UNIT 2 SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Structure **Page Nos.**

2.0 INTRODUCTION

We often come across equations of the forms $x^4 - 3x^3 + x^2 + 6x - 5 = 0$ or $e^x + x - 2 = 0$ etc. Finding one or more values of x which satisfy these equations is one of the important problems in Mathematics.

An equation of the type $f(x) = 0$ is algebraic if it contains power of x, that is, f(x) is a polynomial. The equation is called transcendental, if it contains powers of x, exponential functions, logarithm functions etc.

Example of algebraic equatoins:

$$2x = 5, \quad x^2 + x = 1, \quad x^7 = x(1+2x).$$

Example of transcendental equations

$$x + \sin x = 0, \quad e^{\sqrt{x}} = x, \quad \tan x = x.$$

As we know, direct methods can be used to solve the polynomial equations of fourth or lower orders. We do not have any direct methods for finding the solution of higher order polynomial equations or transcendental equation. In these cases, we use numerical methods to solve them.

In this unit , we shall discuss some numerical methods which give approximate solutions of an equation f(x) = 0. These methods are iterative in nature. An iterative method gives an approximate solution by repeated application of a numerical process. In an iterative method, we start with an initial solution and the method improves this solution until it is improved to acceptable accuracy.

Properties of polynomial equations:

i) The total number of roots of an algebraic equation is the same as its degree.
ii) An algebraic equation can have at most as many positive roots as the number of changes of sign in the coefficients of f(x).

iii) An algebraic equation can have at most as many negative roots as the number of changes of sign in the coefficient of $f(-x)$.

iv) If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ have roots $\alpha_1, \alpha_2, ..., \alpha_n$, then the following hold good:

$$\sum_i \alpha_i = \frac{-a_1}{a_0}, \quad \sum_{i<j} \alpha_i \alpha_j = \frac{a_2}{a_0}, \quad \prod_i \alpha_i = (-1)^n \frac{a_n}{a_0}.$$

2.1 OBJECTIVES

After studying this unit, you should be able to :

- find an initial guess of a root;
- use bisection method;
- use Regula-falsi method;
- use Newton's Method;
- use Secant Method, and
- use successive iterative method.

2.2 INITIAL APPROXIMATION TO A ROOT

All the numerical methods have in common the requirement that we need to make an initial guess for the root. Graphically, we can plot the equation and make a rough estimate of the solution. However, graphic method is not possible to use in most cases. We with to determine analytically an approximation to the root.

Intermediate Value Theorem

This theorem states that if f is a continous function on [a, b] and the sign of f(a) is different from the sign of f(b), that is $f(a)f(b) < 0$, then there exists a point c, in the interval (a, b) such that $f(c) = 0$. Hence, any value $c \in (a, b)$ can be taken as an initial approximation to the root.

Example 1: Find an initial guess to find a root of the equation, $2x - \log_{10} x = 7$.

Solution: Let $f(x) = 2x - \log_{10} x - 7$. The values of function f are as given in *Table 1*.

Table 1

X	1	2	3	4
f(x)	– 5	– 3.301	– 1.477	0.397

We find $f(3)f(4) < 0$. Hence, any values in (3, 4) can be taken as an initial guess.

Example 2: Estimate an initial guess to find a root of the equation, $2x - 3 \sin x - 5 = 0$.

Solution: Let $f(x) = 2x - 3 \sin x - 5$. Note that $f(-x) = -2x + 3 \sin x - 5$ which is always negative. Therefore, the function f(x) has no negative real roots. We tabulate the values of the function for positive x, in Table 2.

Table 2

x	0	1	2	3
f(x)	– 5	– 5.5224	– 3.7278	0.5766

Since f(2) and f(3) are of opposite signs, a root lies between 2 and 3. The initial guess can be taken as any value in (2, 3).

2.3 BISECTION METHOD

This is one of the simplest methods and is based on the repeated application of the intermediate value theorem.

The bisection method is defined as follows:

i) Find an interval (a, b) in which in root lies, using intermediate value theorem.
ii) Direction the interval (a, b). Let c = (a + b)/2. If f(c) = 0, then x = c is the root and the root is determined. Otherwise, use the intermediate value theorem to decide whether the root lies in (a, c) or (c, b).
iii) Repeat step using the interval (a, c).
iv) The procedure is repeated while an length the last interval is less than the desired accuracy. The mid point of this last interval is taken as the root.

Example 3: Use bisection method to find a positive root of the equation
$f(x) = 0.5\ e^x - 5x + 2$

Solution: We find that f(0) = 2.5 and f(1) = −1.6408. Therefore, there is a root between 0 and 1. We apply the bisection method with a=0 and b= 1. The mid point is c = 0.5 and f(0.5) = 0.32436. The root now lies in (0.5, 1.0).

The tabulated values are shown in *Table 3*.

Table 3

a	b	midpoint (c)	f(a)	f(b)	f(c)
0	1	0.5	2.5	-1.6408591	0.32436064
0.5	1	0.75	0.32436064	-1.6408591	-0.6915
0.5	0.75	0.625	0.32436064	-0.6915	-0.190877
0.5	0.625	0.5625	0.32436064	-0.190877	0.06502733
0.5625	0.625	0.59375	0.06502733	-0.190877	-0.063367
0.5625	0.59375	0.578125	0.06502733	-0.063367	0.00072137
0.578125	0.59375	0.5859375	0.00072137	-0.063367	-0.0313502
0.578125	0.5859375	0.58203125	0.00072137	-0.0313502	-0.0153212
0.578125	0.58203125	0.58007813	0.00072137	-0.0153212	-0.0073016
0.578125	0.580078125	0.57910156	0.00072137	-0.0073016	-0.0032906
0.578125	0.579101563	0.57861328	0.00072137	-0.0032906	-0.0012847
0.578125	0.578613281	0.57836914	0.00072137	-0.0012847	-0.0002817
0.578125	0.578369141	0.57824707	0.00072137	-0.0002817	0.00021984
0.57824707	0.578369141	0.57830811	0.00021984	-0.0002817	-3.093E-05
0.57824707	0.578308105	0.57827759	0.00021984	-3.093E-05	9.4453E-05
0.578277588	0.578308105	0.57829285	9.4453E-05	-3.093E-05	3.1762E-05
0.578292847	0.578308105	0.57830048	3.1762E-05	-3.093E-05	4.1644E-07
0.578300476	0.578308105	0.57830429	4.1644E-07	-3.093E-05	-1.526E-05
0.578300476	0.578304291	0.57830238	4.1644E-07	-1.526E-05	-7.42E-06
0.578300476	0.578302383	0.57830143	4.1644E-07	-7.42E-06	-3.502E-06
0.578300476	0.57830143	0.57830095	4.1644E-07	-3.502E-06	-1.543E-06
0.578300476	0.578300953	0.57830071	4.1644E-07	-1.543E-06	-5.631E-07
0.578300476	0.578300714	0.5783006	4.1644E-07	-5.631E-07	-7.333E-08
0.578300476	0.578300595	0.57830054	4.1644E-07	-7.333E-08	1.7156E-07

After 24 iterations we see that the smaller root can be found in the interval [.578300476, .578300595]. Therefore, we can estimate one root to be 0 .5783005. One of the first things to be noticed about this method is that it takes a lot of iterations to get a high degree of precision. In the following error analysis, we shall see method as to why the method is taking so many directions.

2.3.1 Error Analysis

The maximum error after the ith iteration using this process is given by

$$\varepsilon_i = \frac{|b-a|}{2^i}$$

Taking logarithms on both sides and simplfying, we get

$$i \geq \frac{[\log(b-a) - \log\varepsilon_i]}{\log 2} \qquad (1)$$

As the interval at each iteration is halved, we have $(\varepsilon_{i+1}/\varepsilon_i) = (1/2)$. Thus, this method converges linearly .

Example 4 : Obtain the smallest positive root of $x^3 - 2x - 5 = 0$, correct upto 2 decimal places.

Solution : We have $f(x) = x^3 - 2x - 5$, $f(2) = -1$ and $f(3) = 16$. The smallest positive root lies in (2, 3). Therefore, a = 2, b = 3, b − a = 1, we need solution correct to two decimal places, that is,

$\varepsilon \leq 0.5(10^{-2})$, from (1), we get

$$i \geq \frac{\log 1 - \log[0.5(10^{-2})]}{\log 2} = \frac{-\log(0.005)}{\log 2} \approx 8.$$

This shows that 8 iterations are required to obtain the required accuracy. Bisection method gives the iteratived values as $x_1 = 2.5$, $x_2 = 2.25$, …, $x_8 = 2.09$. Then $x \approx 2.09$ is the approximate root.

2.4 REGULA FALSI METHOD

Let the root lie in the interval (a, b). Then, P(a, f(a)), Q(b, f(b)) are points as the curve. Join the points P and Q. The point of intersection of this, with the X-axis, c, line is taken as the next approximation to the root. We deermine by the intermediate value theorem, whether the root now lies in (a, c) or (c, b) we repeat the procedure. If x_0, x_1, x_2,…… are the sequence of approximations, then we stop the iteration when $| x_{k+1} - x_k| <$ given error tolerance.

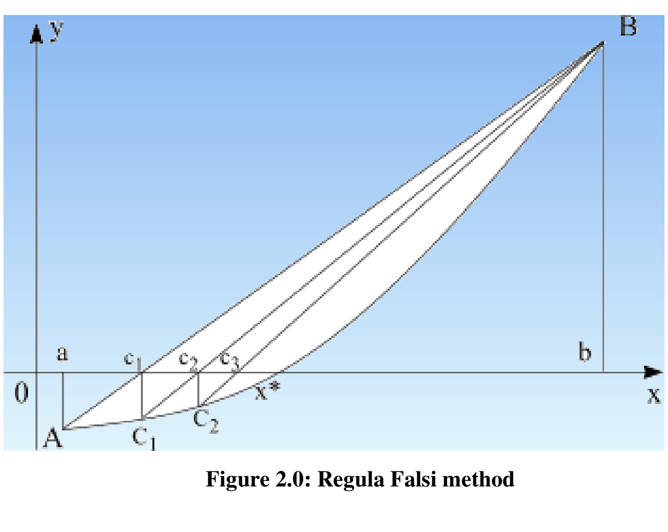

Figure 2.0: Regula Falsi method

The equation of line chord joining (a, f(a)), (b, f(b)) is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}\ (x - a).$$

Setting y = 0, we set the point of intersection with X-axis as givens

$$x = c = a - \frac{b - a}{f(b) - f(a)}\ f(a)$$

$$= \frac{af(b) - bf(a)}{f(b) - f(a)}$$

If we denote $x_0 = a$, $x_1 = b$, then the iteration formula can be written as

$$x_{n+1} = \frac{x_{n+1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \qquad n = 1,2,...... \qquad (2)$$

The rate of convergence is still linear but faster than that of the bisection method. Both these methods will fail if *f* has a double root.

Example 5: Obtain the positive root of the equation $x^2 - 1 = 0$ by Regua Falsi method.

Solution: Let $f(x)=x^2 - 1$. Since f(0) = −1, f(2) = 3, Let us take that the root lies in (0, 2). We have $x_0 = 0$, $x_1 = 2$.

Then, using (2), we get

$$x_2 = \frac{x_0 f(2) - x_1 f(0)}{f(2) - f(0)} = \frac{0 - 2(-1)}{3 + 1} = 0.5, \qquad f(0.5) = -0.75$$

The root lies in (0.5, 2.0), we get

$$x_3 = \frac{0.5 f(2) - 2.0 f(0.5)}{f(2) - f(0.5)} = \frac{0.5(3) - 2.0(-0.75}{3 + 0.75} = 0.8$$

f(0.8) = -0.36. The root lies in (0.8, 2). The next approximation

$$x_4 = \frac{0.8(3) - 2.0(-0.36)}{3 + 0.36} = 0.9286, \qquad f(0.9286) = -0.1377.$$

We obtain the next approximations as x_5 = 0.9756, x_6 = 0.9918, x_7 = 0.9973, x_8 = 0.9990. Since, $|x_8 - x_7| = 0.0017 < 0.005$, the approximation x_8 = 0.9990 is correct to decimal places.

Note that in this problem, the lower end of the interval tends to the root, and the minimum error tends to zero, but the upper limit and maximum error remain fixed. In other problems, the opposite may happen. This is the property to the regula falsi method.

2.5 NEWTON'S METHOD

This method is also called Newton-Raphsan method. We assume that *f* is a differentiable function in some interval [*a*, *b*] containing the root.

We first look at a "pictorial" view of how Newton's method works. The graph of y = f(x) is plotted in *Figure 3.1*. The point of intersection x = r, is the required root.

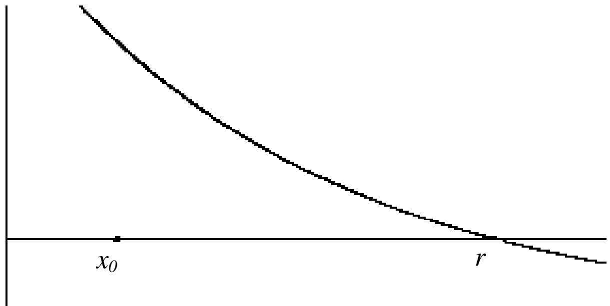

Figure 2.1: Plor of y = f(x)

Let x_0 be an initial approximation of *r*. Then, (x0, f(x0)) is a point as the curve (*Figure 3.2*).

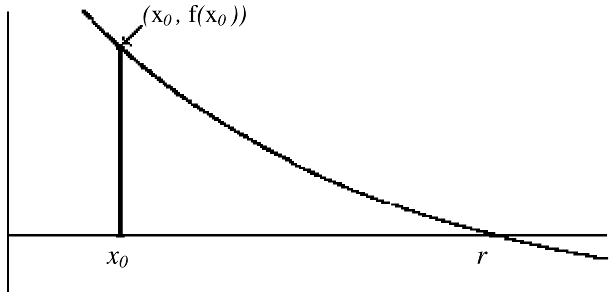

Figure 2.2: Point on the centre

Draw the tangent line to the curve at the point $(x_0, f(x_0))$. This line intersects the *x*-axis at a new point, say x_1 (*Figure 3.3*).

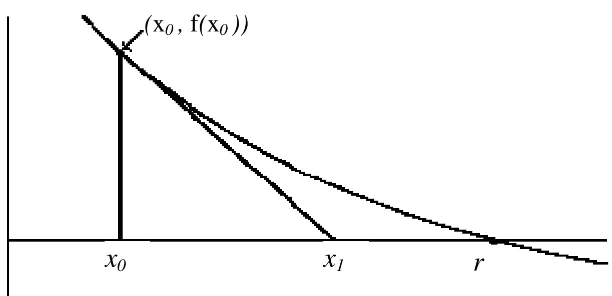

Figure 2.3: Tangent at (x_0, f(x_0))

Now, x_1 is a better approximation to *r*, than x_0 . We now repeat this process, yielding new points x_2 , x_3 , ... until we are "close enough" to *r*. *Figure 3.4* shows one more iteration of this process, determining x_2 .

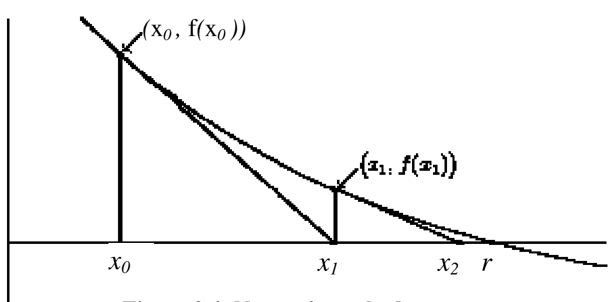

Figure 2.4: Newton's method

Now, we derive this method algebraically. The equation of the tangent at $(x_0, f(x_0))$ is given by

$y - f(x_0) = f'(x_0)(x - x_0)$

Where $f'(x_0)$ is the slope of the curve at $(x_0, f(x_0))$. Setting $y = 0$, we get the point of intersection of the tangent with x-axis as

$y - f(x_0) = f'(x_0)(x - x_0)$, or $x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

But, this is our next approximation, that is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Iterating this process, we get the Newton-Rephson as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{for } n = 0, 1, 2, \ldots\ldots \qquad (3)$$

Example 6: Find the smallest positive root of $x^7 + 9x^5 - 13x - 17 = 0$.

Solution : Let $f(x) = x^7 + 9x^5 - 13x - 17$, we have $f(0) < 0$, $f(1) < 0$ and $f(1)f(2) < 0$. Hence, the smallest positive root lies in (1, 2). We can take any value in (1, 2) or one of the end points as the initial approximation. Let x0 = 1, we have, $f'(x) = 7x^6 + 45x^4 - 13$. The Newton-Raphson method becomes

$$x_{n+1} = x_n - \frac{x_n^7 + 9x_n^5 - 13x_n - 17}{7x_n^6 + 45x_n^4 - 13}, \qquad n = 0,1,2\ldots\ldots\ldots$$

Starting with $x_0 = 1$, we obtain the values given in Table 4.

Table 4

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1	–20	39	1.512820513
1	1.512820513	52.78287188	306.6130739	1.340672368
2	1.340672368	12.33751268	173.0270062	1.269368397
3	1.269368397	1.46911353	133.1159618	1.258332053
4	1.258332053	0.03053547	127.6107243	1.258092767
5	1.258092767	0.00001407	127.4932403	1.258092657

After 6 iterations of Newton's method, we have

$$|x_6 - x_5| = |1.258092657 - 1.258092767| = 0.000000110.$$

Therefore, the root correct to 6 decimal places is $r = 1.258092657$.

Possible drawbacks:

Newton's method may not work in the following cases:

i) The x-values may run away as in *Figure 2.5(a)*. This might occur when the x-axis is an asymptote.

ii) We might choose an x-value that when evaluated, the derivative gives us 0 as in *Figure 2.5(b)*. The problem here is that we want the tangent line to intersect the x-axis so that we can approximate the root. If *x* has a horizontal tangent line, then we can't do this.

iii) We might choose an x, that is the beginning of a cycle as in *Figure 2.5(c)*. Again it is hoped that the picture will clarify this.

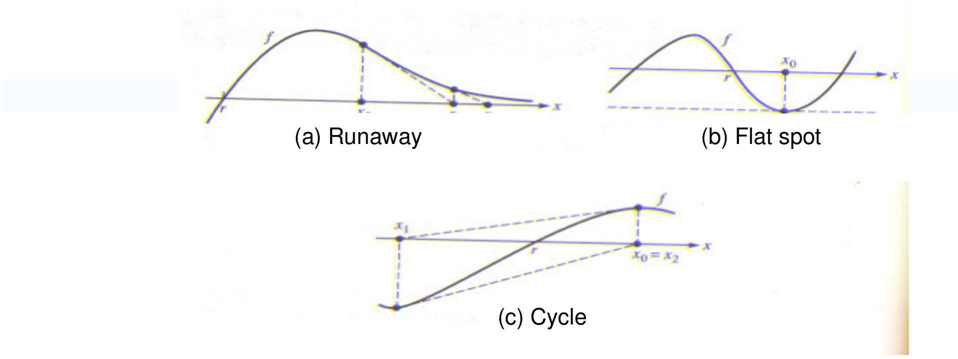

Figure 2.5: Divergence of Newton's method

However, the dificulties posed have one artifical. Normally, we do not encounter such problems in practice. Newton-Raphson methods is one of the powerful methods available for obtaning a simple root of f(x) = 0.

2.5.1 Error Analysis

Let the error at the n^{th} step be defined as

$$e_n = x_n - x$$

Then the error at the next step is

$$x + e_{n+1} = x + e_n - \frac{f(x+e_n)}{f'(x+e_n)}.$$

Explaing in Taylor Series, we obtain

$$e_{n+1} = e_n - \frac{f(x) + e_n f'(x) + \frac{1}{2} e_n^{\ 2} f''(x) + ...}{f'(x) + e_n f''(x) + ...} \tag{4}$$

Since, x is a root, we have f(x) = 0. Then,

$$\begin{aligned}
e_{n+1} &= e_n - \frac{e_n f'(x)[1 + \frac{1}{2} e_n \frac{f''(x)}{f'(x)} + ...]}{f'(x)[1 + e_n \frac{f''(x)}{f'(x)} + ...]} \\
&= e_n - e_n[1 + \frac{1}{2} e_n \frac{f''(x)}{f'(x)} + ...][1 + e_n \frac{f''(x)}{f'(x)} + ...]^{-1} \\
&= e_n - e_n[1 + \frac{1}{2} e_n \frac{f''(x)}{f'(x)} + ...][1 - e_n \frac{f''(x)}{f'(x)} + ...] \\
&= e_n - e_n[1 - \frac{1}{2} e_n \frac{f''(x)}{f'(x)} + ...] \\
&= \frac{1}{2} e_n \frac{f''(x)}{f'(x)} e_n^2 + ...]
\end{aligned} \tag{5}$$

We can neglect the centric and higher powers of e_n, as they are much smallest than e_n^2, (e_n is itself a small number).

Notice that the error is squared at each step. This means that the number of correct decimal places *doubles* with each step, much faster than linear convergence. We call it quadratic convergence.

This sequence will converge if

$$\left|\frac{f''(x)}{f'(x)}e_n^2\right| < |e_n|, \qquad |e_n| < 2\left|\frac{f''(x)}{f'(x)}\right| \tag{6}$$

If f' is not zero at the root (simple root), then there will always be a range round the root where this method converges.

If f' is zero at the root (double root), then the convergence becomes linear.

Example 7: Compute the square root of a, using Newton's method. How does the error behave?

Solution: Let $x = \sqrt{a}$, or $x^2 = a$. Define $f(x) = x^2 - a$. Here, we know the root exactly, so that we can see how well the method converges.

We have the Newton's method for finding a root of f(x) = 0 as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\ 2} - a}{2x_n} = \frac{x_n^{\ 2} + a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \tag{7}$$

Starting with any suitable initial approximation to $\sqrt{a}$, we find x_1, x_2, ….., which convege to the required value.

Error at the n^{th} step is $e_n = x_n - \sqrt{a}$. Substituting, we get

$$\begin{aligned} e_{n+1} &= \frac{(e_n + \sqrt{a})^2}{2(\sqrt{a} + e_n)} - \sqrt{a} \\ &= \frac{2a + 2e_n + \sqrt{a} + e_n^{\ 2}}{2(\sqrt{a} + e_n)} - \sqrt{a} \\ &= \frac{2(\sqrt{a} + e_n)\sqrt{a} + e_n^{\ 2}}{2(\sqrt{a} + e_n)} - \sqrt{a} \\ &= \frac{e_n^{\ 2}}{2(\sqrt{a} + e_n)} \end{aligned} \tag{8}$$

If $a = 0$, this simplifies to $e_n/2$, as expected. Here, we are finding the root of $x^2 = 0$, which gives a double root x = 90.

Since $a>0$, e_{n+1} will be positive, provided e_n is greater than $-\sqrt{a}$, i.e provided x_n is positive. Thus, starting from any positive number, all the errors, except perhaps the first, will be positive.

The method converges when,

$$|\, e_{n+1} \,| = \left|\frac{e_n^{\ 2}}{2(\sqrt{a} + e_n)}\right| < |\, e_n \,| \tag{9}$$

or $\quad e_n < 2(e_n + \sqrt{a})$

which is always true. Thus, the method converges to the square root, *starting from any positive number*, and it does so quadratically.

We now discuss another method, which does not require the knowledge of the derivative of a function.

2.6 SECANT METHOD

Let x_0, x_1 be two initial approximations to the root. We do not require that the root lie in (x_0, x_1) as in Regula Falsi method. Therefore, the approximations x_0, x_1 may lie on the same side of the root. Further, we obtain the sequence of approximations as $x_2, x_3 \ldots\ldots$ At any stage, we do not require or check that the not lies in the interval (x_k, x_{k-1}). The derivation of the method is same as in the Regula Falsi method. (*Figure 2.6*)

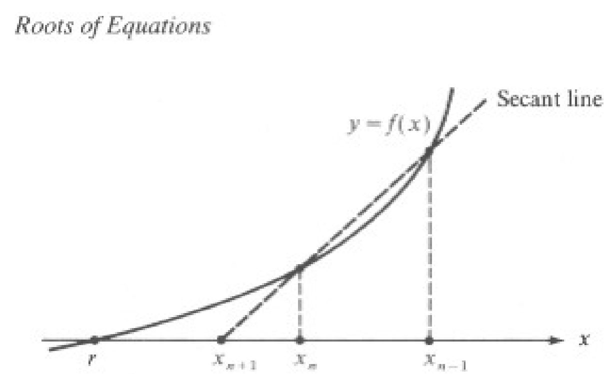

Figure 2.6: Secant method

The method is given by (see Equation (21))

$$x_{n+1} = \frac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}} \qquad (10)$$

We compute x_2 using x_0, x_1; x_3 using x_1, x_2; and so on.

The rate of convergence of the method is super linear (1.618), that is, it works better than the Regula Falsi method.

Example 8: Use secant method to find the roots of the equation $f(x) = 0.5e^x - 5x + 2$.

Solution: We have $f(-x) = 0.5e^{-x} + 5x + 2 > 0$ for all x. Hence, there is no negative root.

We obtain,
$f(0) = 2.5$, $f(1) = -1.6408$, $f(2) = -4.3055$, $f(3) = -2.9572$,
$f(4) = 902990$, $f(x) > 0$ for $x > 4$.

Therefore, the given function has two roots, one root in (0, 1) and the second root in (3, 4).

For finding the first not, we take $x_0 = 0$, $x_1 = 1$, and compute the sequence of approximations $x_2, x_3, \ldots\ldots$

For finding the second root, we take $x_0 = 3$, $x_1 = 4$ and compute the sequence of approximations $x_2, x_3, \ldots\ldots$

The results are given in *Table 5*.

Table 5

x_{n-1}	x_n	x_{n-1}	x_n	x_{n+1}
0	1	2.5	–1.640859086	0.60373945
1	0.603739453	–1.6408591	–0.104224624	0.57686246
0.603739453	0.576862465	–0.1042246	0.002909403	0.57830459
0.576862465	0.578304589	0.0029094	–1.68E-06	0.57830058
0.578304589	0.578300578	–1.65E-05	–2.57E-09	0.57830058
0.578300578	0.578300577	–2.57E-09	1.11E-15	0.57830058
0.578300577	0.578300577	1.11E-15	0	0.57830058
3	4	–2.9572315	9.299075017	3.24128244
4	3.241282439	9.29907502	–1.423168098	3.34198736
3.241282439	3.341987358	–1.4231681	–0.572304798	3.40972316
3.341987358	3.409723161	–0.5723048	0.079817605	3.40173252
3.409723161	3.401432525	0.07981761	–0.003635061	3.40179365
3.401432525	3.401793651	–0.0036351	–2.15E-09	3.4017958
3.401793651	3.401795804	–2.15E-05	5.87E-09	3.4017958
3.401795804	3.401795804	5.87E-09	–7.11E-15	3.4017958

The two roots are 0.57830058, 3.4017958 correct to all decimal places given.

2.7 METHOD OF SUCCESSIVE ITERATION

The first step in this method is to write the equation f(x) = 0 in the form

$$x = g(x). \tag{11}$$

For example, consider the equation $x^2 - 4x + 2 = 0$. We can write it as

$$x = \sqrt{4x-2}, \tag{12}$$

or as $x = (x^2 + 2)/4,$ (13)

or as $x = \dfrac{2}{4-x}$ (14)

Thus, we can choose from (11) in several ways. Since, f(x) = 0 is the same as x = g(x), finding a root of f(x) = 0 is the same as finding a root of x = g(x), i.e. finding a fixed point α of g(x) such that $\alpha = g(\alpha)$. The function g(x) is called an ***iteration function*** for solving f(x) = 0.

If an initial approximation x_0 to a root α is provided , a sequence x_1 , x_2 ,….. may be defined by the iteration scheme

$$x_{n+1} = g(x_n) \tag{15}$$

with the hope that the sequence will converge to α. The successive iterations for solving $x = e^{-x}/3$, by the method $x_{n+1} = e^{-x_n}/3$ is given in *Figure 2.7*.

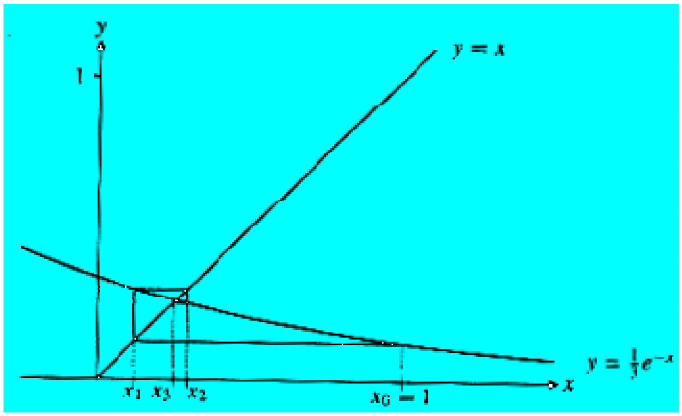

Figure 2.7: Successive iteration method

The method converges if, for some constant M such that $0 < M < 1$, the inequality

$$| g(x) - g(\alpha) | \le M | x - \alpha | \tag{16}$$

holds true whenever $|x - \alpha| \le | x_0 - \alpha |$. For, if (16) holds, we find that

$$| x_{n+1} - \alpha | = | g(x_n) - \alpha | = | g(x_n) - g(\alpha) | \le M | x_n - \alpha | \tag{17}$$

Using this relation recursively, we get

$$| x_{n+1} - \alpha | \le M | x_n - \alpha | \le M^2 | x_{n-1} - \alpha | \le M^n | x_0 - \alpha | \tag{18}$$

Since, $0 < M < 1$, $\lim M^n = 0$ and thus $\lim x_n = \alpha$.

Condition (16) is satisfied if the function g(x) possesses a derivative $g'(x)$ such that $| g'(x) | < 1$ for $| x - \alpha | < | x_0 - \alpha|$ If x_n is close to α, then we have

$$| x_{n+1} - \alpha | = | g(x_n) - g(\alpha) | \le g'(\xi) | x_n - \alpha | \tag{19}$$

for some ξ between x_0 and α. Therefore, condition for convergence is

$$| g'(\xi) | < 1, \text{ or } | g'(x) | < 1. \tag{20}$$

Example 9 : Let us consider the equation $f(x) = x^3 + x - 2$. It has only one real root at $x = 1$. There are several ways in which $f(x)=0$ can be written in the desired form, $x = g(x)$.

For example, we may write $x = x + f(x) = g(x)$ and write the method as

$$x_{n+1} = x_n + f(x_n) = x_n^3 + 2x_n - 2$$

In this case, $g'(x) = 3x^2 + 2$, and the convergence condition is

$$|g'(x)| = 3x^2 + 2 < 1, \qquad or \qquad 3x^2 < -1.$$

Since, this is never true, this arrangement doesn't converge to the root.

An alternate rearrangement is

$$x_{n+1} = 2 - x_n^3$$

This method converges when

$$|g'(x)| = |-3x^2| < 1, \qquad or \qquad x^2 < \frac{1}{3}, \qquad or \qquad |x| < \frac{1}{\sqrt{3}}.$$

Since this range $[-1/\sqrt{3}, 1/\sqrt{3}]$ does not include the root x = 1, this method will not converge either.

Another rearrangement is

$$x_{n+1} = \sqrt[3]{2 - x_n}$$

In this case, the convergence condition becomes

$$\frac{1}{3}|(2 - x_n)|^{-2/3} < 1, \qquad or \qquad (2 - x_n)^{-2} < 3^3, \qquad or \qquad |x_n - 2| > \sqrt{27}.$$

Again, this range does not conain the root.

Another rearrangement is

$$x_{n+1} = \frac{2}{x_n{}^2 + 1} \tag{21}$$

In this case, the convergence condition becomes

$$\frac{4|x|}{(1+x^2)^2} < 1, \qquad 4|x| < (1+x^2)^2$$

This inequality is satisfied when x > 1. Hence, if we start with such an *x*, the method will converge to the root.

Let x_0 = 1.2, Then, from (21), we obtain the sequence of approximations as x_1 = 0.8197, x_2 = 1.1963, x_3 = 0.8227, x_4 = 1.1927, x_5 = 0.8255, x_6 = 1.1894, …….

The approximations oscillate about x = 1 and converge very slowly.

2.8 SUMMARY

In this unit, we have discussed the acutal meaning of root. In general, root determination of an equation is a tedious exercise. So, to handle such tasks, we have discussed some basic, simple, but still powerful methods of root determination. The methods discussed in this unit are Bisection method, Regular falsi method, Newton's method, Secan method and Successive iteration method.

2.9 EXERCISES

E1) In the following problems, find the intervals of length 1 unit, in which the roots lie

(a) $12x^3 - 76x^2 + 131x - 42 = 0$; (b) $4x^2 + 8x - 21 = 0$

(c) $x - e^{-x} = 0$ (d) x = 2 cos x

E2) Find all the roots in Problems 1(a), (b), (c) by ergular falth method, secant method and Newton-Raghson method.

E3) Find the smaller roots in Problems 1(b) and the root in 1(c), by successive iteration method.

E4) Show that the equation $x^3 - 6x - 1 = 0$, has a root in the internal $(-1, 0)$. Obtain this root using the successive iteration method.

2.10 SOLUTIONS TO EXERCISES

E1) (a) (0, 1), (1, 2), (3, 4) (b) (– 4, –3), (1, 2)
(c) (0, 1) (d) (0.5, 1.5)

E2) (a) 0.5, 1.2, 3.5 (b) – 3.5, 1.5
(c) 0.567143

E3) (a) Use $x_{n+1} = x_n - 0.05(4x_n^2 + 8x_n - 21)$ with $x_0 = 1.4$
(b) Write $x_{n+1} = e_n^{-xn}$

E4) Write $x_{n+1} = (x_n^3 - 1) / 6$; $x_0 = -0.5, - 0.167449$