

# Simplex Solver

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## Problem

Given the following linear system and objective function, find the optimal solution.

$$\min(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19})$$

$$\begin{cases} y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_{10} - 9y_{11} - 10y_{12} - 8y_{13} - 4y_{14} - 3y_{15} + 6y_{16} \geq 1 \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_{10} + 9y_{11} - 10y_{12} + 7y_{13} - 4y_{14} + 10y_{15} + 7y_{16} \geq 1 \\ y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_{10} - 9y_{11} - 10y_{12} - 8y_{13} - 4y_{14} - 3y_{15} + 6y_{16} \geq 1 \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_{10} + 9y_{11} - 10y_{12} + 7y_{13} - 4y_{14} + 10y_{15} + 7y_{16} \geq 1 \\ y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_{10} - 9y_{11} - 10y_{12} - 8y_{13} - 4y_{14} - 3y_{15} + 6y_{16} \geq 1 \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_{10} + 9y_{11} - 10y_{12} + 7y_{13} - 4y_{14} + 10y_{15} + 7y_{16} \geq 1 \end{cases}$$

## Solution

Add slack variables to turn all inequalities to equalities.

$$\begin{cases} y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + s_1 = 1 \\ 2y_1 - 5y_2 + 2y_3 - 5y_4 + 2y_5 - 5y_6 + s_2 = 1 \\ -y_1 + 2y_2 - y_3 + 2y_4 - y_5 + 2y_6 + s_3 = 1 \\ -y_1 + 3y_2 - y_3 + 3y_4 - y_5 + 3y_6 + s_4 = 1 \\ 10y_1 - 6y_2 + 10y_3 - 6y_4 + 10y_5 - 6y_6 + s_5 = 1 \\ -7y_1 - 7y_2 - 7y_3 - 7y_4 - 7y_5 - 7y_6 + s_6 = 1 \\ -10y_1 + 7y_2 - 10y_3 + 7y_4 - 10y_5 + 7y_6 + s_7 = 1 \\ -2y_1 - 8y_2 - 2y_3 - 8y_4 - 2y_5 - 8y_6 + s_8 = 1 \\ -3y_1 + 8y_2 - 3y_3 + 8y_4 - 3y_5 + 8y_6 + s_9 = 1 \\ -3y_1 + 3y_2 - 3y_3 + 3y_4 - 3y_5 + 3y_6 + s_{10} = 1 \\ -9y_1 + 9y_2 - 9y_3 + 9y_4 - 9y_5 + 9y_6 + s_{11} = 1 \\ -10y_1 - 10y_2 - 10y_3 - 10y_4 - 10y_5 - 10y_6 + s_{12} = 1 \\ -8y_1 + 7y_2 - 8y_3 + 7y_4 - 8y_5 + 7y_6 + s_{13} = 1 \\ -4y_1 - 4y_2 - 4y_3 - 4y_4 - 4y_5 - 4y_6 + s_{14} = 1 \\ -3y_1 + 10y_2 - 3y_3 + 10y_4 - 3y_5 + 10y_6 + s_{15} = 1 \\ 6y_1 + 7y_2 + 6y_3 + 7y_4 + 6y_5 + 7y_6 + s_{16} = 1 \end{cases}$$

Create the initial tableau of the new linear system.

$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$	$s_7$	$s_8$	$s_9$	$s_{10}$	$s_{11}$	$s_{12}$	$s_{13}$	$s_{14}$	$s_{15}$
1	-1	1	-1	1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	-5	2	-5	2	-5	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	2	-1	2	-1	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
-1	3	-1	3	-1	3	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
10	-6	10	-6	10	-6	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
-7	-7	-7	-7	-7	-7	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
-10	7	-10	7	-10	7	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
-2	-8	-2	-8	-2	-8	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
-3	8	-3	8	-3	8	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
-3	3	-3	3	-3	3	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
-9	9	-9	9	-9	9	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
-10	-10	-10	-10	-10	-10	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
-8	7	-8	7	-8	7	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
-4	-4	-4	-4	-4	-4	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
-3	10	-3	10	-3	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
6	7	6	7	6	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	-1	-3	3	-4	-8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is  $y_6$  and the departing variable is  $s_{15}$ .

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$	$s_7$	$s_8$	$s_9$	$s_{10}$	$s_{11}$	$s_{12}$	$s_{13}$	$s_{14}$	$s_{15}$
7/10	0	7/10	0	7/10	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1/2	0	1/2	0	1/2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
-2/5	0	-2/5	0	-2/5	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
-1/10	0	-1/10	0	-1/10	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
41/5	0	41/5	0	41/5	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
-91/10	0	-91/10	0	-91/10	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
-79/10	0	-79/10	0	-79/10	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
-22/5	0	-22/5	0	-22/5	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
-3/5	0	-3/5	0	-3/5	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
-21/10	0	-21/10	0	-21/10	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
-63/10	0	-63/10	0	-63/10	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
-13	0	-13	0	-13	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
-59/10	0	-59/10	0	-59/10	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
-26/5	0	-26/5	0	-26/5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
-3/10	1	-3/10	1	-3/10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
81/10	0	81/10	0	81/10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-17/5	7	-27/5	11	-32/5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is  $y_5$  and the departing variable is  $s_{16}$ .

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$	$s_7$	$s_8$	$s_9$	$s_{10}$	$s_{11}$	$s_{12}$	$s_{13}$	$s_{14}$	$s_{15}$	
0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	13/81	—
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	44/81	—
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	−19/81	4
0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	−25/81	1
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	106/81	−8
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	−7/81	9
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	−112/81	7
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	34/81	4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	−23/27	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−13/27	7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−13/9	'
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−10/81	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	−98/81	5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	−4/81	5
0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2/27	1
1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−7/81	10
3	7	1	11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	20/81	6

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$s_1 = \frac{29}{27}, s_{10} = \frac{7}{9}, s_{11} = \frac{1}{3}, s_{12} = \frac{67}{27}, s_{13} = \frac{14}{27}, s_{14} = \frac{43}{27}, s_{15} = 0, s_{16} = 0, s_2 = \frac{40}{27}, s_3 = \frac{22}{27}, s_4 = \frac{19}{27}, s_5 = \frac{35}{27},$$