

# Simplex Solver

October 20, 2023

## Problem

Given the following linear system and objective function, find the optimal solution.

$$\min(x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19})$$

$$\left\{ \begin{array}{l} y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_{10} - 9y_{11} - 10y_{12} - 8y_{13} - 4y_{14} - 3y_{15} + 6y_{16} \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_{10} + 9y_{11} - 10y_{12} + 7y_{13} - 4y_{14} + 10y_{15} + 7y_{16} \\ y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_{10} - 9y_{11} - 10y_{12} - 8y_{13} - 4y_{14} - 3y_{15} + 6y_{16} \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_{10} + 9y_{11} - 10y_{12} + 7y_{13} - 4y_{14} + 10y_{15} + 7y_{16} \\ y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_{10} - 9y_{11} - 10y_{12} - 8y_{13} - 4y_{14} - 3y_{15} + 6y_{16} \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_{10} + 9y_{11} - 10y_{12} + 7y_{13} - 4y_{14} + 10y_{15} + 7y_{16} \\ y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_{10} - 9y_{11} - 10y_{12} - 8y_{13} - 4y_{14} - 3y_{15} + 6y_{16} \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_{10} + 9y_{11} - 10y_{12} + 7y_{13} - 4y_{14} + 10y_{15} + 7y_{16} \\ y_1 + 2y_2 - y_3 - y_4 + 10y_5 - 7y_6 - 10y_7 - 2y_8 - 3y_9 - 3y_{10} - 9y_{11} - 10y_{12} - 8y_{13} - 4y_{14} - 3y_{15} + 6y_{16} \\ -y_1 - 5y_2 + 2y_3 + 3y_4 - 6y_5 - 7y_6 + 7y_7 - 8y_8 + 8y_9 + 3y_{10} + 9y_{11} - 10y_{12} + 7y_{13} - 4y_{14} + 10y_{15} + 7y_{16} \end{array} \right.$$

## Solution

Add slack variables to turn all inequalities to equalities.

$$\left\{ \begin{array}{l}
y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7 - y_8 + y_9 - y_1 0 + s_1 = 1 \\
2y_1 - 5y_2 + 2y_3 - 5y_4 + 2y_5 - 5y_6 + 2y_7 - 5y_8 + 2y_9 - 5y_1 0 + s_2 = 1 \\
-y_1 + 2y_2 - y_3 + 2y_4 - y_5 + 2y_6 - y_7 + 2y_8 - y_9 + 2y_1 0 + s_3 = 1 \\
-y_1 + 3y_2 - y_3 + 3y_4 - y_5 + 3y_6 - y_7 + 3y_8 - y_9 + 3y_1 0 + s_4 = 1 \\
10y_1 - 6y_2 + 10y_3 - 6y_4 + 10y_5 - 6y_6 + 10y_7 - 6y_8 + 10y_9 - 6y_1 0 + s_5 = 1 \\
-7y_1 - 7y_2 - 7y_3 - 7y_4 - 7y_5 - 7y_6 - 7y_7 - 7y_8 - 7y_9 - 7y_1 0 + s_6 = 1 \\
-10y_1 + 7y_2 - 10y_3 + 7y_4 - 10y_5 + 7y_6 - 10y_7 + 7y_8 - 10y_9 + 7y_1 0 + s_7 = 1 \\
-2y_1 - 8y_2 - 2y_3 - 8y_4 - 2y_5 - 8y_6 - 2y_7 - 8y_8 - 2y_9 - 8y_1 0 + s_8 = 1 \\
-3y_1 + 8y_2 - 3y_3 + 8y_4 - 3y_5 + 8y_6 - 3y_7 + 8y_8 - 3y_9 + 8y_1 0 + s_9 = 1 \\
-3y_1 + 3y_2 - 3y_3 + 3y_4 - 3y_5 + 3y_6 - 3y_7 + 3y_8 - 3y_9 + 3y_1 0 + s_{10} = 1 \\
-9y_1 + 9y_2 - 9y_3 + 9y_4 - 9y_5 + 9y_6 - 9y_7 + 9y_8 - 9y_9 + 9y_1 0 + s_{11} = 1 \\
-10y_1 - 10y_2 - 10y_3 - 10y_4 - 10y_5 - 10y_6 - 10y_7 - 10y_8 - 10y_9 - 10y_1 0 + s_{12} = 1 \\
-8y_1 + 7y_2 - 8y_3 + 7y_4 - 8y_5 + 7y_6 - 8y_7 + 7y_8 - 8y_9 + 7y_1 0 + s_{13} = 1 \\
-4y_1 - 4y_2 - 4y_3 - 4y_4 - 4y_5 - 4y_6 - 4y_7 - 4y_8 - 4y_9 - 4y_1 0 + s_{14} = 1 \\
-3y_1 + 10y_2 - 3y_3 + 10y_4 - 3y_5 + 10y_6 - 3y_7 + 10y_8 - 3y_9 + 10y_1 0 + s_{15} = 1 \\
6y_1 + 7y_2 + 6y_3 + 7y_4 + 6y_5 + 7y_6 + 6y_7 + 7y_8 + 6y_9 + 7y_1 0 + s_{16} = 1 \\
s_{17} = 1 \\
-2y_1 - 7y_2 - 2y_3 - 7y_4 - 2y_5 - 7y_6 - 2y_7 - 7y_8 - 2y_9 - 7y_1 0 + s_{18} = 1 \\
-6y_1 - 8y_2 - 6y_3 - 8y_4 - 6y_5 - 8y_6 - 6y_7 - 8y_8 - 6y_9 - 8y_1 0 + s_{19} = 1 \\
-4y_1 + 2y_2 - 4y_3 + 2y_4 - 4y_5 + 2y_6 - 4y_7 + 2y_8 - 4y_9 + 2y_1 0 + s_{20} = 1 \\
-7y_1 - 2y_2 - 7y_3 - 2y_4 - 7y_5 - 2y_6 - 7y_7 - 2y_8 - 7y_9 - 2y_1 0 + s_{21} = 1 \\
y_1 - 9y_2 + y_3 - 9y_4 + y_5 - 9y_6 + y_7 - 9y_8 + y_9 - 9y_1 0 + s_{22} = 1 \\
y_1 + 4y_2 + y_3 + 4y_4 + y_5 + 4y_6 + y_7 + 4y_8 + y_9 + 4y_1 0 + s_{23} = 1 \\
9y_1 + 7y_2 + 9y_3 + 7y_4 + 9y_5 + 7y_6 + 9y_7 + 7y_8 + 9y_9 + 7y_1 0 + s_{24} = 1
\end{array} \right.$$

Create the initial tableau of the new linear system.

$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_7$	$y_8$	$y_9$	$y_{10}$	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$	$s_7$	$s_8$	$s_9$	$s_{10}$
1	-1	1	-1	1	-1	1	-1	1	-1	1	0	0	0	0	0	0	0	0	0
2	-5	2	-5	2	-5	2	-5	2	-5	0	1	0	0	0	0	0	0	0	0
-1	2	-1	2	-1	2	-1	2	-1	2	0	0	1	0	0	0	0	0	0	0
-1	3	-1	3	-1	3	-1	3	-1	3	0	0	0	1	0	0	0	0	0	0
10	-6	10	-6	10	-6	10	-6	10	-6	0	0	0	0	1	0	0	0	0	0
-7	-7	-7	-7	-7	-7	-7	-7	-7	-7	0	0	0	0	0	1	0	0	0	0
-10	7	-10	7	-10	7	-10	7	-10	7	0	0	0	0	0	0	1	0	0	0
-2	-8	-2	-8	-2	-8	-2	-8	-2	-8	0	0	0	0	0	0	0	1	0	0
-3	8	-3	8	-3	8	-3	8	-3	8	0	0	0	0	0	0	0	0	1	0
-3	3	-3	3	-3	3	-3	3	-3	3	0	0	0	0	0	0	0	0	0	1
-9	9	-9	9	-9	9	-9	9	-9	9	0	0	0	0	0	0	0	0	0	0
-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	0	0	0	0	0	0	0	0	0	0
-8	7	-8	7	-8	7	-8	7	-8	7	0	0	0	0	0	0	0	0	0	0
-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	0	0	0	0	0	0	0	0	0	0
-3	10	-3	10	-3	10	-3	10	-3	10	0	0	0	0	0	0	0	0	0	0
6	7	6	7	6	7	6	7	6	7	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-2	-7	-2	-7	-2	-7	-2	-7	-2	-7	0	0	0	0	0	0	0	0	0	0
-6	-8	-6	-8	-6	-8	-6	-8	-6	-8	0	0	0	0	0	0	0	0	0	0
-4	2	-4	2	-4	2	-4	2	-4	2	0	0	0	0	0	0	0	0	0	0
-7	-2	-7	-2	-7	-2	-7	-2	-7	-2	0	0	0	0	0	0	0	0	0	0
1	-9	1	-9	1	-9	1	-9	1	-9	0	0	0	0	0	0	0	0	0	0
1	4	1	4	1	4	1	4	1	4	0	0	0	0	0	0	0	0	0	0
9	7	9	7	9	7	9	7	9	7	0	0	0	0	0	0	0	0	0	0
-1	-1	-3	3	-4	-8	2	10	5	-3	0	0	0	0	0	0	0	0	0	0

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is  $y_6$  and the departing variable is  $s_1$ .

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_7$	$y_8$	$y_9$	$y_{10}$	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$	$s_7$	$s_8$
$7/10$	0	$7/10$	0	$7/10$	0	$7/10$	0	$7/10$	0	1	0	0	0	0	0	0	0
$1/2$	0	$1/2$	0	$1/2$	0	$1/2$	0	$1/2$	0	0	1	0	0	0	0	0	0
$-2/5$	0	$-2/5$	0	$-2/5$	0	$-2/5$	0	$-2/5$	0	0	0	1	0	0	0	0	0
$-1/10$	0	$-1/10$	0	$-1/10$	0	$-1/10$	0	$-1/10$	0	0	0	0	1	0	0	0	0
$41/5$	0	$41/5$	0	$41/5$	0	$41/5$	0	$41/5$	0	0	0	0	0	1	0	0	0
$-91/10$	0	$-91/10$	0	$-91/10$	0	$-91/10$	0	$-91/10$	0	0	0	0	0	0	1	0	0
$-79/10$	0	$-79/10$	0	$-79/10$	0	$-79/10$	0	$-79/10$	0	0	0	0	0	0	0	1	0
$-22/5$	0	$-22/5$	0	$-22/5$	0	$-22/5$	0	$-22/5$	0	0	0	0	0	0	0	0	1
$-3/5$	0	$-3/5$	0	$-3/5$	0	$-3/5$	0	$-3/5$	0	0	0	0	0	0	0	0	0
$-21/10$	0	$-21/10$	0	$-21/10$	0	$-21/10$	0	$-21/10$	0	0	0	0	0	0	0	0	0
$-63/10$	0	$-63/10$	0	$-63/10$	0	$-63/10$	0	$-63/10$	0	0	0	0	0	0	0	0	0
-13	0	-13	0	-13	0	-13	0	-13	0	0	0	0	0	0	0	0	0
$-59/10$	0	$-59/10$	0	$-59/10$	0	$-59/10$	0	$-59/10$	0	0	0	0	0	0	0	0	0
$-26/5$	0	$-26/5$	0	$-26/5$	0	$-26/5$	0	$-26/5$	0	0	0	0	0	0	0	0	0
$-3/10$	1	$-3/10$	1	$-3/10$	1	$-3/10$	1	$-3/10$	1	0	0	0	0	0	0	0	0
$81/10$	0	$81/10$	0	$81/10$	0	$81/10$	0	$81/10$	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$-41/10$	0	$-41/10$	0	$-41/10$	0	$-41/10$	0	$-41/10$	0	0	0	0	0	0	0	0	0
$-42/5$	0	$-42/5$	0	$-42/5$	0	$-42/5$	0	$-42/5$	0	0	0	0	0	0	0	0	0
$-17/5$	0	$-17/5$	0	$-17/5$	0	$-17/5$	0	$-17/5$	0	0	0	0	0	0	0	0	0
$-38/5$	0	$-38/5$	0	$-38/5$	0	$-38/5$	0	$-38/5$	0	0	0	0	0	0	0	0	0
$-17/10$	0	$-17/10$	0	$-17/10$	0	$-17/10$	0	$-17/10$	0	0	0	0	0	0	0	0	0
$11/5$	0	$11/5$	0	$11/5$	0	$11/5$	0	$11/5$	0	0	0	0	0	0	0	0	0
$111/10$	0	$111/10$	0	$111/10$	0	$111/10$	0	$111/10$	0	0	0	0	0	0	0	0	0
$-17/5$	7	$-27/5$	11	$-32/5$	0	$-2/5$	18	$13/5$	5	0	0	0	0	0	0	0	0

There are negative elements in the bottom row, so the current solution is not optimal. Thus, pivot to improve the current solution. The entering variable is  $y_5$  and the departing variable is  $s_2$ .

Perform elementary row operations until the pivot element is 1 and all other elements in the entering column are 0.

$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_7$	$y_8$	$y_9$	$y_{10}$	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$	$s_7$	$s_8$	$s_9$	$s_{10}$	$s_{11}$	$s_{12}$	$s_{13}$
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	7	1	11	0	0	6	18	9	5	0	0	0	0	0	0	0	0	0	0	0	0	0

There are no negative elements in the bottom row, so we know the solution is optimal. Thus, the solution is:

$$s_1 = \frac{40}{37}, s_{10} = \frac{28}{37}, s_{11} = \frac{10}{37}, s_{12} = \frac{87}{37}, s_{13} = \frac{17}{37}, s_{14} = \frac{57}{37}, s_{15} = 0, s_{16} = \frac{3}{37}, s_{17} = 1, s_{18} = \frac{67}{37}, s_{19} = \frac{75}{37}, s_2$$