UNIT 2 TIME VALUE OF MONEY

Structure **Page Nos.**

2.0 INTRODUCTION

The notion that money has time value is one of the most basic concepts of investment analysis. For any productive asset it's value will depend upon the future cash flows associated with that particular asset. In order to assess the adequacy of cash flows one of the important parameters is to assess the time value of the cash flows viz., Rs.100 received after one year would not be the same as Rs.100 received after two years. There are several reasons to account for this difference based on the timing of the cash flows, some of which are as follows:

- there is a general preference for current consumption to future consumption,
- capital (savings) can be employed to generate positive returns,
- due to inflation purchasing power of money decreases over time,
- future cash flows are uncertain.

Translating the current value of money into its equivalent future value is referred to as compounding. Translating a future cash flow or value into its equivalent value in a prior period is referred to as discounting. This Unit deals with basic mathematical techniques used in compounding and discounting.

2.1 OBJECTIVES

After going through this unit, you should be able to:

- understand the time value of money;
- understand what gives money its time value;
- understand the methods of calculating present and future value, and
- understand the use of present value technique in financial decisions.

2.2 DETERMINING THE FUTURE VALUE

Let us suppose that you deposit Rs.1000 with a bank which pays 10 per cent interest compounded annually for a period of 3 years. The deposit will grow as follows:

First Year	Principal at the beginning. Interest for the year (1000x.10) Total amount	Rs. 1000 100 1100
Second Year	Principal at the beginning. Interest for the year (1100x.10). Total Amount	1100 110 1210
Third Year	Principal at the beginning. Interest for the year (1210x.10) Total Amount	1210 121 1321

To get the future value from present value for a one year period

$$FV = PV + (PV \times k)$$

where PV = Present Value

k = Interest rate

$$FV = PV(1+k)$$

Similarly for a two year period

$FV = PV$	$+ \quad (PV \times k)$	$+ \quad (PV \times k)$	$+ \quad (PV \times k \times k)$
Principal amount	First period interest on principal	Second period interest on the principal	Second periods interest on the first periods interest

$$\begin{aligned} FV &= PV+PVk+PVk+PVk^2 \\ &= PV+2PVk+PVk^2 \\ &= PV\ (1+2k+K^2) = PV\ (1+k)^2 \end{aligned}$$

Thus, the future value of amount after n periods is (2.1)

$$FV = PV\ (1+k)^n$$

where FV = Future value n years hence

PV = Cash today (present value)

k = Interest rate par year in percentage

n = number of years for which compounding is done

Equation (2.1) is the basic equation for compounding analysis. The factor $(1+k)^n$ is refered to as the compounding factor or the future value interest factor (FVIFk,n). Published tables are available showing the value of $(1+k)^n$ for various combinations of k and n. One such table is given in appendix A of this unit.

Example 2.1 Find out the future value of Rs.1000 compounded annually for 10 years at an interest rate of 10%.

Solution: The future value 10 years hence would be

$$\begin{aligned} FV &= PV\ (1+k)^n \\ FV &= 1{,}000\ (1+.10)^{10} \\ &= 1000 \times (1.10)^{10} \\ &= 1000\ (2.5937) \\ &= 2593.7 \end{aligned}$$

The appreciation in present value of an amount can also be expressed in terms of return. A return is the income on investment over each period divided by the amount of investment at the beginning of the period. From the above example the arithmetic average return would be (2593.7 −1000)/1000=159.37% over the ten year period or 15.937% per year. The main drawback of using arithmetic average is that it ignores the process of compounding. To overcome this, the correct method is to use geometric average return to calculate overage annual return.

Rearranging the equation 2.1 we get

$$k = \sqrt[n]{\frac{FV}{PV}} - 1 \qquad (2.2)$$

using the values from example 2.1

$$= \sqrt[10]{\frac{2593.7}{1,000}} - 1$$

$$= \left(\frac{2593.7}{1000}\right)^{1/10} - 1$$

$$= 1.10 - 1$$

$$= .10 = 10\%$$

2.2.1 Shorter Compounding Period

So far in our discussion we have assumed that the compounding is done annually, now let us consider the case where compounding is done more frequently. In this case the equation (2.1) is modified to factor in the change of frequency of compounding.

$$FV_n = PV\,(1 + \frac{k}{m})^{m \times n} \qquad (2.3)$$

where FV_n = Future value after n years
PV = Present Value
K = nominal annual rate of interest
m = Frequency of compounding done during a year
n = number of years for which compounding is done.

If the interest is payable semiannually frequency of compounding is 2, if it is payable monthly frequency is 12, if it is payable weekly frequency is 52 and so on.

Example 2.2 Calculate the future value of Rs.5000 at the end of 6 years, if nominal interest rate is 12 per cent and the interest is payable quarterly (frequency = 4)

Solution:

$$FV_n = PV\ (1 + \frac{k}{m})^{m \times n}$$

$$FV_6 = 5000(1 + \frac{.12}{4})^{6 \times 4}$$

$$= 5000(1 + .03)^{24}$$

$$= 5000 \times 2.0328$$

$$= 10,164$$

The future value of Rs.5000 after 6 years on the basis of quarterly compounding would be Rs.10 164 whereas in case of semi-annual and annual compounding the future value would be–

$$FV_6 = 5000(1+\frac{.12}{2})^{6\times2}$$
$$= 5000(1+06)^{12}$$
$$= 5000\times2.0122$$
$$= 10{,}061$$

$$FV_6 = 5000\ (1+.12)^6$$
$$= 5000\ (1.9738\ )$$
$$= 9868$$

This difference in future value is due to the fact that interest on interest has been calculated.

2.2.2 Effective vs. Nominal Rates

In the above example we have seen how the future value changes with the change in frequency of compounding. In order to understand the relationship between effective and nominal rate let us calculate the future value of Rs.1000 at the interest rate of 12 per cent when the compounding is done annually, semiannually, quarterly and monthly.

$$FV = 1000\ (1+.12)^1$$
$$= 1120$$
$$FV = 1000\ (1+\frac{.12}{2})^2$$
$$= 1000\ (1.06)^2$$
$$= 1000(1.1236)$$
$$= 1123.6$$
$$FV = 1000\ (1+\frac{.12}{4})^4$$
$$1000 = (1.03)^4$$
$$1000 = (1.1255)$$
$$= 1125.5$$
$$FV = 1000\ (1+\frac{.12}{12})^{12}$$
$$= 1000\ (1.01)^{12}$$
$$= 1000(1.1268)$$
$$= 1126.8$$

From the above calculations we can see that Rs.1000 grows to Rs.1120, Rs.1123.6, Rs.1125.5 and Rs.1126.8 although the rate of interest and time period are the same. In the above case 12.36, 12.55 and 12.68 are known as effective rate of interest. The relationship between the effective and nominal rate of interest is given by

$$r = (1 + \frac{k}{m})^m - 1 \qquad (2.4)$$

where r = effective rate of interest
k = nominal rate of interest
m = frequency of compounding per year

Based on the above stated example the effective interest rate is calculated as follows:

a) Effective interest rate for monthly compounding

$$
\begin{aligned}
r &= (1+\frac{.12}{12})^{12}-1 \\
&= (1.01)^{12}-1 \\
&= 1.1268-1 \\
&= .1268=12.68
\end{aligned}
$$

b) Effective interest rate for quarterly compounding

$$
\begin{aligned}
r &= (1+\frac{12}{4})^{4}-1 \\
r &= (1.03)^{4}-1 \\
r &= 1.1255-1=.1255 \\
&= 12.55\%
\end{aligned}
$$

c) Similarly the effective interest rate for semi-annual compounding is

$$
\begin{aligned}
r &= (1+\frac{12}{2})^{2}-1 \\
r &= (1.06)^{2}-1 \\
r &= 1.1236-1=.1236=12.36
\end{aligned}
$$

Doubling Period

One of the first and the most common questions regarding an investment alternative is the time period required to double the investment. One obvious way is to refer to the table of compound factor from which this period can be calculated. For example the doubling period at 3%, 4%, 5%, 6%, 7%, 8%, 9%, 10%, 12% would be approximately 23 years, 18 years, 14 years, 12 years, 10 years, 9 years, 8 years, 7 years, and 6 years respectively.

If one is not inclined to use future value interest factor tables there is an alternative, known as rule of 72. According to this rule of thumb the doubling period is obtained by dividing 72 by the interest rate. For example, at the interest rate of 8% the approximate time for doubling an amount would be 72/8 = 9 years.

A much more accurate rule of thumb is rule of 69. As per this rule the doubling period is equal to

$$.35+\frac{69}{\text{Interest rate}}$$

Using this rule the doubling period for an amount fetching 10 percent and 15 percent interest would be as follows.

$$.35+\frac{69}{10}=.35+6.9=7.25\text{ years}$$

$$.35+\frac{69}{15}=.35+4.6=4.95\text{ years}$$

2.2.3 Continuous Compounding

The extreme frequency of compounding is continuous compounding where the interest is compounded instantaneously. The factor for continuous compounding for one year is eAPR where e is 2.71828 the base of the natural logarithm. The future value of an amount that is compounded for n years is

$$FV = PV \text{ x } e^{kn}$$

Where k is annual percentage rate and e^{kn} is the compound factor.

Example 2.3: Find the future value of Rs.1000 compounded continuously for 5 year at the interest rate of 12% per year and contrast it with annual compounding.

$$\begin{aligned} \text{Solution: } FV_{\bar{\bar{5}}} &= PVe^{N(APR)} \\ &= 1000 \times 2.71828 \\ &= 1000 \times 2.71828^{.60} \\ &= 1000 \times 1.82212 \\ &= 1822.12 \\ FV_{\bar{\bar{5}}} &= PV(1+k)^n \\ &= 1000(1+.12)^5 \\ &= 1000(1.7623) \\ &= 1762.3 \end{aligned}$$

From this example you can very well see the effects of extreme frequency of compounding.

So far in our discussion we have assumed that the interest rate is going to remain the same over the life of the investment, but now a days we are witnessing an increased volatility in interest rates as a result of which the financial instruments are designed in a way that interest rates are benchmarked to a particular variable and with the change in that variable the interest rates also change accordingly.

In such cases the Future Value is calculated through this equation.

$$FV_n = PV(1+k_1)(1+k_2)(1+k_3)+\ldots \quad (1+k_n) \qquad (2.5)$$

Where k_n is the interest rate for period n.

Example 2.4: Consider a Rs.50, 000 investment in a one year fixed deposit and rolled over annually for the next two years. The interest rate for the first year is 5% annually and the expected interest rate for the next two years are 6% and 6.5% respectively calculate the future value of the investment after 3 years and the average annual interest rate.

Solution:

$$\begin{aligned} FV &= PV\ (1+k_1)(1+k_2)(1+k_3) \\ &= 50{,}000(1+.05)(1+.06)(1+.065) \\ &= 59{,}267.25 \end{aligned}$$

Average annual interest rate

$$\frac{.05+.06+.065}{3}$$

$$= .58333 \text{ (wrong)}$$

By now we know the values of FV, PV, and n. The average annual interest rate would be

$$k = \sqrt[n]{\frac{FV}{PV}}$$

$$k = \sqrt[3]{\frac{59267.25}{50,000}} = \sqrt[3]{1.185345} = 5.8315\%$$

This is also equivalent to

$$k = \sqrt[3]{(1+.05)(1+.06)(1+.065)} - 1$$

$$= 5.8315$$

☞ Check Your Progress 1

1) Calculate the compound value of Rs. 1000, interest rate being 12% per annum, if compounded annually, semi annually, quarterly and monthly for 2 years.

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2) Calculate the future value of Rs. 1000 deposited initially, if the interest is 12% compounded annually for the next five years.

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3) Mr. X bought a share 15 years ago for Rs. 10, the present value of which is Rs. 27.60. Compute the compound growth rate in the price of the share.

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2.3 ANNUITY

An annuity is defined as stream of uniform period cash flows. The payment of life insurance premium by the policyholder to the insurance company is an example of an annuity. Similarly, deposits in a recurring bank account is also an annuity.

Depending on the timing of the cash flows annuities are classified as:

a) Regular Annuity or Deferred Annuity

b) Annuity Due.

The regular annuity or the deferred annuity are those annuities in which the cash flow occur at the end of each period. In case of an annuity due the cash flow occurs at the beginning of the period.

Example 2.5: Suppose Mr. Ram deposits Rs. 10,000 annually in a bank for 5 years, at 10 per cent compound interest rate. Calculate the value of this series of deposits at the end of five years assuming that (i) each deposit occurs at the end of the year (ii) each deposit occurs at the beginning of the year.

Solution: The future value of regular annuity will be

Rs. 1000 $(1.10)^4$ + 1000 $(1.10)^3$ +1000 $(1.10)^2$+1000 (1.10) +1000

= 6105.

The future value of an annuity due will be

Rs. 1000 $(1.10)^5$ + 1000 $(1.10)^4$ +1000 $(1.10)^3$+1000 $(1.10)^2$ +1000 (1.10)

= Rs 1000 (1.611) + 1000 (1.4641) + 1000 (1.331) + 1000 (1.21)+1000 (1.10)

= Rs. 6716.

In the above example you have seen the difference in future value of a regular annuity and annuity due. This difference in value is due to the timing of cash flow. In case of regular annuity the last cash flow does not earn any interest, whereas in the case of annuity due, the cash flows earns an interest for one period.

Formula

In general terms the future value of an annuity (regular annuity) is given by the following formula:

$$\begin{aligned} FVA_n &= A(1+k)^{n-1} + A(1+k)^{n-2} + \dots + A \qquad (2.6) \\ &= A \sum_{t=1}^{n} (1+k)^{n-t} \\ &= A\left[\frac{(1+k)^n - 1}{k}\right] \end{aligned}$$

Future value of an annuity due

$$FVA_{n\,(due)}\ A(1+k)^n + A(1+k)^{n-1} + \dots + A(1+k)$$

$$\begin{aligned} FVA_{n(due)} &= A\sum_{t=1}^{n}(1+k)^{n-t+1} \qquad (2.7) \\ &= A\left[\frac{(1+k)^n - 1}{k}\right](1+k) \end{aligned}$$

Where FVA_n = Future value of an annuity which has a duration of n periods

A = Constant periodic cash flow

k = Interest rate per period

n = duration of the annuity

The term $\left[\frac{(1+k)^n - 1}{k}\right]$ is referred to as the future value interest factor for an annuity ($FVIFA_{k,n}$). The value of this factor for several combinations of k and n are given in the appendix at the end of this unit.

Present Value of an Uneven Series

In real life cash flows occurring over a period of time are often uneven. For example, the dividends declared by the companies will vary from year to year, similarly payment of interest on loans will vary if the interest is charged on a floating rate basis. The present value of a cash flow stream is calculated with the help of the following formula:

$$PV_n = \frac{A_1}{(1+K)} + \frac{A_2}{(1+k)^2} + ... + \frac{A_n}{(1+k)^n} = \sum_{t=1}^{n} \frac{A_t}{(1+k)^t} \qquad (2.8)$$

Where

PV_n = present value of a cash flow stream

A_t = cash flow occurring at the end of the year

k = discount rate

n = duration of the cash flow stream

Shorter Discounting Periods

Sometimes cash flows may have to be discounted more frequently than once a year-semi-annually, quarterly, monthly or daily. The result of this is two fold (i) the number of periods increases (ii) the discount rate applicable per period decreases. The formula for calculating the present value in case of shorter discounting period is

$$PV = FV_n \left[\frac{1}{1+k/m}\right]^{n/m} \qquad (2.9)$$

Where m = number of times per year discounting is done.

Example 2.6: Calculate the present value of Rs. 10,000 to be received at the end of 4 years. The discount rate is 10 percent and discounting is done quarterly.

Solution:

PV = $FV_4 \times$ PVIF k/m, m×n
= 10,000 × PVIF 3%, 16
= 10,000 × 0.623
= Rs. 6230

Determining the Present Value

In the previous sections we have discussed the computation of the future value, now let us work the process in reverse. Let us suppose you have won a lottery ticket worth Rs. 1000 and this Rs. 1000 is payable after three years. You must be interested in knowing the present value of Rs. 1000. If the interest rate is 10 per cent, the present value can be calculated by discounting Rs. 1000 to the present point of time as follows.

$$\text{Value three years hence} = \text{Rs. } 1000 \left(\frac{1}{1.10}\right)$$

$$\text{Value one years hence} = \text{Rs. } 1000 \left(\frac{1}{1.10}\right)\left(\frac{1}{1.10}\right)$$

$$\text{Value now (Present Value)} = \text{Rs. } 1000 \left(\frac{1}{1.10}\right)\left(\frac{1}{1.10}\right)\left(\frac{1}{1.10}\right)$$

Formula

Compounding translates a value at one point in time into a value at some future point in time. The opposite process translates future value into present value. Discounting translates a value back in time. From the basic valuation equation

$$FV = PV\,(1 + k)^n$$

Dividing both the sides by $(1+k)^n$ we get

$$PV = FV \left[\frac{1}{(1+k)}\right]^n \qquad (2.10)$$

The factor $\left[\frac{1}{(1+k)}\right]^n$ is called the discounting factor or the present value interest factor [$PVIF_{k,n}$]

Example 2.7: Calculate the present value of Rs. 1000 receivable 6 years hence if the discount rate is 10 per cent.

Solution: The present value is calculated as follows:

$$\begin{aligned} PV_{kn} &= FV_n \times PVIF_{k,n} \\ &= 1{,}000 \times (\ 0.5645) \\ &= 564.5 \end{aligned}$$

Example 2.8: Suppose you are receiving an amount of Rs.5000 twice in a year for next five years once at the beginning of the year and the other amount of Rs. 5000 at the end of the year, which you deposit in the bank which pays an interest of 12 percent. Calculate the value of the deposit at the end of the fifth year.

Solution: In this problem we have to calculate the future value of two annuities of Rs.5000 having duration of five years. The first annuity is an annuity due and the second annuity is regular annuity, therefore the value of the deposit at the end of five year would be

$$\begin{aligned} & FVA_n + FVA_{n(due)} \\ &= A\left[\frac{(1+k)^n - 1}{k}\right] + A\left[\frac{(1+k)^n - 1}{k}\right](1+k) \\ &= A\,(FVIFA_{12,5}) + A\,(FVIFA_{12,5})\,(1+k) \\ &= 5000\,(6.353) + 5000\,(6.353)(1.12) \\ &= 31{,}765 + 35{,}577 \\ &= 67336 \end{aligned}$$

The value of deposit at the end of the fifth year is Rs. 67,342.

Sinking Fund Factor

Suppose you are interested in knowing how much should be saved regularly over a period of time so that at the end of the period you have a specified amount. To answer this question let us manipulate the equation

$$FVA_n = A\left[\frac{(1+k)^n - 1}{k}\right]$$

which shows the relationship between FVA_n, A, k, and

$$A = \left[\frac{k}{(1+k)^n - 1}\right]^{FVA_n} \tag{2.11}$$

Equation 2.11 helps in answering this question. The periodic deposit is simply A and is obtained by dividing FVAn by $FVIFA_{k,n}$. In eq 2.11 $\left[\frac{k}{(1+k)^n - 1}\right]$ is the inverse of $FVIFA_{k,n}$ and is called the sinking fund factor.

Example 2.9: How much should you save annually so as to accumulate Rs. 20,00,000 by the end of 10 years, if the saving earns an interest of 12 per cent?

Solution:

$$A = FVA_n \left[\frac{k}{(1+k)^n - 1} \right]$$
$$= Rs.20,000 \times \frac{1}{FVIFA_{12\%,10}}$$
$$= Rs.20,00000 \times \frac{1}{17.548}$$
$$= 1,11,400$$

Present value of an annuity

Let us suppose you expect to receive Rs.2000 annually for the next three years. This receipt of Rs.2000 is equally divided. One part viz., Rs.1000 is received at the beginning of the year and the remaining Rs.1000 is received at the end of the year. We are interested in knowing the present value when the discount rate is10 per cent. The cash flows stated above are of two types which are similar to regular annuity and annuity due. The present value of this cash flow is found out as follows:

a) Present value of Rs.1000 received at the end of each year for three years (Regular annuity).

$$Rs.\,1000(\frac{1}{1.10}) + Rs.1000(\frac{1}{1.10})^2 + Rs.1000(\frac{1}{1.10})^3$$
$$1000 \times 09091 + 1000 \times 08264 + 1000 \times 0.7513$$
$$Rs.2479.$$

b) Present value of Rs.1000 received at the beginning of each year for three year (annuity due)

$$Rs.1000 + Rs1000(\frac{1}{1.10}) + Rs.1000(\frac{1}{1.10})^2$$
$$= 1000 + 1000 \times 0.9091 + 1000 \times 08264$$
$$= Rs.2735$$

The present value of this annuity is Rs. 2479+Rs.2735 = Rs. 5214.

Formula

In general terms the present value of a regular annuity may be expressed as follows:

$$PVN_n = \frac{A}{(1+k)} + \frac{A}{(1+k)^2 +} ......+ \frac{A}{(1+k)^n}$$
$$= A\left[\frac{1}{1+k} + \frac{1}{(1+k)^2} + ... \frac{1}{(1+k)^n} \right]$$
$$= A\left[\frac{(1+k)^n - 1}{k(1+k)^n} \right]$$

In case of annuity due

$$PVA_{n(due)} = A\left[\frac{(1+k)^n - 1}{k(1+k)^n}\right](1+k) \qquad (2.12)$$

where PVA_n = Present value of an annuity which has a duration of n periods

A = Constant periodic flows

k = discount rate

Capital Recovery Factor

Equation 2.12 shows the relationship between PVA_n, A, K and n. Manipulating it a bit:

We get

$$A = PVA_n\left[\frac{k(1+k)^n}{(1+k)^n - 1}\right] \qquad (2.13)$$

$\left[\frac{k(1+k)^n}{(1+k)^n - 1}\right]$ in equation 2.13 is inverse of $PVIFA_{k,n}$ and is called the capital recovery factor.

Example 2.10: Suppose you receive a cash bonus of Rs.1,00,000 which you deposit in a bank which pays 10 percent annual interest. How much can you withdraw annually for a period of 10 years.

From eq.2.13

$$A = PVA_n \times \frac{1}{PVIFA_{10\%}10}$$

$$A = \frac{1,00,000}{6.145}$$

$$A = 16,273$$

Present value of perpetuity:

A perpetuity is an annuity of an infinite duration

$$PVA_\infty = A\left[\frac{1}{(1+k)} + \frac{1}{(1+k)^2} + ... + \frac{1}{(1+k)}\infty\right]$$

$$PVA_\infty = A \times PVIFA_{k,\infty}$$

where PVA_∞ = Present value of a perpetuity

A = Constant annual payment

$PVIFA_{k,\infty}$ = Present value interest factor for perpetuity

The value of $PVIFA_{k,\infty}$ is

$$\sum_{t=1}^{\infty} \frac{1}{(1+k)^t} = \frac{1}{k}$$

The present value interest factor of an annuity of infinite duration (perpetuity) is simply 1 divided by interest rate (expressed in decimal form. The present value of an annuity is equal to the constant annual payment divided by the interest rate, for example, the present value of perpetuity of Rs.20, 000 if the interest rate is 10%, is Rs. 2,00,000.

☞ Check Your Progress 2

1) Calculate the present value of Rs. 600 (a) received one year from now (b) received at the end of five years (c) received at the end of fifteen years. Assume a 5% time preference rate.

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2) Mr. Ram is borrowing Rs. 50,000 to buy a motorcycle. If he pays equal installments for 25 years and 4% interest on the outstanding balance, what is the amount of installment? What will be amount of the instalment if quarterly payments are requested to be made?

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3) A bank has offered to pay you an annuity of Rs. 1,800 for 10 years if you invest Rs. 12,000 today. What rate of return would you earn?

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Derivation of Formulas

i) Future Value of an Annuity

Future value of an annuity is

$$FVA_n = A(1+k)^{n-1} + A(1+k)^{n-2} + ...A(1+k) + A \qquad \text{(a1)}$$

Multiplying both sides of the equation a1 by $(1+k)$ gives.

$$(FVA_n)(1+k) = A(1+k)^n + A(1+k)^{n-1} + ...A(1+k)^2 + A(1+k) \qquad \text{(a2)}$$

Subtracting eq. (a1) from eq. (a2) yields

$$FVA_n k = A\left[\frac{(1+k)^n - 1}{k}\right] \qquad \text{(a3)}$$

Dividing both sides of eq. (a3) by k yields

$$FVA_n = A\left[\frac{(1+k)^n - 1}{k}\right]$$

ii) Present Value of an Annuity

The present value of an annuity is

$$PVA_n k = A(1+k)^{-1} + A(1+k)^{-2} + ...+ A(1+k)^{-n} \qquad \text{(a4)}$$

Multiplying both sides of eq (a4) by $(1+k)$ gives:

$$PVA_n(1+k) = A + A(1+k)^{-1} + ...+ A(1+k)^{-n+1} \qquad \text{(a5)}$$

Subtracting eq (a4) from eq (a5) yields:

$$PVa_n k = A\left[1-(1+k)^{-n}\right] = A\left[\frac{(1+k)^n - 1}{k(1+k)^n}\right] \qquad \text{(a6)}$$

Dividing both the sides of eq (a6) by k results in:

$$PVA_n = A\left[\frac{(1+k)^n - 1}{k(1+k)^n}\right]$$

iii) **Present Value of a Perpetuity**

$$PVA_\infty = A(1+k)^{-1} + A(1+k)^{-2} + ...+ A(1+k)^{\infty+1} + A(1+k)^{\infty} \qquad (a7)$$

Multiplying both the sides of eq (a7) by (1+k) gives:

$$PVA_\infty = (1+k) = A(1+k) + A(1+k)^{-1} + ...A(1+k)^{-\infty+2} + A(1+k)^{-\infty+1} \qquad (a8)$$

subtracting eq (a7) from eq (a8) gives:

$$PVA_\infty k = A[1-(1+k)^{\infty}]$$

$$\text{As}\,(1+k)^{-\infty} \to o\ \text{eq.}(a8)\ \text{becomes:}$$

$$PVA_\infty k = A$$

$$\Rightarrow PVA_\infty = \frac{A}{k} \qquad (a9)$$

iv) **Continuous Compounding**

In Section 2.2.2 we had established a relationship between the effective and nominal rate of interest where compounding occur n times a year which is as follows:

$$r = (1+\frac{k}{m})^m - 1 \qquad (a10)$$

Rearranging equation a10, it can be expressed as

$$r = [(1+\frac{k}{m/k})^{m/k}]^k - 1 \qquad (a11)$$

Let us substitute m/k by x om eq (a11)

$$r = [(1+\frac{1}{x})^k] - 1 \qquad (a12)$$

In continuous compounding $m \to \infty$ which implies $x \to \infty$ in eq (a12)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = 2.71828...$$

From equation (a12) results in

$$R = e^k - 1$$

$$\Rightarrow (r+1) = e^k$$

Thus the future value of an amount when continuous compounding is done is as follows:

$$FV_n = PV \times e^{km} \qquad (a13)$$

v) **Continuous Discounting**

From eq (a12)

$$PV = FV_n \times e^{-km}$$

2.4 SUMMARY

Individuals generally prefer possession of cash right now or in the present moment rather than the same amount at some time in the future. This time preference is basically due to the following reasons: (a) uncertainty of cash flows (b) preference for current consumption (c) availability of investment opportunities. In case an investor opts to receive cash in future s/he would demand a risk premium over and above the risk free rate as compensation for time to account for the uncertaininty of cash flows. Compounding and discounting are techniques to facilitate the comparison of cash flows occurring at different time periods. In compounding future value of cash flows at a given interest rate at the end of a given period of time are cash flows at a given interest rate at the beginning of a given period of time is found out. An annuity is a series of periodic cash flows of equal amount. Perpetuity is an annuity of infinite duration. *Table 2.1* depicts the various formulas used for discounting and compounding.

Table 2.1: Summary of Discounting and Compounding Formulas

Purpose compound value of a lump sum	Given PV Present Value	Calculate FV_n Future value n years hence	Formula $FV_n = PV\ (1+k)^n$
Doubling Period Compound value of a lump sum with shorter compounding period	Interest Rate PV and frequency of compounding (m)	Time Required to double an amount Future value after n year (FV_n)	$0.35 + \frac{69}{\text{Interest Rate}}$ $FV_n = PV\,(1 + \frac{k}{m})^{m \times n}$
Relationship between effective and nominal rate	Nominal interest rate (K) and frequency of compounding (m)	Effective interest rate (R)	$r = (1 + \frac{k}{m})^m - 1$
Present value of a single amount	Future value (FV_n)	Present Value (PV)	$PV_n = FV_n\ (\frac{1}{1+k})^n$
Future value of a regular annuity	Constant periodic cash flow (A) interest rate (k) and duration (n)	Further value of a regular annuity (FVA_n)	$FVA_n = A[\frac{(1+k)^n - 1}{k}]$
Future value of a annuity due	Constant periodic cash flow (A) interest rate (k) and duration (n)	Future value of an annuity due FVA_n (due)	$FVA_{n(due)} = A[\frac{(1+k)^n - 1}{k}](1+k)$
Present value of a regular annuity	Constant periodic cash flow (A) interest rate (k) and duration (n)	Present value of a regular annuity PVA_n	$PVA_n = A\left[\frac{(1+k)^n - 1}{k\,(1+k)^n}\right]$
Present value of an annuity due	Constant periodic cash flow (A) interest rate (k) and duration (n)	Present value of an annuity due PVA_n (due)	$PVA_{n(due)} = A[\frac{(1+k)^n - 1}{k(1+k)^n}](1+k)$
Present value of a perpetuity	Constant cash flows (A) and interest rate (k)	Present value of an perpetuity PVA_∞	$PVA_\infty = \frac{A}{K}$

2.5 SELF-ASSESSMENT QUESTIONS

1) If you are offered two investments, one that pays 5% simple interest per year and one that pays 5% compound interest per year, which would you choose? Why?

2) Suppose you make a deposit today in a bank account that pays compounded interest annually. After one, year, the balance in the account has grown.

   a) What has caused it to grow?

   b) After two year's the balance in the account has grown even more what has caused the balance to increase during the second year?

3) The Florida Lottery pays out winnings, after taxes, on the basis of 20 equal annual installments, providing, the first installment at that time when the winning ticket is turned in.

   a) What type of cash flow pattern is the distribution of lottery winnings?

   b) How would you value such winnings?

4) Rent is typically paid at the first of each month. What pattern of cash flow, an ordinary annuity or an annuity due, does a rental agreement follow?

5) a) Under what conditions does the effective annual rate of interest (EAR) differ from the annual percentage rate (APR)?

   b) As the frequency of compounding increases within the annual period what happens to the relation between the EAR and the APR?

6) Using the appropriate table, calculate the compound factor for each of the following combinations of interest rate per period and number of compounding periods:

Number of Periods	Interest rate per Period	Compound Factor
2	2%	-
4	3%	-
3	4%	-
6	8%	-
8	6%	-

7) Using the appropriate table, calculate the discount factor for each of the following combinations of interest rate per period and number of discounting periods.

Number of Periods	Interest Rate per Period	Discount Factor
2	2%	-
4	3%	-
3	4%	-
6	8%	-
8	6%	-

8) Using the appropriate table, calculate the future value annuity factor for each of the following combinations of interest rate per period and number of payments:

Number of Periods	Interest Rate per Period	Discount Factor
2	2%	-
4	3%	-
3	4%	-
6	8%	-
8	6%	-

9) Using the appropriate table, calculate the present value annuity factor for each of the following combinations of interest rate per period and number of payments:

Number of Periods	Interest Rate per Periods	Discount Factor
2	2%	-
4	3%	-
3	4%	-
6	8%	-
8	6%	-

10) Using an 8% compounded interest rate per period calculate the future value of

a) Rs.100 investment
b) one period into the future
c) two periods into the future
d) three periods into the future
e) four periods into the future
f) five periods into the future
g) 40 periods into the future.

11) Suppose you deposit Rs.1,000 into a savings account that earns interest at the rate of 4% compounded annually, what would the balance in the account be:

a) after two years
b) after four years
c) after six years
d) after 20 years

12) You deposit Rs.10,000 in an account that pays 6% compounded interest per period, assuming no withdrawal:

a) What will be the balance in the account after two periods?
b) after the two periods, how much interest has been paid on the principal amount?
c) After the two periods, how much interest has been paid on interest?

13) Using an 8% compounded interest rate, calculate the present value of Rs.100 to be received:

a) one period into the future
b) two periods into the future
c) three periods into the future
d) four periods into the future
e) five periods into the future
f) 40 periods into the future

14) Ted wants to borrow from Fred. Ted is confident that he will have Rs.1, 000 available to pay off Fred in two years. How much will Fred be willing to lend to Ted in return for Rs.1,000 two years from now if he uses a compounded interest rate per year of:
(a) 5% (b) 10% (c) 15%?

15) How much would you have to deposit into a savings account that earns 2% interest compounded quarterly, to have a balance of Rs. 2,000 at the end of four years if you make no withdrawals?

16) What is the present value of Rs.5, 000 to be received five years from now, if the nominal annual interest rate (APR) is 12 % and interest is compounded:
(a) Annually (b) Semiannually (c) Quarterly (d) Monthly

17) Calculate the future value at the end of the second period of this series of end-of period cash flows, using an interest rate of 10% compounded per period:

Year	End of Year Cash Flow
Year 1	Rs. 2,000
Year 2	Rs. 3,000
Year 3	Rs. 4,000
Year 4	Rs. 5,000

18) An investor is considering the purchase of an investment at the end of Year 0 that will yield the following cash flows:

Year	End of Year Cash Flow
Year 1	Rs. 2,000
Year 2	Rs. 3,000
Year 3	Rs. 4,000
Year 4	Rs. 5,000

If the appropriate discount rate for this investment is 10%, what will this investor be willing to pay for this investment.

19) Calculate the present value (that is the value at the end of period 0) of the following series of end of period cash flows:

Year	End of Year Cash Flow
0	Rs.1,000
Year 2	Rs. 200
2	Rs. 400

20) Suppose the investment promises to provide the following cash flows:

Year	End of Year Cash Flow
Year 1	Rs.0
Year 2	Rs.1,000
Year 3	Rs.0
Year 4	Rs.1,000

If interest is compounded annually at 5% what is the value of the investment at the end of: (a) Year 1 (b) Year 0

21) Calculate the future value at the end of the third period of an ordinary annuity consisting of three cash flows of Rs.2,000 each. Use a 5% rate of interest per period.

22) What is the present value of Rs.10 to be received each period forever, if the interest rate is 6%?

23) If an investor is willing to pay Rs.40 today to receive Rs.2 every year forever, what is this investor's opportunity cost used to value this investment?

24) Calculate the present value of an annuity due consisting of three cash flows of Rs.1,000 each, one year apart. Use a 6% compounded interest rate per year.

25) Calculate the future value at the end of the third period of an annuity due, consisting of three cash flows of Rs.1,000 each, each one year apart. Use a 6% compounded interest rate per year.

26) Suppose you have won the Florida Lotto worth Rs.18 million. Further suppose that the State of Florida will pay you the winnings in 20 annual installments,

starting immediately, of Rs.9,00,000 each. If your opportunity cost is 10% what is the value today of these 20 installments?

27) Calculate the required deposit to be made today so that a series of ten withdrawals of Rs.1,000 each can be made beginning five years from today. Assume an interest rate of 5% per period of end of period balances.

28) How much would you need to deposit today so that you can withdraw Rs. 4,000 per year for ten years, starting three years from today?

29) Suppose you wish to invest Rs. 2,000 today so that you have Rs. 4,000 six years from now. What must the compounded annual interest rate be in order to achieve your goal?

2.6 SOLUTIONS/ANSWERS

Check Your Progress 1

1) i) Annual Compounding Rs. 1,254.
   ii) Half year Compounding Rs. 1,262.
   iii) Quarterly Compounding Rs. 1,267.
   iv) Monthly Compounding Rs. 1, 270.

2) Rs. 1,806

3) 7%

Check Your Progress 2

1) a) Rs. 571.20 b) 470.50 c) 288.60

2) Equal yearly instalment = Rs. 3200.61
   Equal quarterly instalment = Rs. 793.28

3) 8.15%