HW 2 Problem 5

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R_B = rate of excess return on bond fund
R_S = rate of excess return on stock fund

σ_B^2 = variance of R_B
σ_S^2 = variance of R_S
$Cov(R_B, R_S)$ = Covariance between R_S and R_B
$= Cov(r_B, r_S)$ since r_f is just a constant

W_B = proportion invested in bond fund
$W_S = 1 - W_B$ = proportion invested in stock fund

expected HPR (average rate of return)

$$\text{Sharpe ratio} = \frac{\text{Risk premium}}{\text{SD excess return}} = \frac{E(r) - r_f}{\sigma}$$

r_f → risk free rate

FIND W_B THAT MAXIMIZES SHARPE RATIO

$$\max_{W_S, W_B} S_p = \frac{E(r_p) - r_f}{\sigma_p} = \frac{E[R_p]}{\sigma_p} = \frac{E[W_S R_S + W_B R_B]}{\sigma_p} = \frac{E[W_S R_S] + E[W_B R_B]}{\sigma_p}$$

$$= \frac{R_S E[1 - W_B] + R_B E[W_B]}{\sigma_p} \longrightarrow \sigma_p = \sqrt{W_B^2 \sigma_B^2 + W_S^2 \sigma_S^2 + 2 W_B W_S Cov(R_B, R_S)}$$

Changing W_B and $E[R_B]$ into W_1 and R_1
W_S and $E[R_S]$ into W_2 and R_2

The S's and B's and R's are confusing

$$S_p = \frac{W_1 R_1 + W_2 R_2}{\sqrt{W_1^2 \sigma_1^2 + W_2^2 \sigma_2^2 + 2 W_1 W_2 \sigma_{12}}}$$

$$S_p^2 = \frac{W_1^2 R_1^2 + 2 W_1 R_1 W_2 R_2 + W_2^2 R_2^2}{W_1^2 \sigma_1^2 + W_2^2 \sigma_2^2 + 2 W_1 W_2 \sigma_{12}}$$

Square both sides

$$= \frac{R_1^2 W_1^2 + 2 R_1 R_2 W_1 W_2 + R_2^2 W_2^2}{\sigma_1^2 W_1^2 + \sigma_2^2 W_2^2 + 2 \sigma_{12} W_1 W_2}$$

Rewrite in order with constants first for sanity (e.g. $3x \rightarrow 3x$ instead of $x3$)

$$= \frac{R_1^2 W_1^2 + 2 R_1 R_2 W_1 (1 - W_1) + R_2^2 (1 - W_1)^2}{\sigma_1^2 W_1^2 + \sigma_2^2 (1 - W_1)^2 + 2 \sigma_{12} W_1 (1 - W_1)}$$

Sub $W_2 = 1 - W_1$

$$= \frac{R_1^2 W_1^2 + 2 R_1 R_2 W_1 (1 - W_1) + R_2^2 (1 - 2W_1 + W_1^2)}{\sigma_1^2 W_1^2 + \sigma_2^2 (1 - 2W_1 + W_1^2) + 2 \sigma_{12} W_1 (1 - W_1)}$$

Sub $(1 - W_1)^2 = 1 - 2W_1 + W_1^2$

$$S_p^2 = \frac{R_1^2 W_1^2 + 2 R_1 R_2 W_1 - 2 R_1 R_2 W_1^2 + R_2^2 - 2 R_2^2 W_1 + R_2^2 W_1^2}{\sigma_1^2 W_1^2 + \sigma_2^2 - 2\sigma_2^2 W_1 + \sigma_2^2 W_1^2 + 2\sigma_{12} W_1 - 2\sigma_{12} W_1^2}$$

Distribute

$$\frac{\partial S_p^2}{\partial W_1} = \frac{\left[\begin{array}{l}(2R_1^2 W_1 + 2R_1 R_2 - 4 R_1 R_2 W_1 - 2R_2^2 + 2R_2^2 W_1)(\sigma_1^2 W_1^2 + \sigma_2^2 - 2\sigma_2^2 W_1 + \sigma_2^2 W_1^2 + 2\sigma_{12} W_1 - 2\sigma_{12} W_1^2) \\ - (R_1^2 W_1^2 + 2R_1 R_2 W_1 - 2 R_1 R_2 W_1^2 + R_2^2 - 2R_2^2 W_1 + R_2^2 W_1^2)(2\sigma_1^2 W_1 - 2\sigma_2^2 + 2\sigma_2^2 W_1 + 2\sigma_{12} - 4\sigma_{12} W_1)\end{array}\right]}{(\sigma_1^2 W_1^2 + \sigma_2^2 - 2\sigma_2^2 W_1 + \sigma_2^2 W_1^2 + 2\sigma_{12} W_1 - 2\sigma_{12} W_1^2)^2}$$

$$= \frac{\left[\begin{array}{l}[W_1 (2R_1^2 - 4R_1 R_2 + 2R_2^2) + 2R_1 R_2 - 2R_2^2][W_1 (\sigma_1^2 - 2\sigma_2^2 + \sigma_2^2 W_1 + 2\sigma_{12} - 2\sigma_{12} W_1) + \sigma_2^2] \\ - [W_1 (R_1^2 W_1 + 2 R_1 R_2 W_1 - 2R_2^2 + R_2^2 W_1) + R_2^2][W_1 (2\sigma_1^2 + 2\sigma_2^2 - 4\sigma_{12}) - 2\sigma_2^2 + 2\sigma_{12}]\end{array}\right]}{(\sigma_1^2 W_1^2 + \sigma_2^2 - 2\sigma_2^2 W_1 + \sigma_2^2 W_1^2 + 2\sigma_{12} W_1 - 2\sigma_{12} W_1^2)^2}$$

This is so long, I'm changing notation again:

$W_1 = x$, $R_1 = a$, $R_2 = b$, $\sigma_1 = c$, $\sigma_2 = d$, $\sigma_{12} = g$

$$= \frac{-2a^2 d^2 x^2 - 2abc^2 x^2 + 2abd^2 x^2 + 2b^2 c^2 x^2 + 2a^2 d^2 x + 2a^2 g x^2 - 4abd^2 x - 2b^2 c^2 x - 2b^2 g x^2 + 2abd^2 + 4b^2 g x - 2b^2 g}{c^4 x^4 + 2c^2 d^2 x^4 + d^4 x^4 - 4c^2 d^2 x^3 - 4c^2 g x^4 - 4d^4 x^3 - 4d^2 g x^4 + 2c^2 d^2 x^2 + 4c^2 g x^3 + 6d^4 x^2 + 12 d^2 g x^3 + 4g^2 x^4 - 4d^2 x - 12 d^2 g x^2 - 8g^2 x^3 + d^4 + 4d^2 g x + 4g^2 x^2}$$

$$0 = -\frac{2[x(b-a) - b] * [x(g(b+a) - ad^2 - bc^2) - bg + ad^2]}{(x^2 (2g - d^2 - c^2) - 2x(g - d^2) - d^2)^2}$$

Solve for x

$$x^* = \frac{-bg + ad^2}{-g(b+a) + ad^2 + bc^2}$$

$$W_1^* = \frac{-R_2 \sigma_{12} + R_1 \sigma_2^2}{-\sigma_{12}(R_2 + R_1) + R_1 \sigma_2^2 + R_2 \sigma_1^2}$$

$$W_B^* = \frac{-E[R_S] Cov(R_S, R_B) + E[R_B] \sigma_S^2}{-Cov(R_S, R_B)(E[R_S] + E[R_B]) + E[R_B] \sigma_S^2 + E[R_S] \sigma_B^2}$$

$$\boxed{W_B^* = \frac{E[R_B] \sigma_S^2 - E[R_S] Cov(R_S, R_B)}{E[R_B] \sigma_S^2 + E[R_S] \sigma_B^2 - Cov(R_S, R_B)(E[R_S] + E[R_B])}}$$