\rightarrow$', we get*
$\sim (A \vee \sim B) \wedge (\sim S \vee T)$
Taking '~' inside we get
$(\sim A \wedge B) \wedge (\sim S \vee T)$ (*using De Morgan's Law*)
Using distributivity of $\wedge$ over $\vee$ we get
$(\sim A \wedge B \wedge \sim S) \vee (\sim A \wedge B \wedge T)$
which is the required form

(ii) Removing outer $\rightarrow$ we get
$\sim (A \rightarrow B) \vee R$
Removing the other '$\rightarrow$' we get
$\sim (\sim A \vee B) \vee R$
Taking ~ inside, we get
$(A \wedge \sim B) \vee R$,
which is the required form

Ex. 6:
(i) Using distributive law in the last formula of 5 (ii) above, we get
$(A \vee R) \wedge (\sim B \vee R)$
which is the required CNF

(ii) Using Left distributivity of $\vee$ over $\wedge$ we get
$((\sim A \wedge B) \vee A) \wedge (\sim A \wedge B) \vee \sim B)$
Using Right distributivity inside each pair of parenthese of $\vee$ over $\wedge$ we get
$((\sim A \vee A) \wedge (B \vee A) \wedge ((\sim A \vee \sim B) \wedge (B \vee \sim B))$
Using $\sim A \vee A = T = B \vee \sim B$, we get
$(T \wedge (B \vee A)) \wedge ((\sim A \vee \sim B) \wedge T)$
which is equivalent to
$(B \vee A) \wedge ((\sim A \vee \sim B) = (A \vee B) \wedge (\sim A \vee \sim B)$
is the required CNF.

Ex. 7: (i) **Consider L.H.S**

Removing inner $\rightarrow$ on L. H.S., we get
$(\sim A \vee B) \rightarrow (A \wedge B)$
removing the other '$\rightarrow$'
$= \sim (\sim A \vee B) \vee (A \wedge B)$
Using De Morgan's Laws, we get
$= (\sim (\sim A) \wedge (\sim B)) \vee (A \wedge B)$
$= (A \wedge \sim B) \vee (A \wedge B)$ (i)
which is in DNF

For R.H.S, removing the two implications, we get
$(\sim (\sim A) \vee B) \wedge (\sim B \vee A)$
$= (A \vee B) \wedge (\sim B \vee A)$
(*which is in CNF, but we require DNF*)
Using Left distributivity of $\wedge$ over $\vee$, we get
$= ((A \vee B) \wedge (\sim B)) \vee ((A \vee B) \wedge A)$
Using Right distributivity of $\wedge$ over $\vee$, we get
$= ((A \wedge \sim B) \vee (B \wedge \sim B)) \vee ((A \wedge A) \vee (B \wedge A))$
Using $B \wedge \sim B = F$ $A \wedge \mathbf{A} = A$
And $P \vee F = P$ we get
$= (A \wedge \sim B) \vee (A \vee (B \wedge A))$
$= (A \wedge \sim B) \vee (A) = (A \wedge \sim B) \vee (A \wedge T)$ (ii)
$= (A \wedge \sim B) \vee (A \wedge (B \vee \sim B))$
$= (A \wedge \sim B) \vee (A \wedge \sim B) \vee (A \wedge B)$
Using $P \vee P = P$, we get
$= (A \wedge \sim B) \vee (A \wedge B)$

(ii) **R.H.S** Applying associative laws, we get
$(\sim A \wedge \sim B) \wedge (A \vee B)$
Using left distributivity of $\wedge$ over $\vee$ we get
$= ((\sim A \wedge \sim B) \wedge A) \vee ((\sim A \wedge \sim B) \wedge B)$
Again using associativity of $\wedge$ and using $\sim A \wedge A = F = \sim B \wedge B$ we get

R.H. S. = F

Consider L.H.S, applying associativity of $\wedge$, we get

$= ((A \wedge B) \wedge (\sim A \vee \sim B))$,

using left distributivity and commutativity of $\wedge$ we get

$= ((A \wedge B) \wedge \sim A) \vee ((A \wedge B) \wedge \sim B)$

Using associativity of $\wedge$ and using $A \wedge \sim A = F = B \wedge \sim B$

$= (B \wedge F) \vee (A \wedge F)$

Using $A \wedge F = F = B \wedge F$

$= F$

Ex. 8: The following table shows that $((A \rightarrow B) \wedge \sim B) \rightarrow \sim A$ is true in every interpretation. Therefore $((A \rightarrow B) \wedge \sim B) \rightarrow \sim A$ is valid and according to the First theorem, $\sim A$ is a logical consequence of $(A \rightarrow B)$ and $\sim B$.

Truth Table of $((A \rightarrow B) \wedge \sim B) \rightarrow \sim A$

A	B	$A \rightarrow B$	$\sim B$	$(A \rightarrow B) \wedge \sim B$	$\sim A$	$(A \rightarrow B) \wedge \sim B) \rightarrow \sim A$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

Ex. 9: (i) From the following table, $((A \rightarrow B) \wedge \sim B \wedge A)$ being False for all interpretations, is inconsistent.

Truth Table of $(A \rightarrow B) \wedge \sim B \wedge A$

A	B	$A \rightarrow B$	$\sim B$	$(A \rightarrow B) \wedge \sim B \wedge A$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

(ii) Prove the inconsistency of $E_1 \wedge E_2 \wedge \sim G$, i.e., of $(A \rightarrow B) \wedge \sim B \wedge A$ by transforming, into a disjunctive normal form:

$(A \rightarrow B) \wedge \sim B \wedge A = (\sim A \vee B) \wedge (\sim B \wedge A)$

$= (\sim A \wedge \sim B \wedge A) \vee (B \wedge \sim B \wedge A)$ (*Distributive Law*)

$= (\sim A \wedge A \wedge \sim B) \vee (F \wedge A)$

$=$ False $\vee$ False $=$ False

Thus $(A \rightarrow B) \wedge \sim B \wedge A$ is inconsistent.

Ex. 10:

Let us symbolize the statements in the problem state of above as follows:

A: The Parliament refuses to act.

B: The strike is over.

R: The president of the firm resigns.

S: The strike lasts more than one year.

Then the facts and the question to be answered in the problem can be symbolized as:

E1: $(A \rightarrow (\sim B \vee (R \wedge S)))$ represents the statement '***If*** *the congress refuses to enact new laws,* ***then*** *the strike will not be over* ***unless*** *it lasts more than one year* ***and*** *the president of the firm resigns.'*

E2 : A, represents the statement '*The congress refuses to act, by and'*

E3: ~ S represent the statement '*The strike just starts.*'

E4: ~ B (to be concluded)

Ex. 10: We solve the problem by showing that the formula P: $((A \rightarrow (\sim B \vee (R \wedge S))) \wedge A \wedge \sim S) \rightarrow \sim B$ is valid by two methods: (i) by reducing to CNF/DNF
(ii) by constructing truth-table of the formula.

Methods (i) Removing the two occurrences of '$\rightarrow$' , we get
$P = \sim ((\sim A \vee (\sim B \vee (R \wedge S))) \wedge A \wedge \sim S) \vee \sim B$
Using De Morgan's Laws, we get
$= (\sim ((\sim A) \vee (\sim B \vee (R \wedge S))) \vee \sim A \vee \sim \sim S) \vee \sim B$
$= (A \wedge (\sim \sim B \wedge \sim (R \wedge S))) \vee \sim A \vee S) \vee \sim B$
$= (A \wedge (B \wedge \sim (R \wedge S))) \vee \sim A \vee S) \vee \sim B$
$P = (A \wedge (B \wedge (\sim R \vee \sim S))) \vee \sim A \vee S \vee \sim B$ ….. (i)

Consider the case R is assigned value F

Then the formula P becomes
$(A \wedge (B \wedge (\sim F \vee \sim S))) \vee (\sim A \vee \sim B \vee S)$
$= ((A \wedge B) \wedge T) \vee (\sim (A \wedge B) \vee S)$
$= (A \wedge B) \vee (\sim (A \wedge B) \vee S)$
By denoting $A \wedge B$ by H we get $P = H \vee (\sim H \vee S) = T$ whether $(A \wedge B)$ is T or F

Consider the case when R is assigned T
Then the formula P given by (i) becomes
$(A \wedge (B \wedge (\sim T \vee \sim S) \vee (\sim A \vee \sim B \vee S)$ (using De Morgan Laws)
$= ((A \wedge B) \wedge \sim S) \vee (\sim (A \wedge B) \vee S)$
$= ((A \wedge B) \wedge \sim S) \vee (\sim (A \wedge B \wedge \sim S))$
Denoting $(A \wedge B \wedge \sim S)$ by K we get
$P = K \vee \sim K = T$
Hence P is valid. Hence, the proof.

Method (ii)
The solution of the problem lies in showing that ~ B logical follows from E_1, E_2, and E_3. This is equivalent to showing that P: $((A \rightarrow (\sim B \vee (R \wedge S))) \wedge A \wedge \sim S) \rightarrow \sim B$ is a valid formula. The truth values of the above formula under all the interpretations are shown in given table

A	B	R	S	~ B	$\sim B \vee (R \wedge S)$
T	T	T	T	F	T
T	T	T	F	F	F
T	T	F	T	F	F
T	T	F	F	F	F
T	F	T	T	T	T
T	F	T	F	T	T
T	F	F	T	T	T
T	F	F	F	T	T
F	T	T	T	F	T
F	T	T	F	F	F
F	T	F	T	F	F
F	T	F	F	F	F
F	F	T	T	T	T
F	F	T	F	T	T
F	F	F	T	T	T
F	F	F	F	T	T

A	B	R	S	E_1	E_2	E_3	~ B	~ B ∨ (R ∧ S)	E_1	(E_1 ∧ E2 ∧ E3) → ~ B
T	T	T	T	T	T	F	F	T	T	T
T	T	T	F	F	T	T	F	F	F	T
T	T	F	T	F	T	F	F	F	F	T
T	T	F	F	F	T	T	F	F	F	T
T	F	T	T	T	T	F	T	T	T	T
T	F	T	F	T	T	T	T	T	T	T
T	F	F	T	T	T	F	T	T	T	T
T	F	F	F	T	T	T	T	T	T	T
F	T	T	T	T	F	F	F	T	T	T
F	T	T	F	T	F	T	F	F	T	T
F	T	F	T	T	F	F	F	F	T	T
F	T	F	F	T	F	T	F	F	T	T
F	F	T	T	T	F	F	T	T	T	T
F	F	T	F	T	F	T	T	T	T	T
F	F	F	T	T	F	F	T	T	T	T
F	F	F	F	T	F	T	T	T	T	T

Under all interpretations formula is True. Hence, the formula P a valid formula. ~ B is a logical consequence of E1, E2 and E3. Hence, the “The strike will not be over” is a valid conclusion.

2.14 FURTHER READINGS

(In the order from *elementary* to *advanced*)

1. McKay, Thomas J., *Modern Formal Logic* (Macmillan Publishing Company, 1989).
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