A SURVEY ON THE GRACEFUL LABELING OF GRAPHS

by

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Thesis directed by Professor Richard Lundgren

ABSTRACT

A graph G is said to be graceful if the numbers used to label its vertices are distinct values of the set $\{0, 1, ..., m\}$, and the edge labels are the set $\{1, 2, ..., m\}$, where the edge labels are the absolute value of the difference of the vertex values. Graceful labeling grew from Ringel's Conjecture, which states that for a given tree T with n vertices and $n-1$ edges, the edges of the complete graph K_{2n-1} can be partitioned into $2n-1$ trees isomorphic to T. Ringel's Conjecture is one of the major unsolved problems of graph theory. The existence of a graceful labeling for a graph implies Ringel's Conjecture is true for that graph. It is this relationship between graceful labeling and Ringel's Conjecture that has created the interest and research in graceful labeling. This is a survey paper exploring the history of the prolonged and prolific interest in the graceful labeling of graphs, as well as the algorithms for gracefully labeling a variety of graphs, including paths, caterpillars, lobsters, k-ary trees, planar grids, cylindrical grids, and balanced trees.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed Richard Lundgren
Richard Lundgren

DEDICATION

To my sons Brian Collison, Greg Collison, and Luke Egbert for their unending patience and support.

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I would like to thank Rich Lundgren for his guidance and patience.

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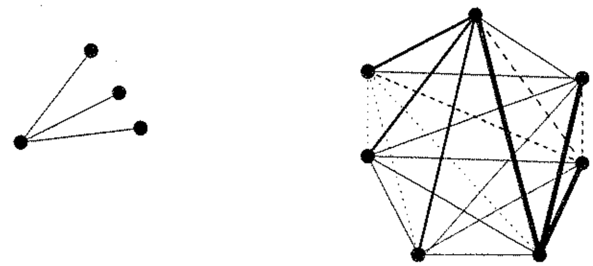
1. Introduction

This is a survey paper of graceful labeling, which has been an area of interest in graph theory for almost forty years. The interest in graceful labeling grew from the discussion of the decomposition of a graph G or a partitioning of the edges $E(G)$ into pairwise edge-disjoint subgraphs. A graph G with n vertices and m edges consists of a vertex set $V(G) = \{v_1, ..., v_n\}$ and an edge set $E(G) = \{e_1, ..., e_m\}$, where each edge consists of two vertices called its endpoints. Further a tree is a connected graph with no cycles and K_n is a complete graph of n vertices such that each of the vertices has an edge connecting it to every other vertex. In the decomposition of a graph the easiest case is to decompose the graph G into single edges. Soon more questions were asked. Can a graph G be decomposed into subgraphs isomorphic to a tree T of more than a single edge? If $\mid E(G) \mid$ is a multiple of $\mid E(T) \mid$, can G be decomposed into isomorphic copies of T? Are there some types of graphs which can be decomposed easier than others? Can we decompose regular graphs, complete graphs, trees or cycles? Are there any patterns?

In 1963, Ringel [1] conjectured the following:

Conjecture 1 Let T be a given tree with n vertices and $n-1$ edges; then the edges of K_{2n-1} can be partitioned into $2n-1$ trees isomorphic to T.

Figure 1.1 shows the conjecture for T, a tree of 3 edges, and K_7. Tree T has 4 vertices or $n = 4$. Per Conjecture 1 there are 7 distinct copies of T in K_7. It



T

K_{2n-1}

Figure 1.1. A complete graph on 7 vertices partitioned into 7 trees with 3 edges

is interesting to consider the number of edges in K_{2n-1} relative to the number of edges in T. The number of edges in K_{2n-1} is equivalent to the number of ways one can choose 2 from the group $2n-1$ or

$$\binom{2n-1}{2} = \frac{(2n-1)(2n-2)}{2} = (2n-1)(n-1).$$

Therefore, the number of distinct edges in K_{2n-1} is always $(2n-1)$ times $\mid E(T) \mid$. It remains to be proven that the partitioning results in trees isomorphic to T, but we know there exist the correct number of edges.

In 1966 Rosa [11], reports Kotzig's stronger conjecture as stated here:

Conjecture 2 Let T be a given tree with n vertices and $n-1$ edges, then the complete graph K_{2n-1} can be cyclically partitioned into $2n-1$ trees isomorphic to T.

This cyclic partitioning is accomplished for an arbitrary T by identifying a suitable set of edges in K_{2n-1} and rotating each node and edge from its original position $2n-1$ times. Conjecture 2 is referred to as the Ringel-Kotzig conjecture. The difference between Conjecture 2 and Conjecture 1 is the idea that the partitioning can be done cyclically, which is a stronger requirement.

Certainly, if Conjecture 2 is true, then Conjecture 1 is also true.

A graph G with m edges and n vertices can be labeled in such a way that the vertices are numbered and each edge is assigned a label equal to the absolute value of the difference of its vertex labels. A graph G is said to be *graceful* if the numbers used to label its vertices are distinct values of the set $\{0, 1, ..., m\}$, and the edge labels are the set $\{1, 2, ..., m\}$, where the edge labels are the absolute value of the difference of the vertex values. The function assigning values to each vertex v of G is denoted by $f(v)$. An example of a graceful labeling is in figure 1.2. It should be noted that these labelings are not usually unique. It should also be noted that the vertex labels are distinct values of the set $\{0, 1, 2, ..., m\}$, not the complete set. It is possible to label the vertices in such a way as to produce two edges with the same label, which does not produce a graceful labeling.

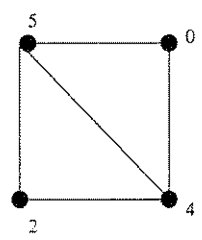


Figure 1.2. Gracefully labeled graph

Alexander Rosa introduced the idea of graceful labeling in 1967 [11]. Rosa [11] then proved the following theorem.

Theorem 1 If a tree T with n vertices and $n-1$ edges has a graceful labeling, then there exists a cyclic partitioning of the complete graph K_{2n-1} into $2n-1$ subgraphs isomorphic to T.

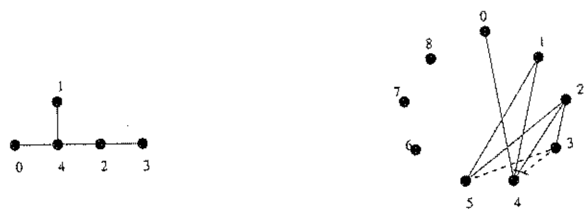


Figure 1.3. A gracefully labeled tree and K_{2n-1}

Proof. View the vertices of K_{2n-1} as the congruence classes mod $2n-1$. The *displacement* between two congruence classes is the number of unit moves needed to get from one to the other. The maximum displacement between two congruence classes mod $2n-1$ is $n-1$. The edges of K_{2n-1} consist $n-1$ "displacement classes", each of size $2n-1$. In other words one displacement class is all edges whose vertices are 1 unit apart and another displacement class is all edges whose vertices are 2 units apart so there are 4 displacement classes in the example (figure 1.3).

From a graceful labeling of T, we define copies of T in K_{2n-1} for $0 \leq k \leq (2n-1)-1$. In the k^{th} copy, the vertices are $k, \ldots, k+(n-1)$ mod $2n-1$, with $k+i$ adjacent to $k+j$ if and only if i is adjacent to j in the graceful labeling. The 0^{th} copy of T looks just like the graceful labeling and has one edge with each displacement. If there is an edge from each displacement, there is an edge of each 'length' and the graceful labeling has been maintained. Moving to the next copy shifts each edge to the next edge in its displacement class. Hence the $2n-1$ copies of T cycle through the $2n-1$ edges from each displacement class, without repetitions, and these $2n-1$ copies of T decompose K_{2n-1}.□

Rosa's work and, in particular, this theorem launched the field of graceful labeling. Then Rosa made the following conjecture.

Conjecture 3 All trees are graceful [10].

Ringel's Conjecture is one of the major unsolved problems of graph theory. The existence of a graceful labeling for a graph implies Ringel's Conjecture is true for that graph. It is this relationship between graceful labeling and Ringel's Conjecture that has created the interest and research in graceful labeling. Conjectures 1 and 2 remain two of the major unsolved problems in graph theory, which is the reason the topic of graceful labeling has such activity.

A path is defined as a graph such that every vertex has degree at most 2 and exactly two vertices have degree 1. Rosa [11] proved that the simplest tree, a path, is graceful. Figure 1.4 shows an example of a gracefully labeled path.

Figure 1.4. A gracefully labeled path

A caterpillar (see figure 1.5) is defined as a graph such that deletion of every vertex of degree one results in a path.

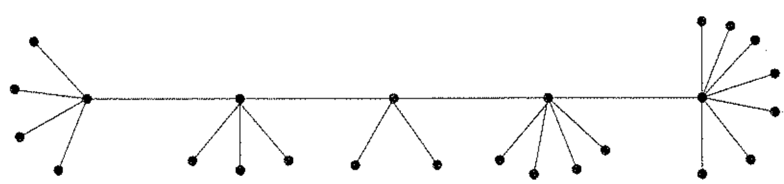


Figure 1.5. A caterpillar

Rosa [11] proved all caterpillars are graceful. Zhao [15] proved all trees of diameter four are graceful. An underlying idea in the graceful labeling of larger graphs is that of joining two or more gracefully labeled graphs and being able to gracefully label the resulting graph. This idea is explored in several chapters of this paper. In this survey paper we will explore algorithms for graphs that have been proven to be graceful and apply these algorithms to examples. The primary area of interest is trees, although we will also look at the graceful labeling of grids, crowns, coronas and cycles with a chord. The interest in graceful labeling of graphs other than trees is tangential to the graceful labeling of trees and the search for patterns leading to the graceful labeling of all trees. We must define a few more terms and the scope of this study before proceeding. A loop is an edge whose endpoints are equal. Parallel edges or multiple edges are edges whose endpoints are the same. A simple graph is a graph with no loops or parallel edges. A connected graph is a graph such that there exists an edge between any two subsets of the graph. In this paper only connected simple graphs are considered. Some related topics not discussed in this thesis are sequential labeling, harmonious labeling, Skolem graceful, edge graceful, and rotatable graceful.

Graceful labeling grew from Ringel's and Kotzig's discussion of graph decomposition. Rosa proved K_{2n-1} has a decomposition into $2n-1$ copies T if T is a tree with n edges and can be gracefully labeled. This thesis explores the history of graceful labeling and discusses the results of graceful labeling different types of graphs. The algorithms for gracefully labeling a variety of graphs are presented for the general cases and demonstrated for specific cases.

2. Paths

A graph is called a path (see figure 2.1) if the degree $d(v)$ of every vertex, v, is ≤ 2 and there are no more than 2 endvertices [14]. An endvertex or leaf is a vertex of degree 1.

Figure 2.1. A path

Theorem 2 Every path is graceful [11].

Proof. We demonstrate an algorithm to gracefully label any path P_n with n vertices. In a path the number of edges is one less that the number of vertices or $m = n - 1$. Labeling can begin at either end without loss of generality. The first vertex at one end is labeled 0, the adjacent vertex is labeled $n - 1$, the next adjacent, non-labeled vertex is labeled 1, and we continue in this manner. Alternate vertices are incremently increasing by 1 while the remaining vertices are incremently decreasing by 1. Consider two cases where $n \equiv 0 (\mathrm{mod}\ 2)$ and $n \equiv 1 (\mathrm{mod}\ 2)$ shown in figures 2.2 and 2.3 respectively. In both cases $k = \lfloor \frac{n}{2} \rfloor$.

For the even case the edge labels beginning with the leftmost edge in figure 2.2 are $\mid (n-1) - 0 \mid, \mid (n-1) - 1 \mid, \mid (n-2) - 1 \mid, \ldots, \mid (n-k) - (k-1) \mid = n-1, n-2, n-3, \ldots, 1$. In determining the last edge value recall $k = \lfloor \frac{n}{2} \rfloor$. For cases where n is even, $k = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ and $(n-k) - (k-1) = n - k - k + 1 = 1$. It is easy to see this is a graceful labeling since all numbers between 1 and $n - 1$

or m are used in the edge labels.

Similarly, when n is odd the edge values beginning on the left are $n-1, n-2, n-3, ..., 1$. In evaluating the right most edge value, recall $k = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ when n is odd. Then $n - k - k = n - \frac{n-1}{2} - \frac{n-1}{2} = 1$. Again every value from 1 to $n-1$ or m is used.□

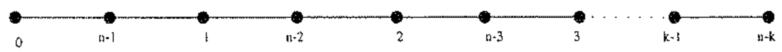


Figure 2.2. A gracefully labeled path P_n where n is even

Figure 2.3. A gracefully labeled path P_n where n is odd

Rosa [13] also introduced α-graceful labeling, which is a stronger standard, and therefore fewer graphs are α-graceful. Graphs that are α-graceful are also graceful. A graceful graph G is said to be α-graceful if there exists a critical value α such that for every edge (u, v), either $f(u) \leq \alpha < f(v)$ or $f(v) \leq \alpha < f(u)$. In each α-graceful graph, α is a positive integer and the vertices are said to have an α-valuation. These α-graceful graphs must be bipartite, which implies that no α-graceful graphs can have an odd cycle [3]. An example of α-graceful labeling is shown in figure 2.4, where $\alpha = 3$. Examine the vertex labels in the path shown in figure 2.4. Beginning at the left end of the path the first vertex is labeled 0 and alternate vertices to the right increase by 1. Call these vertex labels $\{0, 1, 2, 3\}$ set A. While the left most vertex adjacent to vertex 0 is $n-1$ and each alternate vertex to the right decreases by 1. Call these vertex labels $\{6, 5, 4\}$ set B. Sets A and B appear to converge

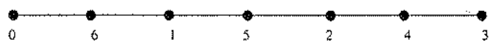


Figure 2.4. An α-gracefully labeled path

to a value from the lower and upper sides respectively. In other words the values belonging to set A are less than or equal to some value and the values belonging to set B are greater that the same value. This is what is meant by α-graceful labeling, where α is this value to which is set A and B converge.

3. Caterpillars

Deleting every endvertex of a caterpillar results in a path. This path is called the spine of the caterpillar. What if the resulting path is one vertex or a point? Then the graph prior to deletion of end vertices is called a star as shown in figure 3.1. A star is a special case of caterpillar.

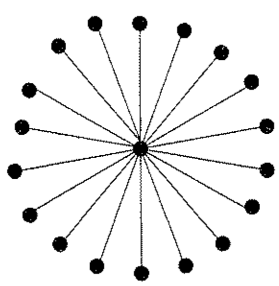


Figure 3.1. A star

Figure 3.2 shows the gracefully labeled caterpillar from figure 1.5. The first vertex on the spine is labeled 0 and the adjacent vertices are labeled $\{24, 23, 22, 21, 20\}$ using the higher values on the leaf neighbors, so that 20 is the label for the next vertex on the spine and there are 4 leaf neighbors of vertex 0. Now the remaining non-labeled neighbors of 20 are labeled $\{1, 2, 3, 4\}$ with the lower values on the leaf neighbors and the value 4 on the next vertex on the spine. Continuing in this manner the caterpillar of 25 vertices is labeled. The edge values are $\{1, 2, 3, ..., 24\}$ and $m = 24$, while the vertex values are from the set $\{0, 1, 2, ...24\}$. Therefore this labeling is α-graceful.

Theorem 3 All caterpillars are α-graceful [11].

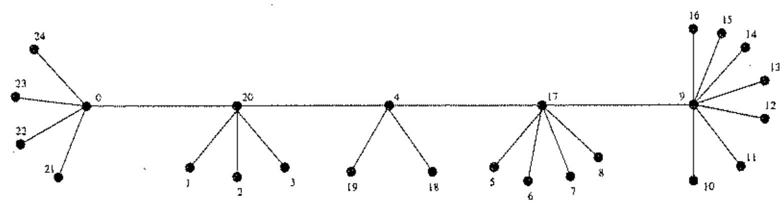


Figure 3.2. A gracefully labeled caterpillar

Proof. We demonstrate an algorithm to α-gracefully label any caterpillar. Let v_i represent the vertices on the spine of the caterpillar and $N(v_i)$ the neighborhood of v_i, that is, the set of all vertices adjacent to v_i, while $|N(v_i)|$ is the number of vertices which are adjacent to v_i.

$$k_i = \begin{cases} \mid N(v_i) \mid -1, & \text{if } i \text{ is the first or the last vertex of the spine} \\ \mid N(v_i) \mid -2, & \text{otherwise} \end{cases}$$

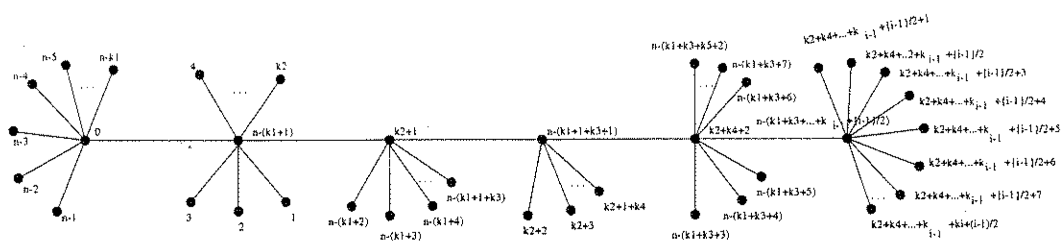


Figure 3.3. A caterpillar which is gracefully labeled.

(1) Begin by labeling the first vertex on the spine 0 (figure 3.3). Alternate vertices of this spine beginning with the first vertex are called odd. All other vertices on the spine are called even.

(2) Leaf neighbors of 0 are labeled beginning with $n-1$ and going in descending order ending with $n-k_1$ where $k_1 = N(v_0) - 1$.

(3) The next vertex on the spine, a neighbor of 0, is labeled $n-(k_1+1)$.

(4) Leaf neighbors of this vertex $n - (k_1 + 1)$ are labeled 1 through k_2.

(5) Continue in this manner. The odd vertices on the spine and the leaf neighbors of the even vertices on the spine are labeled in increasing order, while the leaf neighbors of the the odd vertices on the spine and the even vertices on the spine are labeled in decreasing order. The last vertex on the spine is labeled $n-(k_1+k_3+\cdots+k_{i-1}+\lceil\frac{i}{2}\rceil)$ if it is even. It is $k_2+k_4+\cdots+k_{i-1}+\frac{i-1}{2}$ if odd. The last neighbor of this vertex is labeled $k_2+k_4+\cdots+k_{i-2}+k_i+\frac{i-1}{2}$ or $n-(k_1+k_3+\cdots+k_{i-2}+k_i+\lfloor\frac{i}{2}\rfloor)$ respectively.

(6) $\alpha = \min\,\{x, y\}$ where $x =$ the label of last vertex on the spine and,

$$y = \begin{cases} min\{\text{ labels of leaf neighbors }\}, & \text{if } x < \text{labels of leaf neighbors} \\ max\{\text{ labels of leaf neighbors }\}, & \text{if } x > \text{labels of leaf neighbors} \end{cases}$$

In the example in figure 3.2 $\alpha = min\{9, 10, 11, 12, 13, 14, 15, 16\} = 9.\Box$

4. Grids

A grid is a graph defined as the Cartesian product of two graphs. Grids are not trees and not directly related to the proof of Ringel's conjecture (Conjecture 1). However, interesting work has been done in the area of gracefully labeling grids, as we show in this chapter. The Cartesian product $G \square H$ of graphs G and H whose vertex set is $V(G)$ x $V(H)$ can be defined as follows. Let u be a vertex in $V(G)$ and v be a vertex in $V(H)$. Then (u, v) is an element of $G \square H$ and (u, v) is adjacent to (u', v') if and only if

(1) $u = u'$ and edge vv' belongs to $E(H)$ or

(2) $v = v'$ and edge uu' belongs to $E(G)$ [14] (see figure 4.1). The grid shown is a planar grid $P_3 \square P_5$, where P_m is a path with m vertices.

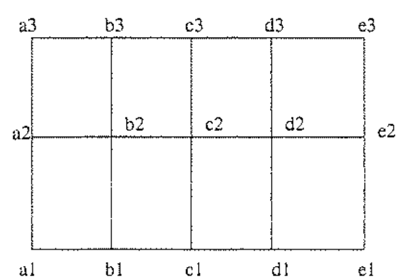


Figure 4.1. A grid on a plane

Another type of grid is a cylindrical grid $C_m \square P_n$ as shown in figure 4.2, where C_m is a cycle of length m. Imagine a cylindrical grid as the grid from figure 4.1 on the curved surface of a cylinder. The last type of grid, $C_m \square C_n$, is on a torus. Following are algorithms for gracefully labeling planar

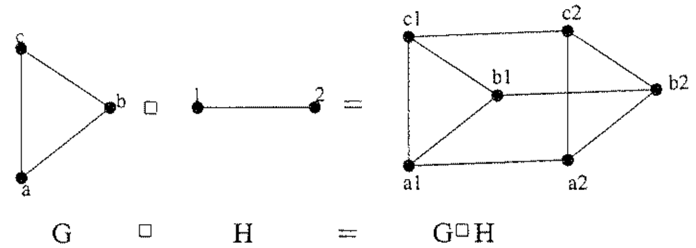


Figure 4.2. $C_3 \square P_2$ - The Cartesian product of a 3-cycle and a path with two vertices

and cylindrical grids respectively.

4.1 Planar Grids

Theorem 4 The graph $P_m \square P_n$ has an α-valuation or α-graceful labeling [9].

Jungreis and Reid [9] developed the following algorithm. Note: This algorithm assumes $m \leq n$. If $m > n$, find a graceful labeling for $P_n \square P_m$ and use the transpose of the resulting labeling as if it were a matrix.

(1) Let J represent the graceful labeling of P_m (figure 4.3).

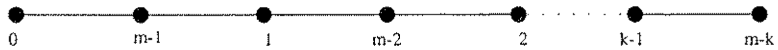


Figure 4.3. Graceful labeling J of P_m

(2) Let $x = k(2m-1)$ and $R(x)$ be the labeling that results from adding $(2mn - 2m - n + 1) - x$ to the higher values in the J labeling and adding x to the lower values (figure 4.4), where $i = 0, 1, 2, ..., \frac{n-1}{2}$.

(3) Let $K = (m-1) - J$ represent another graceful labeling of P_m (figure 4.5). This alternate labeling is also called the complement with respect to $m - 1$.

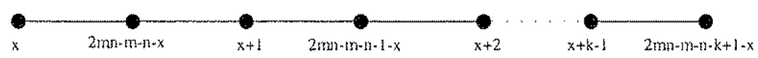


Figure 4.4. Labeling $R(x)$ obtained from J

(4) Let $x = k(2m-1)+(1-m)$ and $S(x)$ be the labeling obtained by adding $(2mn-2m-n+2)-x$ to the higher values in the K labeling and adding x to the lower values (figure 4.6), where $k = 1, 2, ..., \frac{n}{2}$. The definition of x is different for the functions R and S.

$R(x)$ and $S(x)$ are the foundation for the graceful labeling of the grid $P_m \square P_n$. Each column of the grid is a copy of P_m and there are n columns. The first column is the $(2k+1)^{st}$ copy where $k = 0$. The labeling in that column and all odd numbered columns will be determined by the function $R(k(2m-1))$, where $k = 0, 1, \cdots, \frac{n-1}{2}$. The labeling in even numbered columns or $2k^{th}$ copies will be determined by $S(k(2m-1)+(1-m))$, where $k = 1, 2, \cdots, \frac{n}{2}$.(Note: The raised quantities st and th in the previous expressions are not exponents, but suffixes as in 1^{st} or 5^{th}.)

This algorithm produces an α-graceful labeling [9]. Examine the table of values below to see that the odd columns of the grid are labeled according to the function $R(x)$ while the even columns are labeled using the function $S(x)$. The odd rows of the $R(x)$ function begin with 0 and incremently increase. The even rows begin with the highest value in the grid and incremently decrease. For the $S(x)$ function the odd rows begin with a higher value and decrease,

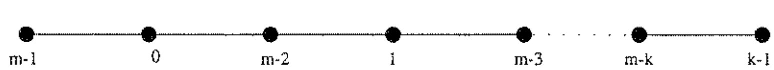


Figure 4.5. Graceful labeling K of P_m

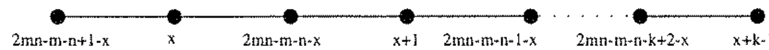


Figure 4.6. Labeling $S(x)$ obtained from K

while the even rows start with smaller values and increase. This means adjacent vertices are approaching a middle value from below and above. This middle value is α.

0	$2mn-2m-n+1$	$2m-1$	$\cdots$	$2mn-n+k-2km$
$2mn-m-n$	m	$2mn-3m-n+1$	$\cdots$	$2km-k-m+1$
1	$2mn-2m-n$	$2m$	$\cdots$	$2mn-n+k-2km-1$
$2mn-m-n-1$	$m+1$	$2mn-3m-n$	$\cdots$	$2km-k-m+2$
2	$2mn-2m-n-1$	$2m+1$	$\cdots$	$2mn-n+k-2km-2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$p-1$	$2mn-2m-n-p+2$	$2m+p-2$	$\cdots$	$2mn-n+k-2km-p+1$
$2mn-m-n-p+1$	$m+p-1$	$2mn-3m-n-p+2$	$\cdots$	$2km-k-m+p-1$

The following example illustrates the process for $P_5 \square P_6$. In $P_5 \square P_6$, $m = 5$ and $n = 6$. First, gracefully label P_5 as shown in figure 4.7.

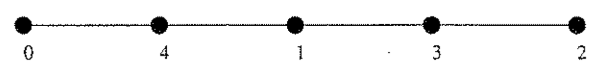


Figure 4.7. Graceful labeling J of P_5

Next calculate $2mn - 2m - n + 1 - x$ and add the result to the higher labels of J and add x to the lower labels of J to obtain labeling $R(x)$ (figure 4.8).

$$2mn - 2m - n + 1 - x = 2(5)(6) - 2(5) - (6) + 1 - x = 45 - x$$

Develop an alternate graceful labeling K of P_5 using $K = (m-1) - J$, which is the complement with respect to $m - 1$ (figure 4.9).

Figure 4.8. Labeling $R(x)$ obtained from J

Now add $2mn - 2m - n + 2 - x$ to all higher values of labeling K and add x to all lower values of K to obtain labeling $S(x)$ (figure 4.10).

$$2mn - 2m - n + 2 - x = 2(5)(6) - 2(5) - (6) + (2) - x = 46 - x$$

If $k = 0, (2k+1) = 1$, use $R(k(2m-1)) = R(0(2(5)-1)) = R(0)$.

$R(0) = 0, 49, 1, 48, 2$

If $k = 1, (2k) = 2$, use $S(k(2m-1) + (1-m)) = S(1(2(5)-1) + (1-5)) = S(5)$.

$S(5) = 45, 5, 44, 6, 43$

If $k = 1, (2k+1) = 3$, use $R(k(2m-1)) = R(1(9)) = R(9)$

$R(9) = 9, 40, 10, 39, 11$

If $k = 2, (2k) = 4$ use $S(k(2m-1) + (1-m)) = S(2(9) - 4)) = S(14)$

$S(14) = 36, 14, 35, 15, 34$

If $k = 2, (2k+1) = 5$ use $R(k(2m-1)) = R(18)$

$R(18) = 18, 31, 19, 30, 20$

If $k = 3, (2k) = 6$ use $S(k(2m-1) + (1-m)) = S(3(9) - 4)) = S(23)$

$S(23) = 27, 23, 26, 24, 25$

The labels form the matrix shown below. Each value represents a vertex in the grid. Examination shows this is a gracefully labeled grid and

Figure 4.9. Graceful labeling K of P_5

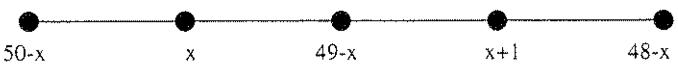


Figure 4.10. Labeling $S(x)$ obtain from K

closer examination reveals that it is α-graceful where $\alpha = 24$.

0	45	9	36	18	27
49	5	40	14	31	23
1	44	10	35	19	26
48	6	39	15	30	24
2	43	11	34	20	25

4.2 Prisms

The prism (also called cylindrical grids) $P_{m,n}$ is defined as the Cartesian product $C_m \square P_n$ where $n \geq 2$ [7].

Jungreis and Reid [9] developed algorithms for the cases $m \equiv 0(\mathrm{mod}\ 4)$ for any n and $m \equiv 2(\mathrm{mod}\ 4)$, $n \equiv 0(\mathrm{mod}\ 2)$. They have not developed an algorithm for $m \equiv 2(\mathrm{mod}\ 4)$ and $n \equiv 1(\mathrm{mod}\ 2)$ or for $m \equiv 1(\mathrm{mod}\ 2)$. Building on this work, Huang and Skiena [7] developed algorithms for all cases where $m \equiv 0(\mathrm{mod}\ 2)$. When $m \equiv 1(\mathrm{mod}\ 2)$ and $3 \leq n \leq 12$ they use these same algorithms for the first $m - 3$ columns and use computers to produce tables for the values in the last two columns.

Frucht stated [3] that all prisms were graceful and he would so show in a later article, but so far, he has not shown this result. Frucht [4] proved that all prisms with $n = 2$ are graceful. Thus far, it has not been proven that all prisms with odd cycles are graceful.

Theorem 5 All prisms are graceful if they have no odd cycles [3].

The following examples use the algorithm given by Huang and Skiena. Graceful labeling of prisms with even cycles falls into two primary cases depending upon whether m is divisible by 4.

CASE 1) When $m \equiv 0 (\text{mod } 4)$ or $C_{4p} \Box P_n$, use the following labeling.

$$f(i,j) = (in - \lfloor \tfrac{i}{2} \rfloor + \lfloor \tfrac{j}{2} \rfloor)\delta(i+j+1) +$$
$$\left((2n-1)m - in + \lfloor \tfrac{i+1}{2} \rfloor - \lfloor \tfrac{j}{2} \rfloor - n\lfloor \tfrac{2i}{m} \rfloor\right) \delta(i+j)$$

where $i\epsilon\{0,1,2,...,m-1\}$ labels the column, $j\epsilon\{0,1,2,...,n-1\}$ labels the row, $\delta(x) = x(\text{mod } 2)$, and each row in the matrix of values represents one copy of C_{4p}.

When the parity $i+j$ is odd, $\delta(i+j) = 1$ and $\delta(i+j+1) = 0$, so that the first term in the labeling function $f(i,j) = 0$ and the value at this vertex is determined by the second term of the labeling function. When the parity of $i+j$ is even, the opposite happens and the first term of $f(i,j)$ determines the value. The table below shows the values for the prism $C_4 \Box P_5$, where $p = 1$ and $n = 5$.

	0	1	2	3
0	0	32	9	18
1	36	5	22	14
2	1	31	10	17
3	35	6	21	15
4	2	30	11	16

CASE 2) When $m \equiv 2(\text{mod } 4)$, there are again two cases, depending on the parity of n. Part A will address the case where $n \equiv 0(\text{mod } 2)$ and Part B will deal with the case for $n \equiv 1(\text{mod } 2)$.

Part A) For $m \equiv 2(\text{mod } 4)$, $m \geq 6$, $n \geq 4$, and $n \equiv 0(\text{mod } 2)$ or $C_{4p+2}\Box P_{2q}$ where $p \geq 1, q \geq 2$ Huang and Skiena use a different presentation of Jungreis' and Reid's algorithm which is shown here.

$$f(i,j) = (in - \lfloor \tfrac{i}{2} \rfloor + \lfloor \tfrac{j}{2} \rfloor)\delta(i+j+1) + \left((2n-1)m - in + \lfloor \tfrac{i+1}{2} \rfloor - \lfloor \tfrac{j}{2} \rfloor - n\lfloor \tfrac{2i}{m} \rfloor\right)\delta(i+j)$$

The example shown in the table below is for the prism $C_6\Box P_4$, where $p = 1$ and $q = 2$ or $m = 6$ and $n = 4$.

	0	1	2	3	4	5
0	0	39	7	28	14	20
1	42	4	35	11	24	17
2	1	38	8	27	15	19
3	41	5	34	12	23	18

Part B) For $m \equiv 2(\text{mod } 4)$ and $n \equiv 1(\text{mod } 2)$ or $C_{4p+2}\Box P_{2q+1}$ where $p \geq 1, q \geq 1$ the labeling is defined as

$$f(i,j) = (in - \lfloor \tfrac{i}{2} \rfloor + \lfloor \tfrac{j}{2} \rfloor + n\lfloor \tfrac{2i}{m} \rfloor)\delta(i+j+1) + \left((2n-1)m - in + \lfloor \tfrac{i+1}{2} \rfloor - \lfloor \tfrac{j}{2} \rfloor\right)\delta(j+1)$$

for $i \leq m-3$. This labels the vertices in all of the columns except the last two. The labels for the vertices in the next to the last column are given by the following labeling:

$$f(m-2,0) = H - n + 2,$$

$$f(m-2,1) = H + 2n, \text{ where } H = \tfrac{(2n-1)m}{2}, \text{ and}$$

$$f(m-2,j) = \left(H + 2n - \lfloor \tfrac{j+1}{2} \rfloor\right) \delta(j) + \left(H + 4n + \lfloor \tfrac{j}{2} - 2 \rfloor\right) \delta(j+1),$$

for $j \geq 2$.

The labels for the vertices in the last column are defined as

$$f(m-1,0) = H - n + 1,$$

$$f(m-1,n-1) = H + \tfrac{3(n-1)}{2}, \text{and}$$

$$f(m-1,j) = \left(H + \lfloor \tfrac{j+1}{2} \rfloor\right) \delta(j) + \left(H + n - \lfloor \tfrac{j}{2} + 1 \rfloor\right) \delta(j+1),$$

for $1 \leq j \leq n-2$.

Using the prism $C_6 \square P_5$ ($p = 1$ and $q = 2$) as an example, the resulting vertex labels are shown in the table below.

	0	1	2	3	4	5
0	0	50	9	41	24	23
1	54	5	45	19	37	27
2	1	49	10	40	46	30
3	53	6	44	20	35	28
4	2	48	11	39	47	33

5. Coronas

Coronas are a class of graphs created by joining two or more graphs. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is the result when the two graphs are joined in the following way. Assuming G_1 has p_1 points, one copy of G_1 is used and p_1 copies of G_2 are used. The i^{th} point of G_1 is joined to every vertex in the i^{th} copy of G_2 [3] (figure 5.1).

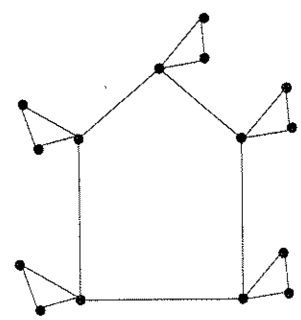


Figure 5.1. Corona $C_5 \odot P_2$

5.1 Wheels

The wheel is the corona $K_1 \odot C_n$ where K_1 is a graph of one vertex. Wheels can also be described as the result of adding a center to an n-gon where $n \geq 3$ and making this center adjacent to every vertex in the n-gon by adding the appropriate edges [3] (see figure 5.2).

Theorem 6 Wheels are graceful graphs [3].

In order to gracefully label wheels consider two cases $n \equiv 0 (\text{mod } 2)$ or $n \equiv 1 (\text{mod } 2)$.

CASE 1) $n = 0 \pmod 2$ First number the vertices of the wheel so that the center of the wheel is v_0 and the remaining vertices are given values $v_1, v_2, ..., v_n$ sequentially. Now use the following algorithm to relabel the vertices and obtain a graceful labeling.

$$f(v_i) = \begin{cases} 2n - i - 1 & \text{if i=2,4,6,...,n-2} \\ 2 & \text{if } i = n-1 \\ i & \text{if } i = 0, 1, 3, 5, ..., n-3 \\ 2n & \text{if } i = n \end{cases}$$

For $n = 8$ use the following example. First label the vertices $v_0, v_1, v_2, ...v_8$ as shown in figure 5.2.

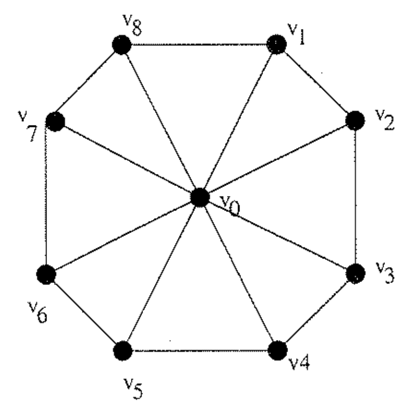


Figure 5.2. Initial labeling of vertices in $K_1 \odot C_8$

Per the algorithm v_0, v_1, v_3, and v_5 are relabeled 0, 1, 3, and 5 respectively. The vertex v_8 is labeled $2n$ or 16 and v_7 is labeled 2. Now the remaining even-numbered vertices v_2, v_4, and v_6 are relabeled with the complement with $2n - 1 - i$ resulting in 13, 11, and 9 respectively (see figure 5.3).

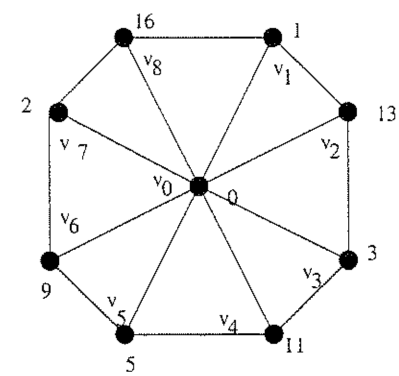


Figure 5.3. Graceful labeling of $K_1 \odot C_8$

CASE 2) Consider $n = 1(\text{mod } 2)$ or odd. Begin by labeling the vertices as in Case 1 with $v_0, v_1, v_2, ...v_n$. The algorithm for relabeling each vertex is only slightly different from Case 1.

$$f(v_i) = \begin{cases} 2i & \text{if } i = 0, 1, \text{or n} \\ n + i & \text{if } i = 2, 4, 6..., n-1 \\ n + 1 - i & \text{if } i = 3, 5, 7, ..., n-2 \end{cases}$$

For $n = 9$ examine the following example as shown in figure 5.4. First label the vertices v_0, v_1, v_2,...v_9 as shown.

For v_0, v_1, and v_9 relabel each vertex with $2i$ or 0, 2, and 18 respectively. For all even vertices add n or 9 to the index to get the value of the label. For all remaining odd vertices use $n + 1 - i$, resulting in 7, 5, and 3 respectively.

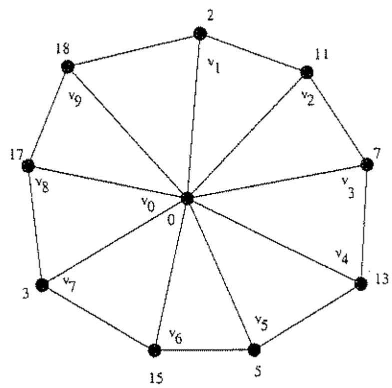


Figure 5.4. Gracefully labeled graph $K_1 \odot C_9$

5.2 Crowns

A crown (figure 5.5) is a cycle C_n plus edges pendant at each vertex [1]. The notation for a crown is $C_n \odot K_1$, which is the reverse order of a wheel.

Frucht showed that all crowns are graceful [3] by using four cases depending upon the parity of n (mod 4).

CASE 1) If $n \equiv 0 (\text{mod } 4)$ a graceful labeling for $C_n \odot K_1$ is obtained by using the following algorithm. First label the vertices of the cycle $v_1, v_2, ..., v_n$ in order and label the vertex pendant to v_i by u_i for $i = 1, 2, ..., n$.

$$f(v_i) = \begin{cases} i-1, & \text{if } i = 1, 3, 5, ..., \frac{n}{2} - 1 \\ i, & \text{if } i = \frac{n}{2} + 1, \frac{n}{3} + 3, ..., n-1 \\ 2n+1-i, & \text{if } i = 2, 4, 6, ..., n \end{cases}$$

and

$$f(u_i) = \begin{cases} 2n - f(v_i), & \text{if } i = 1, 2, 3, ..., \frac{n}{2} \\ 2n + 1 - f(v_i), & \text{if } i = \frac{n}{2} + 1, \frac{n}{2} + 2, ..., n \end{cases}$$

where n is the number of vertices in the cycle. Thus the edges of the cycle are

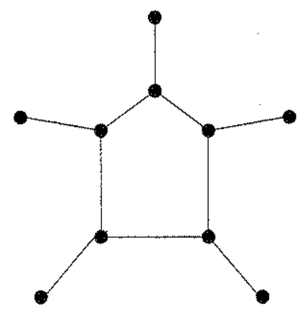


Figure 5.5. The crown $C_5 \odot K_1$

$(v_1, v_2), (v_2, v_3), ..., (v_{n-1}, v_n), (v_n, v_1)$ and the edges joining C_n to the pendant points u_i are (u_i, v_i). An example for $n = 8$ is shown in figure 5.6.

CASE 2) If $n \equiv 1 (\text{mod } 4)$, values for the vertices are given by

$$f(v_i) = \begin{cases} i-1, & \text{for odd } i \neq \frac{n+1}{2} \\ n-3, & \text{if } i = \frac{n+1}{2} \\ 2n+1-i, & \text{if } i = 2, 4, 6, ..., \frac{n-1}{2} \\ 2n-i, & \text{if } i = \frac{n+3}{2}, \frac{n+7}{2}, \frac{n+11}{2}, ..., n-1 \end{cases}$$

and

$$f(u_i) = \begin{cases} 2n - f(v_i), & \text{if } i = 1, 2, 3, ..., \frac{n-3}{2} \\ 2n+1-f(v_i), & \text{if } i = \frac{n-1}{2} \\ 2n-2-f(v_i), & \text{if } i = \frac{n+1}{2} \\ 2n-1-f(v_i), & \text{if } i = \frac{n+3}{2}, \frac{n+5}{2}, ..., n. \end{cases}$$

An example using this algorithm where $n = 5$ is shown in figure 5.7.

CASE 3) When $n \equiv 2 (\text{mod } 4)$, there exists a graceful labeling given by

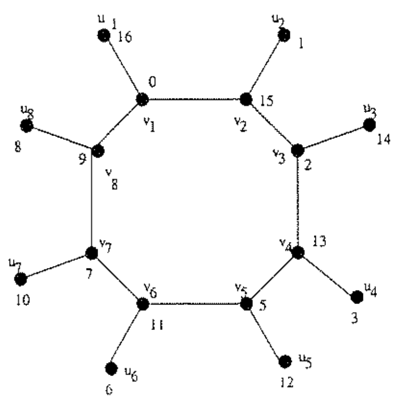


Figure 5.6. Graceful labeling of $C_8 \odot K_1$

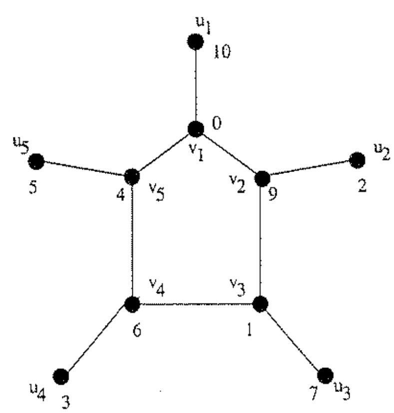


Figure 5.7. Graceful labeling of $C_5 \odot K_1$

$$f(v_i) = \begin{cases} i-1, & \text{for odd } i \neq \frac{n}{2} \\ \frac{n}{2}-2, & \text{if } i = \frac{n}{2} \\ 2n+1-i, & \text{if } i = 2,4,6,...,\frac{n}{2}-1 \\ 2n-1, & \text{if } i = \frac{n}{2}+1, \frac{n}{2}+3, ..., n \end{cases}$$

and

$$f(u_i) = \begin{cases} 2n - f(v_i), & \text{if } i = 1,2,3,...,\frac{n}{2}-2 \\ 2n+1-f(v_i), & \text{if } i = \frac{n}{2}-1 \\ 2n-2-f(v_i), & \text{if } i = \frac{n}{2} \\ 2n-1-f(v_i), & \text{if } i = \frac{n}{2}+1, \frac{n}{2}+2, ..., n. \end{cases}$$

An example of using this algorithm to gracefully label a crown where $n = 6$ is shown in figure 5.8.

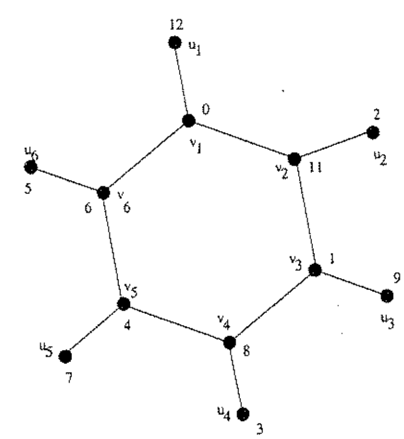


Figure 5.8. Graceful labeling of $C_6 \odot K_1$

CASE 4) When $n \equiv 3 (\text{mod } 4)$, a graceful labeling is determined by

$$f(v_i) = \begin{cases} i-1, & \text{if } i = 1,3,5,...,\frac{n-1}{2} \\ i, & \text{if } i = \frac{n+3}{2}, \frac{n+7}{2}, \frac{n+11}{2}, ..., n \\ 2n+1-i, & \text{if } i = 2,4,6,...n-1 \end{cases}$$

and

$$f(u_i) = \begin{cases} 2n - f(v_i), & \text{if } i = 1,2,3,...,\frac{n+1}{2} \\ 2n+1-f(v_i), & \text{if } \frac{n+3}{2}, \frac{n+5}{2}, ..., n. \end{cases}$$

An example of gracefully labeling a crown where $n = 7$ is pictured in figure 5.9.

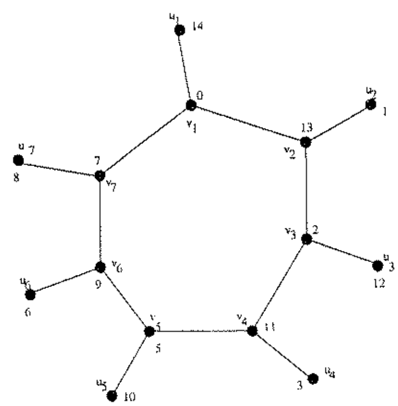


Figure 5.9. Graceful labeling of $C_7 \odot K_1$

6. Cycles with a Chord

A cycle with a chord is a cycle C_k with any two non-adjacent vertices joined. It has been conjectured that these graphs can be gracefully labeled [3], although it has not been proven. These graphs are another example of a class of graphs formed by joining two or more gracefully labeled graphs, where the first graph is a cycle and the second graph is a path of length one. I gracefully labeled graphs of this nature for $4 \leq k \leq 14$ as shown below. There are always $k+1$ edges and in labeling the vertices there are two numbers between 0 and $k+1$ that are not used. A pattern for assigning values to the vertices was not found.

In drawing these graphs the chord can be drawn from any vertex without affecting the labeling. Without loss of generality one vertex is selected as the beginning of the chord, then chords will be drawn to each of the remaining vertices. Due to symmetry it will be necessary to label only some of these graphs to find graceful labelings of all C_k with a chord. In the graph C_4, we only need to consider one chord shown in figure 6.1. In the graph C_6, there are only two configurations we need to consider as shown in figure 6.2 and 6.3. For C_{14}, there are eleven possible chords and with symmetry it is sufficient to examine and label only six of them (see figures 6.4-6.9).

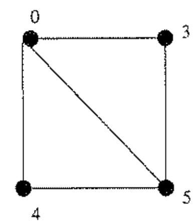


Figure 6.1. Gracefully labeled C_4 with chord

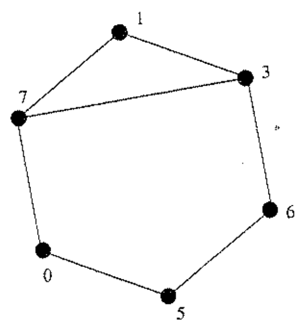


Figure 6.2. Gracefully labeled C_6 with chord skipping one vertex.

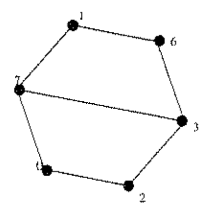


Figure 6.3. Gracefully labeled C_6 with a chord skipping two vertices

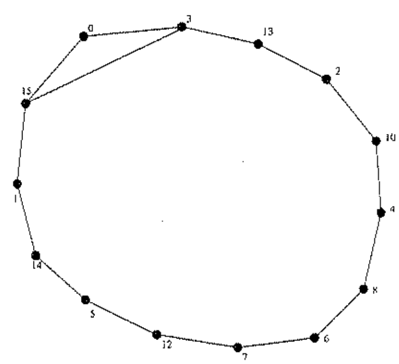


Figure 6.4. Gracefully labeled C_{14} with a chord skipping one vertex

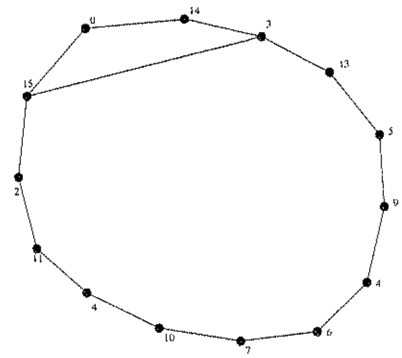


Figure 6.5. Gracefully labeled C_{14} with a chord skipping two vertices

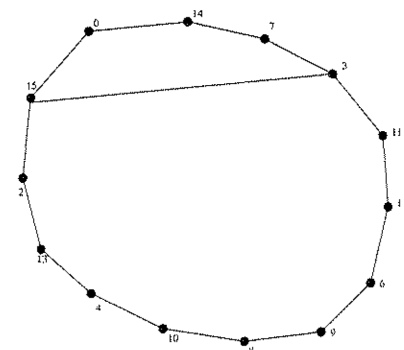


Figure 6.6. Gracefully labeled C_{14} with a chord skipping three vertices

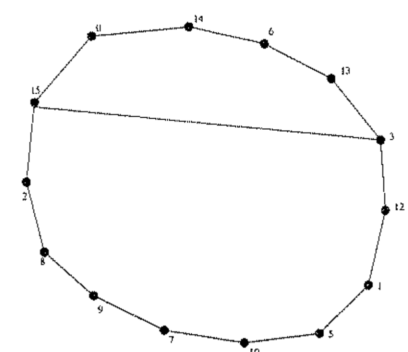


Figure 7.7. Gracefully labeled C_{14} with a chord skipping four vertices

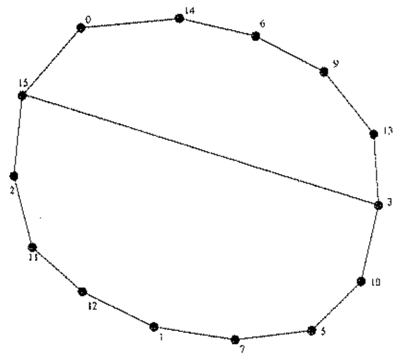


Figure 6.8. Gracefully labeled C_{14} with a chord skipping five vertices

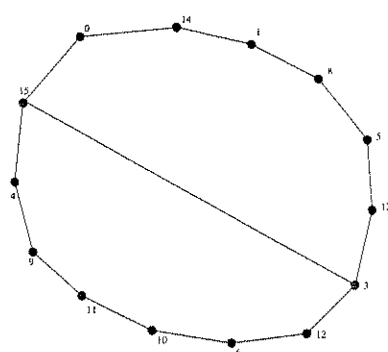


Figure 6.9. Gracefully labeled C_{14} with a chord skipping six vertices

7. Trees

A tree is defined as a connected graph with no cycles [14]. While several classes of trees have been shown to be graceful, Ringel's Conjecture remains open. There are several problems in proving that all trees are graceful. First, there needs to be an approach encompassing all trees instead of proving a specific type of trees is graceful. Second, induction, which works well to prove many characteristics of trees, hasn't worked to prove gracefulness. Induction has traditionally been used on trees in the following manner. Given that a tree of n vertices has characteristic b, delete a pendant vertex and assume the remaining tree has characteristic b. For induction to work we must be able to re-insert the pendant vertex anywhere and prove characteristic b exists in the resulting tree. In α-graceful labeling, and graceful labeling in general, the placement of the vertex is critical to maintaining the graph's gracefulness. Induction does not allow any assumptions about placement of the pendant vertex and therefore, this type of induction cannot be used to prove gracefulness of trees.

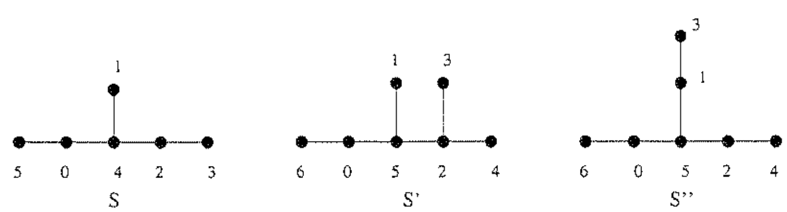


Figure 7.1. Attempting to gracefully label a graph after adding a pendant vertex

In the following example shown in figure 7.1 graph S is α-gracefully labeled. Graph S' and S'' show how a pendant vertex can be added in two different ways. S' is still α-gracefully labeled, while S'' cannot be α-gracefully labeled [2]. Possibly another approach using induction would work. In the following sections we discuss balanced trees, k-ary trees, trees of diameter four, and lobsters.

7.1 Balanced Trees

A balanced tree is defined in either of two ways. In both cases begin with two trees S and T. To create the balanced tree attach a copy of T to every vertex in S [12]. Another way to create a balanced tree is to attach a copy of T to every vertex in S except one [12]. The second type of balanced tree includes a special type of binary, ternary, and all k-ary trees. Figure 7.2 shows trees S and T with the resulting balanced tree U of the first type.

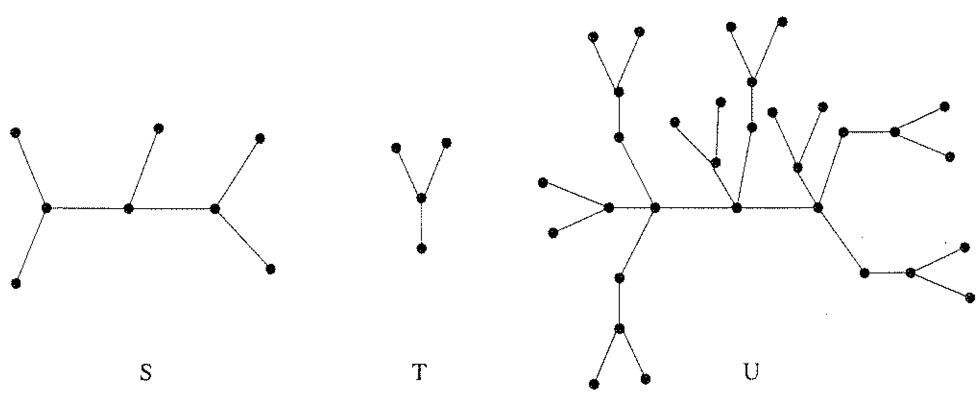


Figure 7.2. Construction of a balanced tree

If trees S and T are gracefully labeled, then U can be gracefully labeled [12]. First, a fixed vertex in T is selected. This is the vertex that will attach to vertices in S and is circled in figure 7.3. Assume $| V(S) |= k$ and vertices of S are labeled $0, 1, ...k-1$. Make k copies of T labeled T_i, $i = 0, 1, ..., k-1$. The fixed vertex in T_i will be labeled $in+1$ where $| V(T) |= n$.

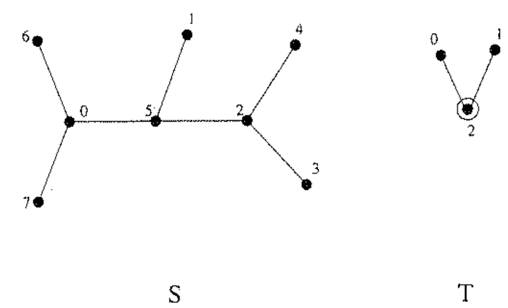


Figure 7.3. Gracefully labeled graphs S and T

$| V(U) |= nk = 24$ so that $V(U)$ will be labeled 1 to nk. The remaining vertices of T will be labeled in the following manner. The highest remaining values will be adjacent to the fixed vertex with lowest value, so that the fixed vertex T_0 will be 1 and the remaining vertices in T_0 will be labeled 24 and 23. The fixed vertex in T_1 is labeled 4 while the remaining vertices in T_1 will be labeled 21 and 20 since 21 and 20 are the highest remaining available values. Continuing this way vertices adjacent to T_7 are labeled 2 and 3. See the values on the 8 copies of T in figure 7.4.

Proceed in the following manner [12]. First attach vertex 1 of T_0 at the end of graph S and call this u_1 as shown in figure 7.5. Next attach to neighbors of u_1 the remaining T_i with maximum value on the fixed vertex and call this u_{22}. Adjacent to u_{22}, the remaining T_i with the minimum value on the fixed vertex and call this u_4. Continue in this manner alternating the maximum and minimum values until all copies of T are attached to a vertex in S. Notice that

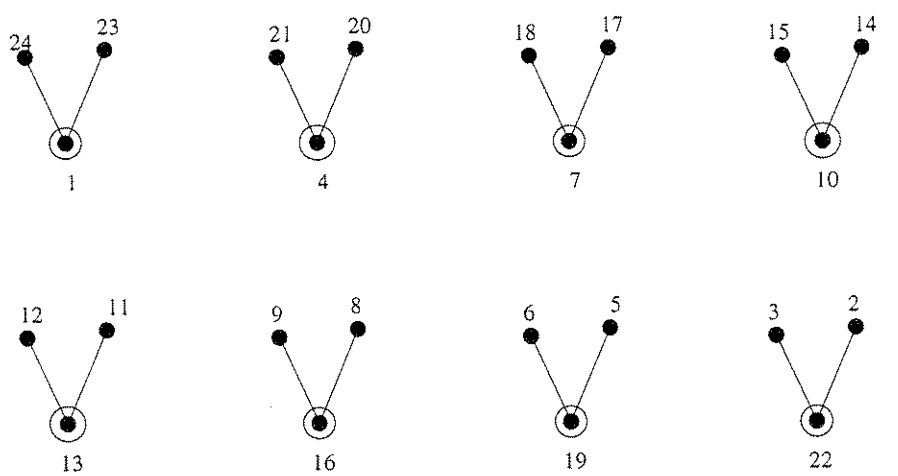


Figure 7.4. Eight copies of T which have been relabeled

the vertices in this balanced tree are labeled 1 to n instead of 0 to $n-1$. This is caused by the notation used in Stanton's and Zarnke's algorithm and could be modified to be consistent with other authors. To achieve this modification, label the fixed vertices in instead of $in+1$ and let the labels of $V(U)$ go from 0 to $|V(U)|-1$.

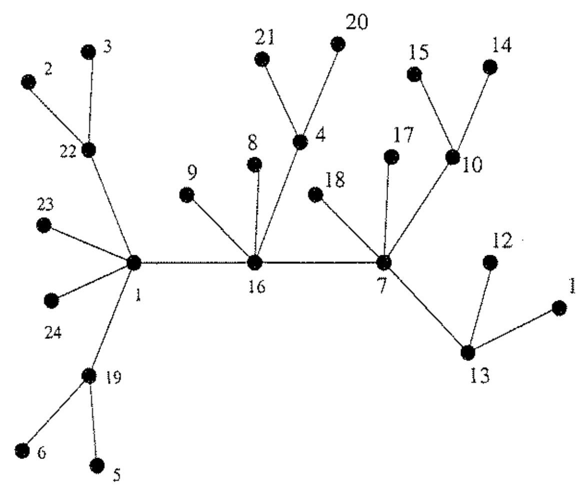


Figure 7.5. Graceful labeling of graph U

7.2 K-ary Trees

In the previous section the algorithm produces a gracefully labeled tree by joining the gracefully labeled graphs S and T. The algorithm gives no instruction for gracefully labeling the graphs S and T. In the case of balanced k-ary trees the algorithm both labels and constructs the tree. A k-ary tree is a rooted plane tree in which each vertex has no more than k children or edges. In a balanced k-ary tree each non-leaf has exactly k children and every child of the same generation has the same number of descendants. The binary tree and all k-ary trees are special cases of the balanced tree. In building a binary tree so that it falls within the definition of a balanced tree, notice that the fixed vertex is always the root of the initial tree.

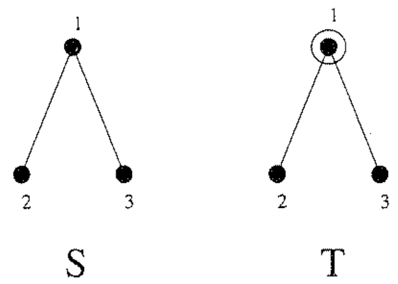


Figure 7.6. Binary trees S and T

The algorithm [12] for the k-ary cases is different from the algorithm used for other balanced trees. We will illustrate the algorithm first with an example involving binary trees. Start with the binary trees S and T which are α-gracefully labeled as shown in 7.6. Vertex 1 in T is the fixed vertex that attaches to the vertices in S while vertex 1 in S is the designated vertex that will not have a copy of T attached. Let $\mid V(T) \mid= n$. Relabel two copies of T using $(n+1) - u$ shown in figure 7.7, where u is the vertex label.

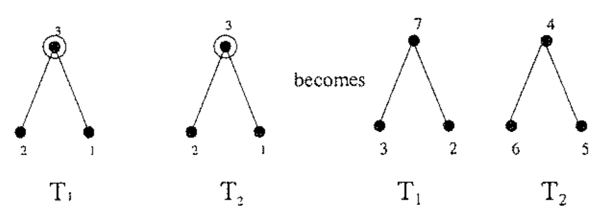


Figure 7.7. Relabeling copies of T

Next add $n+1$ to the values in the odd rows and add 1 to the values in the even rows of one copy of T. In the other copy add $n+1$ to the even rows and 1 to the odd rows (figure 7.7). Now the fixed vertices in the copies of T are attached to the end vertices of S resulting in the new graph U (figure 7.8). The remaining vertex of S is labeled 1.

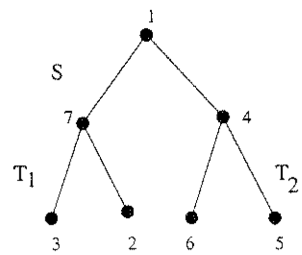


Figure 7.8. Gracefully labeled graph U

To continue the process for a fourth generation, the new graph U is relabeled using $n+1-u$, where $n = |V(U)|$ or 7. Add $n+1$ or 8 to odd rows and add 1 to even rows in one copy of U, while in the other copy add 8 to the values in the even row and add 1 to the values in the odd rows (figure 7.9).

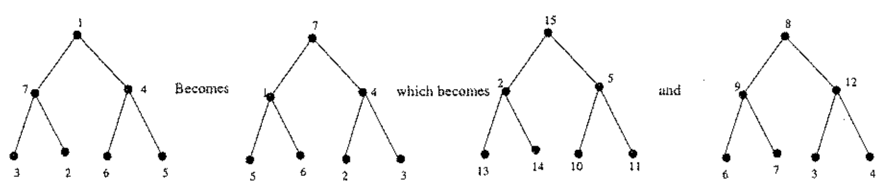


Figure 7.9. Relabeling the gracefully labeled third generation binary tree

The fixed vertex in U is the root and is attached to the end vertices in S resulting in a binary tree of four generations as in figure 7.10.

Gracefully labeling a 4-ary or quaternary tree is the next example. As in the binary case, S and T are initially identical and labeled as shown in figure 7.11. Using $n+1-u$ or 6 to relabel T gives T' as shown in the same figure. Use the idea from labeling balanced graphs in general to determine the value to add to the odd and even rows in the four copies T_i. Calculate $in+1$,

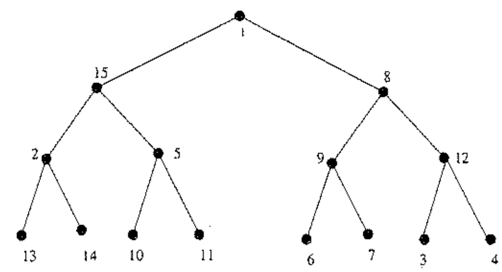


Figure 7.10. Gracefully labeled binary tree of four generations

where $i = 0, 1, ..., n-1$ to get $\{1, 6, 11, 16\}$.

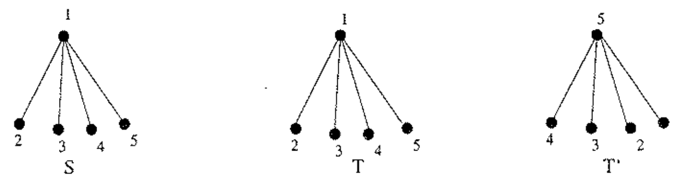


Figure 7.11. Gracefully labeled quaternary trees S and T

Add 1, 6, 11, 16 from this last calculation to the odd rows of T_1, T_2, T_3, and T_4 respectively. In the even rows add to T_1, T_2, T_3 and T_4 respectively 16, 11, 6, and 1. The result is shown in figure 7.12.

This pattern works for k-ary trees where k is even. When k is odd, the middle copy has the same value added to even and odd rows.

7.3 Trees of Diameter four

It has been proven that all trees of diameter four are graceful [15]. Diameter is defined as the maximum distance between any two points in the graph. We start with a discussion of certain graceful trees, which will lead to an interesting result regarding lobsters. A lobster is a tree such that if all pendant vertices were deleted, the remaining tree would be a caterpillar, which we show in figure 1.5. Huang, Kotzig, and Rosa [6] proved the following Lemma, which

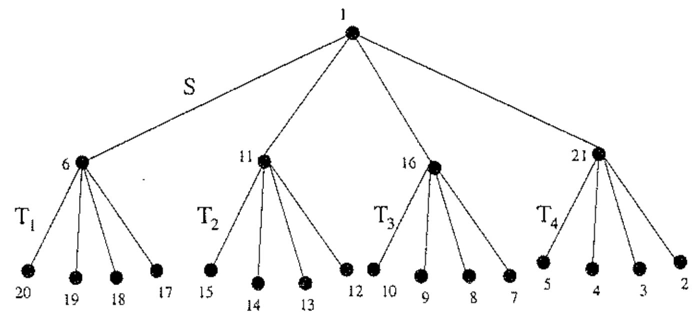


Figure 7.12. Graceful labeling of the third generation quaternary graph U

allows us to prove trees of diameter four are graceful. The uv-join of T_1 and T_2, denoted $T_1^u \cup^v T_2$ is formed by joining the graphs at u and v (vertex u is placed on vertex v) where $u \epsilon V(T_1)$ and $v \epsilon V(T_2)$ and the new graph has one less vertex than $\mid V(T_1 + T_2) \mid$. Lemma 1 states that there exists a graceful labeling for the uv-join of T_1 and T_2, if either T_1 or T_2 is α-graceful and the remaining tree is graceful.

Lemma 1 Let f_1 be an α-labeling of a tree T_1 with $f_1(u) = 0$ (where $u \epsilon V(T_1)$), and let f_2 be a graceful labeling of a tree T_2 with $f_2(v) = 0$ (where $v \epsilon V(T_2)$). Then there exists a graceful labeling of the tree $T = T_1^u \cup^v T_2$.

[Note: If T_2 is α-graceful, then the resulting graph T is α-graceful as well.]

Define a labeling g of T by

$$g(z) = \begin{cases} f_1^*(z), & \text{if } z \epsilon V(T_1) \backslash \{u\}, \quad f_1(u) \leq \alpha, \\ f_1^{**}(z) + m, & \text{if } z \epsilon V(T_1) \backslash \{u\}, \quad f_1(u) > \alpha, \\ f_2(z) + \alpha, & \text{if } z \epsilon V(T_2) \backslash \{v\}, \\ \alpha, & \text{if } z = u \end{cases}$$

where α is the critical value in the α-graceful labeling of T_1, $f_1^*(v) = \mid \alpha - f_1(v) \mid$, $f_1^{**}(v) = n - f_1(v)$, m is the number of vertices in T_2, and n is the number of vertices in T_1. Wang, Ju, Lu, and Zhang refer to $f_1^*(v)$ and $f_1^{**}(v)$ as the inverse labelings of $f_1(v)$, while others use the term "complement with respect to α or n" respectively.

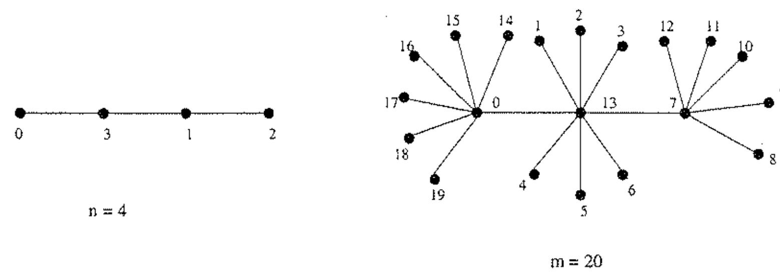


Figure 7.13. Gracefully labeled graphs T_1 and T_2

Examine the gracefully labeled graphs T_1 and T_2 in figure 7.13. Following the algorithm, each vertex in the component from T_2 will have its value increased by 1, which is the value of α from f_1. In the component from T_1 the vertices whose value is less than or equal to α will be relabeled with $\mid \alpha - f_1(u) \mid = \{0, 1, ...\alpha\}$. Those vertices whose value is greater than α will be relabeled with $n - f_1(u) + m = \{1 + m, 2 + m, ..., n - (\alpha + 1) + m\}$. These values are shown in the uv-join of T_1 and T_2 in figure 7.14.

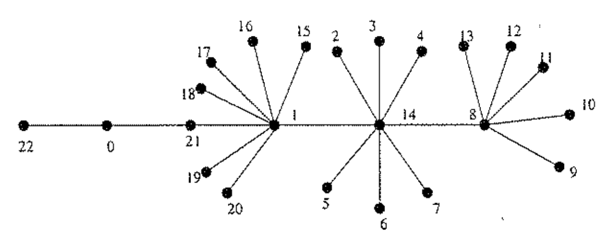


Figure 7.14. Gracefully labeled F, the uv-join of f_1 and f_2

Shi-Lin Zhao [15] showed that all trees of diameter four are graceful by developing an algorithm to gracefully label them. Trees of diameter four are a class of lobsters and an example T is shown in figure 7.15. The approach Zhao uses is also used by Huang [6] in gracefully labeling certain trees of diameter four. The group of Wang, Ju, Lu, and Zhang uses a similar approach as a premise for the graceful labeling of a special class of lobsters. Wang also used this approach with Jin [8] to prove that all trees of diameter four are graceful. This topic has created a lot of interest and many articles.

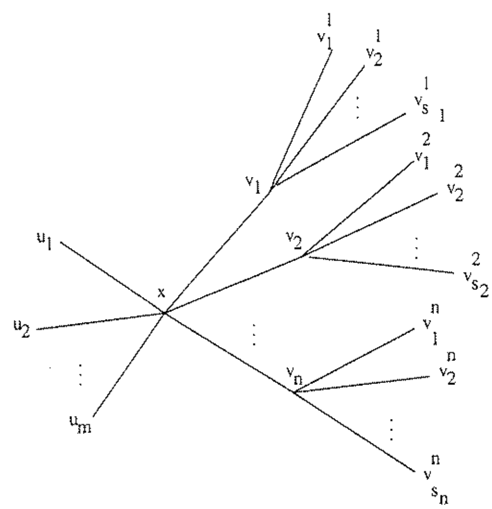


Figure 7.15. A tree T of diameter 4

Theorem 7 All trees of diameter 4 are graceful. [15]

In figure 7.15 call v_i and its neighbors component i. Every tree of diameter 4 must be similar to the one in figure 7.15. That is, there will exist some vertex x that is adjacent to pendant vertices and to other vertices that are not pendant vertices. These vertices can be categorized into three sets.

$A = \{u : d(x,u) = 1, \text{ where } u \text{ is a pendant vertex of } T\}$

$B = \{v : d(x, v) = 1$, where v is not a pendant vertex of $T\}$

$C = \{v : d(x, v) = 2$, where v is a pendant vertex of $T\}$, where $d(u, v)$ is the distance from u to v.

We now describe to to label the vertices in these sets to produce a graceful labeling of T.

Part 1) If n is odd and $A = \emptyset$, let

$$s = \sum_{i=1}^{n} s_i, \text{ where } s_i = \mid N(v_i) \mid -1,$$

$$s_i' = \begin{cases} s_i, & \text{if } s_i \text{ is odd} \\ s_i - 1, & \text{if } s_i \text{ is even} \end{cases}$$

$$D_1(i) = \frac{1}{2}\left(\sum_{r=i+1}^{n} (s_r' - 1) + \sum_{r=1}^{n-i} (s_r' - 1)\right),$$

$$\text{and } D_2(i) = \frac{1}{2}\left(\sum_{r=n+1-i}^{n} (s_r' - 1) + \sum_{r=1}^{i-1} (s_r' - 1)\right).$$

Let $k = \frac{n-1}{2}$ and define the graceful labeling of T as follows:

$$f(x) = s + 2k + 2 \tag{7.1}$$

$$f(v_i) = s + 2k + 2 - i, i = 1, 2, ..., k \tag{7.2}$$

$$f(v_{k+j}) = k + 2 - j, j = 1, 2, ..., k + 1. \tag{7.3}$$

Equation 7.1 labels the center vertex x and equation 7.2 labels the vertices in set B for $i = 1, 2, \ldots k$, while equation 7.3 labels the vertices in set B for $i = k+1, k+2, \ldots k+j$.

Before exploring the labeling of the leaf vertices, we need to examine the notation used for a leaf vertex. The notation v_j^i designates the j_{th} leaf neighbor of v_i in set B. To label the leaf neighbors of set B, that is set C, we consider four cases:

(1) the leaf neighbor for $j = 1, 2, \ldots, \frac{1}{2}(s_i' - 1)$,

(2) the leaf neighbor for $j = \frac{1}{2}(s_i' - 1) + 1, \frac{1}{2}(s_i' - 1) + 2, \ldots, s_i' - 1$,

(3) the leaf neighbor for $j = s_i - 1$, and

(4) the leaf neighbor for $j = s_i$.

Case 1) For $j = 1, 2, \ldots, \frac{1}{2}(s_i' - 1)$ and $i = 1, 2, \ldots, n$,

$$f(v_j^i) = \begin{cases} s - i - D_2(i) + k + 2 - j, & \text{if } i \leq k \\ s + i - D_1(i) - k - j, & \text{if } i > k. \end{cases} \tag{7.4}$$

Case 2) For $j = (\frac{1}{2})(s_i' - 1) + 1, (\frac{1}{2})(s_i' - 1) + 2, \ldots, s_i' - 1$ and $i = 1, 2, \ldots, n$, and also for $j = s_i - 1$ when s_i is odd,

$$f(v_j^i) = \begin{cases} D_2(i) + i + k + 1 + j - \frac{1}{2}(s_i' - 1), & \text{if } i \leq k \\ D_1(i) - i + 3k + 2 + j - \frac{1}{2}(s_i' - 1), & \text{if } i > k. \end{cases} \tag{7.5}$$

Case 3) For $j = s_i - 1$ and s_i is even,

$$f(v^i_{s'_i}) = \begin{cases} i + D_2(i) + k + 1, & \text{if } i \leq k \\ i - D_1(i) - k + s, & \text{if } i > k, \end{cases} \tag{7.6}$$

.

Case 4) The labeling for the last vertex in each v_i is defined by $f(v^i_{s_i})$ in each of the four sub-cases listed below.

Sub-case 1: If each s_i is odd for $i = 1, 2, ..., n$, then

$$f(v^i_{s_i}) = \begin{cases} i + D_2(i) + k + 1, & \text{if } i \leq k \\ i - D_1(i) - k + s, & \text{if } i > k. \end{cases} \tag{7.7}$$

Sub-case 2: If each s_i is even for $i = 1, 2, ..., n$ then

$$f(v^i_{s_i}) = \begin{cases} (\frac{1}{2})s + i + k + 1, & \text{if } i \leq k \\ (\frac{1}{2})s + i - k, & \text{if } i > k. \end{cases} \tag{7.8}$$

Sub-case 3: If s is even, but some s_i in the set $\{s_1, s_2, ..., s_n\}$ are even and the others are odd, then if s_i is odd order the v_i such that $i = 1, 2, ..., a, n - a + 1, n - a + 2, ..., n$ and if s_i is even order the v_i such that $i = a + 1, a + 2, ..., n - a$. This ordering of v_i puts the even v_i in the middle of the sequence and the odd v_i at the beginning or the end so that a is the value of the subscript for the odd s_i preceding the first even s_i, while $n - a + 1$ is the value of the subscript for the first odd s_i following the last even s_i. Define $f(v^i_{s_i})$ for odd s_i where $i = 1, 2, ...a, n - a + 1, n - a + 2, ..., n$ as in Case 1 and define $f(v^i_{s_i})$ for even s_i where $i = a + 1, a + 2, ..., n - a$ as follows.

$$f(v^i_{s_i}) = \begin{cases} (\frac{1}{2})s + i + k - a + 1, & \text{if } i \leq k \\ (\frac{1}{2})s + i - k + a, & \text{if } i > k. \end{cases} \tag{7.9}$$

where $i = a+1, a+2, ..., n-a$.

Sub-case 4: If s is odd, but some s_i in the set $\{s_1, s_2, ..., s_n\}$ are even and the others are odd, then if s_i is odd order the v_i such that $i = 1, 2, ...a, n-a, n-a+1, ..., n$ and if s_i is even order the v_i such that $i = a+1, a+2, ..., n-a-1$. This ordering of v_i puts the even v_i in the middle of the sequence and the odd v_i at the beginning or the end, while a is defined as in Sub-case 3. Define $f(v^i_{s_i})$ for odd s_i where $i = 1, 2, ...a, n-a, n-a+1, ..., n$ as in Case 1 and define $f(v^i_{s_i})$ for even s_i where $i = a+1, a+2, ..., n-a-1$ as shown below.

$$f(v^i_{s_i}) = \begin{cases} (\frac{1}{2})(s+1) + i + k - a, & \text{if } i \leq k \\ (\frac{1}{2})(s+1) + i - k + a, & \text{if } i > k. \end{cases} \tag{7.10}$$

where $i = a+1, a+2, ..., n-a-1$.

To apply these algorithms compare the value of j to s_i. If $j = s_i$, use the appropriate formula from equations 7.7-7.10. If $j = s_i - 1$, use equations 7.6 for even s_i and equations 7.5 for odd s_i. If $\frac{1}{2}(s'_i - 1) < j < s_i - 1$, use equations 7.5 and if $j \leq \frac{1}{2}(s'_i - 1)$, use equations 7.4. The proof that this algorithm does indeed produce a gracefully labeled graph is one of examining the edge values of T. The algorithm only provides a graceful labeling for graphs of diameter four where n is odd and $A = \emptyset$. If n is odd and $A \neq \emptyset$, then the component T' of T that includes vertex x and A is α-graceful since it is a caterpillar. $T - T'$ is graceful from Part 1 and T must be graceful by Lemma 1.

Part 2) If n is even, let T' be A and let T'' be one component incident to x

that does not include vertices of A, then $T' \cup T''$ is a caterpillar and has an α-graceful labeling. $T - T' - T''$ is graceful by results in Part 1 and tree T is graceful by Lemma 1.

We now demonstrate the algorithm by gracefully labeling the example in figure 7.17. There are 5 vertices in set B and each s_i is even.

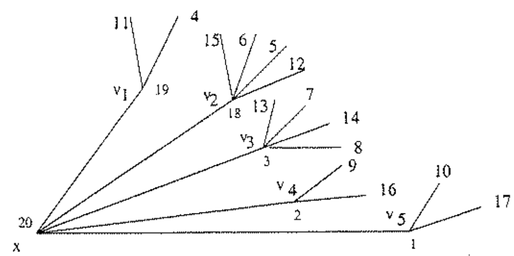


Figure 7.16. Gracefully labeled tree with n odd and s_i even

Begin by calculating the graceful values for vertex x and set B.

$$f(x) = s + 2k + 2 = 14 + 2(2) + 2 = 20$$

$f(v_i) = s + 2k + 2 - i = 14 + 2(2) + 2 - i$ results in 19 and 18 respectively,

$f(v_{k+j}) = k + 2 - j = 2 + 2 - j$ which results in 3, 2, and 1 respectively.

In order to gracefully label the vertices in set C we need to calculate $D_1(i)$ and $D_2(i)$. Since $D_1(i)$ is used only where $i > k$, find $D_1(3), D_1(4)$ and $D_1(5)$ and similarly only find $D_2(1)$ and $D_2(2)$. To calculate $D_1(i)$ and $D_2(i)$ we need the values of s_i and s'_i which are

$$s_1 = 2, s_2 = 4, s_3 = 4, s_4 = 2, \text{ and } s_5 = 2$$

$$\text{while } s'_1 = 1, s'_2 = 3, s'_3 = 3, s'_4 = 1, \text{ and } s'_5 = 1$$

based on the previously shown definitions.

$$D_1(3) = \tfrac{1}{2}((s'_4 - 1) + (s'_5 - 1) + (s'_1 - 1) + (s'_2 - 1)) = \tfrac{1}{2}(0 + 0 + 0 + 1) = 1$$

$$D_1(4) = \frac{1}{2}((s'_5 - 1) + (s'_1 - 1)) = \frac{1}{2}(0+0) = 0$$

$$D_1(5) = \frac{1}{2}(\sum_{r=6}^{5}(s'_r - 1)) = 0$$

$$D_2(1) = \frac{1}{2}((s'_5 - 1) + 0) = \frac{1}{2}(0+0) = 0$$

$$D_2(2) = \frac{1}{2}((0) + (s'_1 - 1)) = \frac{1}{2}(0+0) = 0$$

To assign values to v^i_j the next step is comparing the value of j to s_i and determining which of the equations to use. Begin with the next to last vertex in each s_i and work to the first vertex. The last vertex in each s_i will be labeled using one of the 4 sub-cases.

$$\text{Since } s'_1 = s'_4 = s'_5 = 1 \text{ while } s'_2 = s'_3 = 3,$$

for v^1_1 $j = 1$ and $s_1 - 1 = 1$ so that we use $f(v^i_{s'_i})$ (equation 7.6) for assigning values to the first vertex in s_1,s_4 and s_5 and for assigning values to the third vertex in s_2 and s_3.

$f(v^1_1) = 1+0+2+1 = 4, f(v^4_1) = 4-0-2+14 = 16, f(v^5_1) = 5-0-2+14 = 17$ and $f(v^2_3) = 2+0+2+1 = 5$ and $f(v^3_3) = 3-1-2+14 = 14$.

The second vertex of s_2 and s_3 will be labeled by the second set of equations (equation 7.5) because $\frac{1}{2}(s'_i - 1) < j < s_i - 1$.

$$f(v^2_2) = 0+2+2+2 = 6 \text{ and } f(v^3_2) = 1-3+3(2)+3 = 7.$$

Since $j \leq \frac{1}{2}(s'_i - 1)$ the value for the first vertex in s_2 and s_3 must be assigned using the first set of equations (equation 7.4) so that

$$f(v^2_1) = 14-2-0+2+1 = 15 \text{ and } f(v^3_1) = 14+3-1-2-1 = 13.$$

This leaves the last vertex of each s_i to label, which will be done from equations in Sub-case 2 since the sum of s_i is even and each s_i is even.

$f(v^1_2) = 7+1+2+1 = 11, f(v^2_4) = 7+2+2+1 = 12, f(v^3_4) = 7+3-2 = 8,$

$$f(v_2^4) = 7 + 4 - 2 = 9, f(v_2^5) = 7 + 5 - 2 = 10$$

7.4 Lobsters

Lobsters are defined as trees that result in caterpillars when all pendant vertices are deleted. It has been conjectured that all lobsters are graceful. Wang, working with Jin, Lu, and Zhang, [8] showed gracefulness of a class of lobsters. We examine the results of this study and show an example. To do this Wang, Ju, Lu, and Zhang introduced definitions and lemmas regarding moving components of a graph.

Definition Let u and v be two adjacent vertices of the tree T. By deleting edge uv of T, T turns into two smaller trees $u(T)$ and $v(T)$. These smaller trees are said to be components of T, and $v(T)$ is called a component incident to u.

Lemma 2 Let T be a graceful tree, f its graceful value function, u and v two vertices of T, and u_1 and u_2 two pendant vertices adjacent to u. If

$$f(u_1) + f(u_2) = f(u) + f(v),$$

or

$$2f(u_1) = f(u) + f(v),$$

then the trees

$$T' = T - uu_1 - uu_2 + vu_1 + vu_2,$$

and

$$T'' = T - uu_1 + vu_1$$

are graceful trees with graceful value function$f$.

T' and T'' are called components-moving trees of T.

Lemma 3 Let f be a graceful labeling of $T(V;E)$, $p = |V|$; then

(1) For any $uv \epsilon E$, we have

$$l(uv) = | f(u) - f(v) | = | \phi_p(f(u)) - \phi_p(f(v)) |$$

where $l(vu)$ is the edge label of T;

(2) For every $u_1, u_2, u, v \epsilon V(T)$, if

$$f(u_1) + f(u_2) = f(u) + f(v),$$

or

$$2f(u_1) = f(u) + f(v),$$

then

$$\phi_p(f(u_1)) + \phi_p(f(u_2)) = \phi_p(f(u)) + \phi_p(f(v)),$$

or

$$2\phi_p(f(u_1)) = \phi_p(f(u)) + \phi_p(f(v)),$$

where $\phi_p(f) = p - f(v)$, $v \epsilon V(T)$ and $f(v)$ is the graceful label of graph T.

The authors refer to $\phi(f)$ as an inverse transformation. It is the same function used by Stanton and Zarnke and is called the "complement with respect to p". Lemma 2 states that the components-moving transformation maintains the gracefulness of the graph. Lemma 3 proves that the graph will still be graceful after applying the function ϕ_p. Using these lemmas, a new class of lobsters with diameter greater than four can be gracefully labeled.

Begin with a graph T that is a collection of star trees whose center vertices are all positive evens or positive odds. A star tree is made of a center vertex connected to every other vertex. The star trees are joined by selecting a pendant vertex from each and sticking them together. Label the vertices by the algorithm described for caterpillars in section 3. Use the gracefully labeled graph T from the earlier example in figure 7.17.

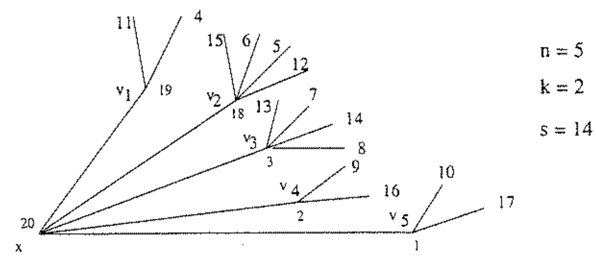


Figure 7.17. Gracefully labeled tree of diameter 4

First apply the function ϕ_p, which will change the label at vertex x from 20 to 0. The label of 0 here is a requirement of Lemma 1. Now a path of length 1 can be joined to graph T at the newly labeled vertex 0 and the second vertex on the path will be labeled according to the algorithm given with Lemma 1 which is 20. Based on Lemma 2, components incident to 0 can be moved. Specifically, the components containing v_1 and v_5 can be moved to the new vertex labeled 20 since v_1 and v_5 are labeled 1 and 19 respectively and add to 20. This process will be repeated. First apply the function ϕ_p, then add a path of length one and finally move the components incident to vertex 20, containing v_{18} and v_2 to vertex 0. Then the two components incident to vertex 0, containing v_1 and v_4 can be moved to vertex 21. The resulting gracefully labeled lobster shown in figure 7.18 has diameter six.

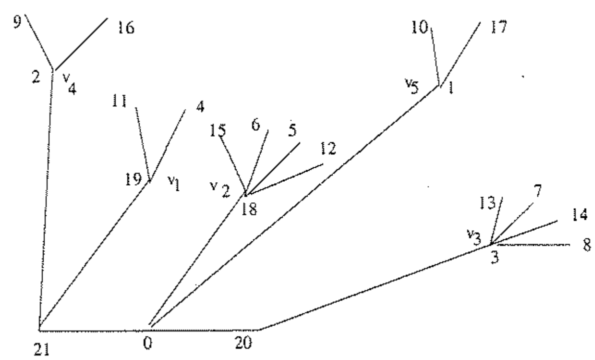


Figure 7.18. Gracefully labeled lobster L of diameter 6

8. Conclusion

The search for gracefully labeled graphs began in an effort to prove Ringle's Conjecture or Conjecture 1 (cf. [1], 1963). Just three years later Kotzig developed the Ringel-Kotzig Conjecture (cf. [11], 1966). Rosa ([11], 1967) introduced the idea of labeling vertices of the graph as a way of approaching the proof of Ringle's conjecture. Graceful labeling and α-graceful labeling came from this study. Kotzig ([10], 1973) then conjectured that all trees are graceful and Rosa (cf. [10], 1973) proved that Kotzig's Conjecture (Conjecture 3) implies Ringle's Conjecture. In an effort to prove trees graceful Rosa proved paths ([11], 1963) and caterpillars ([11], 1963) are graceful. Stanton and Zernke ([12], 1973) proved balanced trees formed from two graceful trees are graceful. K-ary trees are a class of balanced trees proved graceful by Stanton and Zernke ([12], 1973). Lobsters are conjectured to be graceful ([6], 1982). Huang, Kotzig, and Rosa ([6], 1982) proved that a gracefully labeled graph can be joined in a certain way to an α-gracefully labeled graph to produce a gracefully labeled graph. Zhao ([15], 1989)proved all trees of diameter four are graceful. Wang, Ju, Lu, Zhang ([13], 1994) proved lobsters that begin as diameter four and are changed by the components moving tree are graceful.

In an attempt to further study the graceful labeling of graphs formed by joining two or more graphs, Hoede and Kuiper ([5], 1978)proved wheels to be graceful and Frucht ([3], 1979) proved crowns are graceful. It has been conjectured that cycles ([1], 1979) with an added chord are graceful, but this has not been proved. Jungreis and Reid ([9], 1992) proved planar grids are α-graceful, while Huang and Skiena ([7], 1994) proved cylindrical grids are graceful if there are no odd cycles.

Remaining open questions are:

- Ringel's conjecture,
- Ringel-Kotzig conjecture,
- Kotzig's conjecture,
- All lobsters are graceful,
- All trees are graceful, and
- Cycles with an added chord are graceful.

To prove Ringel's conjecture or the Ringel-Kotzig conjecture we need an approach to prove all trees graceful as opposed to proving a class of trees graceful. The typical induction argument used by the graph theorists will not work to prove trees are graceful. Either a different induction approach is needed or an approach which doesn't use induction since the technique applied to individual classes of graphs do not lend themselves to the general case.

It appears some currently used approaches could be extended to cover other cases. One promising approach might be attempting to apply the components-moving tree of Wang, Ju, Lu, Zhang [8] to trees of diameter 4 no matter the parity of the degree of the center vertices of the components. Another approach might be a merging of Huang's, Kotzig's, and Rosa's [6] idea of the uv-join with induction. It is certainly easier to prove graphs are graceful than to prove they are not graceful.

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