UNIT 2 VIEWING TRANSFORMATION

Structure **Page Nos.**

2.0 INTRODUCTION

In unit 1, we have discussed the geometric transformations such as Translation, Rotation, Reflection, Scaling and Shearing. Translation, Rotation and Reflection transformations are used to manipulate the given object, whereas Scaling and Shearing transformations are used to modify the shape of an object, either in 2-D or in 3-Dimensional.

A transformation which maps 3-D objects onto 2-D screen, we are going to call it *Projections.* We have two types of Projections namely, *Perspective projection* and *Parallel projection*. This categorisation is based on the fact whether rays coming from the object converge at the centre of projection or not. If, the rays coming from the object converge at the centre of projection, then this projection is known as *Perspective projection*, otherwise it is *Parallel projection*. In the case of parallel projection the rays from an object converge at infinity, unlike perspective projection where the rays from an object converge at a finite distance (called COP).

Parallel projection is further categorised into *Orthographic* and *Oblique projection*. Parallel projection can be categorized according to the angle that the direction of projection makes with the projection plane If the direction of projection of rays is perpendicular to the projection plane then this parallel projection is known as *Orthographic projection* and if the direction of projection of rays is not perpendicular to the projection plane then this parallel projection is known as *Oblique projection*. The orthographic (perpendicular) projection shows only the front face of the given object, which includes only two dimensions: length and width. The oblique projection, on the other hand, shows the front surface and the top surface, which includes three dimensions: length, width, and height. Therefore, an oblique projection is one way to show all three dimensions of an object in a single view.

Isometric projection is the most frequently used type of *axonometric projection*, which is a method used to show an object in all three dimensions (length, width, and height) in a single view. Axonometric projection is a form of orthographic projection in which the projectors are always perpendicular to the plane of projection.

2.1 OBJECTIVES

After going through this unit, you should be able to:

- define the projection;
- categorize various types of Perspective and Parallel projections;
- develop the general transformation matrix for parallel projection;
- describe and develop the transformation for Orthographic and oblique parallel projections;
- develop the transformations for multiview (front, right, top, rear, left and bottom view) projections;
- define the foreshortening factor and categorize the oblique projection on the basis of foreshortening factors;
- derive the transformations for general perspective projection;
- describe and derive the projection matrix for single-point, two-point and three-point perspective transformations, and
- identify the vanishing points.

2.2 PROJECTIONS

Given a 3-D object in a space, Projection can be defined as a mapping of 3-D object onto 2-D viewing screen. Here, 2-D screen is known as Plane of projection or view plane, which constitutes the display surface. The mapping is determined by projection rays called the projectors. Geometric projections of objects are formed by the intersection of lines (called projectors) with a plane called plane of projection /view plane. Projectors are lines from an arbitrary point, called the centre of projection (COP), through each point in an object. *Figure 1* shows a mapping of point P(x,y,z) onto its image P′(x',y',z') in the view plane.

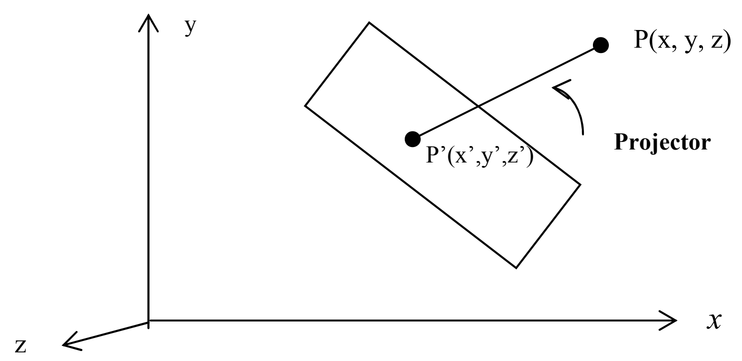

Figure 1

If, the COP (Center of projection) is located at finite point in the three-space, the result is a perspective projection. If the COP is located at infinity, all the projectors are parallel and the result is a parallel projection. *Figure 2(a)-(b)* shows the difference between parallel and perspective projections. In *Figure 2(a)*, **ABCD is projected to A'B'C'D' on the plane of projection and O is a COP**. In the case of parallel projection the rays from an object converges at infinity, the rays from the object become parallel and will have a direction called "direction of projection".

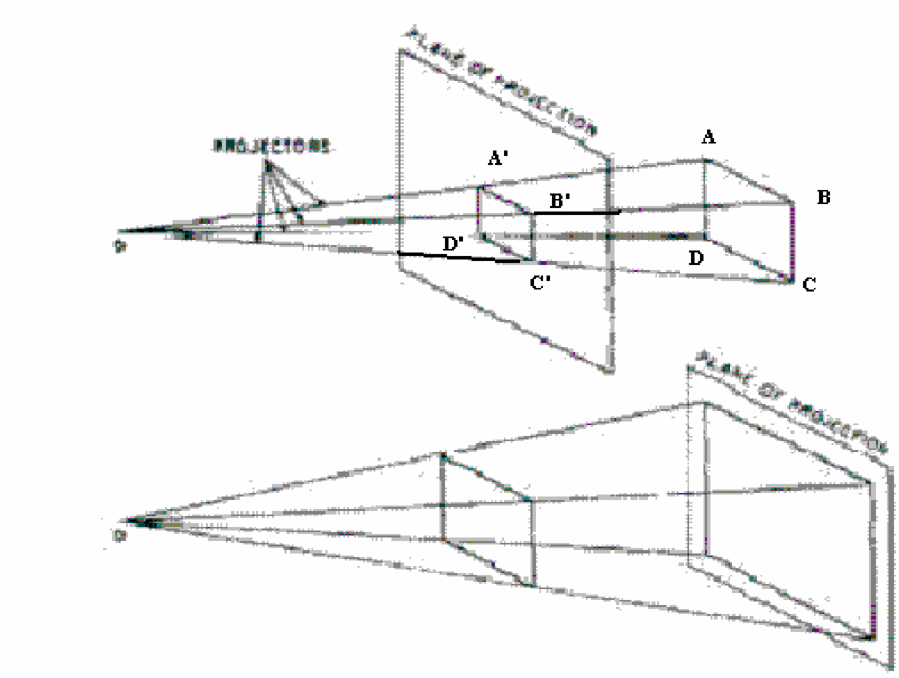

Figure 2(a): Perspective projection

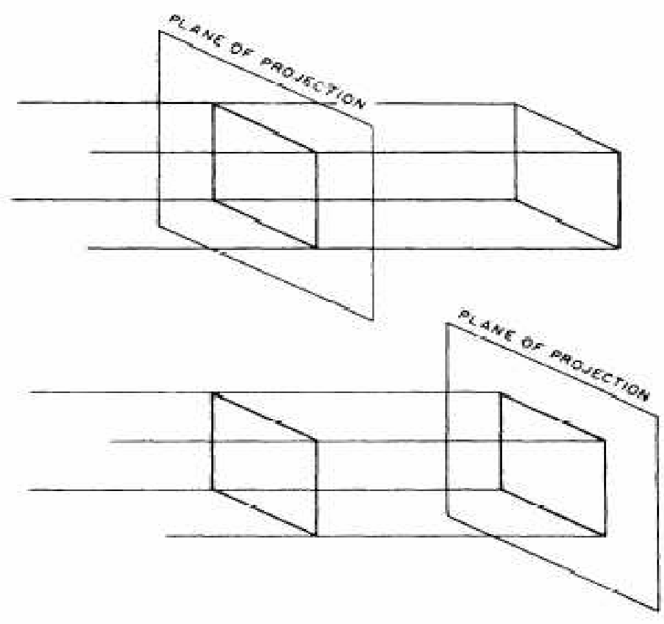

Figure 2(b): Parallel projection

Taxonomy of Projection

There are various types of projections according to the view that is desired. The following *Figure 3* shows taxonomy of the families of *Perspective* and *Parallel* Projections. This categorisation is based on whether the rays from the object converge at COP or not and whether the rays intersect the projection plane perpendicularly or not. The former condition separates the perspective projection from the parallel projection and the latter condition separates the Orthographic projection from the Oblique projection.

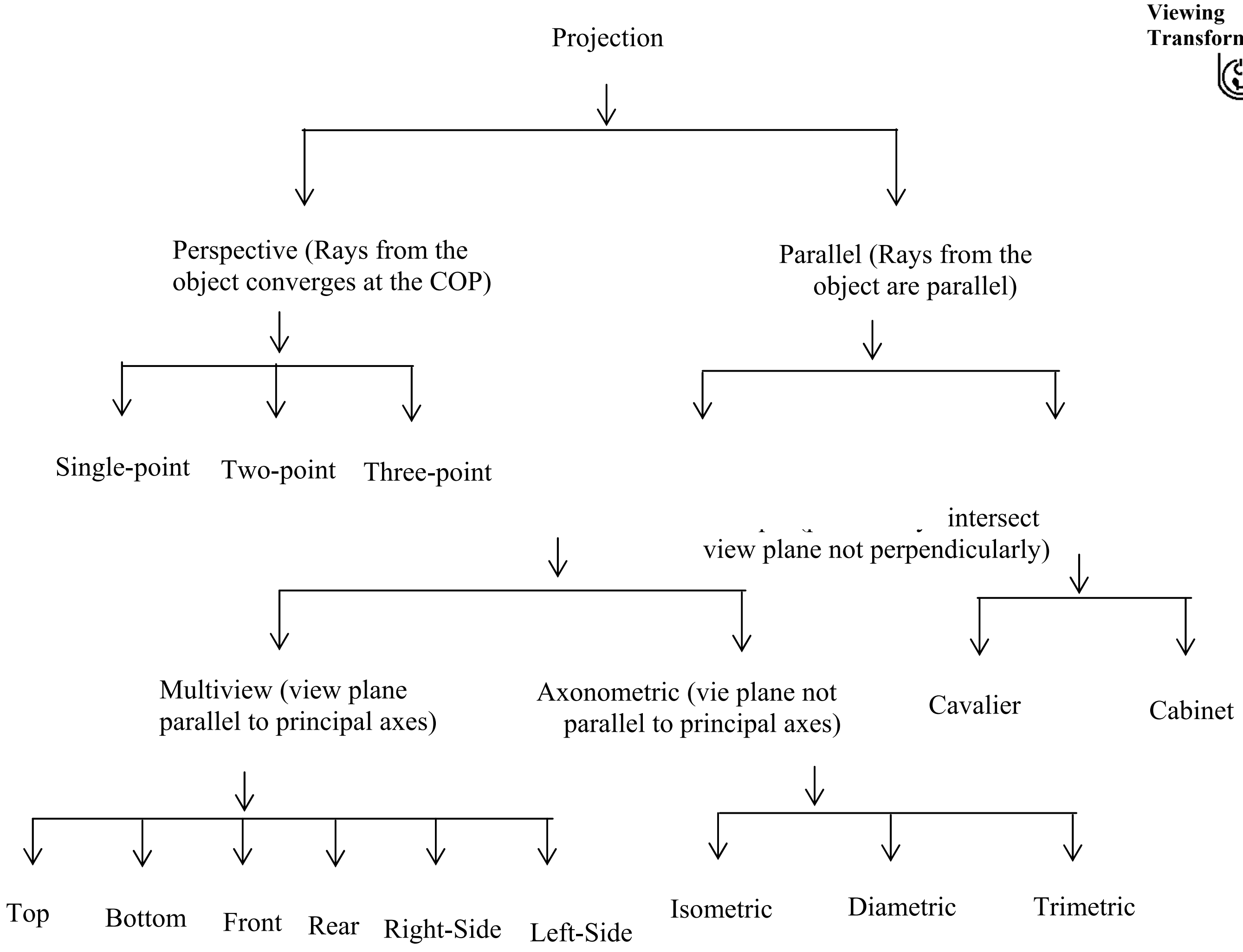

Figure 3: Taxonomy of projection

The direction of rays is very important only in the case of Parallel projection. On the other hand, for Perspective projection, the rays converging at the COP, they do not have a fixed direction i.e., each ray intersects the projection plane with a different angle. For Perspective projection the direction of viewing is important as this only determines the occurrence of a vanishing point.

2.2.1 Parallel Projection

Parallel projection methods are used by engineers to create working drawings of an object which preserves its true shape. In the case of parallel projection the rays from an object converge at infinity, unlike the perspective projection where the rays from an object converse at a finite distance (called COP).

If the distance of COP from the projection plane is infinite then parallel projection (all rays parallel) occurs i.e., when the distance of COP from the projection plane is infinity, then all rays from the object become parallel and will have a direction called "**direction of projection**". It is denoted by **d**=(d1,d2,d3), which means **d** makes unequal/equal angle with the positive side of the x,y,z axes.

Parallel projection can be categorised according to the angle that the direction of projection makes with the projection plane. For example, in Isometric projection, the direction of projection **d**=(d1,d2,d3) makes equal angle (say α) with all the three-principal axes (see *Figure 4*).

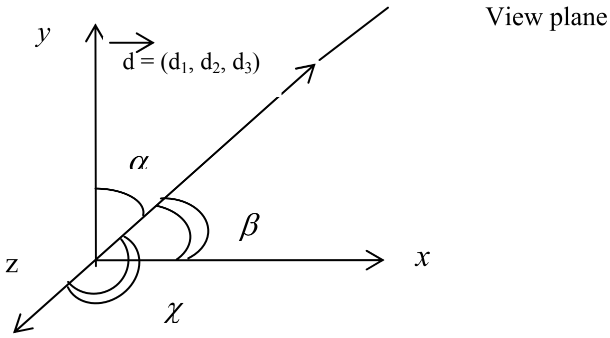

Figure 4: Direction of projection

Rays from the object intersect the plane before passing through COP. In parallel projection, image points are found as the intersection of view plane with a projector (rays) drawn from the object point and having a fixed direction.(see *Figure 5*).

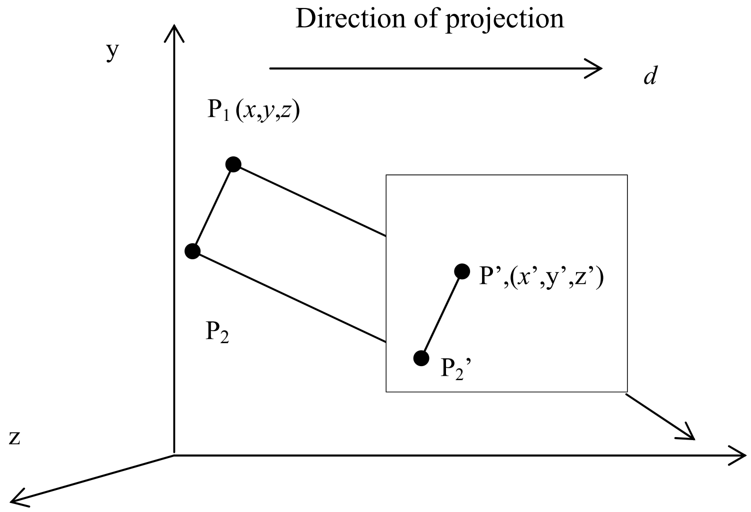

Figure 5: Parallel projection

Parallel rays from the object may be perpendicular or may not be perpendicular to the projection plane. If the direction of projection **d**=(d1,d2,d3) of the rays is perpendicular to the projection plane (or **d** has the same direction as the view plane normal **N**), we have *Orthographic projection* otherwise *Oblique projection.*

Orthographic projection is further divided into *Multiview projection* and *axonometric projection*, depending on whether the direction of projection of rays is parallel to any of the principal axes or not. If the direction of projection is parallel to any of the principal axes then this produces the *front*, *top* and *side views* of a given object, also referred to as *multiview* drawing (see *Figure 8*).

Axonometric projections are orthographic projection in which the direction of projection is not parallel to any of the 3 principle axes. *Oblique* projections are non-orthographic parallel projections i.e., if the direction of projection **d**=(d1,d2,d3) is not perpendicular to the projection plane then the parallel projection is called an *Oblique* projection.

Transformation for parallel projection

Parallel projections (also known as Orthographic projection), are projections onto one of the coordinate planes x = 0, y = 0 or z = 0. The standard transformation for parallel (orthographic) projection onto the xy-plane (i.e. z=0 plane) is:

$P_{par,z}$ =

$$\begin{cases} x' = x \\ y' = y \\ z' = 0 \end{cases}$$

In matrix form:

$$P_{par,z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------------------------(1)}$$

Thus, if P(x,y,z) be any object point in space, then projected point P'(x'y'z') can be obtained as:

$$(x', y', z, 1) = (x, y, z, 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------------------------(2)}$$

$$P'_h = P_h . P_{par,z} \qquad \text{-----------------------------(3)}$$

Example1: Derive the general transformation of parallel projection onto the xy-plane in the direction of projection d=aI+bJ+cK.

Solution: The general transformation of parallel projection onto the xy-plane in the direction of projection d=a**I**+b**J**+c**K,** is derived as follows(see *Figure a*):

Let P(x,y,z) be an object point, projected to P'(x',y',z') onto the z'=0 plane. From *Figure (a)* we see that the vectors **d** and **PP'** have the same direction. This means that

PP'=k.**d** , comparing components, we have:

x'-x=k.a
y'-y=k.b
z'-z=k.c

Since z'=0 on the projection plane, we get k=-z/c.
Thus,

x'=x-a.z/c
y'=y-b.z/c
z'=0

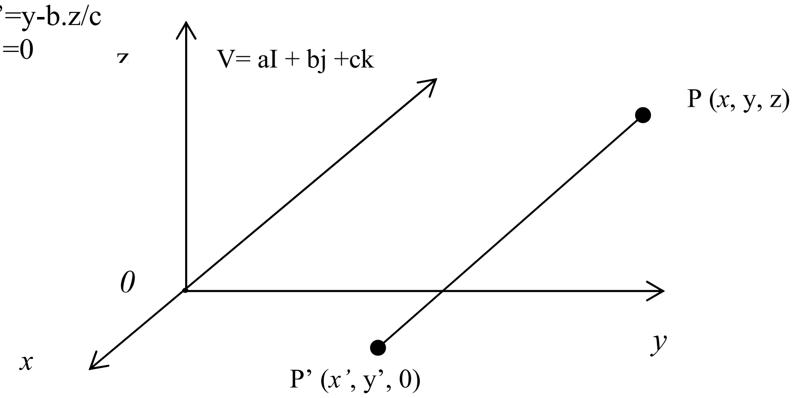

Figure (a)

In terms of homogeneous coordinates, this equation can be written as:

$$(x',y',z'1) = (x, y, z, 1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a/c & -b/c & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{-----------------------------(4)}$$

That is, $P'_h = P_h.P_{par,z}$, where $P_{par,z}$ is the parallel projection with the direction of projection **d** along the unit vector k.

Example 2: Derive the general transformation for parallel projection onto a given view plane, where the direction of projection d=aI+bJ+cK is along the normal $N=n_1I+n_2J+n_3K$ with the reference point $R_0(x_0,y_0,z_0)$.

Solution: The general transformation for parallel projection onto the xy-plane in the direction of projection *Figure (b)*
v = a I + bJ + ck, denoted by P par, V, N, Ro, consists of the following steps:

1) Translate the view reference point R_o of the view plane to the origin, by $T\text{-}R_o$
2) Perform an alignment transformation An so that the view normal vector N of the view points in the direction K of the normal to the xy-plane. The direction of projection vector V is transformed to new vector V' = AnV.
3) Project onto the xy-plane using P par, v'
4) Align k back to N, using An.
5) Translate the origin back to Ro, by T_{Ro}

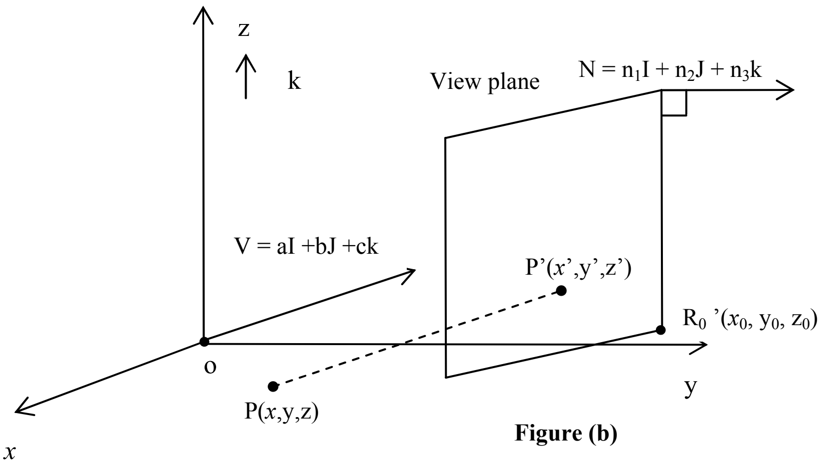

Figure (b)

So
P_{par}, V, N, R_o = T–Ro A_N^{-1} . $P_{par,\,v}$'. An. T_{Ro}

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_0 & -y_0 & -z_0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\lambda}{|N|} & \frac{-n_1n_2}{|N|} & \frac{-n_1n_3}{} & 0 \\ 0 & \frac{n_3}{\lambda} & \frac{n_2}{\lambda} & 0 \\ \frac{n_1}{|N|} & \frac{n_2}{|N|} & \frac{n_3}{|N|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-a}{c} & 1 & 0 & 0 \\ \frac{-a}{c} & \frac{-b}{c} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\lambda}{|N|} & 0 & \frac{n_2}{|N|} & 0 \\ \frac{-n_1 n_2}{|N|} & \frac{n_3}{\lambda} & \frac{n_2}{|N|} & 0 \\ \frac{-n_1 n_3}{\lambda\,|N|} & \frac{n_2}{|N|} & \frac{n_3}{|N|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_0 & y_0 & z_0 & 1 \end{pmatrix}$$

where λ =

$\sqrt{n_2^2 + n_3^2}$ and $\lambda \neq 0$.

After multiplying all the matrices, we have:

$$P\ par, V, N, R_0 = \begin{pmatrix} d_1 - an_1 & -bn_1 & -cn_1 & 0 \\ -an_2 & d_1 - bn_2 & -cn_2 & 0 \\ -an_3 & -bn_3 & d_1 - cn_3 & 0 \\ ad_0 & bd_0 & c.do & d_1 \end{pmatrix} \text{---------------------------(5)}$$

Where $d_0 = n_1 x_0 + n_2 y_0 + n_3 z_0$ and
$d_1 = n_1 a + n_2 b + n_3 c$

Note: Alignment transformation, An, refer any book for computer graphic.

2.2.1.1 Orthographic and Oblique Projections

Orthographic projection is the simplest form of parallel projection, which is commonly used for engineering drawings. They actually show the 'true' size and shape of a single plane face of a given object.

If the direction of projection **d**=(d1,d2,d3)has the direction of view plane normal **N** (or **d** is perpendicular to view plane**)**, the projection is said to be *orthographic*. Otherwise it is called *Oblique* projection. The *Figure 6* shows the orthographic and oblique projection.

We can see that the orthographic (perpendicular) projection shows only front surface of an object, which includes only two dimensions: length and width. The oblique projection, on the other hand, shows the front surface and the top surface, which includes three dimensions: length, width, and height. Therefore, an oblique projection is one way to show all three dimensions of an object in a single view

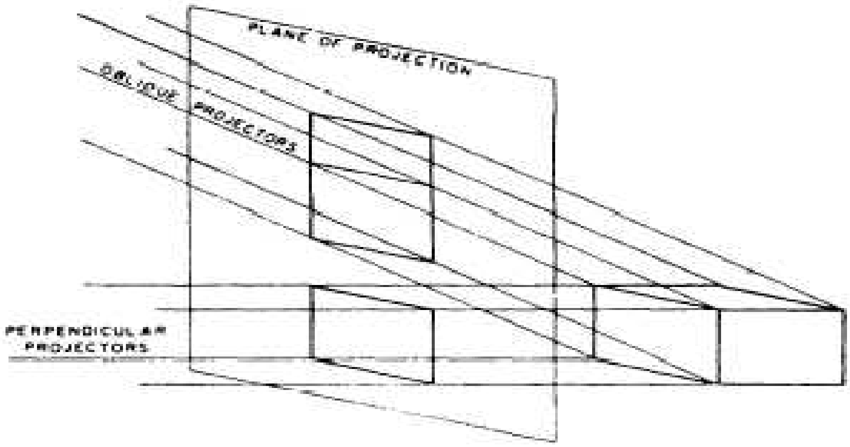

Figure 6: Orthographic and oblique projection

Orthographic projections are projections onto one of the coordinate planes x=0, y=0or z=0. The matrix for orthographic projection onto the z=0 plane (i.e. xy-plane) is:

$$P_{par,z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{------------------------------}(6)$$

Note that the z-column (third column) in this matrix is all zeros. That is for orthographic projection onto the z=0 plane, the z-coordinates of a position vector is set to zero. Similarly, we can also obtain the matrices for orthographic projection onto the x=0 and y=0 planes as:

$$P_{par,x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{------------------------------}(7)$$

and

$$P_{par,y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{------------------------------}(8)$$

For example, consider the object given in *Figure 6(a).* The orthographic projections of this object onto the x=0, y=0 and z=0 planes from COP at infinity on the +x-, +y- and +z-axes are shown in *Figure 7* (b)-(d).

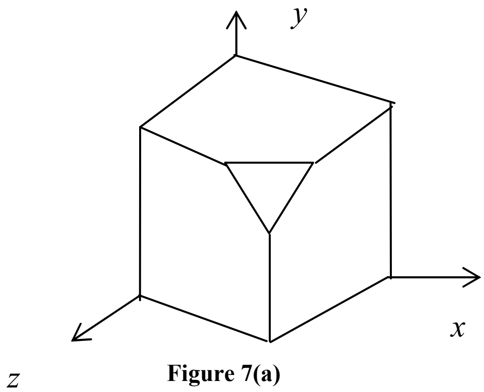

Figure 7(a)

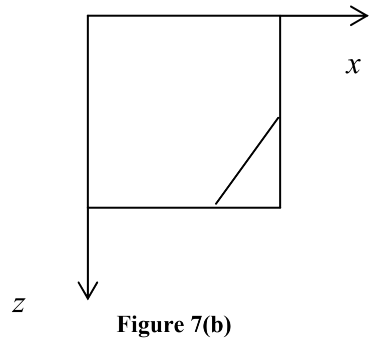

Figure 7(b)

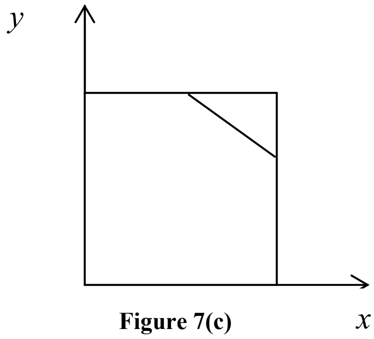

Figure 7(c)

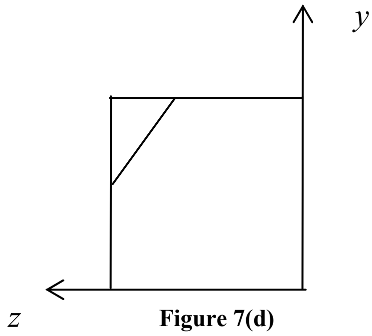

Figure 7(d)

A single orthographic projection does not provide sufficient information to visually and practically reconstruct the shape of an object. Thus multiple orthographic projections are needed (known as *multiview drawing*). In all, we have 6 views:

1) Front view
2) Right-side view
3) Top-view
4) Rear view
5) Left-side view
6) Bottom view

The *Figure 8* shows all 6 views of a given object.

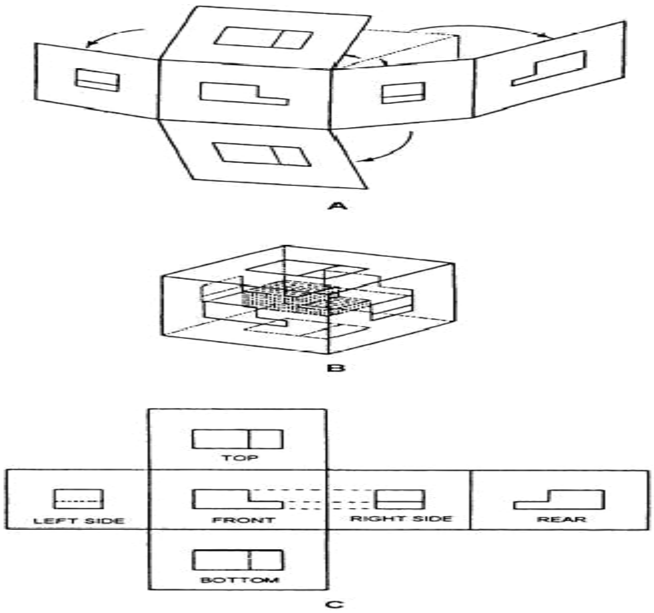

Figure 8: Multiview orthographic projection

The front, right-side and top views are obtained by projection onto the z=0, x=0 and y=0 planes from COP at infinity on the +z-, +x-, and +y-axes.

The rear, left-side and bottom view projections are obtained by projection onto the z=0, x=0, y=0 planes from COP at infinity on the –z-, -x and –y-axes(see *Figure 8*). All six views are normally not required to convey the shape of an object. The front, top and right-side views are most frequently used.

The direction of projection of rays is shown by arrows in *Figure 9*.

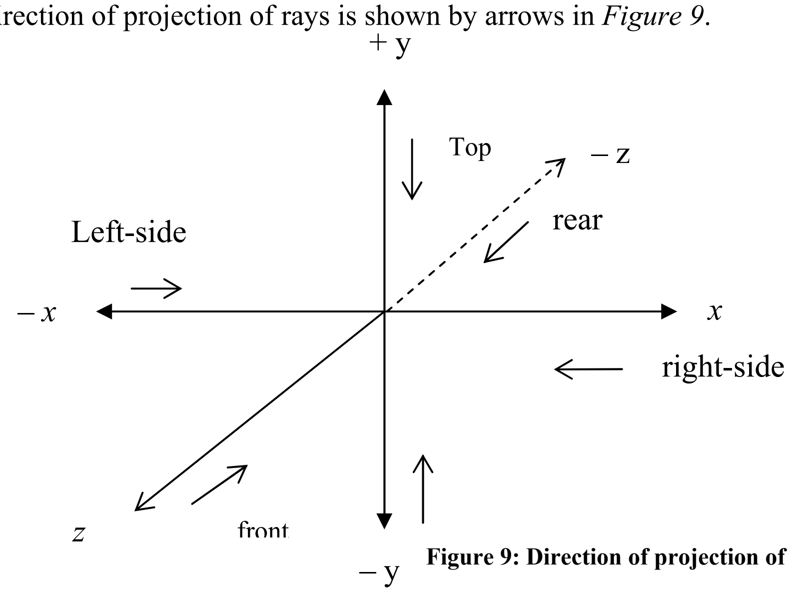

Figure 9: Direction of projection of rays in multiview drawing

The projection matrices for the front, the right-side and top views are given by:

$P_{front} = P_{par,z} = diag(1,1,0,1)$
$P_{right} = P_{par,x} = diag(0,1,1,1)$
$P_{top} = P_{par,y} = diag(1,0,1,1)$

It is important to note that the other remaining views can be obtained by combinations of reflection, rotation and translation followed by projection onto the z=0 plane from the COP at infinity on the +z-axis. For example: the rear view is obtained by reflection through the z=0 plane, followed by projection onto the z=0 plane.

$P_{rear} = M_{xy}.\ P_{par,z}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{-------------------------(9)}$$

Similarly, the left-side view is obtained by rotation about the y-axis by $+90^0$, followed by projection onto the z=0 plane.

$P_{left} = [R_y]_{90^0}.P_{par,z}$

$$= \begin{pmatrix} \cos 90 & 0 & -\sin 90 & 0 \\ 0 & 1 & 0 & 0 \\ \sin 90 & 0 & \cos 90 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{------------------(10)}$$

And the bottom view is obtained by rotation about the x-axis by -90^0, followed by projection onto the z=0 plane.

$P_{bottom} = [R_x]_{90^0}.P_{par,z}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-90) & \sin(-90) & 0 \\ 0 & -\sin(-90) & \cos(-90) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{-----------(11)}$$

Example 3: Show all the six views of a given object shown in following *Figure*. The vertices of the object are A(4,0,0), B(4,4,0), C(4,4,8), D(4, 0, 4), E (0,0,0), F(0,4,0), G(0,4,8), H(0,0,4).

Solution: We can represent the given object in terms of Homogeneous-coordinates of its vertices as:

$$V = [ABCDEFGH] = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{pmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{pmatrix}$$

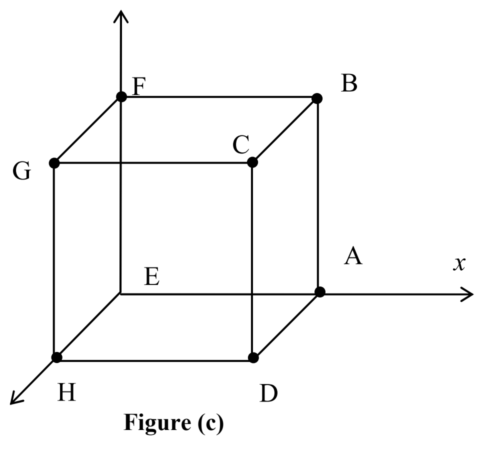

Figure (c)

(1) If we are viewing from the front, then the new coordinate of a given object can be found as:

$P'_{n,z} = P_n . P_{front}$

$$\begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array}\begin{pmatrix} x'1 & y'1 & 1 \\ x'2 & y'2 & 1 \\ x'3 & y'3 & 1 \\ x'4 & y'4 & 1 \\ x'5 & y'5 & 1 \\ x'6 & y'6 & 1 \\ x'7 & y'7 & 1 \\ x'8 & y'8 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array}\begin{pmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & \end{pmatrix}$$

from matrix, we can see that
A′ = D′, B′ = C′, E′ = H′, F′ = G′, Thus we can see only C′D′G′H′
as shown in *Figure d*

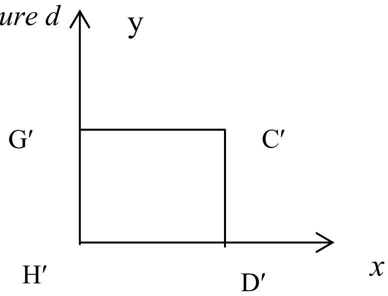

Figure d

(2) If we are viewing from right-side, then

$$P'_{n,x} = V._{\text{Pright}} = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}\begin{pmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 4 & 1 \end{pmatrix} . \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D \\ E' \\ F' \\ G' \\ H' \end{array}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{pmatrix}$$

Here, we can see that A’ =E’, B’ = F’, C’ = G’ and D’ = H’.
Thus, we can see only A’B’C’D’ as shown in *Figure e*.

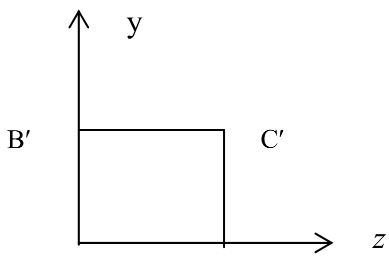

Figure e

(3) if we are viewing from top, then

$$P'_{n,y} = P_n . P_{top} = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{bmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{bmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 4 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 4 & 0 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

Here, we can see that A' = B', E' = F', C' $\neq$ D' and G' $\neq$ H'
Thus we can see only the square B'F'G'C' but the line H'D' is hidden an shown in *Figure f.*

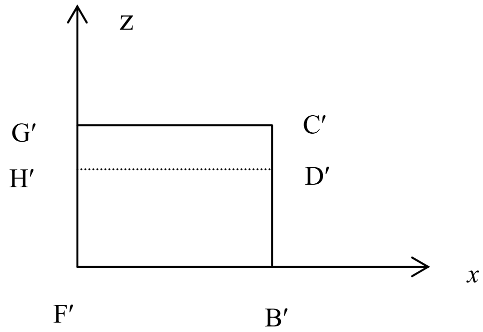

Figure f

Similarly we can also find out the other side views like, rear left-side and bottom using equation – 1, 2, 3

☞ Check Your Progress 1

1) Define the following terms related with Projections with a suitable diagram:

a) Center of Projection (COP)
b) Plane of projection/ view plane
c) Projector
d) Direction of projection

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2) Categories the various types of parallel and perspective projection.

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3) In orthographic projection
   a) Rays intersect the projection plane.
   b) The parallel rays intersect the view plane not perpendicularly.
   c) The parallel rays intersect the view plane perpendicularly.
   d) none of these

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Oblique projection

If the direction of projection **d**=(d1,d2,d3) of the rays is not perpendicular to the view plane(or **d** does not have the same direction as the view plane normal **N**), then the parallel projection is called an *Oblique projection* (see *Figure 10*).

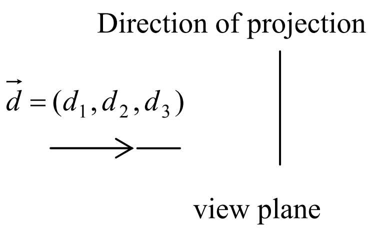

Figure 10 (a): Oblique projection

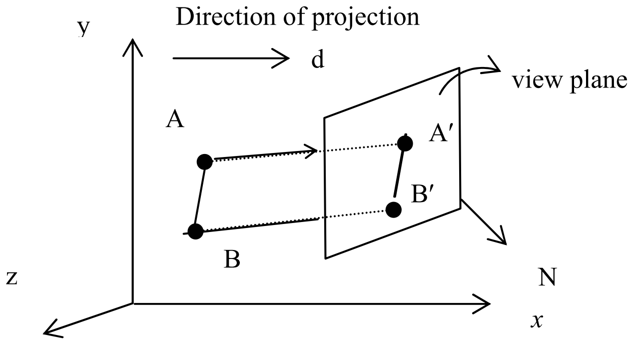

Figure 10 (b): Oblique projection

In oblique projection only the faces of the object parallel to the view plane are shown at their true size and shape, angles and lengths are preserved for these faces only. Faces not parallel to the view plane are discarded.

In Oblique projection the line perpendicular to the projection plane are *foreshortened* (shorter in length of actual lines) by the direction of projection of rays. The direction of projection of rays determines the amount of foreshortening. The change in length of the projected line (due to the direction of projection of rays) is measured in terms of foreshortening factor, f.

Foreshortening factors w.r.t. a given direction

Let AB and CD are two given line segments and direction of projection **d**=(d1,d2,d3). Also assumed that AB $\|$ CD $\|$ **d** . Under parallel projection, let AB and CD be projected to A’B’ and C’D’, respectively.

Observation:

i) A’B’ $\|$ C’D’ will be true, i.e. Parallel lines are projected to parallel lines, under parallel projection.

ii) |A’B’|/|AB| = |C’D’|/|CD| must be true, under parallel projection.

This ratio (projected length of a line to its true length) is called the foreshortening factor w.r.t. a given direction.

Mathematical description of an Oblique projection (onto xy-plane)

In order to develop the transformation for the oblique projection, consider the *Figure 10*. This *figure* shows an oblique projection of the point A (0, 0, 1) to position A'(x',y',0) on the view plane (z=0 plane). The direction of projection **d=**(d1,d2,d3).

Oblique projections (to xy-plane) can be specified by a number f and an angle θ. The number f, known as foreshortening factor, indicates the ratio of projected length OA'of a line to its true length. Any line L perpendicular to the xy-plane will be foreshortened after projection.

θ is the angle which the projected line OA'(of a given line L perpendicular to xy-plane) makes with the positive x-axis.

The line OA is projected to OA'. The length of the projected line from the origin =|OA'|

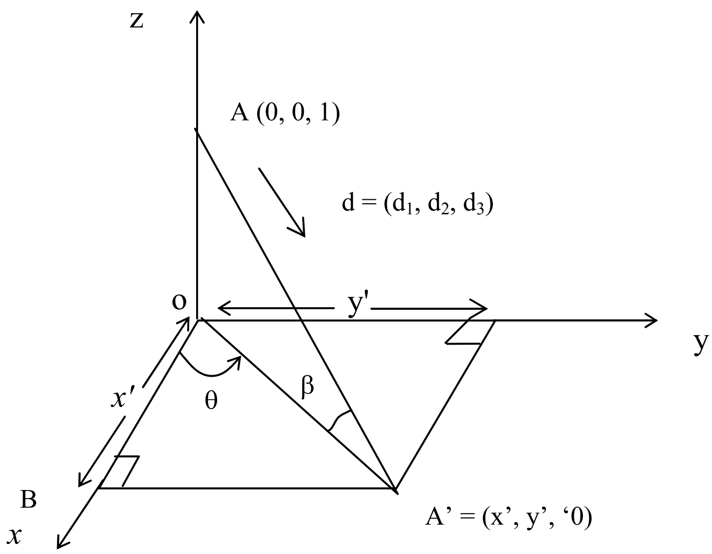

Figure 11: Oblique projection

Thus, foreshortening factor, f=|OA'|/|OA|=|OA'|, in the z-direction
From the triangle OAP', we have,

OB=x'=f.cosθ
BA'=y'=f.sinθ

When f = 1, then obligue projection is known as Cavalier projection

Given θ = 45°, then we have

$$P_{cav} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

When f = ½ then obligue projection is called a cabinet projection.

Here θ = 30° (Given), we have

$$
\text{Pcab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{3}/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

we can represent a given unit cube in terms of Homogeneous coordinates of the

$$
\text{vertices as:} V = [A\ B\ C\ D\ E\ F\ G\ H] = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}
$$

i) To draw the cavalier projection, we find the image coordinates of a given unit cube as follows:

$$
P' = V.\, P_{cav} = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \sqrt{2}/2 & (1+\frac{\sqrt{2}}{2} & 0 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 1 \\ (1+\sqrt{2}/2 & \sqrt{2}/2 & 0 & 1 \\ (1+\sqrt{2}/2 & (1+\sqrt{2}/2 & 0 & 1 \end{bmatrix}
$$

Hence, the image coordinate are:

A' = (0, 0, 0), B' = (1, 0, 0), C' = (1, 1, 0), D' = (0, 1, 0) E' = (√2/2 , 1 + √2/2, 0)
F' = (√2/2, √2/2, 0), G' = (1 + √2/2, √2/2, 0), H' = (1 + √2/2, 1 + √2/2, 0)

Thus, cavalier projection of a unit cube is shown in *Figure 11(a)*.

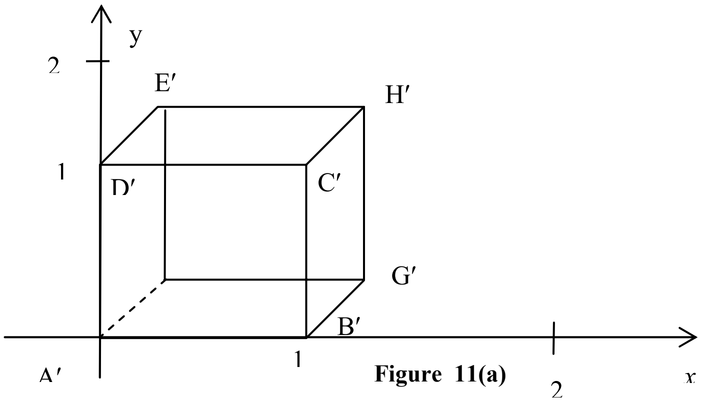

Figure 11(a)

To determine projection matrix for oblique projection, we need to find the direction vector **d.** Since vector **PP'** and vector **d** have the same direction. Thus, **PP'=d**

Thus, $x' - 0 = d_1 = f.\cos\theta$
$y' - 0 = d_2 = f.\sin\theta$
$z' - 1 = d_3$

As $z'=0$ on the xy-plane, $d_3 = -1$,

Since, Oblique projection is a special case of parallel projection, thus, we can transform the general transformation of parallel projection for Oblique projection as follows:

$$P_{Oblique} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -d_1/d_3 & -d_2/d_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f.\cos\theta & f.\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{-----------------(12)}$$

Where, f=foreshortening factor, i.e., the projected length of the z-axis unit vector. If β is the angle

Between the Oblique projectors and the plane of projection then,
$1/f=\tan(\beta)$, i.e., $f=\cot(\beta)$ ------------ (13)

θ=angle between the projected line with the positive x-axis.

Special cases:

1) If f=0, then $\cot(\beta)=0$ that is $\beta=90^0$, then we have an Orthographic projection.
2) If f=1, the edge perpendicular to projection plane are not foreshortened, then $\beta=\cot^{-1}(1)=45^0$ and this Oblique projection is called *Cavalier* projection.
3) If f=1/2 (the foreshortening is half of unit vector), then $\beta=\cot^{-1}(1/2)=63.435^0$ and this Oblique projection is called *Cabinet* projection.

Note: The common values of θ are **30° and 45°. the values of (180° –** θ) is also acceptable.

The *Figure 12* shows an Oblique projections for foreshortening factor f=1,7/8,3/4,5/8,1/2, with $\theta=45^0$

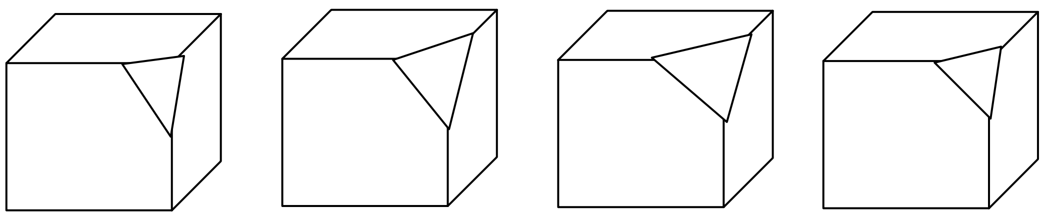

Figure 12: Oblique projections for f=1,7/8,3/4,5/8,1/2, with $\theta=45^0$ (from left to right)

Example4: Find the transformation matrix for a) cavalier projection with $\theta=45^0$, and b) cabinet projection with $\theta=30^0$ c) Draw the projection of unit cube for each transformation.

Solution: We know that cavalier and cabinet projections are a special case of an oblique projection. The transformation matrix for oblique projection is:

$$P_{obligue} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f.\cos\theta & f.\sin\theta & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) To draw the cabinet projection, we find the image coordinates of a unit cube as:

$$P'\ V.\ Pcab = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \sqrt{3}/4 & 5/4 & 0 & 1 \\ \sqrt{3}/4 & 1/4 & 0 & 1 \\ (1+\sqrt{3}/4) & 1/4 & 0 & 0 \\ (1+\sqrt{3}/4) & 5/4 & 0 & 1 \end{bmatrix}$$

Hence, the image coordinates are:
A' (0, 0, 0), B' = (1,0,0), C' = (1, 1, 0), D' = (0, 1, 0), E' = ($\sqrt{3}/4$, 5/4, 0)
F' = ($\sqrt{3}/4$, 1/4, 0), G' = (1 + $\sqrt{3}/4$, 1/4, 0), H' = (1 + $\sqrt{3}/4$, 5/4, 0)

The following *Figure (g)* shows a cabinet projection of a unit cube.

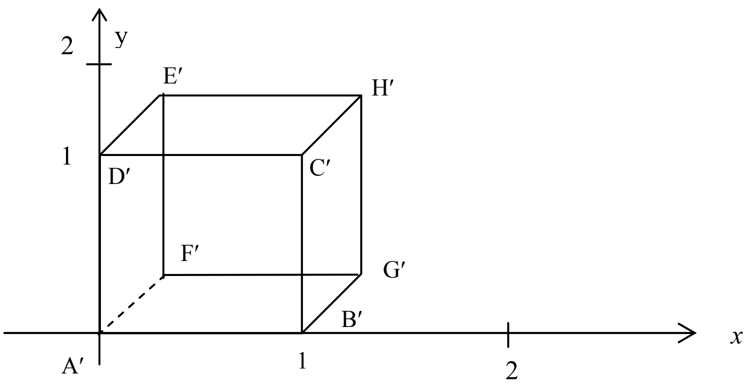

Figure (g)

2.2.1.2 Isometric Projection

There are 3 common sub categories of Orthographic (axonometric) projections:

1) *Isometric*: The direction of projection makes equal angles with all the three principal axes.
2) *Dimetric*: The direction of projection makes equal angles with exactly two of the three principal axes.
3) *Trimetric*: The direction of projection makes unequal angles with all the three principal axes.

Isometric projection is the most frequently used type of *axonometric* projection, which is a method used to show an object in all three dimensions in a single view. Axonometric projection is a form of orthographic projection in which the projectors are always perpendicular to the plane of projection. However, the object itself, rather than the projectors, are at an angle to the plane of projection.

Figure13 shows a cube projected by isometric projection. The cube is angled so that all of its surfaces make the same angle with the plane of projection. As a result, the length of each of the edges shown in the projection is somewhat shorter than the actual length of the edge on the object itself. This reduction is called foreshortening. Since, all of the surfaces make the angle with the plane of projection, the edges foreshorten in the same ratio. Therefore, one scale can be used for the entire layout; hence, the term *isometric* which literally means the same scale.

Construction of an Isometric Projection

In isometric projection, the direction of projection $d = (d_1,d_2,d_3)$ makes an equal angles with all the three principal axes. Let the direction of projection $d = (d_1,d_2,d_3)$ make equal angles (say α) with the positive side of the x,y, and z axes(see *Figure 13*).

Then
$i.d=d_1=|i|.|d|.\cos\alpha \Rightarrow \cos\alpha=d_1/|d|$
similarly
$d_2=j.d=|j|.|d|.\cos\alpha \Rightarrow \cos\alpha=d_2/|d|$
$d_3=k.d=|k|.|d|.\cos\alpha \Rightarrow \cos\alpha=d_3/|d|$

so $\cos\alpha=d_1/|d| = d_2/|d| = d_3/|d|$
⇨ $d_1= d_2 = d_3$ is true
we choose $d_1=d_2=d_3=1$

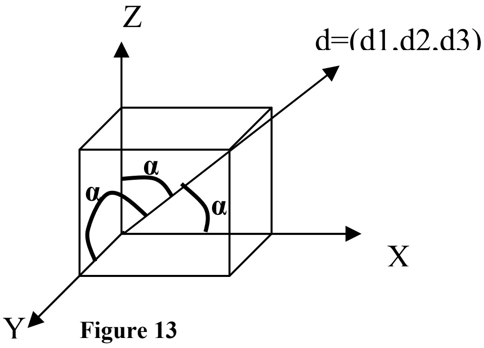

Figure 13

Thus, we have **d =(1, 1, 1)**

Since, the projection, we are looking for is an isometric projection => orthographic projection, i.e, the plane of projection, should be perpendicular to d, so $d = n = (1,1,1)$. Also, we assume that the plane of projection is passing through the origin.

⇨ We know that the equation of a plane passing through reference point $R(x_0,y_0,z_0)$ and having a normal $N = (n_1,n_2,n_3)$ is: $(x - x_0).n_1 + (y - y_0).n_2 + (z - z_0).n_3=0$ -------------------------(14)

Since $(n_1,n_2,n_3)=(1,1,1)$ and
$(x_0,y_0,z_0)=(0,0,0)$
From equation (14), we have $\mathbf{x + y + z = 0}$
Thus, we have the equation of the plane: $\mathbf{x + y + z = 0}$ and **d = (1,1,1)**

Transformation for Isometric projection

Let P(x,y,z) be any point in a space. Suppose a given point P(x,y,z) is projected to P'(x'y',z') onto the projection plane $x + y + z = 0$. We are interested to find out the projection point P'(x',y',z').

The parametric equation of a line passing through point P(x, y, z) and in the direction of d (1, 1, 1) is:

P + t.**d** = (x, y, z) + t. (1,1,1) = (x + t, y + t, z + t) is any point on the line, where $-\infty< t < \infty$. The point P' can be obtained, when $t = t^*$.
Thus $P'=(x',y',z')=(x + t^*,y + t^*,z + t^*)$, since P' lies on $x + y + z = 0$ plane.

⇨ $(x + t^*)+(y + t^*) + (z + t^*)=0$
⇨ $3.t^*=-(x + y + z)$
⇨ $t^*=-(x + y + z)/3$ should be true.
⇨ $x'= (2.x - y - z)/3$, $y'=(-x +2.y - z)/3$, $z'=(- x - y +2.z)/3$

Thus, P'=(x',y',z')=[(2.x –y–z)/3, (–x +2.y– z)/3, (-x-y+2.z)/3] --------------------(15)
In terms of homogeneous coordinates, we obtain

$$(x', y', z, 1) = (x, y, z, 1)\begin{pmatrix} 2/3 & -1/3 & 1/3 & 0 \\ -1/3 & 2/3 & -1/3 & 0 \\ -1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note: We can also verify this Isometric transformation matrix by checking all the foreshortening factors, i.e., to check whether all the foreshortening factors (fx, fy, fz) are equal or not. Consider the points A,B and C on the coordinate axes (see *Figure14*).

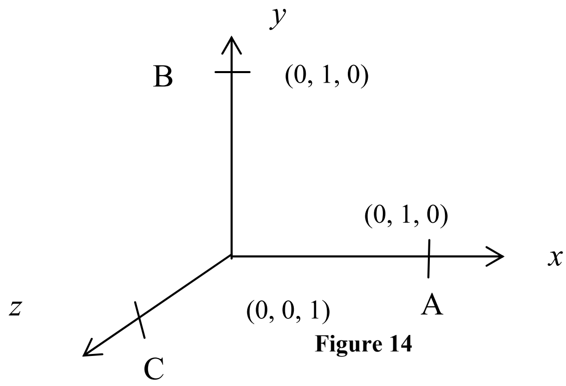

Figure 14

i) Take OA, where O=(0,0,0) and A (1,0,0). Suppose O is projected to O' and A is projected to A'
Thus, by using equation (15), we have O'=(0,0,0) and A'=(2/3,-1/3,-1/3).
So |O'A'| = $\sqrt{(2/3)^2}$ $+(-1/3)^2+(-1/3)^2$ = $\sqrt{2/3}$ = fx ------------(16)

ii) Take OB, where O = (0,0,0) and B (0,1,0). Suppose O is projected to O' and B is projected to B'. Thus by using equation (15), we have O'=(0,0,0) and B'= (-1/3,2/3,-1/3).
So |O'B'| = $\sqrt{-(1/3)^2}$ $+(2/3)^2+(-1/3)^2$ = $\sqrt{2/3}$ = fy -------------(17)

iii) Take OC, where O=(0,0,0) and C(0,0,1). Suppose O is projected to O' and C is projected to C'

Thus, by using equation(15), we have O'=(0,0,0) and C'=(-1/3,-1/3,2/3).

So |O'C'|=$\sqrt{}(-1/3)^2+(-1/3)^2+(2/3)^2$ = $\sqrt{}2/3$=fz --------------(18)

Thus, we have **fx=fy=fz**, which is true for Isometric projection.

Example 5: Obtain the isometric view of a cuboid, shown in *figure*. The size of cuboid is 10x8x6, which is lying at the origin.

Solution: The given cuboids can be represented in terms of Homogeneous coordinates of vertices as shown in *Figure (h):*

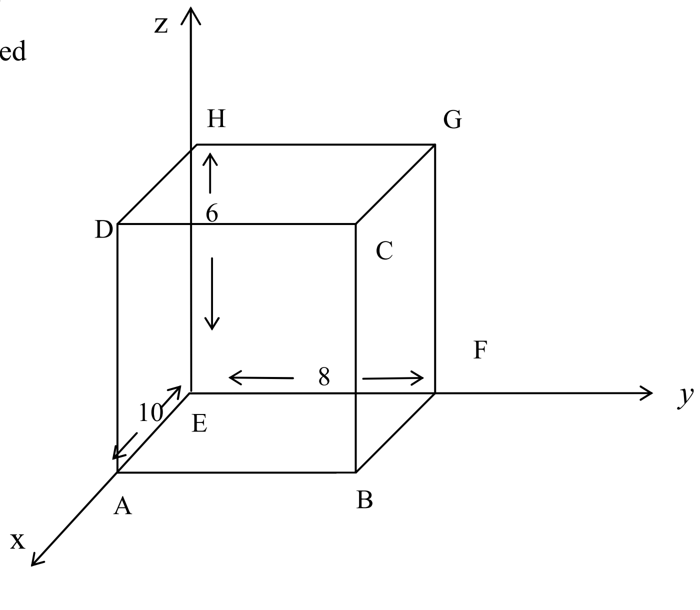

Figure (h)

$$V = [A\ B\ C\ D\ E\ F\ G\ H] = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{pmatrix} 10 & 0 & 0 & 1 \\ 10 & 8 & 0 & 1 \\ 10 & 8 & 6 & 1 \\ 10 & 8 & 6 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 1 \\ 0 & 8 & 6 & 1 \\ 0 & 0 & 6 & 1 \end{pmatrix}$$

To draw an Isometric projection, we find the image coordinate of a given cuboid as follows:

$$P' = V.\, P_{ISO} = \begin{pmatrix} 10 & 0 & 0 & 1 \\ 10 & 8 & 0 & 1 \\ 10 & 8 & 6 & 1 \\ 10 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 1 \\ 0 & 8 & 6 & 1 \\ 0 & 0 & 6 & 1 \end{pmatrix} . \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} =$$

$$\begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 20 & -10 & -10 & 3 \\ 12 & 8 & -18 & 3 \\ 6 & 0 & -6 & 3 \\ 14 & -16 & 2 & 3 \\ 0 & 0 & 0 & 3 \\ -8 & 16 & -8 & 3 \\ -14 & 10 & 4 & 3 \\ -6 & -6 & 12 & 3 \end{bmatrix} = \begin{bmatrix} 6.66 & -3.33 & -3.33 & 1 \\ 4.0 & 2.66 & -6.0 & 1 \\ 2 & 0 & -2.0 & 1 \\ 4.66 & -5.33 & 0.66` & 1 \\ 0 & 0 & 0 & 1 \\ -2.66 & 5.33 & 1.33 & 1 \\ -4.66 & 3.33 & 1.33 & 1 \\ -2.0 & -2.0 & 4.0 & 1 \end{bmatrix}$$

Thus, by using this matrix, we can draw an isometric view of a given cuboids.

☞ **Check Your Progress 2**

1) When all the foreshortening factors are different, we have
   a) Isometric b) Diametric c) Trimetric Projection d) All of these.

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2) Distinguish between Orthographic and Oblique parallel projection.

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3) What do you mean by foreshortening factor. Explain Isometric, Diametric and Trimetric projection using foreshortening factors.

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4) Show that for Isometric projection the foreshortening factor along x, y and z-axes must be $\sqrt{2/3}$, i.e. fx = fy = fz = $\sqrt{2/3}$

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5) Consider a parallel projection with the plane of projection having the normal (1,0,–1) and passing through the origin O(0,0,0) and having a direction of projection **d** = (–1,0,0). Is it orthographic projection? Explain your answer with reason.

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6) Compute the cavalier and cabinet projections with angles of 45^0 and 30^0 respectively of a pyramid with a square base of side 4 positioned at the origin in the xy-plane with a height of 10 along the z-axis.

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2.2.2 Perspective Projections

In a perspective projection the center of projection is at finite distance. This projection is called perspective projection because in this projection faraway objects look small and nearer objects look bigger. *See Figure 15 and 16.*

In general, the plane of projection is taken as Z=0 plane.

Properties

1) Faraway objects look smaller.
2) Straight lines are projected to straight lines.
3) Let l_1 and l_2 be two straight lines parallel to each other. If l_1 and l_2 are also parallel to the plane of projection, then the projections of l_1 and l_2 (call them l'_1 and l'_2), will also be parallel to each other.
4) If l_1 and l_2 be two straight lines parallel to each other, but are not parallel to the plane of projection, then the projections of l_1 and l_2 (call them l'_1 and l'_2), will meet in the plane of projection (see *Figure 16*).

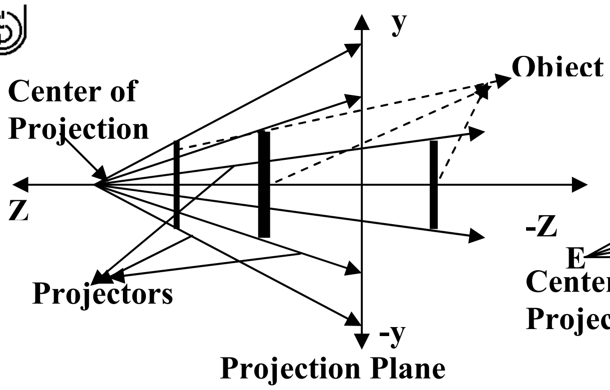

Figure 15

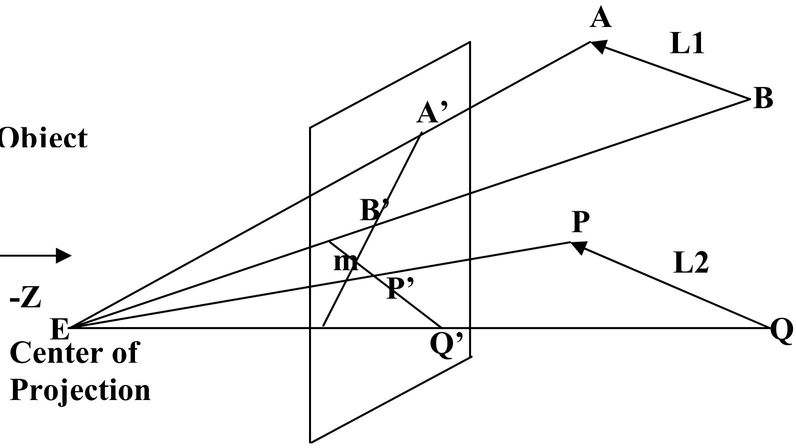

Figure 16

The infinite lines AB and PQ will be projected to the lines A'B' and P'Q' respectively on the plane of projection. That is all points of the line AB is projected to all points of the line A'B'. Similarly all points of the line PQ is projected to all points of the line P'Q'. But A'B' and P'Q' intersect at M and M is the projection of some point on the line AB as well as on PQ, but AB ‖ PQ, which implies that M is the projection of point at infinity where AB and PQ meet. In this case M is called a **Vanishing point.**

Principle Vanishing point

Suppose l_1 and l_2 be two straight lines parallel to each other, which are also parallel to x-axis. If the projection of l_1 and l_2 (call them l'_1 and l'_2), appears to meet at a point (point at infinity), then the point is called a Principle vanishing point w.r.t. the x-axis. Similarly we have Principle vanishing point w.r.t. the y-axis and z-axis.

Remark

A Perspective projection can have at most 3 Principle Vanishing points and at least one Principle vanishing point.

To understand the effects of a perspective transformation, consider the *Figure 17.* This *figure* shows the perspective transformation on z=0 plane of a given line AB which is parallel to the z-axis. The A*B* is the projected line of the given line AB in the z=0 plane. Let a centre of projection be at (0,0,-d) on the z-axis. The *Figure (A)* shows that the line A'B' intersects the z=0 plane at the same point as the line AB. It also intersects the z-axis at z=+d. It means the perspective transformation has transformed the intersection point.

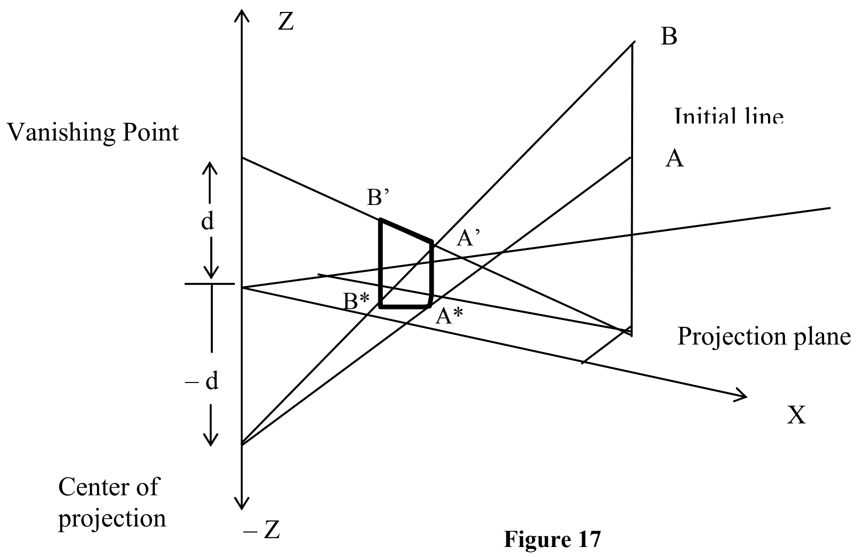

Figure 17

Mathematical description of a Perspective Projection

A perspective transformation is determined by prescribing a C.O.P. and a viewing plane. Let P(x,y,z) be any object point in 3D and C.O.P. is at E(0,0,-d). The problem is to determine the image point coordinates P'(x',y',z') on the Z=0 plane (see *Figure 18)*.

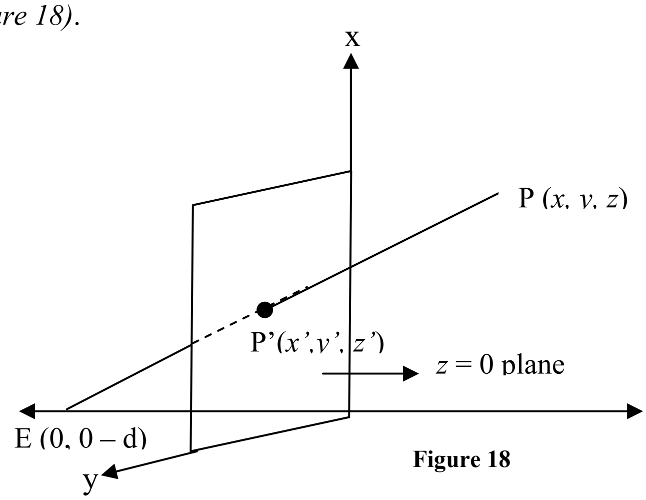

Figure 18

The parametric equation of a line EP, starting from E and passing through P is:

$E+t(P-E) \quad 0<t<\infty$

$=(0,0,-d)+t[(x,y,z)-(0,0,-d)]$

$=(0,0,-d)+t(x,y,z+d)$

$=[t.x,\ t.y,\ -d+t.(z+d)]$

Point P' is obtained, when t=t*

That is, $P'=(x',y',z') =[t^*.x,\ t^*.y,\ -d+t^*.(z+d)]$

Since P' lies on Z=0 plane implies $-d+t^*.(z+d)=0$ must be true, that is $t^*=d/(z+d)$ is true.

Thus $x'=t^*.x=x.d/(z+d)$
$y'=t^*.y=y.d/(z+d)$
$z'=-d+t^*(z+d)=0$

thus $P'=(\ x.d/(z+d),\ y.d/(z+d),\ 0)$
$=(x/((z/d)+1),y/((z/d)+1),0)$
in terms of Homogeneous coordinate system $P'=(x,y,0,(z/d)+1)$. ---------(5)

The above equation can be written in matrix form as:

$$P(x',y',z',1)=(x,y,z,1)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1/d\\0&0&0&1\end{pmatrix} = [x,y,0,(z/d)+1] \qquad \text{------------(1)}$$

That is, $P'_h = P_h.P_{per,z}$ ---------------- (2)

Where $P_{per,z}$ in equation (4.6) represents the *single point perspective transformation on z-axis*.

The Ordinary coordinates are:
$[x',y',z',1]=[x/(r.z+1),y/(r.z+1),0,1]$ where $r=1/d$ ---------------- (3)

Vanishing Point

The vanishing point is that point at which parallel lines appear to converge and vanish. A practical example is a long straight railroad track.

To illustrate this concept, consider the *Figure 17* which shows a perspective transformation onto z=0 plane. The *Figure17* shows a Projected line A*B* of given line AB parallel to the z-axis. The center of projection is at (0,0,-d) and z=0 be the projection plane.

Consider the perspective transformation of the point at infinity on the +z-axis, i.e.,

$$[0,0,1,0] \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0,0,0,1/d) \text{ -------------- } (4)$$

Thus, the ordinary coordinates of a point (x',y',z',1)=(0,0,0,1), corresponding to the transformed point at infinity on the z-axis, is now a finite point. This means that the entire semi-infinite positive space(0<=z<=∞) is transformed to the finite positive half space 0<=z'<=d.

Single point perspective transformation

In order to derive the single point perspective transformations along x and y-axes, we construct *Figures (19) and (20)* similar to *Figure 18*, but with the corresponding COP's at E(-d,0,0) and E(0,-d,0) on the negative x and y-axes respectively.

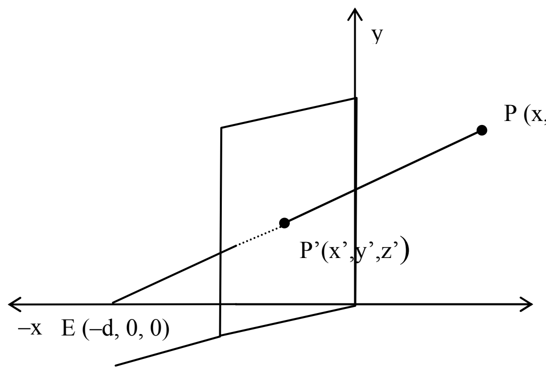

Figure 19

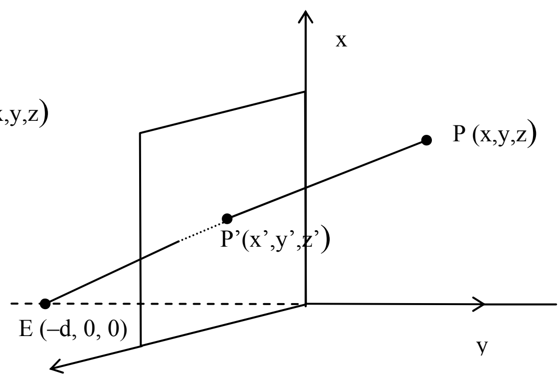

Figure 20

The parametric equation of a line EP, starting from E and passing through P is:

E+t(P-E) 0<t<∞

=(-d,0,0)+t[(x,y,z)-(-d,0,0)]

=(-d,0,0)+t[x+d,y,z]

=[-d+t.(x+d), t.y, t.z]

Point P' is obtained, when t=t*

That is, P'=(x',y',z') =[-d+t*.(x+d), t*.y, t*.z]

Since, P' lies on X=0 plane implies -d+t*.(x+d)=0 must be true, that is t*=d/(x+d) is true.

Thus, x'=-d+t*(x+d)=0
y'=t*.y=y.d/(x+d)
z'=t*.z=z.d/(x+d)
thus P'=(0, y.d/(z+d), z.d/(x+d))
=(0,y/((z/d)+1), z/((x/d)+1))

in terms of Homogeneous coordinate system P'=(0,y,z,(x/d)+1).

The above equation can be written in matrix form as:

$$P(x',y',z',1)=(x,y,z,1)\begin{pmatrix} 0 & 0 & 0 & 1/d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =[0,y,z,(x/d)+1]$$

=[0,y/((z/d)+1), z/((x/d)+1),1] -------- (5)

That is, $P'_h = P_h.P_{per,x}$ --------------------------------(6)

Where $P_{per,z}$ in equation (5) represents the *single point perspective transformation* w.r.t. x-axis.

Thus, the ordinary coordinates(projected point P' of a given point P) of a *single point perspective transformation* w.r.t. x-axis is:

(x',y',z',1)= [0,y/((z/d)+1), z/((x/d)+1),1] has a center of projection at [-d,0,0,1] and a vanishing point located on the x-axis at [0,0,0,1]

Similarly, the *single point perspective transformation w.r.t. y-axis* is therefore:

$$P(x',y',z',1)=(x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =[x,0,z,(y/d)+1]$$

=[x/((y/d)+1),0, z/((y/d)+1),1]

That is, $P'_h = P_h.P_{per,y}$ -----------------------------(7)

Where $P_{per,y}$ in equation (5) represents the *single point perspective transformation* w.r.t. y-axis.

Thus, the ordinary coordinates(projected point P' of a given point P) of a *single point perspective transformation* w.r.t. y-axis is:

(x',y',z',1)=[x/((y/d)+1),0, z/((y/d)+1),1] has a center of projection at [0,-d,0,1] and a vanishing point located on the y-axis at [0,0,0,1].

Example 6: Obtain a transformation matrix for perspective projection for a given object projected onto x=3 plane as viewed from (5,0,0).

Solution: Plane of projection: *x* = 3 (given)
Let P (*x, y, z*) be any point in the space. We know the
Parametric equation of a line AB, starting from A and passing

through B is

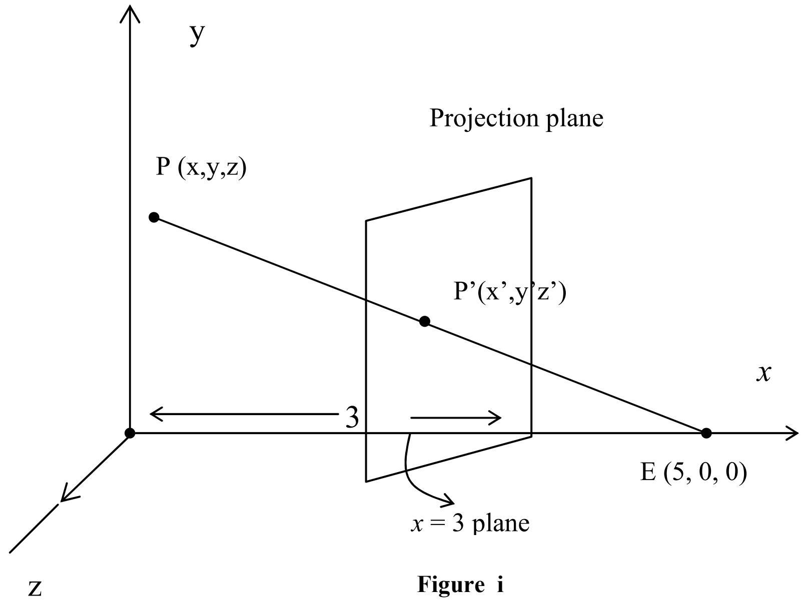

Figure i

$P(t) = A + t.(B - A), o < t < \infty$

So that parametric equation of a line starting from E (5,0,0) and passing through P (x, y, z) is:

$E + t(P - E), o < t < \infty.$
$= (5, 0, 0) + t[(x, y, z) - (5, 0, 0)]$
$= (5, 0, 0) + [t(x - 5), t.y, t.z]$
$= [t.(x - 5) + 5, t.y, t.z]$. Assume

Point P' is obtained, when t = t*

$\therefore P' = (x', y', z') = [t^*(x - 5) + 5, t^*y, t^*.z]$

Since, P' lies on $x = 3$ plane, so
$t^*(x - 5) + 5 = 3$ must be true;

$$t^* = \frac{-2}{x-5}$$

$$P' = (x', y', z') = \left(3, \frac{-2.y}{x-5}, \frac{-2.z}{x-5}\right)$$
$$= \left(\frac{3x-15}{x-5}, \frac{-2.y}{x-5}, \frac{-2.z}{x-5}\right)$$

In Homogeneous coordinate system

$$P' = (x', y', z', 1) = \left(\frac{3x-15}{x-5}, \frac{-2.y}{x-5}, \frac{2.z}{x-5}, 1\right)$$
$$= (3x - 15, -2.y, -2.z, x - 5) \quad \text{--------------(1)}$$

In Matrix form:

$$(x', y', z', 1) = (x, y, z, 1)\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -15 & 0 & 0 & -5 \end{bmatrix} \text{-------------- (2)}$$

Thus, equation (2) is the required transformation matrix for perspective view from (5, 0, 0).

Example 7: Consider the line segment AB in 3D parallel to the z-axis with end points A (– 5,4,2) and B (5,-6,18). Perform a perspective projection on the X=0 plane, where the eye is placed at (10,0,10).

Solution: Let P (x, y, z) be any point in the space.

The parametric equation of a line starting from E and passing through P is:

$$E + t.\,(P - E),\ o < t < 1.$$
$$= (10,0,10) + t.\,[(x, y, z) - (10, 0, 10)]$$
$$= (10, 0,10) + t\,[(x - 10)],\ y\,(z - 10)]$$
$$= (t.\,(x - 10) + 10,\ t.\,y,\ t\,(z - 10) + 10)$$

Assume point P' can be obtained, when t = t*

$\therefore$ P' = $(x', y', z') = (t^*\,(x - 10) + 10,\ t^*.y,\ t^*.\,(z - 10) + 10)$

since point P' lies on $x = 0$ plane

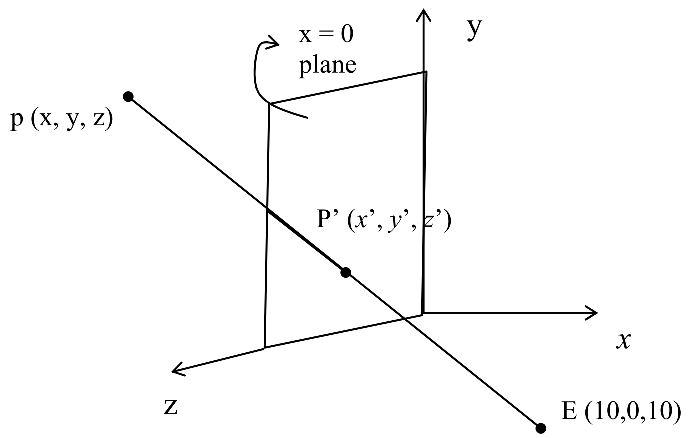

Figure j

$= t^*\,(x - 10) + 10 = 0$

$= t^* = \dfrac{-10}{x-10}$

$= P' = (x', y', z') = \left(0, \dfrac{-10.y}{x-10}, \dfrac{-10(z-10)}{x-10} + 10\right)$

$$\left(0, \frac{-10.y}{x-10}, \frac{10.x - 10.z}{x-10}\right)$$

In terms of Homogeneous coordinate system;

$$P' = (x', y', z', 1) = \left(0, \frac{-y}{\frac{x}{10}-1}, \frac{x-z}{\frac{x}{10}-1}, 1\right) = (0, -y, x-z, \frac{x}{10} - 1)$$

In Matrix form

$$(x', y', z', 1) = (x, y, z, 1)\begin{bmatrix} 0 & 0 & 1 & 1/10 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{------------------(1)}$$

This equation (1) is the required perspective transformation, which gives a coordinates of a projected point P' (x', y', z') onto the $x = 0$ plane, when a point p (x, y, z) is viewed from E (10, 0, 10)

Now, for the given points A (–5, 4, 2) and B (5, –6, 18), A' and B' are their projection on the $x = 0$ plane.

Then from Equation (1).

$$A' = (x'_1{}' y'_1, z'_1, 1) = (-5, 4, 2, 1).\begin{bmatrix} 0 & 0 & 1 & 1/10 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \quad (0, -4, -7, \frac{-5}{10} - 1)$$

$$= (0, -4, -7, \frac{-15}{10})$$

$$= (0, -40, -70, -15)$$

$$= (0, \frac{40}{15}, \frac{70}{15}, 1)$$

Hence $x_1' = 0$; $y_1' = 2.67$; $z_1' = 4.67$

similarly $$B' = (x_2', y_2', z_2', 1) = (5, -6, 18, 1) . \begin{bmatrix} 0 & 0 & 1 & 1/10 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= (0, 60, -130, -5)$$

$$= (0, -12, 26, 1)$$

Hence $x_2' = 0$; $y_2' = -12$; $z_2' = 26$

Thus the projected points A' and B' of a given points A and B are:

$$A' = (x_1', y_1' z_1') = (0, 2.67, 4.67) \quad \text{and} \quad B' = (x_2', y_2', z_2') = (0, -12, 26, 1)$$

Example 8: Consider the line segment AB in *Figure k*, parallel to the z-axis with end points A (3, 2, 4) and B (3, 2, 8). Perform a perspective projection onto the z = 0 plane from the center of projection at E (0, 0, – 2). Also find out the vanishing point.

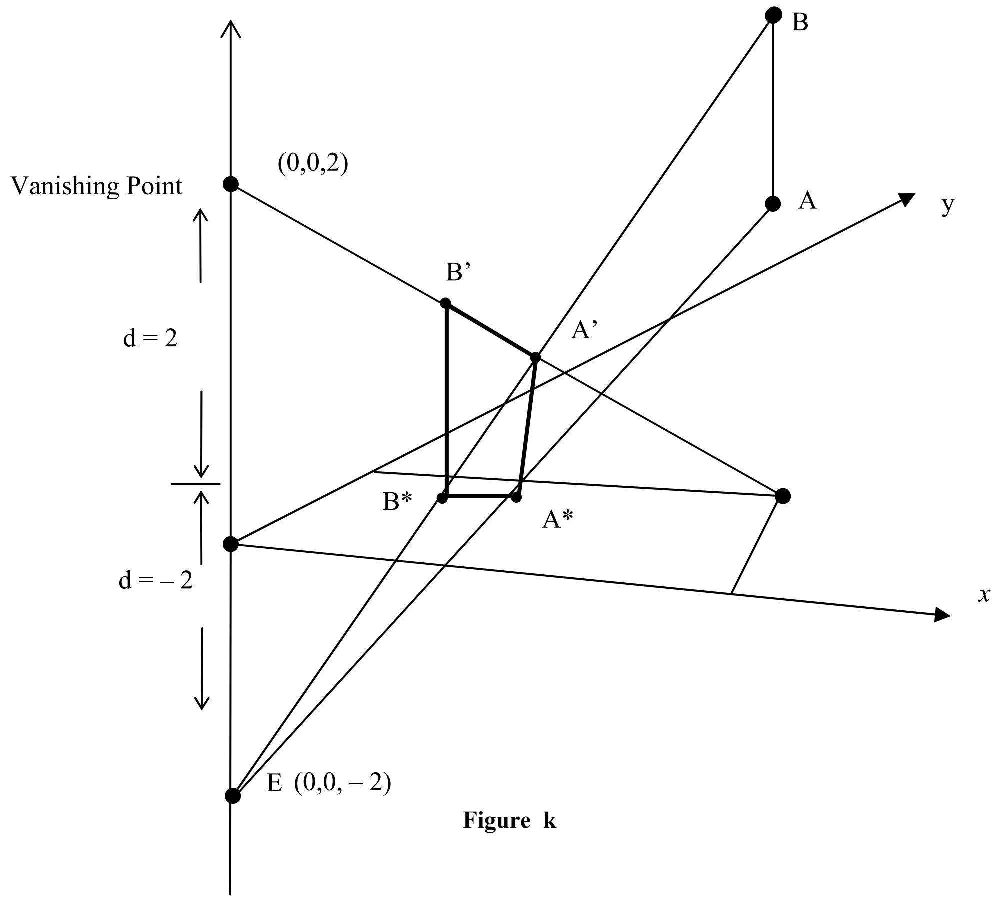

Figure k

Solution. We know that (from Equation (1)), the center of single point perspective transformation: of a point P (x, y, z) onto z = 0 plane, where center of projection is at (0, 0, – d) is given by:

$$(x', y', z', 1) = (x, y, z, 1) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P'n = Pn.\ P_{per,z} \quad \text{------------------------(I)}$$

Thus the perspective transformation of a given line AB to A* B* with d = 2 is given by:

$$V'_n = V_n.\ P_{per,z}$$

$$\begin{array}{c} A* \\ B* \end{array} \begin{bmatrix} x_1^* & y_1^* & z_1^* & 1 \\ x_1^* & y_2^* & z_1^* & 1 \end{bmatrix} = \begin{array}{c} A \\ B \end{array} \begin{bmatrix} 3 & 2 & 4 & 1 \\ 3 & 2 & 8 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c} A* \\ B* \end{array} \begin{bmatrix} 1 & 0.667 & 0 & 1 \\ 0.6 & 0.4 & 0 & 1 \end{bmatrix}$$

Hence, the projected points of a given line AB is:

A* = (1, 0.667, 0)
B* = (0.6, 0.4, 0)

The vanishing point is (0, 0, 0).

Example 9: Perform a perspective projection onto the z = 0 plane of the unit cube, shown in *Figure (l)* from the cop at E (0, 0, 10) on the z-axis.

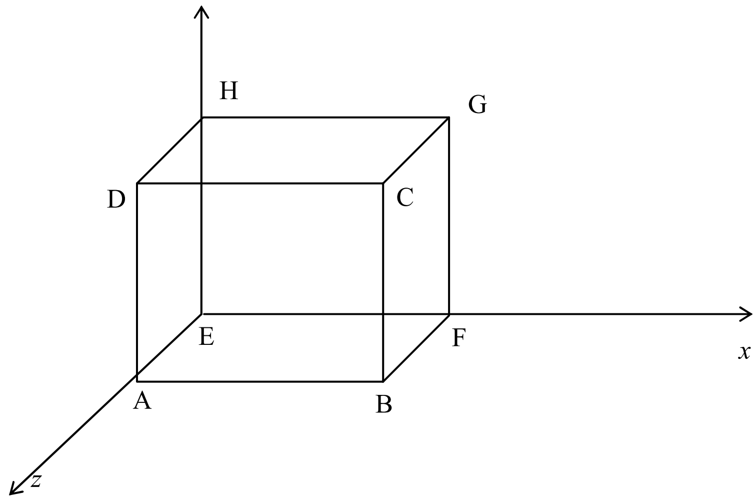

Figure (l)

01: Here center of projection

$E = (0, 0, -d) = (0, 0, 10).$

$\therefore\ d = -10$

we know that (from equation – 1), the single point perspective transformation of the projection with $z = 0$, plane, where cop is at (0, 0, –d) is given by:

$$(x', y', z', 1) = (x, y, z, 1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{-----------(I)}$$

$$P_n' = P.\ P_{per,\,z} \qquad \text{--------------- (II)}$$

Thus the perspective transformation of a given cube v = [ABCDEFGH] to V' = [A'B'C'D'E'F'G'H'] with d = – 10 is given by:

$$[V'] = [V]\ .\ [P_{per,\,z}]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V' = \begin{bmatrix} 0 & 0 & 0 & 0.9 \\ 1 & 0 & 0 & 0.9 \\ 1 & 1 & 0 & 0.9 \\ 0 & 1 & 0 & 0.9 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1.11 & 0 & 0 & 1 \\ 1.11 & 1.11 & 0 & 1 \\ 0 & 1.11 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Thus the projected points of a given cube V = [ABCDEFGH] are:
A' =(0, 0, 0), B' = (1.11, 0, 0), C' = (1.11, 1.11, 0), D' = (0, 1.11, 0), E' = (0, 0, 0)
F' = (1, 0, 0), G' = (1, 1, 0) and H' = (0, 1, 0).

☞ Check Your Progress 3

1) Obtain the perspective transformation onto z = d plane, where the c. o. p. is at the origin.

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2) Consider a cube given in example – 4, the cube can be centered on the z-axis by translating it $-\frac{1}{2}$ units in the x y directions perform a single point perspective transformation onto the z = 0 plane, with c. o. p. at Zc = 10 on the z-axis.

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3) A unit cube is placed at the origin such that its 3-edges are lying along the x, y and z-axes. The cube is rotated about the y-axis by 30°. Obtain the perspective projection of the cube viewed from (80, 0, 60) on the z = 0 plane.

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Two-Point and Three-Point Perspective transformations

The 2-point perspective projection can be obtained by rotating about one of the principal axis only and projecting on X=0 (or Y=0 or Z=0) plane. To discuss the phenomenon practically consider an example for 3-point perspective projection (given below) some can be done for 2-point aspect.

Example 10: Find the principal vanishing points, when the object is first rotated w.r.t. the y-axis by – 30° and x-axis by 45°, and projected onto z = 0 plane, with the center of projection being (0, 0, – 5).

Solution: Rotation about the y-axis with angle of rotation
$\theta = (-30°)$ is

$$[R_y] = [Ry]_{\theta=-30} = \begin{bmatrix} \cos(30°) & 0 & -\sin(-30°) \\ 0 & 1 & 0 \\ \sin(-30°) & 0 & \cos(-30°) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}$$

Similarly Rotation about the x-axis with angle of Rotation ϕ 45° is:

$$[R_x] = [R_x]_{45°} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore\ [R_y].[R_x] = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2\sqrt{2} & 1/2\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1/2 & -3/2\sqrt{2} & \sqrt{3}/2\sqrt{2} \end{bmatrix} \text{-----------------(1)}$$

Projection: Center of projection is E (0, 0, – 5) and plane of projection is z = 0 plane.

For any point p (x, y, z) from the object, the Equation of the ray starting from E and passing through the point P is:

$$E + t\,(P - E),\ t > 0$$

i.e. $(0, 0, -5) + t\,[(x, y, z) - (0, 0, -5)]$
$= (0, 0, -5) + t\,(x, y, z + 5)$
$= (t.x,\ t.\,y,\ -5 + t\,(z + 5))$

for this point to be lie on z = 0 plane, we have:
$-5 + t\,(z + 5) = 0$

$$\therefore\ t = \frac{5}{z+5}$$

$\therefore$ the projection point of p (x, y, z) will be:

$$P' = (x', y', z') = \left(\frac{5.x}{z+5}, \frac{5.y}{z+5}, 0\right)$$

In terms of homogeneous coordinates, the projection matrix will become:

$$[P] = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \text{-------------(2)}$$

$$\therefore [R_y].\,[R_x].[P] = \begin{bmatrix} \sqrt{3}/2 & -1/2\sqrt{2} & 1/2\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1\sqrt{2} & -\sqrt{3}/2\sqrt{2} & \sqrt{3}/2\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5\sqrt{3}}{2} & \frac{-5}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ 0 & \frac{5}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-5}{2} & \frac{-5\sqrt{3}}{2\sqrt{2}} & 0 & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad \text{----------------(3)}$$

Let (x, y, z) be projected, under the combined transformation (3) to (x', y', z'), then

$$(x', y', z', 1) = (x, y, z, 1) \begin{bmatrix} \frac{5\sqrt{3}}{2} & \frac{-5}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ 0 & \frac{5}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-5}{2} & \frac{-5\sqrt{3}}{2\sqrt{2}} & 0 & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= x' = \frac{\left(\frac{5\sqrt{3}}{2}.x - \frac{5}{2}.z\right)}{\left(\frac{x}{2\sqrt{2}} + \frac{y}{\sqrt{2}} + \frac{\sqrt{3}.z}{2\sqrt{2}} + 5\right)}$$

and

$$\left.y' = \frac{\left(\frac{-5}{2\sqrt{2}}.x + \frac{5}{\sqrt{2}}.y - \frac{5\sqrt{3}}{2\sqrt{2}}.z\right)}{\left(\frac{x}{2\sqrt{2}} + \frac{y}{\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}.z + 5\right)}\right\} \text{-----------(4)}$$

Case 1: Principal vanishing point w.r.t the x-axis.

By considering first row of the matrix (Equation – (3)), we can claim that the principal vanishing point (w.r.t) the x-axis) will be:

$$\left(\frac{\frac{5\sqrt{3}}{2}}{\frac{1}{2\sqrt{2}}}, \frac{\frac{-5}{2\sqrt{2}}}{\frac{1}{2\sqrt{2}}}, 0\right)$$

i.e., $(5\sqrt{6}, -5, 0)$ -------------------(I)

In order to varify our claim, consider the line segments AB, CD, which are parallel to the x-axis, where A = (0, 0, 0), B = (1, 0, 0), C = (1, 1, 0), D = (0, 1, 0)

If A', B', C', D' are the projections of A, B, C, D, respectively, under the projection matrix (3), then

$$A' = (0, 0, 0),\ B' = \left(\frac{5\sqrt{3}}{\frac{1}{2\sqrt{2}} + 5}, \frac{\frac{-5}{2\sqrt{2}}}{\frac{1}{2\sqrt{2}} + 5}, 0 \right)$$

$$C' = \left(\frac{\frac{5\sqrt{3}}{2}}{\left(\frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} + 5 \right)}, \frac{\left(-\frac{5}{2\sqrt{2}} + \frac{5}{\sqrt{2}} \right)}{\left(\frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} + 5 \right)}, 0 \right)$$

$$D' = \left(0, \frac{5/\sqrt{2}}{\left(\frac{1}{\sqrt{2}} + 5 \right)}, 0 \right) \qquad \{\text{Using Equation (4)}\}$$

$$A' = (0,0,0)\ ,\ B' = \left(\frac{5\sqrt{6}}{1+10\sqrt{2}}, \frac{-5}{1+10\sqrt{2}}, 0 \right),$$

$$C' = \left(\frac{5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}}, 0 \right) \text{ and}$$

$$D' = \left(0, \frac{5}{1+5\sqrt{2}}, 0 \right)$$

Consider the line equation of A'B': The parametric Equation is:

$$A' + t\,(B' - A')$$

i.e. $(0, 0, 0) + t \left(\frac{5\sqrt{6}}{1+10\sqrt{2}}, \frac{-5}{1+10\sqrt{2}}, 0 \right)$

$$= \left(\frac{5t\sqrt{6}}{1+10\sqrt{2}}, \frac{-5.t}{1+10\sqrt{2}}, 0 \right)$$

we will verify that the vanishing point (I) lies on this line:

i.e. $\left(\frac{5t\sqrt{6}}{1+10\sqrt{2}}, \frac{-5.t}{1+10\sqrt{2}}, 0 \right) = (5\sqrt{6}, -5, 0)$

$$= \frac{5.t.\sqrt{6}}{1+10\sqrt{2}} = 5\sqrt{6}$$

and $\frac{-5t}{1+10\sqrt{2}} = -5$ -----------------(5)

must be true for some 't' value.

$t = (1 + 10\sqrt{2})$

then the equation (5) is true and hence (I) lies on the line A'B'.
Similarly consider the line equation C'D': The parametric Equation is:

$$C' + s\,(D' - C') \qquad \text{i.e.}$$

$$= \left(\frac{5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}}, 0\right) + s\left[\left(0, \frac{5}{1+5\sqrt{2}}, 0\right) - \left(\frac{5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}} 0\right)\right]$$

$$= \left(\frac{5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}}, 0\right) + s\left(\frac{-5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{1+5\sqrt{2}} - \frac{5}{3+10\sqrt{2}}, 0\right) \text{ and}$$

$$= \left(\frac{5\sqrt{6}}{3+10\sqrt{2}} \frac{-5.s.\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}} + \frac{5.s.(2+5\sqrt{2})}{(1+5\sqrt{2})(3+10\sqrt{2})}, 0\right), \text{ but}$$

we have to verify that the vanishing point (I) lies on C'D'.

i.e. we have to show

$$\left(\frac{5\sqrt{6}}{3+10\sqrt{2}}(1-s) \frac{5}{3+10\sqrt{2}} \left(1 + \frac{s(2+5\sqrt{2})}{(1+5\sqrt{2})}\right), 0\right) = (5\sqrt{6}, -5, 0)$$

for some 's' value This holds true if

$$\left.\begin{aligned} &\frac{5\sqrt{6}}{3+10\sqrt{2}}(1-s) = 5\sqrt{6} \\ \text{and } &\frac{5}{3+10\sqrt{2}}\left(1 + \frac{s(2+5\sqrt{2})}{(1+5\sqrt{2})}\right) = -5 \end{aligned}\right\} \qquad \text{------------(6)}$$

must holds simultaneously for some 's' value.

If we choose $s = -2(1 + 5\sqrt{2})$, then both the conditions of (6) satisfied
$\therefore (5\sqrt{6}, -5, 0)$ lies on C'D'

$= (5\sqrt{6}, -5, 0)$ is the point at intersection of A'B' and C'D'.
$(5\sqrt{6}, -5, 0)$ is the principal vanishing point w.r.t. the x-axis.

Case 2: Principal vanishing point w.r.t y-axis:-

From the 2nd Row of the matrix (Equation (3)), the principal vanishing point w.r.t y-axis will be:

$$\left(0, \frac{5}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \text{ in homogeneous system.}$$

The vanishing point in Cartesian system is:

$$\left(0, \frac{5/\sqrt{2}}{1/\sqrt{2}}, 0\right) = (0, 5, 0) \qquad \text{-----------------------(II)}$$

similar proof can be made to verify our claim:

Case 3: Principal vanishing point w.r.t z-axis:

From the 3rd row of matrix equation (3), we claim that the principal vanishing point w.r.t z-axis will be: $\left(-\frac{5}{2}, \frac{-5\sqrt{3}}{2\sqrt{2}}, 0, \frac{\sqrt{3}}{2\sqrt{2}}\right)$ in Homogeneous system.

In Cartesian system, the vanishing point is:

$$\left(\frac{(-5/2)}{\frac{\sqrt{3}}{2\sqrt{2}}}, \frac{\left(\frac{-5}{2}\frac{\sqrt{3}}{\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2\sqrt{2}}\right)}, 0 \right) = \left(\frac{-5\sqrt{2}}{\sqrt{3}}, -5,\ 0 \right) \quad \text{------------------(III)}$$

A similar proof can be made to verify (III)

General Perspective transformation with COP at the origin

Let the given point P(x,y,z) be projected as P'(x',y',z') onto the plane of projection. The COP is at the origin, denoted by O(0,0,0). Suppose the plane of projection defined by the normal vector $\mathbf{N}=n_1\mathbf{I}+n_2\mathbf{J}+n_3\mathbf{K}$ and passing through the reference point $R_0(x_0,y_0,z_0)$. From *Figure 21*, the vectors **PO** and **P'O** have the same direction. The vector **P'O** is a factor of **PO**. Hence they are related by the equation: **P'O** = α **PO**, comparing components we have x'=α.x y'=α.y z'=α.z we now find the value of α.

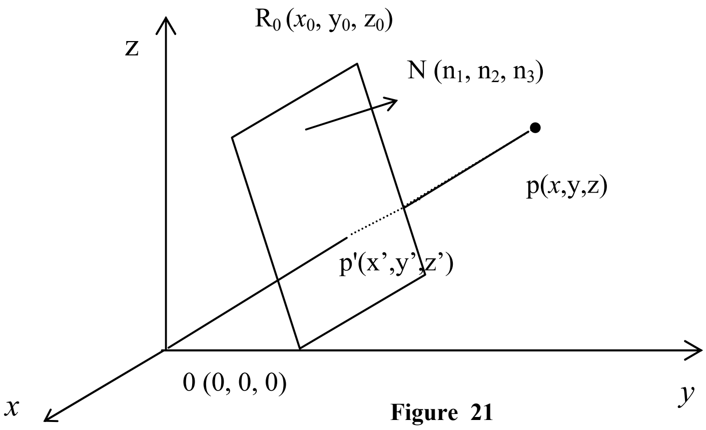

Figure 21

We know that the equation of the projection plane passing through a reference point R_0 and having a normal vector $\mathbf{N}=n_1\mathbf{I}+n_2\mathbf{J}+n_3\mathbf{K}$ is given by $\mathbf{PR_0}.\mathbf{N}=0$, that is

$(x-x_0,y-y_0,z-z_0).(n_1,n_2,n_3)=0$ i.e. $n_1.(x-x_0)+n_2.(y-y_0)+n_3.(z-z_0)=0$ ---------( )

since P'(x',y',z') lies on this plane, thus we have: $n_1.(x'-x_0)+n_2.(y'-y_0)+n_3.(z'-z_0)=0$
After substituting x'=α.x ; y'=α.y ; z'=α.z, we have :

$$\alpha = (n_1.x_0+n_2.y_0+n_3.z_0)/(n_1.x+n_2.y+n_3.z) = d_0/(n_1.x+n_2.y+n_3.z)$$

This projection transformation cannot be represented as a 3x3 matrix transformation. However, by using the homogeneous coordinate representation for 3D, we can write this projection transformation as:

$$P_{per,N,Ro} = \begin{pmatrix} d_0 & 0 & 0 & n_1 \\ 0 & d_0 & 0 & n_2 \\ 0 & 0 & d_0 & n_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the projected point $P'_h(x',y',z',1)$ of given point $P_h(x, y, z, 1)$ can be obtained as

$$P'h = Ph.\ Pper,N,\ Ro = [x, y, z, 1]\begin{pmatrix} d_0 & 0 & 0 & n_1 \\ 0 & d_0 & 0 & n_2 \\ 0 & 0 & d_0 & n_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{-----------------(16)}$$

$= [d_0.x,\ d_0.y,\ d_0z,\ (n_1.x + n_2.y + n_3.z)]$

Where $d_0 = n_1.x_0 + n_2.y_0 + n_3.\ z_0$.

General Perspective transformation w.r.t. an arbitrary COP

Let the COP is at C(a,b,c), as shown in *Figure 22*.

From *Figure 7*, the vectors **CP** and **CP'** have the same direction. The vector **CP'** is a factor of **CP**, that is **CP'**=α.**CP**

Thus, (x'-a)= α.(x-a) z
(y'-b)= α.(y-b)
(z'-c)= α.(z-c) -------------(17)

$N = (n_1, n_2, n_3)$

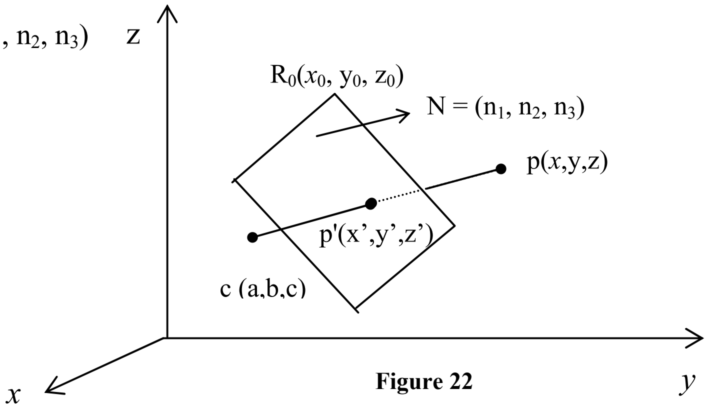

Figure 22

We know that the projection plane passing through a reference point $R_0(x_0,y_0,z_0)$ and having a normal vector N= n1I+n2J+n3K, satisfies the following equation:

n1.(x-x0)+n2.(y-y0)+n3.(z-z0)=0

Since P'(x',y',z') lies on this plane, we have:

n1.(x'-x0)+n2.(y'-y0)+n3.(z'-z0)=0

Substituting the value of x', y' and z', we have:

α= (n1.(x0-a)+n2.(y0-b)+n3.(z0-c))/(n1.(x-a)+n2.(y-b)+n3.(z-c))
=((n1.x0+n2.y0+n3.z0)-(n1.a+n2.b+n3.c))/(n1.(x-a)+n2.(y-b)+n3.(z-c))
=(d0-d1)/(n1.(x-a)+n2.(y-b)+n3.(z-c))
=d/(n1.(x-a)+n2.(y-b)+n3.(z-c))

Here, d=d0-d1= (n1.x0+n2.y0+n3.z0)- (n1.a+n2.b+n3.c) represents perpendicular distance from COP, C to the projection plane.

In order to find out the general perspective transformation matrix, we have to proceed as follows:

Translate COP, C(a,b,c) to the origin. Now, R'0=(x0-a,y0-b,z0-c) becomes the reference point of the translated plane.(but Normal vector will remain same).

Apply the general perspective transformation $P_{per,N,R'o}$

Translate the origin back to C.

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a & -b & -c & 1 \end{pmatrix} \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ n_1 & n_2 & n_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} d+n_1.a & n_1.b & n_1.c & n_1 \\ n_2.a & d+n_2.b & n_2.c & n_2 \\ n_3.a & n_3.b & d+n_3.c & n_3 \\ -a.d_0 & -b.d_0 & -c.d_0 & -d_1 \end{pmatrix} \quad \text{------------ (18)}$$

Where d = N.CR’ 0 = d0 – d1 = (n1. x0 + n2. Y0 + n3.z0) – (n1.a+n2.b +n3.c)
= n1. (x0 – a) + n2. (y0 – b) + n3. (z0 – c)

And d1 = n1.a + n2.b + n3.c

Example 11: Obtain the perspective transformation onto z = – 2 Plane, where the center of projection is at (0, 0, 18).

Solution: Here centre of projection, C (a, b, c) = (0, 0, 18)

$\therefore$ $(n_1, n_2, n_3) = (0, 0, 1)$

and Reference point R_0 $(x_0, y_0, z_0) = (0, 0, -2)$

$\therefore$ $d_0 = (n_1x_0 + n_2.y_0 + n_3z_0) = -2$

$d_1 = (n_1.a + n_2.b + n_3.\ c) = 18$

we know that the general perspective transformation when cop is not at the origin is given by:

$$\begin{pmatrix} d+n_1.a & n_1.b & n_1.c & n_1 \\ n_2.a & d+n_2.b & n_2.c & n_2 \\ n_3.a & n_3.b & d+n_3.c & n_3 \\ -a.d_0 & -b.d_0 & -c.d_0 & -d_1 \end{pmatrix}$$

$$= \begin{pmatrix} -20 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 36 & -18 \end{pmatrix} = \begin{pmatrix} -20 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

Example 12: Find the perspective transformation matrix on to z = 5 plane, when the c.o.p is at origin.

Solution. Since z = 5 is parallel to z = 0 plane, the normal is the same as the unit vecter ‘k’.

$\therefore$ $(n_1, n_2, n_3) = (0, 0, 1)$

and the Reference point R0 $(x_0, y_0, z_0) = (0, 0, 5)$

$d_0 = n_1.\ x_0 + n_2.\ y_0 + n_3.\ z_0) = 5$

we know the general perspective transformation, when cop is at origin is given by:

$$\begin{pmatrix} d_0 & 0 & 0 & n_1 \\ 0 & d_0 & 0 & n_2 \\ 0 & 0 & d_0 & n_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

☞ **Check Your Progress 4**

1) Determine the vanishing points for the following perspective transformation matrix:

$$\begin{bmatrix} 8.68 & 5.6 & 0 & 2.8 \\ 0 & 20.5 & 0 & 4.5 \\ 7.0 & 800 & 0 & 2.0 \\ 5.3 & 7.3 & 0 & 3.0 \end{bmatrix}$$

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2) Find the three-point perspective transformation with vanishing points at $V_x = 5$, $V_y = 5$ and $V_z = -5$, for a Given eight vertices of a cube A (0, 0, 1), B (1, 0, 1), C (1, 1, 1) D (0, 1, 1), E (0, 0, 0), F (1, 0, 0), G (1, 1, 0), H (0, 1, 0).

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2.3 SUMMARY

- Projection is basically a transformation (mapping) of 3D objects on 2D screen.
- Projection is broadly categorised into Parallel and Perspective projections depending on whether the rays from the object converge at the COP or not.
- If the distance of COP from the projection plane is finite, then we have Perspective projection. This is called perspective because faraway objects look smaller and nearer objects look bigger.
- When the distance of COP from the projection plane is infinite, then rays from the objects become parallel. This type of projection is called parallel projection.
- Parallel projection can be categorised according to the angle that the direction of projection makes with the projection plane.
- If the direction of projection of rays is perpendicular to the projection plane, we have an Orthographic projection, otherwise an Oblique projection.
- Orthographic (perpendicular) projection shows only one face of a given object, i.e., only two dimensions: length and width, whereas Oblique projection shows all the three dimensions, i.e. length, width and height. Thus, an Oblique projection is one way to show all three dimensions of an object in a single view.
- In Oblique projection the line perpendicular to the projection plane are *foreshortened* (Projected line length is shorter then actual line length) by the direction of projection of rays. The direction of projection of rays determines the amount of foreshortening.
- The change in length of the projected line (due to the direction of projection of rays) is measured in terms of foreshortening factor, f, which is defined as the ratio of the projected length to its true length.

- In Oblique projection, if foreshortening factor f=1, then we have cavalier projection and if f=1/2 then cabinet projection.
- The plane of projection may be perpendicular or may not be perpendicular to the principal axes. If the plane of projection is perpendicular to the principal axes then we have *multiview* projection otherwise *axonometric* projection.
- Depending on the foreshortening factors, we have three different types of Axonometric projections: Isometric (all foreshortening factors are equal), Dimetric (any two foreshortening factors equal) and Trimetric (all foreshortening factors unequal).
- In perspective projection, the parallel lines appear to meet at a point i.e., point at infinity. This point is called vanishing point. A practical example is a long straight railroad track, where two parallel railroad tracks appear to meet at infinity.
- A perspective projection can have at most 3 principal vanishing points (points at infinity w.r.t. x, y, and z-axes, respectively) and at least one principle vanishing point.
- A single point perspective transformation with the COP along any of the coordinate axes yields a single vanishing point, where two parallel lines appear to meet at infinity.
- Two point perspective transformations are obtained by the concatenation of any two one-point perspective transformations. So we can have 3 two-point perspective transformations, namely P_{per-xy}, P_{per-yz}, P_{per-xz} .
- Three point perspective transformations can be obtained by the composition of all the three one-point perspective transformations.

2.4 SOLUTIONS/ANSWERS

Check Your Progress 1

1) Consider a following *Figure m*, where a given line AB is projected to A' B' on a projection plane.

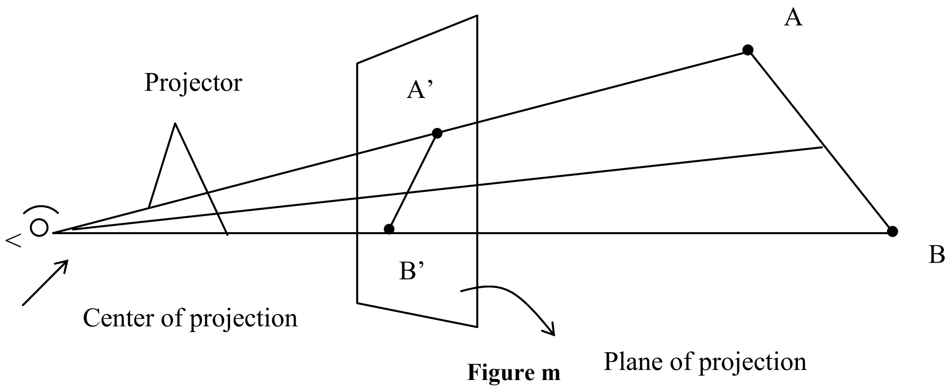

Figure m

a) **Center of projection (cop)**: In case of perspective projection, the rays from an object converge at the finite point, known as center of projection (cop). In *Figure 1*, 0 is the center of projection, where we place our eye to see the projected image on the view plane.

b) **Plane of projection**: Projection is basically a mapping of 3D-object on to 2D-screen. Here 2D-screen, which constitutes the display surface, is known as plane of projection/view plane. That a plane (or display surface), where we are projecting an image of a given 3D-object, is called a plane of projection/view plane. *Figure 1* shows a plane of projection where a given line AB is projected to A'B'.

c) **Projector**: The mapping of 3D-objects on a view plane are formed by projection rays, called the projectors. The intersection of projectors with a view plane form the projected image of a given 3D-object (see *Figure 1*).

d) **Direction of projection**: In case of parallel projection, if the distance of cop from the projection plane is infinity, then all the rays from the object become parallel and will have a direction called "direction of projection". It is denoted by d = (d_1,d_2,d_3), where d_1, d_2 and d_3 make an angle with positive side of *x*, *y* and *z* axes, respectively (see *Figure n*)

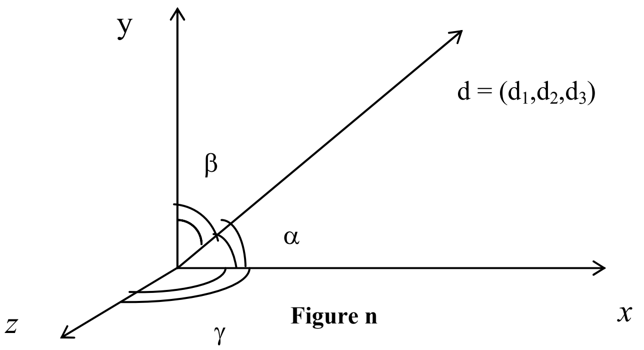

Figure n

The Categorisation of parallel and perspective projection is based on the fact whether coming from the object converge at the cop or not. If the rays coming from the object converges at the centre of projection, then this projection is known as perspective projection, otherwise parallel projection.

Parallel projection can be categorized into orthographic and Oblique projection.

A parallel projection can be categorized according to the angle that the direction of projection $\overline{d}$ makes with the view plane. If $\overline{d}$ is $\perp^r$ to the view plane, then this parallel projection is known as orthographic, otherwise Oblique projection.
Orthographic projection is further subdivided into multiview view plane parallel to the principal axes)

Axonometric projection (view plane not to the principal axes).

Oblique projection is further subdivided into cavalier and canbinet and if f = ½ then cabinet projection.

The *Figure O* shows the Taxonomy of projections:

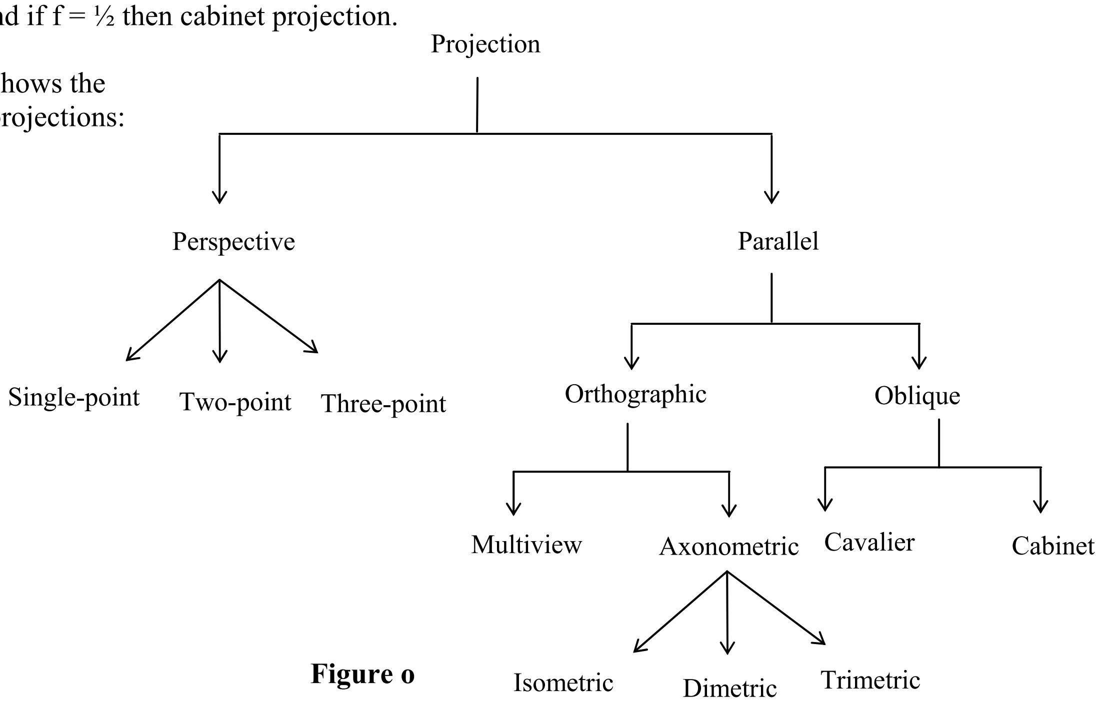

Figure o

3) C

Check Your Progress 2

1) C

2) We know that, the parallel projections can be categorized according to the angle that the direction of projection $\overline{d} = (d_1, d_2, d_3)$ makes with the projection plane. Thus, if direction of projection $\overline{d}$ is $\perp^r$ to the projection plane then we have orthographic projection and if the $\overline{d}$ is not $\perp^r$ to the projection plane then we have oblique projection.

3) The ratio of projected length of a given line to its true length is called the foreshortening factor w.r.t. a given direction.
Let AB is any given line segment
Also assume AB || $\vec{a}$.

Then Under parallel projection, AB is projected to A'B'; The change in the length of projected line is measured in terms of foreshortening factor. f.

$$\therefore\ f = \frac{|A'B'|}{|AB|}$$

Depending on foreshortening factors, we have (3) different types of Axonometric projections:

- Isometric
- Diametric
- Trimetric

When all foreshortening factors along the x-, y- and z-axes are equal, i.e., $f_x = f_y = f_z$, then we have Isometric projection, i.e., the direction of projection makes equal angle with all the positive sides of x, y, and z-axes, respectively.

Similarly, if any two foreshortening factors are equal, i.e., $f_x = f_y$ or $f_y = f_z$ or $f_x = f_z$ then, we have Diametric projection. If all the foreshortening factors are unequal $\overline{d}$ makes unequal angles with *x*, *y*, and *z*-axes/, then we have Trimetric projection.

4) Refer 2. 3. 1. 2 Isometric projection.

5) For orthographic projection, Normal vector N should be parallel to the direction of projection vecter, $\overline{d}$.

i.e. $\overline{d} = k\overline{N}$ where k is a constant.

$(-1, 0, 0) = k(1, 0, -1)$

This is not possible

Hence, the projection plane is not perpendicular to the direction of projection. Hence it is not an orthographic projection.

6) The transformation matrix for cavalier and cabinet projections are given by:

$$P_{cav} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f.\cos\theta & f.\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cos 45° & \sin 45° & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{-----1)}$$

$$\&P_{cab}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ f.\cos\theta & f.\sin\theta & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ \frac{1}{2}.\sin 30° & \frac{1}{2}.\cos 30° & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0.43 & 0.25 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\text{---(2)}$$

The given pyromid can be shown by the following *Figure p*.

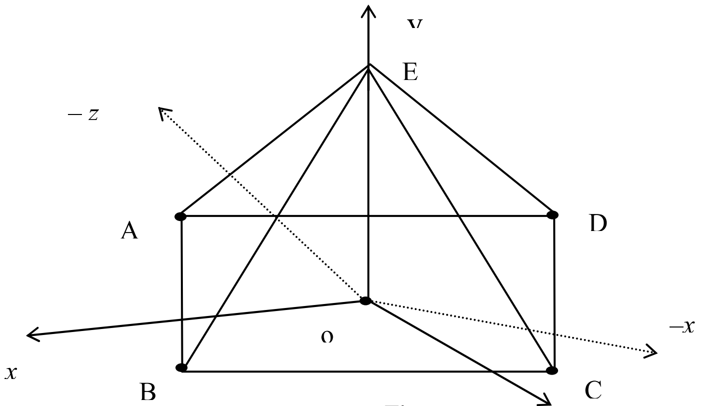

Figure p

The vertices of the pyramid are:

A (2, 0, – 2), B (2, 0, 2), C (– 2, 0, 2)
D (– 2, 0, – 2), E (0, 10, 0)

Using the projection matrices from (1) and (2), we can easily compute the new vertices of the pyramid for cavalier and cabinet projections. (refer Example 4).

Check Your Progress 3

1) Let p (*x, y, z*) be any point in 3D and the cop is E (0, 0, 0).

The parametric equation of the ray, starting from E and passing through p is:

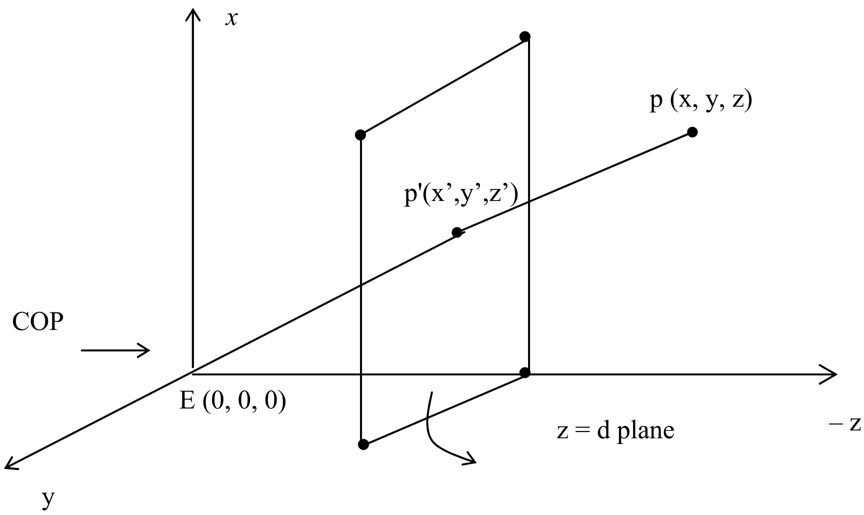

Figure q

E + t (P – E), t > 0
= (0, 0, 0) + t [(*x, y, z*) – (0, 0, 0)]
= (t.x, t. y, t. z)

For this projected point of p (x, y, z) will be:

$$t.\ z = d$$

$$= \qquad t = \frac{d}{z} \text{ must be true.}$$

Hence the projected point of p (x, y, z) will be:

$$P' = (x', y', z') = \left(\frac{d.x}{z}, \frac{d.y}{z}, d\right) \Rightarrow \text{in homogenous Coordinates } \left(\frac{dx}{z}, \frac{dy}{z}, d, 1\right)$$

$$= (dx, dy, dz, z)$$

In matrix form:

$$(x', y', z', 1) = (x, y, z, 1)\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2) Since the cube is first translated by –0.5 units in the x and y-directions, to get the centred cube on the z-axis.

The transformation matrix for translation is:

$$[T_{x,y}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix} \qquad \text{----------(1)}$$

A single-point perspective transformation onto the z = 0 plane is given by:

$$P_{per,z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{----------(2)}$$

It has a center of projection on the z-axis: at $d = -10 \Rightarrow \frac{1}{d} = -0.1$

From equation (2)

$$P_{per,z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The resulting transformation can be obtained as:

$$[T] = [T_{x,y}].\ [P_{per,z}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix}$$

Thus, the projected points of the centred cube V = [ABCDEFGH] will be:

$$[V'] = [V].\,[T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -0.5 & 0 & 0.9 \\ 0.5 & -0.5 & 0 & 0.9 \\ 0.5 & 0.5 & 0 & 0.9 \\ -0.5 & 0.5 & 0 & 0.9 \\ -0.5 & -0.5 & 0 & 1 \\ 0.5 & -0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \\ -0.5 & 0.5 & 0 & 1 \end{bmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} -0.56 & -0.56 & 0 & 1 \\ 0.56 & -0.56 & 0 & 1 \\ 0.56 & 0.56 & 0 & 1 \\ -0.56 & -0.56 & 0 & 1 \\ -0.5 & -0.5 & 0 & 1 \\ 0.5 & -0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \\ -0.5 & 0.5 & 0 & 1 \end{bmatrix}$$

3) A unit cube is placed at the origin such that its 3-edges are lying along the x,y, and z-axes. The cube is rotated about the y-axis by 30°. Obtain the perspective projection of the cube viewed from (80, 0, 60) on the z= 0 plane.

3) Rotation of a cube by 30° along y-axis,

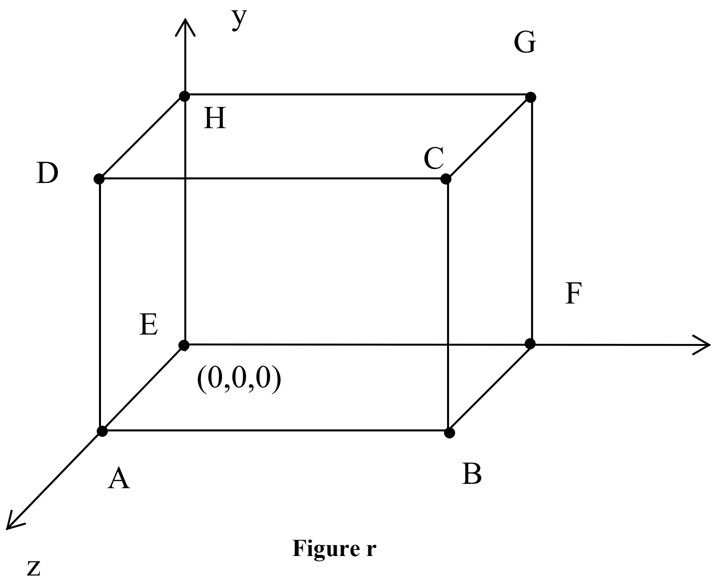

Figure r

$$[R_y]_{30°} = \begin{bmatrix} \cos 30° & 0 & -\sin 30° & 0 \\ 0 & 1 & 0 & 0 \\ \sin 30° & 0 & \cos 30° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.86 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 8.6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let p (x, y, z) be any point of a cube in a space and p' (x', y', z') is its projected point onto z = 0 plane.

The pametric equation of a line, starting from E (80, 0, 60) and passing through P (x, y, z) is:

E + t (P – E), 0 < t ∞.
= (80, 0, 60) + t [(x, y, z) – (80, 0, 60)]
= (80, 0, 60) + t[(x – 80), y, (z – 60)]
= [t (x – 80) + 80, t.y, t. (z – 60) + 60]

Assume point P' can be obtained, when t = t*

⇒ P' = (x', y', z') = [t* (x – 80) + 80, t*.y, t* (z – 60) + 60]

Since point p' lies on z = 0 plane, so

$t^* (z - 60) + 60 = 0 \Rightarrow t^* = \dfrac{-60}{z-60}$

$\Rightarrow p' = (x', y', z') = \left(\dfrac{-60.x + 80.z}{z-60}, \dfrac{-60.y}{z-60}, 0\right)$

In Homogeneous coordinates system:

$$P'(x', y', z', 1) = \left(\frac{-60.x + 80.z}{z-60}, \frac{-60.y}{z-60}, 0, 1\right)$$
$$= (-60.\ x + 80.\ z, -60.\ Y, 0, z - 60)$$

In Matrix form:

$$(x', y', z', 1) = (x, y, z, 1). \begin{bmatrix} -60 & 0 & 0 & 0 \\ 0 & -60 & 0 & 0 \\ 80 & 0 & 0 & 1 \\ 0 & 0 & 0 & -60 \end{bmatrix} \text{------------(1)}$$

$$P'_n = P_n.\ P_{par,\ z} \qquad \text{-------------(2)}$$

Since a given cube is rotated about y-axis by 30°, so the final projected point p' (of a cube on z = 0 plane) can be obtained as follows:

$$P'_n = P_n.\ [R_y]_{30°}.\ P_{par,\ z}$$

$$(x', y', z', 1) = (x, y, z, 1) . \begin{bmatrix} 0.86 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.86 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} -60 & 0 & 0 & 0 \\ 0 & -60 & 0 & 0 \\ 80 & 0 & 0 & 1 \\ 0 & 0 & 0 & -60 \end{bmatrix}$$

$$(x', y', z', 1) = (x, y, z, 1) \begin{bmatrix} 91.9 & 0 & 0 & -0.5 \\ 0 & -60 & 0 & 0 \\ 38.8 & 0 & 0 & 0.86 \\ 0 & 0 & 0 & -60 \end{bmatrix} \text{-------------(3)}$$

$P''n = P.\ P_{par}, z, 30°$

This equation (3) is the required perspective transformation. Which gives a coordinates of a projected point P' (x', y', z') onto the z = 0 plane, when a point P (x, y, z) is viewed from E (80, 0, 60).

Thus, all the projected points of a given cube can be obtained as follows:

$$P' = V.\ Ppar,\ z,\ 30° = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 90.9 & 0 & 0 & -0.5 \\ 0 & -60 & 0 & 0 \\ 38.8 & 0 & 0 & 0.86 \\ 0 & 0 & 0 & -60 \end{bmatrix}$$

$$= \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array}\begin{bmatrix} 38.8 & 0 & 0 & -59.14 \\ 129.7 & 0 & 0 & -59.64 \\ 129.7 & -60 & 0 & -59.64 \\ 38.8 & -60 & 0 & -60.86 \\ 0 & 0 & 0 & -60.0 \\ 90.9 & 0 & 0 & -60.5 \\ 90.9 & -60 & 0 & -60.5 \\ 0 & -60 & 0 & -60.0 \end{bmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array}\begin{bmatrix} -0.72 & 0 & 0 & 1 \\ -2.17 & 0 & 0 & 1 \\ -2.17 & 1.01 & 0 & 1 \\ -0.64 & 0.99 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1.50 & 0 & 0 & 1 \\ -1.50 & 0.99 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Hence, A' = (– 0.72, 0, 0), B' = (– 2.17, 0, 0), C' = (– 2.17, 1.01, 0)
D' = (– 0.64, 0.99, 0), E' = (0, 0, 0), F' = (– 1.5, 0, 0)
G' = (– 1.50, 0.99, 0) and H' = (0, 1, 0).

Check Your Progress 4

1) The given perspective transformation matrix can be written as:

From Rows one, two and three from equation matrix (I), the vanishing point w.r.t. x, y and z axis, will be:

$C_x = (3.1, 2.0, 0)$
$C_y = (0, 4.56, 0)$
$C_z = (3.5, 4.0, 0)$

2) From the given V.P., we can obtain the corresponding center of projections. Since vanishing points: $V_x = 5$, $V_y = 5$ and $V_z = -5$, hence center of projections is at:

$C_x = -5$, $C_y = -5$ and $C_z = 5$

$$\therefore 1/d_1 = \frac{1}{5} = 0.2,\ \frac{1}{d_2} = \frac{1}{5} = 0.2 \text{ and } \frac{1}{d_3} = \frac{-1}{5} = -0.2$$

Hence, the 3 – point perspective transformation is:

$$P_{per-xyz} = \begin{bmatrix} 1 & 0 & 0 & 0.2 \\ 0 & 1 & 0 & 0.2 \\ 0 & 0 & 1 & -0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{------------(I)}$$

Thus by multiplying v = [ABCDEFGH] with projection matrix (I), we can obtain the transformed vertices of a given cube.