Simplex Solver

October 20, 2023

Problem

Given the following linear system and objective function, find the optimal solution.

$$min(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_10 + x_11 + x_12 + x_13 + x_14 + x_15 + x_16 + x_17 + x_18 + x_1$$

$$\begin{cases} y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_10 - 9y_11 - 10y_12 - 8y_13 - 4y_14 - 3y_15 + 6y_16 \geq \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_10 + 9y_11 - 10y_12 + 7y_13 - 4y_14 + 10y_15 + 7y_16 \\ y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_10 - 9y_11 - 10y_12 - 8y_13 - 4y_14 - 3y_15 + 6y_16 \geq \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_10 + 9y_11 - 10y_12 + 7y_13 - 4y_14 + 10y_15 + 7y_16 \\ y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_10 - 9y_11 - 10y_12 - 8y_13 - 4y_14 - 3y_15 + 6y_16 \geq \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_10 + 9y_11 - 10y_12 + 7y_13 - 4y_14 + 10y_15 + 7y_16 \end{cases}$$

Solution

Add slack variables to turn all inequalities to equalities.

$$\begin{cases} y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + s_1 = 1 \\ 2y_1 - 5y_2 + 2y_3 - 5y_4 + 2y_5 - 5y_6 + s_2 = 1 \\ -y_1 + 2y_2 - y_3 + 2y_4 - y_5 + 2y_6 + s_3 = 1 \\ -y_1 + 3y_2 - y_3 + 3y_4 - y_5 + 3y_6 + s_4 = 1 \\ 10y_1 - 6y_2 + 10y_3 - 6y_4 + 10y_5 - 6y_6 + s_5 = 1 \\ -7y_1 - 7y_2 - 7y_3 - 7y_4 - 7y_5 - 7y_6 + s_6 = 1 \\ -10y_1 + 7y_2 - 10y_3 + 7y_4 - 10y_5 + 7y_6 + s_7 = 1 \\ -2y_1 - 8y_2 - 2y_3 - 8y_4 - 2y_5 - 8y_6 + s_8 = 1 \\ -3y_1 + 8y_2 - 3y_3 + 8y_4 - 3y_5 + 8y_6 + s_9 = 1 \\ -3y_1 + 3y_2 - 3y_3 + 3y_4 - 3y_5 + 3y_6 + s_10 = 1 \\ -9y_1 + 9y_2 - 9y_3 + 9y_4 - 9y_5 + 9y_6 + s_11 = 1 \\ -10y_1 - 10y_2 - 10y_3 - 10y_4 - 10y_5 - 10y_6 + s_12 = 1 \\ -8y_1 + 7y_2 - 8y_3 + 7y_4 - 8y_5 + 7y_6 + s_13 = 1 \\ -4y_1 - 4y_2 - 4y_3 - 4y_4 - 4y_5 - 4y_6 + s_14 = 1 \\ -3y_1 + 10y_2 - 3y_3 + 10y_4 - 3y_5 + 10y_6 + s_15 = 1 \\ 6y_1 + 7y_2 + 6y_3 + 7y_4 + 6y_5 + 7y_6 + s_16 = 1 \end{cases}$$

Create the initial tableau of the new linear system.

$$
\begin{array}{cccccccccccccccccccccc}
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} & s_{15} \\
\hline
1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -5 & 2 & -5 & 2 & -5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 2 & -1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 3 & -1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & -6 & 10 & -6 & 10 & -6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-7 & -7 & -7 & -7 & -7 & -7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 7 & -10 & 7 & -10 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & -8 & -2 & -8 & -2 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 8 & -3 & 8 & -3 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 3 & -3 & 3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-9 & 9 & -9 & 9 & -9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-10 & -10 & -10 & -10 & -10 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-8 & 7 & -8 & 7 & -8 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-4 & -4 & -4 & -4 & -4 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-3 & 10 & -3 & 10 & -3 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
6 & 7 & 6 & 7 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & -1 & -3 & 3 & -4 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is y_6 and the departing variable is s_{15}.

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$$
\begin{array}{cccccccccccccccccccc}
y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 & s_{10} & s_{11} & s_{12} & s_{13} & s_{14} \\
\hline
7/10 & 0 & 7/10 & 0 & 7/10 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2/5 & 0 & -2/5 & 0 & -2/5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/10 & 0 & -1/10 & 0 & -1/10 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
41/5 & 0 & 41/5 & 0 & 41/5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-91/10 & 0 & -91/10 & 0 & -91/10 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-79/10 & 0 & -79/10 & 0 & -79/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-22/5 & 0 & -22/5 & 0 & -22/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3/5 & 0 & -3/5 & 0 & -3/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-21/10 & 0 & -21/10 & 0 & -21/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-63/10 & 0 & -63/10 & 0 & -63/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-13 & 0 & -13 & 0 & -13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-59/10 & 0 & -59/10 & 0 & -59/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-26/5 & 0 & -26/5 & 0 & -26/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-3/10 & 1 & -3/10 & 1 & -3/10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
81/10 & 0 & 81/10 & 0 & 81/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-17/5 & 7 & -27/5 & 11 & -32/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is y_5 and the departing variable is s_{16}.

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Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

y_1	y_2	y_3	y_4	y_5	y_6	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_10	s_11	s_12	s_13	s_14	s_15	
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	$13/81$	$-$
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	$44/81$	$-$
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	$-19/81$	4
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	$-25/81$	1
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	$106/81$	-8
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	$-7/81$	9
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	$-112/81$	7
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	$34/81$	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	$-23/27$	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	$-13/27$	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	$-13/9$	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	$-10/81$	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	$-98/81$	59
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	$-4/81$	5
0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$2/27$	1
1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-7/81$	10
3	7	1	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$20/81$	64

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$s_1 = \frac{29}{27}, s_10 = \frac{7}{9}, s_11 = \frac{1}{3}, s_12 = \frac{67}{27}, s_13 = \frac{14}{27}, s_14 = \frac{43}{27}, s_15 = 0, s_16 = 0, s_2 = \frac{40}{27}, s_3 = \frac{22}{27}, s_4 = \frac{19}{27}, s_5 = \frac{35}{27},$$