UNIT 2 INTERPOLATION

Structure **Page Nos.**

2.0 INTRODUCTION

The interpolation has been defined as the art of reading between the lines of a table, and in elementary mathematics the term usually denotes the process of computing intermediate values of a function from a set of given values of that function. Suppose the value of the function f(x) (instead of the analytical formula representing the function) is tabulated at a discrete set of values of the argument x at $x_0, x_1, \ldots, x_n$ $x_0 \leq x_1 \leq \ldots \leq x_n$. If the value of f(x) is to be found at some point ξ in the interval $[x_o, x_n]$ and ξ is not one of the nodes x_i, then value is estimated by using the known values of f(x) at the surrounding points. Such estimates of f(x) can be made using a function that fits the given data. If the point ξ is outside the interval $[x_o, x_n]$, then the estimation of $f(\xi)$ is called extrapolation.

The general problem of interpolation, however, is much more complex than this. In higher mathematics we often have to deal with functions whose analytical form is either totally unknown or else is of such a nature, complicated or otherwise, that the function cannot easily be subjected to certain operations like differentiation and integration etc. In either case, it is desirable to replace the given function by another which can be more readily handled.

We derive various forms of the interpolating polynomial. Polynomials are used as the basic means of approximation in nearly all areas of numerical analysis. We have divided our discussion on polynomial interpolation into 3 sections. First we discuss Lagrange form of interpolating polynomial for unequally spaced nodes. Also general expression for the error (truncation) of polynomial interpolation is obtained which provides the estimates of the error in polynomial approximation. In next section, we deal with another very useful form of interpolating polynomial called the Newton form of interpolating polynomial. Also we obtain another expression for the error term in term of divided difference.

Finally we deal with some useful forms of interpolating polynomial for equally spaced nodes like Newton's forward difference form Newton's backward difference form after introducing the concepts of forward and backward differences.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- find the Lagrange's form and Newton's divided difference form of interpolating polynomials interpolating f(x) at n + 1 distinct nodal points;
- compute the approximate value of f a non-tabular point;
- compute the value of $\bar{x}$ (approximately) given a number $\bar{y}$ such that f($\bar{x}$) = $\bar{y}$ (inverse interpolation);
- compute the error committed in interpolation, if the function is known, at a non-tabular point of interest;
- find an upper bound in the magnitude of the error;
- form a table of divided differences and find divided differences with a given set of arguments from the table;
- write a forward (backward) difference in terms of function values from a table of forward (backward) differences and locate a difference of given order at a given point from the table; and
- obtain the interpolating polynomial of f(x) for a given data by Newton's forward (backward) difference formula.

2.2 LAGRANGE'S FORM

2.2.1 Problem of Interpolation

We are here concerned with a real-valued function f(x) defined on the interval [a, b] such that the analytical formula representing the function is unknown, but the values of the function f(x) are given for a given set of n + 1 distinct values of x = x_0, x_1, x_2, …, x_n where $x_0 < x_1 < x_2, \ldots, < x_n$ lying in the interval [a, b]. We denote $f(x_k)$ by f_k. The technique of determining an approximate value of f(x) for a non-tabular value of x which lies in the interval is called **interpolation**. The process of determining the value of f(x) for a value of x lying outside the interval [a, b] is called **extrapolation**. However, we shall assume that f(x) is defined in $(-\infty, \infty)$ in which it is continuously differentiable a sufficient number of times.

2.2.2 Lagrange's Form of Interpolating Polynomial

In this section, we derive a polynomial P(x) of degree ≤ n which agrees with values of f at the given (n + 1) distinct points, called the nodes or abscissas. In other words, we can find a polynomial P(x) such that $P(x_i) = f_i$, i = 0, 1, 2, …, n. Such a polynomial P(x) is called the interpolating polynomial of f(x).

First we prove the existence of an interpolating polynomial by actually constructing one such polynomial having the desired property. The uniqueness of the interpolating polynomial is proved by invoking the corollary of the fundamental theorem of algebra. Then we derive a general expression for error in approximating the function by the interpolating polynomial at a point and this allows us to calculate a bound on the error over an interval.

In polynomial interpolation the approximating function is taken to be a polynomial $P_n(x)$ of degree ≤ n such that

$$P_n(x_i) = f(x_i) \qquad i = 0,1,2 \ldots..n \qquad (2.2.1)$$

Let $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x\, x + a_0$. Then, by conditions (4.3.1) we have

$$a_n x_i^n + a_{n-1} x_i^{n-1} + \ldots + a_1 x_i + a_0 = f(x_i), \quad i = 0, 1, 2\ldots, n.$$

This is a system of n + 1 linear equations in n + 1 unknowns $a_0, a_1, \ldots, a_n$. Since the determinant of the co-efficients

$$\begin{vmatrix} 1 & x_0 & \ldots & x_0^n \\ 1 & x_1 & \ldots & x_1^n \\ \ldots & & \ldots & \\ 1 & x_n & \ldots & x_n^n \end{vmatrix} = \prod_{i>j} (x_i - x_j) \neq 0$$

as $x_o, x_1 \ldots\ldots x_n$ are distinct points/nodes, the values of $a_0, a_1, \ldots, a_n$ can be uniquely determined so that the polynomial $P_n(x)$ exists and is unique. But this does not give us the explicit form of $P_n(x)$. Hence in the following we first show the existence of $P_n(x)$ by constructing one such polynomial and then prove its uniqueness.

Theorem 1: Let $x_0, x_1, \ldots, x_n$ be n + 1 distinct points on the real line and let f(x) be a real-valued function defined on some interval I = [a, b] containing these points. Then, there exists exactly one polynomial $P_n(x)$ of degree $\leq$ n, which interpolates f(x) at $x_o, x_1, \ldots, x_n$, that is, $P_n(x_i) = f(x_i) = f_i$ i = 0, 1, 2, …, n.

Proof: Consider the problem of determining a polynomial of degree 1 that passes through the distinct points $(x_0 y_0)$ and $(x_{1,} f_1)$. That is, we are approximating the function f by means of a first degree polynomial interpolating f(x) at $x = x_0$ and x_1.

Consider the polynomial

$$P_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f_0 + \frac{(x - x_0)}{(x_1 - x_0)} f_1$$

Then $P_1(x_0) = f_0$ and $P_1(x_1) = f_1$. Hence $P_1(x)$ has the required properties. Let

$$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} \text{ and } L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}, \text{ then } P_1(x) = L_0(x)\, f_0 + L_1(x)\, f_1.$$

Also we note that $L_0(x_0) = 1, L_1(x_0) = 0, L_0(x_1) = 0$ and $L_1(x_1) = 1$.

For the general case, let the required polynomial be written as

$$P_n(x) = L_0(x) f_0 + L_1(x) f_1 + \ldots + L_n(x) f_n \qquad (2.2.2)$$

$$= \sum_{i=0}^{n} L_i(x) f_i$$

Setting $x = x_j$, we get

$$f_j = P_n(x_j) = \sum_{i=0}^{n} L_i(x_j) f_i$$

Since the polynomial fits the data exactly, we must have

$$L_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The polynomial $L_i(x)$ which are of degree $\leq$ n are called the Lagrange fundamental polynomials. It is easily verified that these polynomials are given by

$$L_i(x) = \frac{(x-x_0)(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1}).....(x_i-x_n)}$$

$$= \prod_{\substack{j=0\\j\neq i}}^{n}(x-x_j) \Big/ \prod_{\substack{j=0\\j\neq i}}^{n}(x_i-x_j), \text{ i = 0, 1, 2, ..., n.} \qquad (2.2.3)$$

Substitution of (4.3.2) in (4.3.1) gives the required **Lagrange** form of the interpolating polynomial. The uniqueness of the interpolating polynomial can be established easily with the help of following results.

Lemma 1. If $x_1, x_2, \ldots, x_k$ are distinct zeros of the polynomial P(x), then $P(x) = (x - x_1)\ (x - x_2) \ldots (x - x_k)\ R(x)$ for some polynomial R(x)

Corollary. If $P_k(x)$ and $Q_k(x)$ are two polynomials of degree ≤ k which agree at the k +1 distinct points then $P_k(x) = Q_k(x)$ identically.

Example 1: Find the Lagrange's interpolating polynomial for the following data:

X	$\frac{1}{4}$	$\frac{1}{3}$	1
f(x)	–1	2	7

Solution:

$$L_0(x) = \frac{\left(x-\frac{1}{3}\right)(x-1)}{\left(\frac{1}{4}-\frac{1}{3}\right)\left(\frac{1}{4}-1\right)} = 16\left(x-\frac{1}{3}\right)(x-1)$$

$$L_1(x) = \frac{\left(x-\frac{1}{4}\right)(x-1)}{\left(\frac{1}{3}-\frac{1}{4}\right)\left(\frac{1}{3}-1\right)} = -18\left(x-\frac{1}{4}\right)(x-1)$$

$$L_2(x) = \frac{\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)}{\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)} = 2\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)$$

Hence $P_2(x) = L_0(x)(-1) + L_1(x)(2) + L_2(x)(7)$

$$P_2(x) = -16\left(x-\frac{1}{3}\right)(x-1) - 36\left(x-\frac{1}{4}\right)(x-1) + 14\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)$$

Example 2: If f(1) = –3, f(3) = 9, f(4) = 30 and f(6) = 132, find the Lagrange's interpolation polynomial of f(x). Also find the value of f when x = 5.

Solution: We have $x_0 = 1, x_1 = 3, x_2 = 4, x_3 = 6$ and
$f_0 = -3, f_1 = 9, f_2 = 30, f_3 = 132.$
The Lagrange's interpolating polynomial $P_3(x)$ is given by

$$P_3(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2 + L_3(x) f_3 \qquad (2.2.4)$$

Where

$$L_0(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} = -\frac{1}{30}(x^3 - 13x^2 + 54x - 72)$$

$$L_1(x) = \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} = \frac{1}{6}(x^3 - 11x^2 + 34x - 24)$$

$$L_2(x) = \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} = -\frac{1}{6}(x^3 - 10x^2 + 27x - 18)$$

$$L_3(x) = \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} = \frac{1}{30}(x^3 - 8x^n + 19x - 12)$$

Substituting these in (2.2.4) we have:

$$P_3(x) = -\frac{1}{30}(x^3 - 13x^2 + 54x - 72)\,(-3) + \frac{1}{6}(x^3 - 11x^2 + 34x - 24)$$
$$\times (9) - \frac{1}{6}(x^3 - 10x^2 + 27x - 18)\,(30)$$
$$+ \frac{1}{30}(x^3 - 8x^2 + 19x - 12\,(132)$$

which gives on simplification

$P_3(x) = x^3 - 3x^2 + 5x - 6.$

$$f(5) \approx P_3(5) = (5)^3 - 3(5)^2 + 5\times 5 - 6 = 125 - 75 + 25 - 6 = 69$$

You may now solve the following exercises

E1) Prove the uniqueness of the interpolating polynomial using corollary of Lemma 1.

E2) Find Lagrange's interpolating polynomial for the data. Hence obtain f(2).

x	0	1	4	5
f(x)	8	11	68	123

E3) Using the Lagrange's interpolation formula, find the value of y when x=10

x	5	6	9	11
f(x)	12	13	14	16

2.2.3 Inverse Interpolation

In inverse interpolation in a table of values of x find y = f(x), one is given a number $\bar{y}$ and wishes to find the point $\bar{x}$ so that $f(\bar{x}) = \bar{y}$, where f(x) is the tabulated function. For this, we naturally assume that the function y = f(x) has a unique inverse x = g(y) in the range of the table. The problem of inverse interpolation simply reduces to interpolation with x-row and y-row in the given table interchanged so that the interpolating points now become $y_0, y_1, \ldots, y_n$ (same as $f_0, f_1, \ldots, f_n$ i.e. $f(x_i) = f_i = y_i$) and the corresponding function values are $x_0, x_1, \ldots, x_n$ where the function is x = g(y). Since the points $y_0, y_1, \ldots, y_n$ are invariably unequally spaced, this interpolation can be done by Lagrange's form of interpolation (also by Newton's divided difference form discussed later). By Lagrange's formula

$$P_n(y) = \sum_{i=0}^{n} x_i \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(y - y_j)}{(y_i - y_j)} \text{ and}$$

$\bar{x} \approx P_n(\bar{y})$. This process is called inverse interpolation

Example 3: Find the value of x when y = 3 from the following table of values

x	4	7	10	12
y	–1	1	2	4

Solution: The Lagrange's interpolation polynomial of x is given by

$$P(y) = \frac{(y-1)(y-2)(y-4)}{(-2)(-3)(-5)}(4) + \frac{(y+1)(y-2)(y-4)}{(2)(1)(-3)}(7)$$

$$+\frac{(y+1)(y-1)(y-4)}{(3)(1)(-2)}(10) + \frac{(y+1)(y-1)(y-2)}{(5)(3)(2)}(12)$$

So $P(3) = \frac{(2)(1)(-1)}{-(2)(3)(5)} \cdot (4) + \frac{(4)(1)(-1)}{(2)(3)} \cdot (7)$

$$+\frac{(4)(2)(-1)}{-(3)(2)}(10) + \frac{(4)(2)(1)}{(5)(3)(2)}(12)$$

$$= \frac{4}{15} - \frac{14}{3} + \frac{40}{3} + \frac{48}{15} = \frac{182}{15} = 12.1333$$

$\therefore x(3) \approx P(3) = 12.1333.$

You may now solve the following exercises

E4) Using Lagranges interpolation formula, find the value of f(4) from the following data:

x	1	3	7	13
f(x)	2	5	12	20

E5) From the following table, find the Lagrange's interpolating polynomial, which agrees with the values of x at the given value of y. Hence find the value of x when y = 2.

x	1	19	49	101
y	1	3	4	5

E6) Using the inverse Lagrange interpolation, find the value of x when y = 3 from the following table of values:

x	36	54	72	144
y	– 2	1	2	4

2.2.4 General Error Term

The next step is to estimate/calculate a remainder term or bound for the error involved in approximating a function by an interpolating polynomial.
Let $E_n(x) = f(x) - P_n(x)$ be the error involved in approximating the function f(x) by an interpolating polynomial. We derive an expression for $E_n(x)$ in the following

theorem. We shall just indicate the proof. This result helps us in estimating a useful bound on the error as explained through an example.

Theorem 2: Let $x_0, x_1, ..., x_n$ be distinct numbers in the interval [a, b] and f has (continuous) derivatives up to order (n + 1) in the open interval (a, b). If $P_n(x)$ is the interpolating polynomial of degree ≤ n, which interpolating f(x) at the points $x_0, x_1, \ldots, x_n$, then for each x $\in$ [a, b], a number $\xi(x)$ in (a, b) exists such that

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n).$$

Proof (Indication): If x $\neq x_k$ for any k = 0, 1, 2, …, n, define the function g for t in [a, b] by $g(t) = f(t) - P_n(t) - [f(x) - P_n(x)] \prod_{j=0}^{n} \frac{(t-x_j)}{(x-x_j)}$.

g(t) has continuous derivatives up to (n + 1) order. Now, for k = 0, 1, 2, …, n, we have $g(x_k) = 0$ and g(x) = 0.

Thus, g has continuous derivatives up to order (n + 1) and g vanishes at the (n + 2) distinct points x, $x_0, x_1, \ldots, x_n$. By generalized Rolle's Theorem stated below, there exists ξ $(\xi(x))$ in (a, b) for which $g^{(n+1)}(\xi) = 0$. That is

$$0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - (n+1)! \frac{[f(x) - P_n(x)]}{\prod_{i=0}^{n}(x-x_i)}$$

where the differentiation is with respect to t.

Simplifying above we get the error at x = $\bar{x}$

$$E_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x}) = \frac{f^{(n+1)}(\xi(\bar{x}))}{(n+1)!}\prod_{i=0}^{n}(\bar{x}-x_i) \qquad (2.2.5)$$

By repeated application of Rolle's Theorem the following theorem can be proved.

Theorem 3 (Generalized Rolle's Theorem):

Let f be a real-valued function defined on [a, b] which is n times differentiable on (a, b). If f vanishes at the (n + 1) distinct prints $x_0, x_1, \ldots, x_n$, in [a, b] then a number c in (a, b) exists such that $f^{(n)}(c) = 0$.

The error formula derived above, is an important theoretical result and error formula and interpolating polynomial will be used in deriving important formula for numerical differentiation and numerical integration

It is to be noted that $\xi = \xi(\bar{x})$ depends on the point $\bar{x}$ at which error estimate is required. This error formula is of limited utility since $f^{(n+1)}(x)$ is not known (when we are given a set of data at specific nodes) and the point ξ is hardly known. But the formula can be used to obtain a bound on the error of interpolating polynomial as illustrated below:

Example 4: The following table given the values of $f(x) = e^x$, $.1 \leq x \leq 2$. If we fit an interpolating polynomial of degree four to the data given below, find the magnitude of the maximum possible error in the computed value of f(x) when x = 1.25.

x	1.2	1.3	1.4	1.5	1.6
f(x)	3.3201	3.6692	4.0552	4.4817	4.9530

Solution: From Equation, the magnitude of the error associated with the 4th degree polynomial approximation is given by

$$\left|E_4(x)\right| = \left|(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\frac{f^{(5)}(\xi\xi}{5!}\right|$$

$$= \left|(x-1.2)(x-1.3)(x-1.4)(x-1.5)(x-1.6)\frac{f^{(5)}(\xi\xi}{5!}\right|$$

$f^{(5)}(x) = e^x$ and $\underset{1.2\le x\le 1.6}{Max}\left|f^{(5)}(x)\right|$ = $e^{1.6}$ = 4.9530

Since f(x) = e^x is increasing in [1.2, 1.6]. Hence

$$\left|E_4(1.25)\right| \le (1.25-1.2)(1.25-1.3)(1.25-1.4)(1.25-1.5)\times\frac{4.9530}{120} = 0.00000135$$

When nodes are equipspaced, we shall get another bound later.

2.3 NEWTON FORM OF THE INTERPOLATING POLYNOMIAL

The Lagrange's form of the interpolating polynomial discussed in previous section has certain drawbacks. One generally calculates a linear polynomial $P_1(x)$, a quadratic polynomial $P_2(x)$ etc. by increasing the number of interpolating points, until a satisfactory approximation $P_k(x)$ to f(x) has been found. In such a situation, Lagrange form does not take any advantage of the availability of $P_{k-1}(x)$ in calculating $P_k(x)$. In Newton form, this advantage is taken care of.

Before deriving Newton's general form of interpolating polynomial, we introduce the concept of divided difference and the tabular representation of divided differences. Also the error term of the interpolating polynomial in this case is derived in terms of divided differences. Using the two different expressions for the error term we establish a relationship between nth order divided difference and the nth order derivative.

Suppose we have determined a polynomial $P_{k-1}(x)$ of degree ≤ k – 1 which interpolates f(x) at the points $x_0, x_1, \ldots, x_{k-1}$. In order to make use of $P_{k-1}(x)$ in calculating $P_k(x)$ we consider the following problem. What function g(x) should be added to $P_{k-1}(x)$ to get $P_k(x)$? Let g(x) = $P_k(x) - P_{k-1}(x)$. Now, g(x) is a polynomial of degree ≤ k and
$g(x_i) = P_k(x_i) - P_{k-1}(x_i) = f(x_i) - f(x_i) = 0$ for = 0, 1, …, k – 1.

Hence g(x) can be written as $A_k(x-x_0).....(x-x_{k-1})$
Where A_k is a constant depending on $x_0, x_1, \ldots, x_{k-1}$.

Suppose that $P_n(x)$ is the Lagrange polynomial of degree at most n that agrees with the function f at the distinct numbers $x_0, x_1, \ldots, x_n$. The divided difference of f with respect to $x_0, x_1, \ldots, x_n$ can be obtained by proving that P_n has the representation, called Newton form.

$$P_n(x) = A_0 + A_1(x-x_0) + A_2(x-x_0)(x-x_1) + ...$$
$$+A_n(x-x_0)......(x-x_{n-1}) \qquad (2.3.1)$$

for appropriate constants $A_0, A_1, \ldots.. A_n$.

Evaluating $P_n(x)$ [Equation (4.4.1)] at x_o we get $A_0 = P_n(x_0) = f(x_0)$.

Similarly when $P_n(x)$ is evaluated at x_1, we get $A_1 = \dfrac{f(x_1) - f(x_1)}{x_1 - x_0}$. Let us introduce the notation for divided differences and define it at this stage. The divided difference of the function f, with respect to x_i, is denoted by $f[x_i]$ and is simply the evaluation f at x_i, that is, $f[x_i]=f(x_i)$. The first divided difference of f with respect to x_i and x_{i+1} is denoted by $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

The remaining divided differences of higher orders are defined inductively as follows. The kth divided differences relative to $x_i, x_{i+1}, \ldots, x_{i+k}$ is defined as follows:

$$f[x_i, x_{i+1}, .., x_{i+k}] = \frac{f[x_{i+1,\ldots} x_{i+k}] - f[x_{i,} \ldots.. x_{i+k-1}]}{x_{i+k} - x_i}$$

where the (k –1)st divided differences $f[x_i,\ldots, x_{i+k-1}]$ and $f[x_{i+1},\ldots, x_{i+k}]$ have been determined. This shows that the kth divided difference is the divided differences of (k –1)st divided differences justifying the name. It can be shown that the divided difference $f[x_0, x_{1-1}, x_k]$ is invariant under all permutations of argument $x_0, x_1,\ldots, x_k$.

The constant $A_2, A_3, \ldots, A_n$ can be consecutively obtained in similar manner like the evaluation of A_0 and A_1. As shown in the evaluation of A_0 and A_1 , the required constants $A_k=f[x_0, x_1,\ldots x_k]$.

This shows that $P_n(x)$ can be constructed step by step with the addition of the next term in Equation (4.4.1), as one constructs the sequence $P_0(x)$, $P_1(x)$ with $P_k(x)$ obtained from $P_{k-1}(x)$ in the form

$$P_k(x) = P_{k-1}(x) + A_k(x - x_0)\ldots..(x - x_{k-1}) \qquad (2.3.2)$$

That is, g(x) is a polynomial of degree ≤ k having (at least) k distinct zeros $x_0,\ldots..x_{k-1}$. This constant A_k is called the kth divided difference of f(x) at the points $x_0, x_1, \ldots, x_k$ and is denoted by $f[x_0, x_1\ldots, x_k]$. This coefficient depends only on the values of f(x) at the points $x_0, x_1, \ldots, x_k$. The following expressions can be proved for $f[x_0, x_1, \ldots..x_k]$.

$$f[x_0, x_1 \ldots.., x_k] = \sum_{i=0}^{k} \frac{f(x_i)}{(x_i - x_0)\ldots(x_i - x_{i-1})\ldots.(x_i - x_{i+1})\ldots(x_i - x_k)} \qquad (2.3.3)$$

and

$$f[x_0, x_1, \ldots.. x_k] = \frac{f[x_1, \ldots.. x_k] - f[x_0, \ldots.. x_{k-1}]}{x_k - x_0} \qquad (2.3.4)$$

Equation (4.4.2) can now be rewritten as

$$P_k(x) = P_{k-1}(x) + f[x_{0,} x_1, \ldots, x_k](x - x_0), \ldots, (x - x_{k-1}) \qquad (2.3.5)$$

Using Equation (4.4.5), Equation (4.4.1) can be rewritten as

$$P_k(x) = f[x_0] + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_)) f[x_0, x_1, x_2]$$
$$+ \ldots + (x - x_0)(x - x_1)\ldots(x - x_{n-1}) f[x_0, \ldots, x_n] \qquad (2.3.6)$$

This can be written compactly as follows

$$P_n(x) = \sum_{i=0}^{n} f[x_{0,} x_1 \ldots x_i] \prod_{j=0}^{i-1} (x - x_j) \qquad (2.3.7)$$

This is Newton's form of interpolating polynomial or Newton's interpolatory divided-difference formula.

Methods for determining the explicit representation of an interpolating polynomial from tabulated data are known as divided-difference methods. These methods can be used to derive techniques for approximating the derivatives and integrals of functions, as well as for (approximating the solutions) to differential equations.

2.3.1 Table of Divided Differences

Suppose we denote, for convenience, a first order divided difference of f(x) with any two arguments by f[., .], a second order divided difference with any three arguments by f[., ., .] and so on. Then the table of divided difference can be written as follows:

Table 1

X	f[.]	f[., .]	f[., ., .]	f[[., ., ., .]	f[[., ., ., .,.]
x_0	f_0				
		$f[x_0, x_1]$			
x_1	f_1		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
x_2	f_2		$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$	
x_3	f_3		$f[x_2, x_3, x_4]$		
		$f[x_3, x_4]$			
x_4	f4				

Example: If $f(x) = x^3$, find the value of $f[a, b, c]$.

Solution: $f[a, b] = \dfrac{f(b) - f(a)}{b - a} = \dfrac{b^3 - a^3}{b - a} = b^2 + ba + a^2$

Similarly $f[b, c] = b^2 + bc + c^2$

Hence $f[a, b, c] = \dfrac{f[b,c] - f[a,b]}{c - a}$

$$= \frac{b^2 + bc + c^2 - b^2 - ba - a^2}{c - a}$$

$$= \frac{(c - a)(c + a + b)}{(c - a)}$$

$$= a + b + c$$

You may now solve the following exercises

E7) If $f(x)=\dfrac{1}{x}$, show that $f[a, b, c, d] = -\dfrac{1}{abcd}$.

E8) Using divided difference show that the following data

x	1	2	3	5	6
f(x)	1	3	7	21	31

represents a second degree polynomial. Obtain this polynomial. Hence, find the approximate value of f(4).

Example 5: Form the following table of values, find the Newton's form of interpolating polynomial approximating f(x).

x	–1	0	3	6	7
f(x)	3	–6	39	822	1611

Also find the approximate value of the function f(x) at x = 2.

Solution: We note that the value of x, that is x_i are not equally spaced. To find the desired polynomial, we form the table of divided differences of f(x)

Table 2

	x	f[.]	f[., .]	f[., ., .]	f[[., ., ., .]	f[[., ., ., .,.]
x_0	–1	3				
			–9			
x_1	0	–6		6		
			15		5	
x_2	3	39		41		1
			261		13	
x_3	6	822		132		
			789			
x_4	7	1611				

Newton's interpolating polynomial $P_4(x)$ is given by

$$P_4(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$
$$+(x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] +$$
$$(x - x_0)(x - x_1)(x - x_2)(x - x_3)f[x_0, x_1, x_2, x_3] \qquad (2.3.8)$$

The divided differences $f[x_0]$, $f[x_0, x_1]$, $f[x_1, x_1, x_2]$, $f[x_0, x_1, x_2, x_3]$, $f[x_0, x_1, x_2, x_3]$ are those which lie along the diagonal at $f(x_0)$ as shown by the dotted line. Substituting the values of x_i and the values of the divided differences in Equation (4.4.8), we get

$$P_4(x) = 3 + (x + 1)(-9) + (x + 1)(x - 0)(6) + (x + 1)(x - 0)(x - 3)(5)$$
$$+(x + 1)(x - 0)(x - 3)(x - 6)(1)$$

This on simplification gives

$$P_4(x) = x^4 - 3x^3 + 5x^2 - 6$$
$$\therefore f(x) \approx P_4(x)$$
$$f(2) \approx P_4(2) = 16 - 24 + 20 - 6 = 6$$

You may now solve the following exercises.

E9) From the table of values given below, obtain the value of y when x = 1 using
(a) divided difference interpolation formula.
(b) Lagrange's interpolation formula

x	0	2	3	4
f(x)	– 4	6	26	64

We now give another expression for the error term, that is, the error committed in approximating f(x) by $P_n(x)$.

Let $P_n(x)$ be the Newton form of interpolating polynomial of degree ≤ n which interpolates f(x) at $x_0, \ldots, x_n$. The Interpolating error $E_n(x)$ of $P_n(x)$ is given by

$$E_n(x) = f(x) - P_n(x). \qquad (2.3.9)$$

Let $\bar{x}$ be any point different from $x_0, x_1, \ldots x_n$. If $P_{n+1}(x)$ is the Newton form of interpolating polynomial which interpolates f(x) at $x_0, x_1, \ldots x_n$ and $\bar{x}$, then $P_{n+1}(\bar{x}) = f(\bar{x})$. Then by Equation (4.4.5) we have

$$P_{n+1}(x) = P_n(x) + f[x_0, x_1, \ldots, x_n, \bar{x}]\prod_{j=0}^{n}(\bar{x} - x_j)$$

Putting x = $\bar{x}$, in the above, we have

$$f(\bar{x}) = P_{n+1}(\bar{x}) = P_n(\bar{x}) + f[x_0, x_1, \ldots x_n, \bar{x}]\prod_{j=0}^{n}(\bar{x} - x_j)$$

That is, $E_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x}) = f[x_0, x_1, \ldots, x_n, \bar{x}]\prod_{j=0}^{n}(\bar{x} - x_j)$

This shows that the error is like the next term in the Newton form.

Comparing the two expressions derived for $E_n(x)$, we establish a relationship between divided differences and the derivatives of the function as follows:

$$E_n(\bar{x}) = \frac{f^{n+1}(\xi(\bar{x}))}{(n+1)}\prod_{j=0}^{n}(\bar{x} - x_i)$$

$$= f[x_0, x_1 \ldots x_n, \bar{x}]\prod_{j=0}^{n}(\bar{x} - x_i).$$

Comparing, we have

$$f[x_0, x_1, \ldots, x_{n+1}) = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

(Considering $\bar{x} = x_{n+1}$). Further, it can be shown that $\xi \in$ (min x_i, max x_i).
We state these results in the following theorem.

Theorem 4: Let f(x) be a real-valued function, defined on [a, b] and n time differentiable in (a, b). If $x_0, x_1, \ldots x_n$ are n+1 distinct points in [a, b], then there exists $\xi \in (a, b)$ such that

$$f[x_0, \ldots x_n] = \frac{f^{(n)}(\xi)}{n!}$$

Corollary 1: If $f(x) = x^n$, then $f[x_0, x_1, \ldots, x_n] = \frac{n!}{n!} = 1$

Corollary 2: If $f(x) = x^k$, $k < n$, then $f[x_0, \ldots, x_k] = 0$
(Since the nth derivative of x^k, $k < n$, is zero).

Example 6: If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, then find $f[x_0, x_1, \ldots, x_n]$.

Solution: By corollaries 1 and 2, we have $f[x_0, x_1, \ldots, x_n] = a_n \frac{n_1}{n_1} + 0 = a_n$

Let us consider the error formula

$$E_n(x) = f(x) - P_n(x) = \frac{f^{n+1}(\xi(x))}{(n+1)!}\prod_{j=0}^{n}(x - x_i)$$

We want to obtain a bound on the error. For this we have

$$|E_n(x)| = |f(x) - P_n(x)| \le \frac{\max_{t \in I}|f^{(n+1)}(t)|}{(n+1)!}\max_{t \in I}|\psi_n(t)| \qquad (2.3.10)$$

where [a, b] = I, $\psi_n(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$.

Example 7: Find a bound for the error in linear interpolation.

Solution: For the case n = 1, we have linear interpolation. If $x \in [x_{i-1}, x_i]$ then we approximate f(x) by $P_1(x)$ which interpolates at x_{i-1}, x_i. From Equation (4.4.10). We have

$$|E_1(x)| \le \frac{1}{2} \max_{x\in I} .|f''(x)| \max_{x\in I} |\psi_1(x)|$$

where $\Psi_1(x) = (x - x_{i-1})(x - x_i)$.

Now $\frac{d\Psi_1}{dx} = x - x_{i-1} + x - x_i = 0 \Rightarrow x = (x_{i-1} + x_i)/2$.

Hence, the maximum value of $|(x - x_{i-1})(x - x_i)|$ occurs at $x = x^* = (x_{i-1} + x_i)/2$.

$$\therefore \max_{x_{i-1}\le t\le x_i} |\psi_1(x)| = \frac{(x_i - x_{i-1})^2}{4} = \frac{h^2}{4}.$$

Thus $|E_1(x)| \le \frac{(x_i - x_{i-1})^2}{4} \cdot \frac{1}{2} \max_{x\in I} |F''(x)| = \frac{h^2}{8} \cdot M$

Where $h = x_i - x_{i-1}$ and $|f''(x)| \le M$ on $I = [x_{i-1}, x_i]$.

Example 8: Determine the largest step-size h that can be used in the tabulation of the function $f(x) = (1 + x^2)^2$ on [0, 1] so that the error in the linear interpolation is less than 5×10^{-5}.

Solution: We have to determine the spacing h in a table of equally spaced values of the function of $f(x) = (1 + x^2)^2$ between 0 and 1. By assumption, the table will contain $f(x_i)$ with $x_i = 1 + ih$, i = 0, …, N, where $N = \frac{1-0}{h} = \frac{1}{h}$.

If $\bar{x} \in [x_{i-1}, x_i]$, we approximate $f(\bar{x})$ by $P_1(\bar{x})$, where $P_1(x)$ is the linear polynomial which interpolates f(x) at x_{i-1} and x_i. By Equation (4.4.10)

$$|E_1(\bar{x})| \le \frac{h^2}{8} \cdot M \le \frac{h^2}{8} \cdot \max_{x\in[0,1]} |f''(x)|$$

When $M = \max_{x\in[x_{i-1}, x_i]} |f''(x)|$.

$$|E_1(\bar{x})| \le \frac{h^2}{8} . M' \text{ where } M' = \max_{0\le x\le 1} |f''(x)|$$

When $\bar{x} \in (0, 1)$.

Since $f''(x) = 12x^z + 4$ and it is an increasing function on [0, 1], $\max_{x\in[0,1]} |f''(x)| = 16$.

Thus $|E_1(\bar{x})| \le \frac{h^2}{8}.16 = 2h^2$

We have $2h^2 < 5 \times 10^{-5}$ or $h \le .005$.

That is $N = \frac{1}{.005} = \frac{1000}{5} = 200$.

You may now solve the following exercises.

E10) Determine the spacing h in a table of equally spaced points of the function $f(x) = x\sqrt{x}$ between 1 and 2, so that the error in the linear interpolation is less than 5×10^{-6}.

2.4 INTERPOLATION AT EQUALLY SPACED POINTS

Suppose the value of f(x) at (n + 1) equally spaced values of x are known or given, that is, (x_i, y_i), i = 0,…, n are known where $x_i - x_{i-1} = h$ (fixed), i = 1, 2, …, x and y = f(x). Suppose we are required to approximate value of f(x) or its derivative f'(x) for some values of x in the interval of interest. The methods for solving such problems are based on the concept of finite differences. However, our discussion will be confined to obtain Newton's forward and backward difference forms only. Each is suitable for use under specific situation.

2.4.1 Differences – Forward and Backward Differences

Suppose that we are given a table of values (x_i, y_i) i = 0,…, N where $y_i = f(x_i) = f_i$. Let the nodal points be equidistant. That is $x_i = x_0 + ih$, i = 0, 1, …, N with N = (b − a)/h (I = [a, b]).

For simplicity we introduce a linear change of variables $s = s(x) = \dfrac{x - x_0}{h}$, so that x = x(s) = x_0 + sh and introduce the notation.

$$f(x)=f(x_0 + sh)=f_s.$$

The linear change of variables transforms polynomials of degree n in x into polynomial of degree n in s. For equally spaced nodes, we shall deal with two types of differences, namely forward and backward and discuss their representation in the form of a table. We shall also derive/give the relationship of these differences with divided differences and their interrelationship.

Forward Differences

We denote the forward differences of f(x) of ith order at x = x_0 + sh by $\Delta^i f_s$ and define it as follows:

$$\Delta^i f_s = \begin{cases} f_s, & i=0 \\ \Delta(\Delta^{i-1} f_s) = \Delta^{i-1} f_{s+1} - \Delta^{i-1} f_s, & i > o \end{cases} \qquad (2.4.1)$$

where Δ denotes forward difference operator.

When s = k, that is x = s_k, we have

for i = 1 $\quad \Delta f_k = f_{k+1} - f_k$

for i =2 $\quad \Delta^2 f_k = \Delta f_{k+1} - \Delta f_k = f_{k+2} - f_{k+1} - [f_{k+1} - f_k]$

$\qquad = f_{k+2} - 2f_{k+1} + f_k$

Similarly, you can obtain

$$\Delta^3 f_k = f^{k+3} - 3\, f_{k+2} + 3f_{k+1} - f_k.$$

We recall the binomial theorem

$$(a+b)^s = \sum_{j=0}^{s} \binom{s}{j} a^j b^{s-j.} \qquad (2.4.2)$$

where $\binom{s}{j} = C(s, j)$

and s is a real and non-negative integer.
The shift operator E is defined as

$$E f_i = f_{i+1} \qquad (2.4.3)$$

In general $Ef(x) = f(x + h)$. We have $E^s f_i = f_{i+s}$

For example $E^4 f_i = f_{i+4}$. $E^{1/2} f_i = f_{i+1/2}$ and $E^{-1/2} f_i = f_{i-\frac{1}{2}}$

Now $\Delta f_i = f_{i+1} - f_i = Ef_i - f_i = (E-1) f_i$

Hence the shift and forward difference operators are related by
$\Delta = E - 1$ or $E = 1 + \Delta$.
Operating s times, we get

$$\Delta^s = (E-1)^s = \sum_{j=0}^{s} \binom{s}{j} E^j (-1)^{s-j} \qquad (2.4.4)$$

by Equation (4.5.2) (4.5.3), we get

$$\Delta^s f_i = \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j} \binom{s}{j} f_{j+1}$$

We now give in the following table, the forward differences of various orders using 5 values.

Table 3 : Forward Difference Table

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

Note that the forward difference $\Delta^k f_0$ lie on a straight line slopping down ward to the right.

Now we give Lemma 1 the relationship between the forward and divided differences, which can be proved by induction. This relation is utilized to derive the Newton's forward-difference formula which interpolates f(x) at
$x_k + ih$, i =0, 1, …, n.

Lemma 1: For all in ≥ 0

$$f[x_k, x_{k+1}, x_{k+i}] = \frac{1}{i! h^i} \Delta^i f_k$$

This has an easy corollary.

Corollary: If $P_n(x)$ is a polynomial of degree n with leading co-efficients a_n, and x_0 is an arbitrary point, then

$\Delta^n P_n(x_0) = a_n \, n! \, h^n$ and

$\Delta^{n+1} P_n(x_0) = 0$, i.e., all higher differences are zero.

Backward Differences

The backward differences of f(x) of ith order at $x_s = x_0 + sh$ are denoted by $\nabla^i f_s$. They are defined as follows:

$$\nabla^i f_s = \nabla^i f_s = \begin{cases} f_s, & i=0 \\ \nabla^{i-1}(\nabla f_s) = \nabla^{i-1}[f_s - f_{s-1}], & i \ge 1 \end{cases} = \nabla^{t-1} f_s - \nabla^{t-1} f_{s-1} \quad (2.5.5)$$

where ∇ denotes backward difference operator. When s = k, that is
$x = x_0 + kh = x_k$, we have for

$i = 1$ $\quad \nabla f_k = f_k - f_{k-1}$

$i = 2$ $\quad \nabla^2 f_k = \nabla(f_k - f_{k-1}) = \nabla f_k - \nabla f_{k-1} = f_k - f_{k-1} + f_{k-2}$

Similarly for

$i = 3$ $\quad \nabla^3 f_k = f_k - 3f_{k-1} + 3f_{k-2} - f_{k-3}$

The relation between the backward difference operator ∇ and the shift operator E is given by

$$\nabla = 1 - E^{-1} \qquad E = (I - \nabla)^{-1}$$

Also $\Delta = E - 1 = E(I - E^{-1}) = E\nabla$

Since $\nabla f_k = f_k - f_{k-1} = f_k - E^{-1} f_k = (1 - E^{-1}) f_k$

Operating s time, we get

$$\nabla^s f_k = (1 - E^{-1})^s f_k = \sum_{j=0}^{s} (-1)^j \binom{s}{j} f_{k-j}$$

We can extend the binomial coefficient notation to include negative numbers, by letting

$$\binom{-s}{i} = \frac{-s(-s-1)(-s-2)\ldots(-s-i+1)}{i!} = (-1)^i \frac{s(s+1)\ldots(s+i-1)}{i!}$$

The backward differences of various orders with 5 nodes are given in the following table:

Table 4: Backward Difference Table

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0				
		∇f_1			
x_1	f_1		$\nabla^2 f_2$		
		∇f_2		$\nabla^3 f_3$	
x_2	f_2		$\nabla^2 f_3$		$\nabla^4 f_4$
		∇f_3		$\nabla^3 f_4$	
x_3	f_3		$\nabla^2 f_4$		
		∇f_4			
x_4	f_4				

Note that the backward difference $\nabla^k f_4$ lie on a straight line slopping upward to the right. Also note that $\Delta f_k = \nabla f_{k+1} = f_{k+1} - f_k$.

2.4.2 Newton's Forward-Difference and Backward-Difference Formulas

The Newton's form of interpolating polynomial interpolating at x_k, x_{k+1} ., x_{k+n} is

$$P_n(x) = \sum_{i=0}^{n} (x - x_k)(x - x_{k+1}) \ldots (x - x_{k+i-1}) f[x_{k+1}, x_{k+2}, \ldots, x_{k+1}] \qquad (2.4.6)$$

In this we make use of the following:

$f[x_k, \ldots, x_{k+n}] = \dfrac{1}{n!\,h^n} \Delta^n f_k$ and get

$$P_n(x) = \sum_{i=0}^{n} (x - x_k)(x - x_{k+1}) \ldots (x - x_{k+i-1}) \frac{1}{i!\,h^n} \Delta^i f_k \qquad (2.4.7)$$

Setting k = 0, we have

Here $x_s = x_0 + sh$ may be introduced, where

$$s = \frac{x_s - x_0}{h}$$

Also f(x) = can be derived straight from $E^s f_0 = (1 + \Delta)^s f_0$

$$P_n(x) = f_0 + \frac{(x - x_0)}{1!} \frac{\Delta f_0}{h} + \frac{(x - x_0)(x - x_1)}{2!} \frac{\Delta^2 f_0}{h^2} + \ldots \frac{(x - x_0) \ldots (x - x_{n-1})}{n!} \frac{\Delta^n f_0}{h^n} \qquad (2.4.8)$$

Also we have $x - x_{k+j} = x_0 + sh - [x_0 + (k + j)h] = (s - k - j)h$

Substituting this in Equation (4.5.8) and simplifying we get

$$P_n(x) = p(x_0 + sh) = \sum_{i=0}^{n} (s - k)(s - k - 1) \ldots (s - k - i + 1) \Delta^i f_k$$

$$= \sum_{i=0}^{n} \Delta^i f_k \binom{s-k}{i}$$

$$= f_k + (s - k)\Delta)_k + \frac{(s - k)(s - k - 1)}{2!} \Delta^2 f_k + \ldots$$

$$+ \frac{(s - k) \ldots (s - n - 1)}{n!} \Delta^n f_k \qquad (2.4.9)$$

of degree ≤ n.

Setting k = 0, in Equation (4.5.9) we get the formula

$$P_n(x_0 + sh) = \sum_{i=0}^{n} \Delta^i f_0 \binom{s}{i}$$

This form Equation (4.5.1)..(4.5.9)…. is called the Newton's forward-difference formula.

The error term is now given by

$$E_n(x) = \binom{s}{n+1} h^{n+1} f^{n+1}(\xi\xi$$

For deriving the Newton's Backward-Difference formula, we reorder the interpolating nodes as $x_n, x_{n-1}, \ldots x_0$ and apply the Newton's divided difference form. This gives

$$P_n(x) = f[x_n] + (x - x_n) f[x_{n-1}, x_n] + (x - x_n)(x - x_{n-1}) f[x_{n-2}, x_{n-1}, x_n]$$

$$+ (x - x_n) \ldots (x - x_0) f[x_n \ldots x_0] \qquad (2.4.10)$$

Set $x = x_n + sh$, then

$$x - x_j = x_n + sh - [x_n - (n - i)h] = (s + n - j)h$$

$$x - x_{n-j} = x_n + sh - [x_n - (n - n + j)h] = (s + j)h$$

and

$$(x - x_n)(x - x_{n-1}) \ldots (x - x_{n-i+1}) = s(s + 1) \ldots (s + i - 1) h^x$$

Also we have $f[x_{n-k}, \ldots, x_n] = \frac{1}{k!\,h^k} \Delta^k (x_n) = \frac{1}{k!\,h^k} \Delta^k f(x_n)$

Substituting those in Equation (2.4.10) and simplifying we get the following two expressions

$$P_n(x) = \sum_{i=0}^{n} \frac{1}{i!\,h^i} (x - x_n)(x - x_{n-i}) \ldots (x - x_{n-i+1}) \Delta^i f_n$$

$$= f_n + \frac{(x - x_n)}{1!} \frac{\Delta f_n}{h} + \frac{(x - x_n)(x - x_{n-i})}{2!} \frac{\Delta^2 f_n}{h^2} + \ldots$$

$$+\frac{(x-x_n)(x-x_{n-i})\ldots(x-x_n)}{n!}\frac{\Delta^n f_n}{h^n} \qquad (2.4.11)$$

$$=f_n+s\Delta\Delta_n+\frac{s(s+1)}{2!}\Delta^2 f_n+\ldots+\frac{s(s+1)\ldots(s+n-1)}{n!}\Delta^n f_n$$

Since $\binom{-s}{k}=\dfrac{(-1)^k s(s+1)\ldots(s+k-1)}{k!}$, we have

$$P_n(x)=f(x_n)+(-1)\binom{-s}{1}\nabla f(x_n)+(-1)^2\binom{-s}{2}\nabla^2 f(x_n)+\ldots+(-1)^n\binom{-s}{n}\nabla^n f(x_n) \qquad (2.4.12)$$

or

$$P_n(x)=\sum_{k=0}^{n}(-1)^k\binom{-s}{k}\nabla^k f(x_n)$$

This is called Newton's backward-difference form.

In this case, error is given by

$$E_n(x)=(-1)^{n+1}\frac{s(s+1)\ldots(s+n)}{(n+1)!}h^{n+1}f^{n+1}(\xi\xi)$$

The forward-difference formula (4.5.7) is suitable for approximating the value of the function at x that lies towards the beginning of the table and the backward-difference form is suitable for approximating the value of the function at x that lies towards the end of the table.

Example 9: Find the Newton's forward-difference interpolating polynomial which agrees with the table of values given below. Hence obtain the value of f(x) at x = 1.5.

x	1	2	3	4	5	6
f(x)	10	19	40	79	142	235

Solution:

Table 5: Forward Differences

x	f(x)	Δf	Δ^2f	Δ^3f		
1	10					
		9				
2	19		12			
		21		6		
3	40		18		0	
		39		6		0
4	79		24		0	
		63		6		
5	142		30			
		93				
6	285					

The Newton's forward-differences interpolation polynomial is

$$f(x)\approx f_0+(x-1)\Delta)_0+\frac{(x-1)(x-2)}{2!}\Delta^2 f_0+\frac{(x-1)(x-2)(x-3)}{3!}\Delta^3 f_0$$

$$=10+(x-1)(9)+\frac{(x-1)(x-2)}{2}(12)+\frac{(x-1)(x-2)(x-3)}{6}(6)$$

On simplification we get

$f(x) \approx x^3 + 2x + 7$

$\therefore f(1.5) = (1.5)^3 + 2(1.5) + 7 = 13.375$

Example 10: Find Newton's backward difference form of interpolating polynomial for the data

x	4	6	8	10
f(x)	19	40	79	142

Hence interpolate f(9).

Solution: We have

Table 6: Backward Difference

x	f(x)	∇f	$\nabla^2 f$	$\nabla^3 f$
4	19			
		21		
6	40		18	
		39		6
8	79		24	
		63		
10	142			

$$P_n(x) = 142 + (x-10)\frac{63}{2} + \frac{(x-10)(x-8)}{2!}\cdot\frac{24}{4} + \frac{(x-10)(x-8)(x-6)}{3!}\cdot\frac{6}{8}$$

$$f(9) \approx P_n(9) = 142 - \frac{63}{2} - 3 - \frac{3}{8} = 107.125$$

You may now solve the following exercises.

E11) Find the Newton's backward differences interpolating polynomial for the data of Example 9.

E12) Using forward differences, show that the following data represents a third degree polynomial.

x	– 3	– 2	– 1	0	1	2	3
f(x)	– 29	– 9	– 1	1	3	11	31

Find the polynomial and obtain the value of f(0.5).

E13) Using forward Differences, show that the following data:

x	– 1	0	1	2	3	4
f(x)	6	1	0	3	10	21

represents a second degree polynomial. Find this polynomial and an approximate value of f(2.5)

E14) Estimate the value of f(1.45) from the data given below

x	1.1	1.2	1.3	1.4	1.5
f(x)	1.3357	1.5095	1.6984	1.9043	2.1293

E15) Evaluate the differences

(i) $\nabla^3[a_2x^2 + a_1x + a_0]$

(ii) $\nabla^3[a_3x^3 + a_2x^2 + a_1x + a_0]$

(iii) $\Delta^3[a_3x^3 + a_2x^2 + a_1x + a_0]$

E16) Show that the nth order divided differences of f(x) = $\frac{1}{x}$ is

$(-1)^n/(x_0.x_1...x_n)$.

E17) A table of values is to be constructed for the function f(x) given by f(x) = x^4 + 1 in the interval [3, 4] with equal step-length. Determine the spacing h such that linear interpolation gives results with accuracy 1×10^{-4}.

E18) Find the Newton's divided difference form of interpolating polynomial for the data.

x	– 4	–1	0	2	5
f(x)	1245	33	5	9	1335

Also approximate f(1) from the polynomial.

E19) Construct Newton's forward difference table for the data

x	3	5	7	9
f(x)	6	24	38	108

Hence approximate f(4) from Newton's forward difference interpolating polynomial.

E20) If f(x) = ax^2 + bx + c (a, b, c, ∈, R), then show that f[1, 2, 3] = a.

2.5 SUMMARY

In the first section apart from deriving the Lagrange's form of interpolating polynomial for a given data, it has been shown that the interpolating polynomial for a given data is unique. We have also seen how the Lagrange's interpolation formula can be applied with y as the independent variable and x as the dependent variable so that the value of x corresponding to a given value of y can be calculated approximately when some conditions are satisfied. Finally, we have derived the general error formula and its use has been illustrated to judge the accuracy of our calculation. In the next section, we have derived a form of interpolating polynomial called Newton's general form (divided difference form) which has some advantages over the Lagrange's form discussed in section 1. We have introduced the concept of divided differences and discussed some of its properties before deriving Newton's general form. The error term also has been derived and we have established a relationship between the divided difference and the derivative of the function f(x) using the two different expressions of the error terms. In section 3, we have derived interpolation formulas for data with **equally spaced** values of the argument. The application of the formulas derived in this section is easier compared to the application of the formulas derived in first and second sections.

The mathematical formulas derived in the unit are listed below for your easy reference.

1. Lagrange's Form

$$P_n(x) = \sum_{i=0}^{n} f(x_i)\,L_i(x) \quad \text{where} \quad L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

2. Inverse Interpolation

$$P_n(y) = \sum_{i=0}^{n} x_i \left[\prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(y - y_j)}{(y_i - y_j)} \right]$$

3. Interpolation Error

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi\xi(x)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

4. Divided Difference

$$f[x_0, x_1, \ldots x_j] = \frac{f[x_1, \ldots\ldots x_j] - f[x_0, \ldots\ldots x_{j-1}]}{x_j - x_0}$$

5. Newton's Form

$$P_n(x) = \sum_{i=0}^{n} f[x_0, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

6. Interpolation Error (In terms of divided difference)

$$E_n(x) = f[x_0, x_1, \ldots\ldots x_n, x] \prod_{j=0}^{n} (x - x_j)$$

7. $f[x_0, x_1, \ldots, x_n] = \dfrac{f^{(n)}(\xi\xi}{n!}, \xi \in$ (min x_i, max x_i)

8. Newton's Forward Difference Formula

$$P_n(x) = P_n(x_0 + sh) = \sum_{i=0}^{n} \binom{s}{i} \Delta^i f_0$$

$$= f_0 + s\Delta\Delta_0 + \frac{s(s-1)}{2!} \Delta^2 f_0 + \ldots + \frac{s(s-1)\ldots\ldots(s-n+1)}{n!} \Delta^n f_0$$

When $s = (x - x_0)/h$ and

$$E_n(x) = \binom{s}{n+1} h^{n+1} f^{(n+1)}(\xi\xi$$

9. Newton's Backward Difference Formula:

$$P_n(x) = P_n(x_n + sh) = \sum_{k=0}^{n} (-1)^k \binom{-s}{k} \nabla^k f_n$$

where $s = (x - x_n)/h$ and

$$E_n(x) = (-1)^{n+1} \frac{s(s+1)\ldots(s+n)}{(n+1)!} h^{n+1} f^{(n+1)}(\xi)$$

10. $f[x_k, \ldots, x_{k+n}] = \dfrac{1}{n!h^n} \Delta^n f_k$ and

$$f[x_{n-k}, \ldots, x_n] = \frac{1}{k!h^k} \nabla^k f(xn).$$

2.6 SOLUTIONS/ANSWERS

E1) We show that if P(x) and Q(x) are two polynomials of degree $\leq$ k which agree at the k + 1 distinct points $x_0, x_1, \ldots x_k$, then P(x) = Q(x) identically. Let ψ (x) = P(x) = Q(x). Then ψ (x) is a polynomial of degree $\leq$ k, and by Lemmal, it can be written as ψ (x) = $(x - x_0)(x - x_1)\ldots(x - x_k)$ R(x) with R(x) some polynomial. Let R(x) = $c_m x^m + = c_{m-1}x^{m-1} + \ldots + c_1 x + c_0$ such that

$c_m \neq 0$. Then $k \geq$ degree of $\psi = k + 1 + m$ which is absurd. Hence $\psi(x) = 0$ identically, so P(x) = Q(x).

E2) $x^3 - x^2 + 3x + 8,$ 18

E3) 14.6667

E4) 6.6875

E5) Let x = g(y). The Lagrange's interpolating polynomial P(y) of g(y) is given by

$$P(y) = -\frac{1}{24}(y^3 - 12y^2 + 47y - 60) + \frac{19}{4}(y^3 - 10y^2 + 29y - 20)$$
$$-\frac{49}{3}(y^3 - 9y^2 + 23y - 15) + \frac{101}{8}(y^3 - 8y^2 + 19y - 12)$$

which, on simplification gives

$P(y) = y^3 - y^2 + 1$ when y = 2, x $\approx P(2) = 5$.

E6) x = g(y)

$$P(y) = \frac{(y-1)(y-2)(y-4)}{(-3)(-4)(-6)}.(36) + \frac{(y+2)(y-2)(y-4)}{(3)(-1)(-3)}.(54)$$
$$+\frac{(y+2)(y-1)(y-4)}{(4)(1)(-2)}.(72) + \frac{(y+2)(y-1)(y-2)}{(6)(3)(2)}.(144)$$
$$x \approx P(3) = \frac{(2)(1)(-1)}{-72} \times (36) + \frac{(5)(1)(-1)}{9} \times (54)$$
$$+\frac{(5)(2)(1)}{8} \times (72) + \frac{(5)(2)(1)}{36} \times (144)$$
$$= 1 - 30 + 90 + 40 = 101$$

E7) $$f[a,b] = \frac{\frac{1}{b} - \frac{1}{a}}{b-a} = -\frac{1}{ab}$$

$f[b,c] = -\frac{1}{bc}$ and $f[c,d] = -\frac{1}{cd}$

$$f[a,b,c] = \frac{1}{abc}, f[b,c,d] = \frac{1}{bcd}$$
$$f[a,b,c,d] = \frac{f[b,c,d,] - f[a,b,c]}{d-a} = -\frac{1}{abcd}$$

E8) Divided Difference Table

x	f[x]	f[., .]	f[., ., .]		
1	1				
		2			
2	3		1		
		4		0	
3	7		1		0
		7		0	
5	21		1		
		10			
6	31				

Since third and higher order divided differences are zeros, given data represents a second degree polynomial

$$P_2(x) = f[x_0] + (x - x_0]f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

$= 1 + (x-1).2 + (x-1)(x-2).1 = x^2 - x + 1 f(4) \approx P_2(4) = 13$

$f(4) \approx P_2(4) = 13$

E9) Lagrange's interpolation polynomial

$$P(x) = \frac{1}{6}(x^3 - 9x^2 + 26x - 24) + \frac{3}{2}(x^3 - 7x^2 + 12x)$$

$$-\frac{26}{3}(x^3 - 6x^2 + 8x) + 8(x^3 - 5x^2 + 6x)$$

$= x^3 + x - 4$ on simplification.

Divided difference interpolation polynomial:

x	f[x]	f[., .]	f[., ., .]	f[., ., .,.]
0	– 4			
		5		
2	6		5	
		20		1
3	26		9	
		38		
4	64			

P(x) = – 4 +(x – 0)5 + x(x – 2)(5) + x(x – 2)(x – 3)(1)
= x^3 +x – 4
$f(1) \approx p(1) = -2$.

E10) $|E_1(\bar{x})| \le \frac{h^2}{8}M$ where M $= \max\limits_{1 \le x \le 2}$ = |f'' (x)|

and $\bar{x} \varepsilon (1,2) . f^1(x) = \frac{3}{2}x^{1/2}, f''(x) = \frac{3}{4}x^{-1/2}$.

$$\max_{1 \le x \le 2} |f''(x)| = \frac{3}{4}$$

$$|E_1(\bar{x})| \le \frac{h^2}{8} \cdot \frac{3}{4} \le 5 \cdot 10^{-6} = \frac{1}{2} \cdot 10^{-5}$$

i.e. $h^2 \le \frac{32}{2 \times 3} \cdot 10^{-5} = \frac{16}{3} 10^{-5}$ $\Rightarrow$ ***h*** = 0.0073

E11) **Backward Difference Table**

	x	f(x)	∇ f	∇^2f	∇^3f	
x_0	1	10				
			9			
x_1	2	19		12		
			21		6	
x_2	3	40		18	0	
			39		6	… 0
x_3	4	79		24	0	
			63		6	

x_4	5	142		30
			93	
x_5	6	235		

$$P(x) = f_5 + (x - x_5)\nabla f_5 + \frac{(x-x_5)(x-x_4)}{2!}\nabla^2 f_5$$

$$+\frac{(x-x_5)(x-x_4)(x-x_3)}{3!}\nabla^3 f_5 = 235 + 93(x-6) + 15(x-6)(x-5) + (x-4)(x-5)(x-6) = x^3 + 2x + 7$$

E12) **Forward Difference Table**

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
–3	–29				
		20			
–2	– 9		–12		
		8		6	
–1	– 1		– 6		0
		2		6	
0	1		0		0
		2		6	
1	3		6		0
		8		6	
2	11		12		
		20			
3	31				

$$f(x) \approx P_3(x) = f(x_0) + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{3!h^3}\Delta^3 f_0$$

$$= x^3 + x + 1 f(0.5) \approx P_3(0.5) = 1.625$$

E13) **Forward Difference Table**

x	f(x)	Δf	$\Delta^2 f$	. . .
–1	6			
		–5		
0	1		4	
		–1		0
1	0		4	
		3		0
2	3		4	
		7		0
3	10		4	
		11		
4	21			

Since third and higher order differences are zeros, f(x) represents a second order polynomial

$$f(x) \approx P_2(x) = f_0 + \frac{(x-x_0)}{h}\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!h^2}\Delta^2 f_0$$

$$= 6 + (x+1)(-5) + \frac{(x+1)(x-0)}{2}(4) = 2x^2 - 3x + 1$$

$f(2.5) \approx 6.$

E14) Backward Difference Table

x	f(x)	∇ f	∇ ²f	∇ ³f	∇ ³f
1.1	1.3357				
		0.1738			
1.2	1.5095		0.0151		
		0.1889		0.0019	
1.3	1.6984		0.0170		0.0002
		0.2059		0.0021	
1.4	1.9043		0.0910		
		0.2050			
1.5	2.1293				

Here $x_n = 1.5, x = 1.45, h = 0.1 \therefore s = \frac{x - x_n}{h} = \frac{1.45 - 1.5}{0.1} = -0.5$

$$f(x) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n$$

$$+ \frac{s(s+1)(s+2)(s+3)\Delta^4 f_n}{4!}$$

$$= 2.1293 - 0.1125 - 0.00239 - 0.00013 - 0.0000075$$

$$= 2.01427 \approx 2.0143$$

E15) i) 0

ii) $a_3\ 3!\ h^2$ (Recall that $f(x_{0,\ldots\ldots}x_i] = \frac{f^i(\xi)}{i!}$

iii) $a_3\ 3!h^3$ and consider x fixed)

E16) Prove this by induction.

E17) $|E_1(x)| \le \frac{h^2}{8}M_2$ where $M_2 = \underset{3\le x\le 4}{Max}\ |f''(x)|$

We have $f'(x) = 4x^3$, $f''(x) = 12x^2$ $M_2 = 12 \times 16 = 192$

We have $\frac{h^2}{8} \times 192 \le 10^{-4} \Rightarrow h \le \frac{1}{\sqrt{24}}10^{-2} \approx 0.002$

E18) **Divided Difference Table**

x	f[.]	f[., .]	f[., ., .]	f[[., ., ., .]	f[[., ., ., .,.]
–4	1245				
		–404			
– 1	33		94		
		– 28		– 14	

0	5		10		3
		2		13	
2	9		88		
		442			
5	1355				

$$P_4(x) = 1245 - 404(x+4) + 94(x+4)(x+1) - 14(x+4)(x+1)x + 3(x+4)(x+1)x(x-2)$$

$f(1) \approx P_4(1) = -5$

E19) **Forward Difference Table**

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$
3	6			
		18		
5	24		–4	
		14		60
7	38		56	
		70		
9	108			

$$s = \frac{x - x_0}{h} = \frac{4-3}{2} = \frac{1}{2}$$

$$f(x_0 + sh) \approx P_4(x_0 + sh) = f_0 + \frac{1}{2}\Delta f_0 + \frac{1}{8}\Delta^2 f_0 + \frac{1}{16}\Delta^3 f_0$$

$$\text{i.e., } f(4) = 6 + \frac{1}{2}\cdot 18 - \frac{1}{8}\cdot -4 + \frac{1}{16}.60$$

$$= 6 + 9 + \frac{1}{2} + \frac{15}{4} = 19.25$$

E20) We have $f[x_0, \ldots\ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi\xi) \in (\min x_i, \max x_i)$

$$f[1,2,3] = \frac{1}{2!} f''(\xi\xi = \frac{1}{2!} \frac{d^2}{dx^2} \frac{(ax^2 + bc + c)}{x = \xi} = \frac{1}{2}.2a = a$$