UNIT 1 THE FIRST-ORDER PREDICATE LOGIC (FOPL)

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1.0 INTRODUCTION

In the previous unit, we discussed how propositional logic helps us in solving problems. However, one of the major problems with propositional logic is that, sometimes, it is unable to capture even elementary type of reasoning or argument as represented by the following statements:

Every man is mortal.
Raman is a man.
Hence, he is mortal.

The above reasoning is intuitively correct. However, if we attempt to simulate the reasoning through Propositional Logic and further, for this purpose, we use symbols P, Q and R to denote the statements given above as:

P: Every man is mortal,
Q: Raman is a man,
R: Raman is mortal.

Once, the statements in the argument in English are symbolised to apply tools of propositional logic, we just have three symbols P, Q and R available with us and apparently no link or connection to the original statements or to each other. The connections, which would have helped in solving the problem become invisible. In Propositional Logic, there is no way, to conclude the *symbol* R from the *symbols* P and Q. However, as we mentioned earlier, even in a natural language, the conclusion of the *statement* denoted by R from *the statements* denoted by P and Q is obvious. Therefore, we search for some **symbolic** system of reasoning that helps us in discussing *argument forms* of the above-mentioned type, in addition to those forms which can be discussed within the framework of propositional logic. **First Order Predicate Logic (FOPL)** is the most well-known symbolic system for the pourpose.

The symbolic system of FOPL treats an atomic statement *not as an indivisible unit*. Rather, FOPL not only treats an atomic statement divisible into subject and predicate but even further deeper structures of an atomic statement are considered in order to handle larger class of arguments. How and to what extent FOPL symbolizes and establishes *validity/invalidity* and *consistency/inconsistency* of *arguments* is the subject matter of this unit.

In addition to the baggage of concepts of propositional logic, FOPL has the following additional concepts: terms, predicates and quantifiers. These concepts will be introduced at appropriate places.

In order to have a glimpse at how FOPL extends propositional logic, let us again discuss the earlier argument.

Every man is mortal. Raman is a man.
Hence, he is mortal.

In order to derive the validity of above simple argument, in stead of looking at an atomic statement as indivisible, to begin with, we divide each statement into *subject* and *predicate*. The two predicates which occur in the above argument are:
'*is mortal'* and *'is man'*.

Let us use the notation
IL: *is_mortal* and
IN: *is_man*.

In view of the notation, the argument on para-phrasing becomes:
For all x, if IN (x) then IL (x).
IN (Raman).
Hence, IL (RAMAN)

More generally, relations of the form *greater-than (x, y)* denoting the phrase 'x is greater than y', *is_brother_ of* (x, y) denoting '*x is brother of y,' Between (x, y, z)* denoting the phrase that '*x lies between y and z*', and *is_tall (x)* denoting '*x is tall*' are some **examples of predicates**. The variables x, y, z etc which appear in a predicate are called **parameters** of the predicate.

The parameters may be given some appropriate values such that after substitution of appropriate value from all possible values of each of the variables, the predicates become *statements*, for each of which we can say whether it is 'True' or it is 'False'.

For example, for the predicate *greater-than (x, y),* if x is given value 3 then we obtain *greater-than (3, y),* for which still it is not possible to tell whether it is True or False. Hence, '*greater-than (3, y)*' is also a predicate. Further, if the variable y is given value 5 then we get *greater (3, 5)* which , as we known, is False. Hence, it is possible to give its Truth-value, which is *False* in this case. Thus, from the *predicate greater-than (x, y)*, we get the *statement greater-than (3, 5)* by assigning values 3 to the variable x and 5 to the variable y. These values 3 and 5 are called parametric values or *arguments* of the predicate *greater-than*.

(Please note 'argument of a function/predicate' is a mathematical concept, different from logical argument)

Similarly, we can represent the phrase *x likes y* by the *predicate LIKE (x, y)*. Then *Ram likes Mohan* can be represented by the statement *LIKE (RAM, MOHAN).*

Also *function symbols* can be used in the first-order logic. For example, we can use *product (x, y)* to denote $x * y$ and *father (x)* to mean the *'father of x'*. The statement: *Mohan's father loves Mohan* can be symbolised as *LOVE (father (Mohan), Mohan)*. Thus, we need not know name of father of Mohan and still we can talk about him. A function serves such a role.

We may note that LIKE (Ram, Mohan) and LOVE (father (Mohan),Mohan) are atoms or atomic statements of PL, in the sense that, one can associate a truth-value *True* or

False with each of these, and each of these does not involve a logical operator like ~, $\wedge$, $\vee$, $\rightarrow$ or $\leftrightarrow$.

Summarizing in the above discussion, *LIKE (Ram, Mohan)* and *LOVE (father (Mohan) Mohan)* are **atoms;** where as GREATER, LOVE and LIKE are **predicate symbols;** x and y are **variables and 3**, *Ram* and *Mohan* are **constants;** and *father* and *product* are **function symbols**.

1.1 OBJECTIVES

After studying this unit, you should be able to:

- explain why FOPL is required over and above PL;
- define, and give appropriate examples for, each of the new concepts required for FOPL including those of quantifier, variable, constant, term, free and bound occurrences of variables, closed and open wff;
- check consistency/validity, if any, of closed formulas;
- reduce a given formula of FOPL to normal forms: Prenex Normal Form (PNF) and (Skolem) Standard Form, and
- use the tools and techniques of FOPL, developed in the unit, to solve problems requiring logical reasoning.

1.2 SYNTAX OF FOPL

In the introduction of the unit, we had a bird's eye view of:

(i) How analysis of an atomic statement of PL can and should be carried out.
(ii) What are the new concepts and terms that are required to discuss the subject matter of FOPL.
(iii) How (i) and (ii) above will prove useful in solving problems using FOPL over and above the set of problems solvable using only PL.

Also, in the introduction to the previous unit, we mentioned that a symbolic logic is a formal language and hence, all the rules for building constructs of the language must be specified clearly and unambiguously.

Next, we discuss how various constructs are built up from the alphabet.

For this purpose, from the discussion in the Introduction, we need at least the following concepts.

i) ***Individual symbols or*** *constant symbols*: These are usually names of objects, such as Ram, Mohan, numbers like 3, 5 etc.

ii) **Variable symbols**: These are usually lowercase unsubscripted or subscripted letters, like x, y, z, x_3.

iii) **Function symbols:** These are usually lowercase letters like f, g, h,….or strings of lowercase letters such as *father* and *product*.

iv) **Predicate symbols***:* These are usually uppercase letters like P, Q, R,….or strings of lowercase letters such as *greater-than*, *is_tall* etc.

A function symbol or predicate symbol takes a fixed number of arguments. If a *function symbol f* takes *n* arguments, *f* is called an *n-place function symbol*. Similarly, if a predicate symbol Q takes *m* arguments, P is called an m-*place predicate symbol*. For example, father is a one-place *function* symbol, and GREATER and LIKE are two-place *predicate* symbols. However, *father-of* in father_of (x, y) is a, *two place predicate* symbol.

The symbolic representation of an argument of a function or a predicate is called a *term* where a **term** is defined recursively as follows:

i) A variable is a term.
ii) A constant is a term.
iii) If *f* is an n-place function symbol, and $t_1 \ldots t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.
iv) Any term can be generated only by the application of the rules given above.

For example: Since, *y* and 3 are both terms and *plus* is a two-place function symbol, *plus (y, 3)* is a term according to the above definition.

Furthermore, we can see that *plus (plus (y, 3), y)* and *father (father (Mohan))* are also terms; the former denotes $(y + 3) + y$ and the later denotes *grandfather of Mohan*.

A predicate can be thought of as a function that maps a list of constant arguments to T or F. For example, GREATER is a predicate with GREATER (5, 2) as T, but GREATER (1, 3) as F.

We already know that in PL, an atom or atomic statement is an indivisible unit for representing and validating arguments. Atoms in PL are denoted generally by symbols like P, Q, and R etc. But in FOPL,

Definition: An Atom is

(i) either an atom of Propositional Logic, or
(ii) is obtained from an n-place predicate symbol P, and terms $t_1, \ldots t_n$ so that $P(t_1, \ldots, t_n)$ is an atom.

Once, the atoms are defined, by using the logical connectives defined in Propositional Logic, and assuming having similar meaning in FOPL, we can build complex formulas of FOPL. Two special symbol $\forall$ and $\exists$ are used to denote qualifications in FOPL. The symbols $\forall$ and $\exists$ are called, respectively, the *universal* quantifier and *existential* quantifier. For a variable x, $(\forall x)$ is read as *for all x,* and $(\exists x)$ is read as *there exists an x.* Next, we consider some examples to illustrate the concepts discussed above.

In order to symbolize the following statements:

i) There exists a number that is rational.
ii) Every rational number is a real number
iii) For every number *x*, there exists a number *y*, which is greater than *x*.

let us denote *x is a rational number* by Q(x), *x is a real number* by R(x), and *x is less than y* by *LESS(x, y)*. Then the above statements may be symbolized respectively, as

(i) $(\exists x)\ Q(x)$
(ii) $(\forall x)\ (Q(x) \rightarrow R(x))$
(iii) $(\forall x)\ (\exists y)$ LESS(x, y).

Each of the expressions (i), (ii), and (iii) is called a **formula** or a well-formed formula or **wff.**

Next, we discuss three new concepts, viz **Scope** of occurrence of a quantified variable, Bound occurrence of a quantifier variable or quantifier and *Free occurrence* of a variable.

Before discussion of these concepts, we should know the *difference between a **variable** and **occurrence of a variable** in a quantifier expression*.

The variable *x* has THREE occurrences in the formula
$(\exists x)\ Q(x) \rightarrow P(x, y)$.

Also, the variable *y* has only one occurrence and the variable *z* has zero occurrence in the above formula. Next, we define the three concepts mentioned above.

Scope of an occurrence of a quantifiers is the smallest but complete formula following the quantifier sometimes delimited by pair *f* parentheses. For example, Q(x) is the scope of $(\exists x)$ in the formula
$(\exists x)\ Q(x) \rightarrow P(x, y)$.
But the scope of $(\exists x)$ in the formula: $(\exists x)\ (Q(x) \rightarrow P(x, y))$ is $(Q(x) \rightarrow P(x, y))$.
Further in the formula:
$(\exists x)\ (P(x) \rightarrow Q(x, y)) \wedge (\exists x)\ (P(x) \rightarrow R(x, 3))$,
the scope of **first** occurrence of $(\exists x)$ is the formula $(P(x) \rightarrow Q\ (x, y)$ and the scope of **second** occurrence of $(\exists x)$ is the formula
$(P(x) \rightarrow R(x, 3))$.
As another example, the scope of the only occurrence of the quantifier $(\forall y)$ in
$(\exists x)\ ((\ P(x) \rightarrow Q(x) \leftrightarrow (\forall y)\ (Q\ (x) \rightarrow R\ (y)))$ is $(\ Q\ (x) \rightarrow R(y))$. But the scope of the only occurrence of the existential variable $(\exists x)$ in the same formula is the formula:
$(P(x) \rightarrow Q(x))\ P\ \ \leftrightarrow (\forall y)\ (Q\ (x) \rightarrow R(y))$

*An **occurrence** of a variable in a formula* is **bound** if and only if the occurrence is within the scope of a quantifier employing the variable, or is the occurrence in that quantifier. An occurrence of a variable in a formula is **free** if and only if this occurrence of the variable is not bound.

Thus, in the formula $(\exists x)\ P(x, y) \rightarrow Q\ (x)$, there are three occurrences of *x*, out of which first two occurrences of *x* are *bound*, where, the last occurrence of *x is free*, because scope of $(\exists x)$ in the above formula is P(x, y). The only occurrence of *y* in the formula is free. Thus, *x* is both a bound and a free variable in the above formula and y is only a free variable in the formula so far, we talked of an *occurrence of a variable* as free or bound. Now, we talk of (only) a *variable* as free or bound. *A variable is **free*** in a formula if at least one occurrence of it is free in the formula. A variable is **bound** in a formula if at least one occurrence of it is bound.

It may be noted that a variable can *be **both free and bound** in a formula*. In order to further elucidate the concepts of *scope, free and bound occurrences of a variable,* we consider a similar but different formula for the purpose:
$(\exists x)\ (P(x, y) \rightarrow Q(x))$.
In this formula, *scope* of the only occurrence of the quantifier $(\exists x)$ is the whole of the rest of the formula, viz. scope of $(\exists x)$ in the given formula is $(P(x, y) \rightarrow Q\ (x))$
Also, all three occurrence of variable *x* are bound. The only occurrence of *y* is free.

Remarks: It may be noted that a bound variable *x* is just a **place holder** or a **dummy variable** in the sense that all occurrences of a bound variable *x* may be replaced by another free variable say *y*, which does not occur in the formula. However, once, *x* is replaced by *y* then *y* becomes bound. For example, $(\forall x)\ (f\ (x))$ is the same as $(\forall y)\ f\ (y)$. It is something like

$$\int_1^2 x^2 dx = \int_1^2 y^2 dy = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

Replacing a bound variable *x* by another variable *y* under the restrictions mentioned above is called **Renaming of a variable x**

Having defined an atomic formula of FOPL, next, we consider the definition of a general formula formally in terms of atoms, logical connectives, and quantifiers.

Definition A well-formed formula, **wff** a just or formula in FOPL is defined recursively as follows:

i) An atom or atomic formula is a *wff.*
ii) If E and G are wff, then each of ~ (E), (E ∨ G), (E ∧ G), (E → G), (E ↔ G) is a **wff.**
iii) If E is a *wff* and x is a free variable in E, then (∀x)E is a wff.
iv) A *wff* can be obtained only by applications of (i), (ii), and (iii) given above.

We may drop pairs of parentheses by agreeing that quantifiers have the least scope. For example, (∃x) P(x, y) → Q(x) stands for
((∃x) P(x, y)) → Q(x)

We may note the following two cases of translation:

(i) for all x, P(x) is Q(x) is translated as
(∀x) (P(x) → Q(x))
(the other possibility (∀x) P(x) ∧ Q(x) is not valid.)

(ii) for some x, P(x) is Q (x) is translated as (∃x) P(x) ∧ Q(x)
(the other possibility (∃x) P(x) → Q(x) is not valid)

Example

Translate the statement: *Every man is mortal. Raman is a man. Therefore, Raman is mortal.*

As discussed earlier, let us denote "*x is a man*" by MAN (x), and "*x is mortal*" by MORTAL(x). Then "*every man is mortal*" can be represented by

(∀x) (MAN(x) → MORTAL(x)),

"Raman is a man" by

MORTAL(Raman).

The whole argument can now be represented by

(∀x) (MAN(x) → MORTAL(x)) ∧ MAN (Roman)→ MORTAL (Roman).
as a single statement.
In order to further explain symbolisation let us recall the axioms of natural numbers:

(1) For every number, there is one and only one immediate successor,
(2) There is no number for which 0 is the immediate successor.
(3) For every number other than 0, there is one and only one immediate predecessor.

Let the *immediate successor* and *predecessor of x*, respectively be denoted by f(x) and g(x).

Let E (*x*, *y)* denote *x is equal to y*. Then the axioms of natural numbers are represented respectively by the formulas:

(i) $(\forall x)\ (\exists y)\ (E(y, f(x)) \wedge (\forall z)\ (E(z, f(x)) \rightarrow E(y, z)))$
(ii) $\sim ((\exists x)\ E(0, f(x)))$ and
(iii) $(\forall x)\ (\sim E(x, 0) \rightarrow ((y)\exists, g(x)) \wedge (\forall z)\ (E(z, g(x)) \rightarrow E(y, z))))).$

From the semantics (for meaning or interpretation) point of view, the **wff of FOPL** may be divided into two categories, each consisting of

(i) wffs, in each of which, **all** occurrences of variables are **bound**.
(ii) wffs, in each of which, at **least one** occurrence of a variable is **free**.

The wffs of FOPL in which there is no occurrence of a free variable, are like *wffs* of PL in the sense that we can call each of the wffs as **True, False, consistent, inconsistent, valid, invalid etc**. Each such a formula is called **closed formula**. However, when a wff involves a free occurrence, then it is not possible to call such a wff as True, False etc. **Each of such a formula is called an open formula**.

For example: Each of the formulas: greater (x, y), greater (x, 3), $(\forall y)$ greater (x, y) has one free occurrence of variable x. Hence, each is an **open** formula.
Each of the formulas: $(\forall x)\ (\exists y)$ greater (x, y), $(\forall y)$ greater (y, 1), greater (9, 2), does not have free occurrence of any variable. Therefore each of these formulas is a closed formula.

Next we discuss some equivalences, and inequalities

The following equivalences hold for any two formulas P(x) and Q(x):
(i) $(\forall x)\ P(x) \wedge (\forall x)\ Q(x) = (\forall x)\ (P(x) \wedge Q(x))$
(ii) $(\exists x)\ P(x) \vee (\exists x)\ Q(x) = (\exists x)\ (P(x) \vee Q(x)$
But the following inequalities hold, in general:
(iii) $(\forall x)\ (P(x) \vee Q(x) \neq (\forall x)\ P(x) \vee (\forall x)\ Q(x)$
(iv) $(\exists x)\ (P(x) \wedge Q(x) \neq (\exists x)\ P(x) \wedge (\exists x)\ Q(x)$

We justify (iii) & (iv) below:
Let P(x): x is odd natural number,
Q(x): x is even natural number.
Then L.H.S of (iii) above states *for every natural number it is either odd or even,* **which is correct**. But R.H.S of (iii) states that *every natural number is odd* or *every natural number is even*, **which is not correct.**
Next, L.H.S. of (iv) states that: there is a natural number which is both even and odd, **which is not correct.** However, R.H.S. of (iv) says *there is an integer which is odd and there is an integer which is even,* correct.

Equivalences involving Negation of Quantifiers

(v) $\sim (\forall x)\ P(x) = (\exists x) \sim P(x)$
(iv) $\sim (\exists x)\ P(x) = (\forall x) \sim P(x)$

Examples: For each of the following closed formula, Prove
(i) $(\forall x)\ P(x) \wedge (\exists y) \sim P(y)$ is inconsistent.
(ii) $(\forall x)\ P(x) \rightarrow (\exists y)\ P(y)$ is valid

Solution: (i) Consider
$(\forall x)\ P(x) \wedge (\exists y) \sim P(y)$
$= (\forall x)\ P(x) \wedge \sim (\forall y)\ P(y)$ (*taking negation out*)

But we know for each bound occurrence, a variable is dummy, and can be replaced in the whole scope of the variable uniformly by another free variable. Hence,

R = (∀x) P(x) ∧ ~ (∀x) P(x)
Each conjunct of the formula is either
True of False and, hence, can be thought of as a formula of PL, in stead of formula of FOPL, Let us replace (∀x) (P(x) by Q , a formula of PL.

R = Q ∧ ~ Q = False
Hence, the proof.

(ii) Consider
(∀x) P(x) → (∃y) P(y)
Replacing '→' we get
= ~ (∀x) P(x) ∨ (∃y) P(y)
= (∃x) ~ P(x) ∨ (∃y) P(y)
= (∃x) ~ P(x) ∨ (∃x) P(x) *(renaming x as y in the second disjunct)*

In other words,
= (∃x) (~ P(x) ∨ P(x)) *(using equivalence)*
The last formula states: *there is at least one element say b, for ~ P(b) ∨ P(b) holds i.e., for b, either P(b) is False or P(b) is True.*
But, as P is a predicate symbol and b is a constant ~ P(b) ∨ P(b) must be True. Hence, the proof.

Ex. 1 Let P(x) and Q(x) represent "x is a rational number" and "x is a real number," respectively. Symbolize the following sentences:

(i) Every rational number is a real number.
(ii) Some real numbers are rational numbers.
(iii) Not every real number is a rational number.

Ex. 2 Let C(x) mean "x is a used-car dealer," and H(x) mean "x is honest." Translate each of the following into English:

(i) (∃x)C(x)
(ii) (∃x) H(x)
(iii) (∀x)C(x) → ~ H (x))
(iv) (∃x) (C(x) ∧ H(x))
(v) (∃x) (H(x) → C(x)).

Ex. 3 Prove the following:

(i) P(a) → ~ ((∃x) P(x)) is consistent.
(ii) (∀x) P(x) ∨ ((∃y) ~ P(y)) is valid.

1.3 PRENEX NORMAL FORM

In order to facilitate problem solving through PL, we discussed two normal forms, viz, the conjunctive normal form **CNF** and the disjunctive normal form **DNF**. In **FOPL**, there is a normal form called the **prenex normal form**. The use of a prenex normal form of a formula simplifies the proof procedures, to be discussed.
Definition A formula G in FOPL is said to be in a **prenex normal form** if and only if the formula G is in the form

$(Q_1x_1)\ldots.(Q_n\ x_n)$ P

where each (Q_ix_i), for i = 1, ….,n, is either $(\forall x_i)$ or $(\exists x_i)$, and P is a quantifier free formula. The expression $(Q_1x_1)\ldots.(Q_n\ x_n)$ is called the **prefix** and P is called the **matrix of the formula G**.

Examples of some formulas in prenex normal form:

(i) $(\exists x)\ (\forall y)\ (R(x, y) \lor Q(y))$, $(\forall x)\ (\forall y)\ (\sim P(x, y) \to S(y))$,
(ii) $(\forall x)\ (\forall y)\ (\exists z)\ (P(x, y) \to R\ (z))$.

Next, we consider a method of transforming a given formula into a prenex normal form. For this, first we discuss equivalence of formulas in FOPL. Let us recall that two formulas E and G are **equivalent**, denoted by **E = G**, *if and only if the truth values of F and G are identical under every interpretation.* The pairs of equivalent formulas given in Table of equivalent Formulas of previous unit are still valid as these are quantifier–free formulas of FOPL. However, there are pairs of equivalent formulas of FOPL that contain quantifiers. Next, we discuss these additional pairs of equivalent formulas. We introduce some notation specific to FOPL: *the symbol G denote a formula that does not contain any free variable x.* Then we have the following pairs of equivalent formulas, where Q denotes a quantifier which is either ∀ or ∃. Next, we introduce four laws for **pairs of equivalent formulas**.

In the rest of the discussion of FOPL, P[x] is used to denote the fact that *x* is a free variable in the formula P, for example, P[x] = $(\forall y)$ P (x, y). Similarly, R [x, y] denotes that variables *x* and *y* occur as free variables **in the formula** R Some of these equivalences, we have discussed earlier.

Then, the following laws involving quantifiers hold good in FOPL
(i) $(Qx)\ P[x] \lor G = (Qx)\ (P[x] \lor G)$.
(ii) $(Qx)\ P[x] \land G = (Qx)\ (P[x] \land G)$.

In the above two formulas, Q may be either ∀ or ∃.

(iii) $\sim ((\forall x)\ P[x]) = (\exists x)\ (\sim P[x])$.
(iv) $\sim ((\exists x)\ P[x]) = (\forall x)\ (\sim P[x])$.
(v) $(\forall x)\ P[x] \land (\forall x)\ H[x] = (\forall x)\ (P[x] \land H[x])$.
(vi) $(\exists x)\ P[x] \lor (\exists x)\ H[x] = (\exists x)\ (P[x] \lor H[x])$.

That is, the universal quantifier ∀ and the existential quantifier ∃ can be distributed respectively over ∧ and ∨.

But we must be careful about *(we have already mentioned these inequalities)*
(vii) $(\forall x)\ E[x] \lor (\forall x)\ H[x] \neq (\forall x)\ (P[x] \lor H[x])$ and
(viii) $(\exists x)\ P[x] \land (\exists x)\ H[x] \neq (\exists x)\ (P[x] \land H[x])$

Steps for Transforming an FOPL Formula into Prenex Normal Form

Step 1 Remove the connectives '↔' and '→' using the equivalences
$P \leftrightarrow G = (P \to G) \land (G \to P)$
$P \to G = \sim P \lor G$

Step 2 Use the equivalence to remove even number of ~'s
$\sim (\sim P) = P$

Step 3 Apply De Morgan's laws in order to bring the negation signs immediately before atoms.

$$\sim (P \vee G) = \sim P \wedge \sim G$$
$$\sim (P \wedge G) = \sim P \vee \sim G$$

and the quantification laws

$$\sim ((\forall x)\ P[x]) = (\exists x)\ (\sim P[x])$$
$$\sim ((\exists x)\ P\ [x]) = (\forall x)\ (\sim F[x])$$

Step 4 rename bound variables **if necessary**

Step 5 Bring quantifiers to the left before any predicate symbol appears in the formula. This is achieved by using (i) to (vi) discussed above.

We have already discussed that, if all occurrences of a bound variable are replaced uniformly throughout by another variable not occurring in the formula, then the equivalence is preserved. Also, we mentioned under (vii) that $\forall$ does not distribute over $\wedge$ and under (viii) that $\exists$ does not distribute over $\vee$. In such cases, in order to bring quantifiers to the left of the rest of the formula, we may have to first rename one of bound variables, say x, may be renamed as z, which does not occur either as free or bound in the other component formulas. And then we may use the following equivalences.

$$(Q1\ x)\ P[x] \vee (Q2\ x)\ H[x] = (Q1\ x)\ (Q2\ z)\ (P[x] \vee H[z])$$
$$(Q3\ x)\ P[x] \wedge (Q4\ x)\ H[x] = (Q3\ x)\ (Q4\ z)\ (P[x] \wedge H[z])$$

Example: Transform the following formulas into prenex normal forms:

(i) $(\forall x)\ (Q(x) \rightarrow (\exists x)\ R\ (x, y))$
(ii) $(\exists x)\ (\sim (\exists y)\ Q(x, y) \rightarrow ((\exists z)\ R(z) \rightarrow S\ (x)))$
(iii) $(\forall x)\ (\forall y)\ ((\exists z)\ Q(z, y, z) \wedge ((\exists u)\ R\ (x, u) \rightarrow (\exists v)\ R\ (y, v)))$.

Part (i)

Step 1: *By removing '$\rightarrow$', we get*

$$(\forall x)\ (\sim Q\ (x) \vee (\exists x)\ R\ (x, y))$$

Step 2: *By renaming x as z in ($\exists$x) R (x, y) the formula becomes*

$$(\forall x)\ (\sim Q\ (x) \vee (\exists z)\ R\ (z, y))$$

Step 3: *As ~ Q(x) does not involve z, we get*

$$(\forall x)\ (\exists z)\ (\sim Q\ (x) \vee R\ (z, y))$$

Part (ii)

$(\exists x)\ (\sim (\exists y)\ Q\ (x, y) \rightarrow ((\exists z)\ R\ (z) \rightarrow S\ (x)))$

Step 1: Removing outer '$\rightarrow$' we get

$$(\exists x)\ (\sim (\sim ((\exists y)\ Q\ (x, y))) \vee ((\exists z)\ R\ (z) \rightarrow S\ (x)))$$

Step 2: Removing inner '$\rightarrow$' , and simplifying ~ (~ ( )) we get

$$(\exists x)\ ((\exists y)\ Q\ (x, y) \vee (\sim ((\exists z)\ R(z)) \vee S\ (x)))$$

Step 3: Taking '~' inner most, we get

$$(\exists x)\ (\exists y)\ Q\ (x, y) \vee ((\forall z) \sim R(z) \vee S(x)))$$

As first component formula Q (x, y) does not involve z and S(x) does not involve both y and z and ~ R(z) does not involve y. Therefore, we may take out ($\exists$ y) and ($\forall$z) so that, we get

$(\exists x)\ (\exists y)\ (\forall z)\ (Q\ (x, y) \vee (\sim R(z) \vee S\ (x)\ )$, which is the required formula in prenex normal form.

Part (iii)
$(\forall x)\ (\forall y)\ ((\exists z)\ Q\ (x, y, z) \wedge ((\exists u)\ R\ (x, u) \rightarrow (\exists v)\ R\ (y\ v)))$
Step 1: Removing '→', we get
$(\forall x)\ (\forall y)\ ((\exists z)\ Q\ (x, y, z) \wedge (\sim ((\exists u)\ R\ (x, u)) \vee (\exists v)\ R\ (y, v)))$

Step 2: Taking '~' inner most, we get
$(\forall x)\ (\forall y)\ ((\exists z)\ Q\ (x, y, z) \wedge ((\forall u) \sim R\ (x, u) \vee (\exists v)\ R\ (y, v)))$

Step 3: As variables z, u & v do not occur in the rest of the formula except the formula which is in its scope, therefore, we can take all quantifiers outside, preserving the order of their occurrences, Thus we get
$(\forall x)\ (\forall y)\ (\exists z)\ (\forall u)\ (\exists v)\ (Q\ (x, y, z) \wedge (\sim R\ (x, u) \vee R\ (y, v)))$

Ex: 4 (i) Transform the formula $(\forall x)\ P(x) \rightarrow (\exists x)\ Q(x)$ into prenex normal form.

(ii) Obtain a prenex normal form for the formula
$(\forall x)\ (\forall y)\ ((\exists z)\ (P(x, y) \wedge P(y, z)) \rightarrow (\exists u)\ Q\ (x, y, u))$

1.4 (SKOLEM) STANDARD FORM

A further refinement of Prenex Normal Form (PNF) called (Skolem) Standard Form, is the basis of problem solving through Resolution Method. The Resolution Method will be discussed in the next unit of the block.

The **Standard Form of a formula of FOPL is** obtained through the following three steps:

(1) The given formula should be converted to Prenex Normal Form (PNF), and then
(2) Convert the Matrix of the PNF, i.e, quantifier-free part of the PNF into conjunctive normal form
(3) Skolomization: Eliminate the existential quantifiers using skolem constants and functions

Before illustrating the process of conversion of a formula of FOPL to Standard Normal Form, through examples, we discuss briefly skolem functions.

Skolem Function

We in general, mentioned earlier that $(\exists x)\ (\forall y)\ P(x,y) \neq\ (\forall y)\ (\exists x)\ P(x,y)$…….(1)

For example, if P(x,y) stands for the relation 'x>y' in the set of integers, then the L.H.S. of the inequality (i) above states: *some (fixed) integer (x) is greater than all integers (y)*. This statement is False.

On the other hand, R.H.S. of the inequality (1) states: *for each integer y, there is an integer x so that x>y*. This statement is True.

The difference in meaning of the two sides of the inequality arises because of the fact that on L.H.S. *x* in (∃x) is independent of *y* in (∀y) **whereas** on R.H.S *x* of dependent on *y*. In other words, *x* on L.H.S. of the inequality can be replaced by some constant say 'c' whereas on the right hand side *x* is some function, say, f(y) of *y*.

Therefore, the two parts of the inequality (i) above may be written as
LH.S. of (1) $= (\exists x)\ (\forall y)\ P\ (x,y) = (\forall y)\ P(c,y)$,

Dropping x because there is no x appearing in ($\forall$y) P(c,y)

R.H.S. of (1) = $(\forall y)\ (\exists x)\ P(f(y),y) = (\forall y)\ P(f(y), y)$

The above argument, in essence, explains what is meant by each of the terms viz. *skolem constant, skolem function and skolomisation.*

The constants and functions which replace existential quantifiers are respectively called **skolem constants and skolem functions.** The process of replacing all existential variables by skolem constants and variables is called **skolemisation**.

A form of a formula which is obtained after applying the steps for

(i) reduction to PNF and then to
(ii) CNF and then
(iii) applying skolomization is called **Skolem Standard Form** or just **Standard Form.**

We explain through examples, the skolomisation process after PNF and CNF have already been obtained.

Example: Skolomize the following:

(i) $(\exists x_1)\ (\exists x_2)\ (\forall y_1)\ (\forall y_2)(\exists x_3)(\forall y_3)\ P(x_1, x_2, x_3, y_1, y_2, y_3)$

(ii) $(\exists x_1)(\forall y_1)(\exists x_2)(\forall y_2)\ (\exists x_3)P(x_1, x_2, x_3, y_1, y_2\ ) \wedge (\exists x_1)(\forall y_3)(\ \exists x_2)\ (\forall y_4)Q(x_1, x_2, y_3, y_4)$

Solution (i) As existential quantifiers x_1 and x_2 precede all universal quantifiers, therefore, x_1 and x_2 are to be replaced by *constants*, but by distinct constants, say by 'c' and 'd' respectively. As existential variable x_3 is preceded by universal quantifiers y_1 and y_2, therefore, x_3 is replaced by some function $f(y_1, y_2)$ of the variables y_1 and y_2. After making these substitutions and dropping universal and existential variables, we get the skolemized form of the given formula as
$(\forall y_1)\ (\forall y_2)\ (\forall y_3)\ (c, d, f\ (y_1, y_2), y_1, y_2, y_3)$.

Solution (ii) As a first step we must bring all the quantifications in the beginning of the formula through Prenex Normal Form reduction. Also,

$$(\exists x)\ldots P(x,\ldots) \wedge (\exists x)\ldots.\ Q\ (x,\ldots.) \neq (\exists x)\ (\ldots.P(x) \wedge \ldots Q(x,\ldots.),$$

therefore, we rename the second occurrences of quantifiers $(\forall x_1)$ and $(\forall x_2)$ by renaming these as x_5 and x_6. Hence, after renaming and pulling out all the quantifications to the left, we get

$$(\exists x_1)\ (\forall y_1)\ (\exists x_2)\ (\forall y_2)\ (\exists x_3)\ (\exists x_5)\ (\forall y_3)\ (\exists x_6)\ (\forall y_4)$$
$$(P(x_1, x_2, x_3, y_1, y_2) \wedge Q\ (x_5, x_6, y_3, y_4)$$

Then the existential variable x_1 is independent of all the universal quantifiers. Hence, x_1 may be replaced by a constant say, 'c'. Next x_2 is preceded by the universal quantifier y_1 hence, x_2 may be replaced by $f\ (y_1)$. The existential quantifier x_3 is preceded by the universal quantifiers y_1 and y_2. Hence x_3 may be replaced by $g\ (y_1, y_2)$. The existential quantifier x_5 is preceded by again universal quantifier y_1 and y_2. In other words, x_5 is also a function of y_1 and y_2. But, we have to use a different function symbol say h and replace x_5 by $h\ (y_1, y_2)$. Similarly x_6 may be replaced by $j\ (y_1, y_2, y_3)$.

Thus, (Skolem) Standard Form becomes
$(\forall y_1)\ (\forall y_2)\ (\forall y_3)\ (P\ (c, f(y_1), g(y_1, y_2), y_1, y_2) \wedge Q\ (h\ (y_1, y_2), j\ (y_1, y_2, y_3)))$.

Ex 5. Obtain a (skolem) standard form for each of the following formula:

(i) $(\exists x)\ (\forall y)\ (\forall v)\ (\exists z)\ (\forall w)\ (\exists u)\ P\ (x, y, z, u, v, w)$

(ii) $(\forall x)\ (\exists y)\ (\exists z)\ ((P\ (x, y) \vee \sim Q\ (x, z)) \rightarrow R\ (x, y, z))$

1.5 APPLICATIONS OF FOPL

We have developed tools of FOPL for solving problems requiring logical reasoning.

Now, we attempt solve the problem mentioned in the introduction the unit to show insufficiency of Propositional Logic.

Example: Every man is mortal. Raman is a man. Show that Raman is mortal. The problem can be symbolized as:

(i) $(\forall x)\ (MAN(x) \rightarrow MORTAL\ (x))$.

(ii) MAN (Roman).

To show

(iii) Mortal (Raman)

Solution:

By Universal Instantiation of (i) with constant Raman, we get

(iv) Man (Raman) $\rightarrow$ Mortal (Raman)

Using Modus Ponens with (iii) & (iv) we get Mortal (Raman)

Ex: 6 No used-car dealer buys a used car for his family. Some people who buy used cars for their families are absolutely dishonest. Conclude that some absolutely dishonest people are not used-car dealers.

Ex: 7 Some patients like all doctors. No patient likes any quack. Therefore, no doctor is a quack.

When it is convenient, we shall regard a set of literals as synonymous with a clause. For example, $P \vee Q \vee \sim R = \{P, Q, \sim R\}$. A clause consisting of *r* literals is called an r-literal clause. A one-literal clause is called a unit clause. When a clause contains no literal, we call it the empty clause. Since, the empty clause is always false. We customarily denote the empty clause by **False**.

The disjunctions $(\sim P(x, f(x)) \vee R(x, f(x), g(x)))$ and $Q(x, g(x)) \vee R(x, f(x), g(x))$ of the standard form

$$\sim P(x, f(x) \vee R(x, f(x), g(x)) \wedge Q(x, g(x)) \vee R(x, f(x), g(x))$$

are clauses. A set S of clauses is regarded as a conjunction of all clauses in S, (1) with the condition that every variable that occurs in S is considered governed by a universal quantifier. By this convention, a standard form can be simply represented by a set of clauses.

For example, the standard form the above mentioned formula of (1) can be represented by the set.

$$\{P(x, f(x)) \vee R(x, f(x), g(x)), Q(x, g(x)) \vee R(x, f(x), g(x))\}.$$

Example:

Find a standard form for the following formula:

a) $\sim ((\forall x)\ P(x) \rightarrow (\exists y)\ (\forall z)\ Q(y, z))$

Solution: Removing the logical symbol '$\rightarrow$', we get

$\sim (\sim (\forall x)\ P(x)) \vee (\exists y)\ (\forall y)\ Q\ (y, z))$

Taking '~' inside, we getr

$= (\forall x)\ P(x) \wedge \sim (\exists y)\ (\forall z)\ Q\ (y\ z)$

Again taking ~ inside, we get
$= (\forall x)\ P(x) \wedge (\forall y)(\exists z) \sim Q(y, z)$

As variables *x*, *y* and *z* do not occur any where else, expect within their respective scopes, therefore, the quantifiers may be taken in the beginning of the formula without any changes. Hence, we get
$(\forall x)(\forall y)(\exists z)(P(x) \wedge \sim Q(y, z))$
which is the required standard form.

Notation: Once, the standard form is obtained, there are no existential quantifications left in the formula. Also universal quantifications are dropped, because whatever variables appear in the rest of the formula can have only universal quantification and hence universal quantifications are implied. Next, if the standard form (without any quantifiers appearing) which is by definition also in CNF, is of the form:
$C_1 \wedge C_2 \wedge \ldots.. \wedge C_n$ then, we may denote the standard form as a set $\{c_1, c_2, \ldots., c_n\}$.
For example, in the previous example, the standard normal form was
$(\forall x)(\forall y)(\forall z)(P(x) \wedge \sim Q(y, z))$, after dropping quantifiers becomes
$P(x) \wedge \sim Q(y, z)$.
The last expression can be written as { P(x), ~ Q (y, z)}.

Ex: 8 Obtain a standard form of the formula
$(\exists x)(\forall y)(\forall z)(\exists u)(\forall v)(\exists w)\ P(x, y, z, u, v, w)$.

Ex: 9 Obtain a standard form of the formula
$(\forall x)(\exists y)(\exists z)((\sim P(x, y) \wedge Q(x, z)) \vee R(x, y, z))$.

Ex: 10 Using Show that G is logical conclusion of H_1 and H_2, where

$H_1 : (\forall x)(C(x) \rightarrow (W(x) \wedge R(x)))$
$H_2 : (\exists x)(C(x) \wedge O(x))$
$G : (\exists x)(O(x) \wedge R(x))$

Ex: 11 Using resolution method, solve the following logic problem.
(i) Some patients like all doctors.
(ii) No patient likes any quake.
(iii) Therefore, no doctor is a quake.

Ex: 12 Conclude that some of the officials were drug pushers where we know the following

(i) The custom officials searched everyone who entered this country who was not a VIP.
(ii) Some of the drug pushers entered this country and they were only searched by drug pushers. (iii) No drug pusher was a VIP.
(iv) Some of the officials were drug pushers.

Ex: 13 From the given statement: *Everyone who saves money earns interest*, conclude that if there is no interest, then nobody saves money.

1.6 SUMMARY

In this unit, initially, we discuss how inadequacy of PL to solve even simple problems, requires some extension of PL or some other formal inferencing system so as to compensate for the inadequacy. First Order Predicate Logic (FOPL), is such an extension of PL that is discussed in the unit.

Next, syntax of proper structure of a formula of FOPL is discussed. In this respect, a number of new concepts including those of quantifier, variable, constant, term, free and bound occurrences of variables; closed and open wff, consistency/validity of wffs etc. are introduced.

Next, two normal forms viz. Prenex Normal Form (PNF) and Skolem Standard Normal Form are introduced. Finally, tools and techniques developed in the unit, are used to solve problems involving logical reasoning.

1.7 SOLUTIONS/ANSWERS

Ex. 1 (i) $(\forall x)\ (P\ (x) \rightarrow Q(x))$
(ii) $(\exists x)\ (P(x) \wedge Q(x))$
(iii) $\sim (\forall x)\ (\ Q\ (x) \rightarrow P(x))$

Ex. 2

(i) There is (at least) one (person) who is a used-car dealer.
(ii) There is (at least) one (person) who is honest.
(iii) All used-car dealers are dishonest.
(iv) (At least) one used-car dealer is honest.
(v) There is at least one thing in the universe, (for which it can be said that) if that something is Honest then that something is a used-car dealer

Note: the above translation is not the same as: Some no gap one honest, is a used-car dealer.

Ex 3: (i) After removal of '$\rightarrow$' we get the given formula

$= \sim P(a) \vee \sim ((\ \exists x)\ P(x))$
$= \sim P(a) \vee (\forall x)\ (\sim P(x))$

Now P(a) is an atom in PL which may assume any value T or F. On taking P(a) as F the given formula becomes T, hence, consistent.
(ii) The formula can be written
$(\forall x)\ P(x) \vee \sim (\forall x)\ (P(x))$, *by taking negation outside the second disjunct and then renaming.*
The $(\forall x)$ P(x) being closed is either T or F and hence can be treated as formula of PL. Let $\forall x$ P(x) be denoted by Q. Then the given formula may be denoted by $Q \vee \sim Q$ = True (always) Therefore, formula is valid.

Ex: 4 (i) $(\forall x)\ P(x) \rightarrow (\exists x)\ Q(x) = \sim ((\forall x)\ P(x)) \vee (\exists x)\ Q(x)$ (*by removing the connective*$\rightarrow$)

$= (\exists x)\ (\sim P(x)) \vee (\exists x)\ Q(x)$ (by taking '~' inside)

$= (\exists x)\ (\sim P(x) \vee Q(x))$ (By taking distributivity of $\exists x$ over $\vee$)

Therefore, a prenex normal form of $(\forall x)\ P(x) \rightarrow (\exists x)\ Q(x)$ is $(\exists x)\ (\sim P(x) \vee Q(x))$.

(ii) $(\forall x)\ (\forall y)\ ((\exists z)\ (P(x, y) \wedge P(y, z)) \rightarrow (\exists u)\ Q\ (x, y, u))$ (*removing the connective*$\rightarrow$)

$= (\forall x)\ (\forall y)\ (\sim ((\exists z)\ (P(x, z) \wedge P(y, z)))$
$\vee (\exists u)\ Q\ (x, y, u))$ (*using De Morgan's Laws*)
$= (\forall x)\ (\forall y)\ ((\forall z)\ (\sim P(x, z) \vee \sim P(y, z))$
$\vee (\exists u)\ Q(x, y, u))$
$= (\forall x)\ (\forall y)\ (\forall z)\ (\sim P(x, z)$

$\vee \sim P(y, z) \vee Q(x, y, u)$ (*as z and u do not occur in the rest of the formula except their respective scopes*)

Therefore, we obtain the last formula as a prenex normal form of the first formula.

Ex 5 (i) In the given formula $(\exists x)$ is not preceded by any universal quantification. Therefore, we replace the variable *x* by a (skolem) constant *c* in the formula and drop $(\exists x)$.

Next, the existential quantifier $(\exists z)$ is preceded by two universal quantifiers viz., *v* and *y*. we replace the variable *z* in the formula, by some function, say, f (v, y) and drop $(\exists z)$. Finally, existential variable $(\exists u)$ is preceded by three universal quantifiers, viz., $(\forall y)$, $(\forall y)$ and $(\forall w)$. Thus, we replace in the formula the variable *u* by, some function g(y, v, w) and drop the quantifier $(\exists u)$. Finally, we obtain the standard form for the given formula as

$(\forall y)\ (\forall v)\ (\forall w)\ P(x, y, z, u, v, w)$

(ii) First of all, we reduce the matrix to CNF.

$= (P(x, y) \vee \sim Q(x, z)) \rightarrow R(x, y, z)$

$= (\sim P(x, y) \wedge Q(x, z)) \vee R(x, y, z)$

$= (\sim P(x, y) \vee R(x, y, z)) \wedge (Q(x, z) \vee R(x, y, z))$

Next, in the formula, there are two existential quantifiers, viz., $(\exists y)$ and $(\exists z)$. Each of these is preceded by the only universal quantifier, viz. $(\forall x)$.

Thus, each variable *y* and *z* is replaced by a function of *x*. But the two functions of *x* for *y* and *z* must be different functions. Let us assume, variable, *y* is replaced in the formula by f(x) and the variable *z* is replaced by g(x). Thus the initially given formula, after dropping of existential quantifiers is in the standard form:

$(\forall x)\ ((\sim P(x, y) \vee R(x, y, z)) \wedge (Q(x, z) \vee R(x, y, z)))$

Ex: 6 Let

(i) U(x), denote *x is a used-car dealer,*
(ii) B(x) denote *x buys a used car for his family*, and
(iii) D(x) denote *x is absolutely dishonest*,

The given problem can be symbolized as

(i) $(\forall x)\ (U(x) \rightarrow \sim B(x))$
(ii) $(\exists x)\ (B(x) \wedge D(x))$.
(iii) $(\exists x)\ (D(x) \wedge \sim U(x))$ (to be shown)

By Existential Instantiation of (iii) we get that for some fixed c

(iv) $B(c) \wedge D(c)$

Using Universal Instantiation of (i), with c we get

(v) $U(c) \rightarrow \sim B(c)$

Using simplification of (iv) we get

(vi) B (c) (vii) D(c)

Using Modus Tollens with (v) and (vi) taking B (c) = ~ (~ (B (c)), we get

(vii) $\sim U(c)$

Using conjunction of (vii) & (viii), we get

(ix) $D(c) \wedge \sim U(c)$

Using Existential Generalization of (ix) we get

$(\exists x)\ (D(x) \wedge \sim U(x))$,

which is required to be established.

Ex: 7

Let us use the notation for the predicates of the problem as follows:

P(x) : x is a patient,
D(x): x is a doctor,
Q(x): x is a quack,
L(x, y): x likes y.

The problem can be symbolized as follows:

(i) $(\exists x)\ ((P(x) \wedge (\forall y)\ (D(y) \rightarrow L(x, y)))$
(ii) $(\forall x)\ (P(x) \rightarrow (\forall y)\ (Q(y) \rightarrow \sim L\ (x, y)))$
(iii) $(\forall x)\ (D(x) \rightarrow \sim Q(x))$. (*to be shown*)

Taking Existential Instantiation of (i), we get a specific c such that
(iv) $P\ (c) \wedge (\forall y)\ (\ D\ (y) \rightarrow L\ (c, y))$
By simplification of (iv), we get
(v) $P(c)$
(vi) $(\forall y)\ (D\ (y) \rightarrow L\ (c, y))$

By Universal Instantiation of (ii) with x as c (because the fact in (ii) is true for all values of x and for the already considered value c also. This type of association of an already used value c may not be allowed in Existential Instantiation)
we get
(vii) $P\ (c) \rightarrow \forall(y)\ (Q\ (y) \rightarrow \sim L\ (c, y))$
Using Modus Ponens with (v) and (vii) we get
(viii) $\forall(y)\ (Q(y) \rightarrow \sim L\ (c, y))$
As $(\forall y)$ is the quantifier appearing in both (vi) and (viii),
we can say that for an arbitrary a, we have
(ix) $D\ (a) \rightarrow L\ (c, a)$ for every a (from (vi)) and
(x) $Q\ (a) \rightarrow \sim L\ (c, a)$ for every a (from (viii))
(Using the equivalent $P \rightarrow Q = \sim Q \rightarrow \sim P$, we get from (x):
(xi) $L\ (c, a) \rightarrow \sim Q\ (a)$

Ex: 8 In the formula, $(\exists x)$ is preceded by no universal quantifiers, $(\exists u)$ is preceded by $(\forall y)$ and $(\forall z)$, and $(\exists w)$ by $(\forall y)$, $(\forall z)$ and $(\forall z)$ and $(\forall v)$. Therefore, we replace the existential variable *x* by a constant *a*, *u* by a two-place function f(y, z), and *w* by a three-place function g(y, z, v). Thus, we obtain the following standard form of the formula:

$$(\forall y)\ (\forall z)\ (\forall v)\ P(a, y, z, f\ (y, z), v, g(y, z, v)).$$

Ex: 9 As a first step, the matrix is transformed into the following conjunctive normal form:

$$(\forall x)\ (\exists y)\ (\exists z)\ ((\sim P(x, y) \vee R(x, y, z)) \vee R\ (x, y, z)) \wedge (Q(x, z) \vee R(x, y, z))).$$

As the existential variables $(\exists y)$ and $(\exists z)$ are both preceded by $(\forall x)$, the variables *y* and *z* are replaced, by one-place function f(x) and g(x) respectively.

In this way, we obtain the standard form of the formula as:

$$(\forall x)\ ((\sim P(x, f(x)) \vee R(x, f(x), g(x))) \wedge (Q(x, g(x)) \vee R(x, f(x), g(x)))).$$

Ex: 10 As we are going to use resolution method, we consider ~ G: $\sim (\exists x)\ (O\ (x) \wedge R\ (x)$ as an axiom.

We use the resolution method to show these clauses as unsatisfiable.
As $\mathbf{H_1}$ unsolves only quantifier at extreme left, its standard form is:
$H_1 \sim (x) \vee (W (x) \wedge R (x)) = (\sim ((x) \vee W (x)) \wedge (\sim C (x) \vee R (x))$
$H_2 : C (a) \wedge O (a)$ (a for $\exists x$)
For standard form of ~ G
$\sim G = \sim (\exists x) (O (x) \wedge R (x)) = (\forall x) (\sim O (x) \vee \sim R (x))$
Standard form of $\sim G = \sim O (x) \vee \sim R (x)$
Thus, the clauses of the wffs of the problem are:

(i) $\sim C(x) \vee W(x)$
(ii) $\sim C(x) \vee R(x)$ from H_1
(iii) C(a)
(iv) O(a) from H_2
(v) $\sim O(x) \vee \sim R(x)$ from ~ G.
Resolving (ii) and (iii), we get
(vi) R(a) Resolving (iv) and (v), we get
(vii) ~ R(a) Resolvent (v) and (iv)
Resolving (vi) and (vii) we get
(viii) False
Hence, G is a logical consequence of F_1 and F_2.

Ex : 11 Let us use the symbols:
P(x): x is a patient
D(x): x is a doctor
Q(x): x is a quake
L (x, y): x likes y.
Therefore, the given statements in the problem are symbolized as:

(i) $(\exists x) (P(x) \wedge (\forall y) (D(y) \rightarrow L(x, y)))$
(ii) $(\forall x) (P(x) \rightarrow (\forall y) (Q(y) \rightarrow \sim L(x, y)))$
(iii) $(\forall x) (D(x) \rightarrow \sim Q(x))$.

The clauses which are obtained after reducing to standard form are:

(iv) P(a) from (i) to standard
(v) $\sim D(y) \vee L(a, y)$ from (i) form are:
(vi) $\sim P(x) \vee \sim Q(y) \vee \sim L(x, y)$ from (ii)
(vii) D(b) from ~ (iii)
(viii) Q(b) from ~ G from ~ (iii)

Resolving (v) and (vii), we get
(ix) L(a, b)
Resolving (iv) and (vi) we get
(x) $\sim Q(y) \vee \sim L(a, y)$
Resolving (viii) and (x), we get
(xi) ~L (a, b)
Resolving (ix) and (xi), we get
(xii) False

Ex: 12 E(x): x *entered this country*

V(x): *x was VIP*,
S(x, y): *y searched x*,
C(x): *x was a custom official*, and
P(x): *x was a drug pusher*.

When symbolized, the known facts become:

(i) $(\forall x)\,(E(x) \wedge \sim V(x) \rightarrow (\exists y)\,(S(x, y) \wedge C(y)))$
(ii) $(\exists x)\,(P(x) \wedge E(x) \wedge (\forall y)\,(S(x, y) \rightarrow P(y)))$
(iii) $(\forall x)\,(P(x) \rightarrow \sim V(x))$

and on symbolization, the conclusion becomes
(iv) $(\exists x)\,(P(x) \wedge C(x))$.

As resolution method is to be used, we assume ~ (iv)
After converting to standard form, we get the clauses:

(v) $\sim E(x) \vee V(x) \vee S(x, (f(x))$	(from (i))
(vi) $\sim E(x) \vee V(x)\ C(f(x))$	(from (i))
(vii) $P(a)$	(from (ii))
(viii) $E(a)$	(from (ii))
(ix) $\sim S(a, y) \vee P(y)$	(from (ii))
(x) $\sim P(x) \vee \sim V(x)$	(from (iii))
(xi) $\sim P(x) \vee \sim C(x)$	(from (iv))

Resolving (vii) and (x) we get
(xii) $\sim V(a)$
Resolving (vi) and (viii), we get
(xiii) $V(a) \vee C(f(a))$
Resolving (xii) and (xiii), we get
(xiv) $Cf(a))$
Resolving (v) and (viii), we get
(xv) $V(a) \vee S(a, f(a))$
Resolving (xii) and (xv) we get
(xvi) $S(a, f(a))$
Resolving (xvi) and (ix)
(xvii) $P(f(a))$
Resolving (xvii) and (xi), we get
(xviii) $\sim C(f(a))$
Resolving (xiv) and (xviii)
(xix) False

Hence, the proof by resolution method.

Ex: 13 Let us use the symbols:

S (x, y): x saves y,
M (x): x is money,
I (x): x is interest,
E (x, y): x Earns y.

Then the given statement on symbolization becomes:
(i) $(\forall x)\,((\exists y)\,(S(x, y) \wedge M(y)) \rightarrow (\exists y)\,(I(y) \wedge E(x, y)))$
the conclusion on
(ii) $\sim (\exists x)\, I(x) \rightarrow (\forall x)\,(\forall y)\,(S(x, y) \rightarrow \sim M(y))$
(iii) $\sim (\sim (\exists x)\, I(x) \rightarrow (\forall x)\,(\forall y)\,(S(x, y) \rightarrow \sim M(y)))$

The negation of (ii) becomes (iii) $\sim ((\exists x)\, I(x) \rightarrow (\forall x)\,(\forall y)\,(S(x, y) \rightarrow \sim M(y)))$

After converting to standard form, we get the clauses:

(iv) $\sim S(x, y) \vee \sim M(y) \vee I(f(x))$	(from (i))
(v) $\sim S(x, y) \vee \sim M(y) \vee E(x, f(x))$.	(from (ii))

(vi) ~I(z) (from (iii))
(vii) S(a, b) (from (iii))
(viii) M(b) (from (iii))

Resolving (iv) and (vi), we get

(ix) $\sim S(x, y) \vee \sim M(y)$

Resolving (vii) and (ix) we get

(x) ~ M(b)

Resolving (viii) and (x), we get

(xi) False

1.8 FURTHER READINGS

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