Graceful Labeling Algorithms

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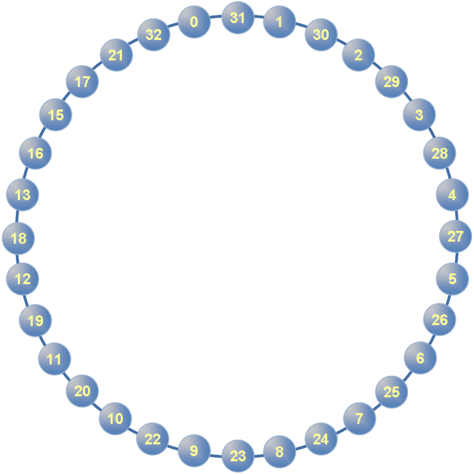
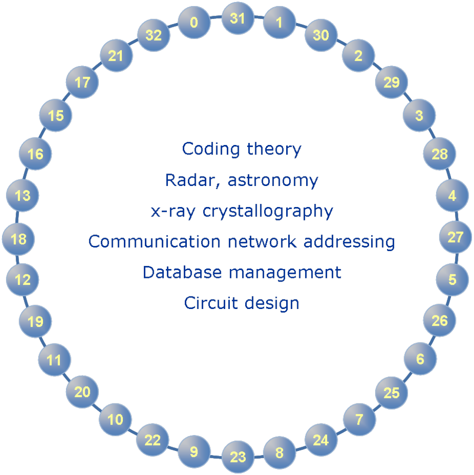
- Since then many different types of graph labeling techniques have been investigated.
- Over 800 papers have been published in this area.

A Graceful Cycle



Applications



Coding theory

Radar, astronomy

x-ray crystallography

Communication network addressing

Database management

Circuit design

Graceful Labeling

- For a connected graph G with q edges, a vertex labeling $f : V(G) \rightarrow \{0, 1, 2, ..., q\}$ such that distinct vertices have distinct labels induces an edge labeling where an edge uv gets the label $|f(u) - f(v)|$.

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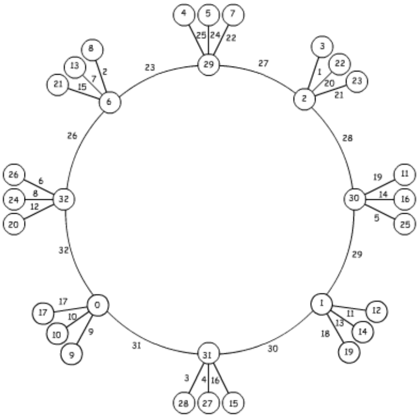
- Such a labeling is called *graceful* if the edges are labeled $1, 2, ..., q$.
- G is called *graceful* if it has a graceful labeling.

An Example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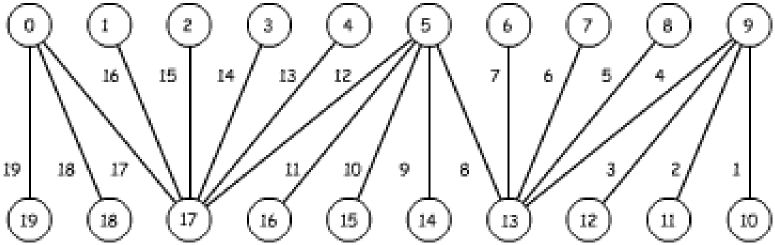
Another Example



Nongraceful Examples

Theorem

The complete graph K_n is graceful iff $n \leq 4$.

Graceful Trees

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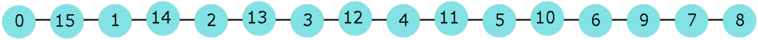
- Extended to $n = 28$ by Nikoloski et al, and to $n = 29$ by M. Horton (2003)
- Some results are known. For example, paths and caterpillars are graceful.

Unlabeled rootless trees

Tree size (n)	Number of unlabeled rootless trees of this size	Otter (*Ann. Math. 1948)*
7	11	
8	23	
9	47	
10	106	
11	235	
12	551	
13	1,301	
14	3,159	
15	7,741	
16	19,320	
17	48,629	
18	123,867	
19	317,955	
20	823,065	
21	2,144,505	
22	5,623,756	
23	14,828,074	
24	39,299,897	
25	104,636,890	
26	279,793,450	
27	751,065,460	
28	2,023,443,032	
29	5,469,566,585	
30	14,830,871,802	
31	40,330,829,030	
32	109,972,410,221	

Paths, Caterpillars



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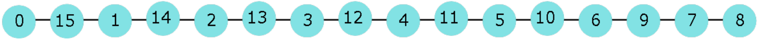

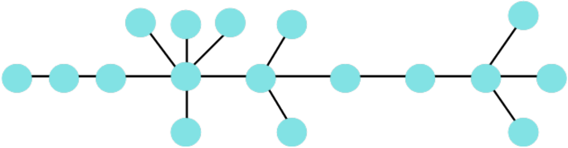

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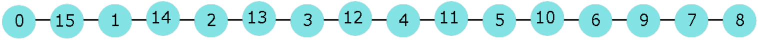
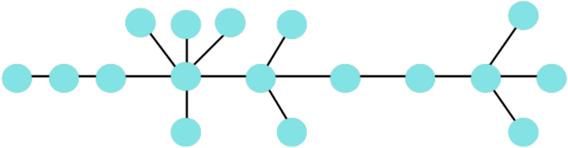
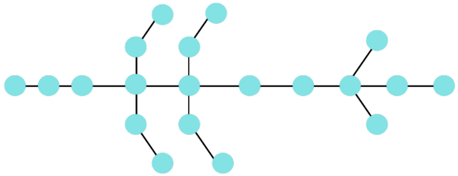




Wikipedia

Lobsters



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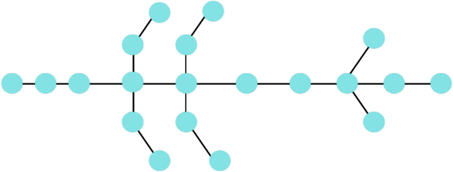

Graph Decomposition

- A *decomposition* of a graph $G = (V, E)$ is a partition of the edge set E into subsets $E_1, E_2, ..., E_r$.

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- If H_i is the subgraph of G induced by E_i, then we also say the G decomposes into subgraphs
- If the subgraphs H_i are all isomorphic to a single graph (say) H then we say that G is $H - decomposable$ and we write $H|G$.

Graph Decompositions and Graceful Labelings

- Conjecture (Ringel, 1963) If T is a tree with m edges, then K_{2m+1} decomposes into $2m+1$ copies of T.

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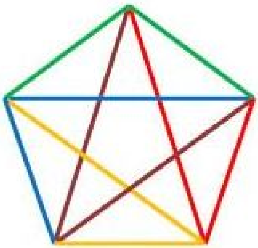

Graph Decompositions and Graceful Labelings

- Theorem (Rosa) If a tree T with m edges has a graceful labeling, then K_{2m+1} decomposes into $2m+1$ copies of T.

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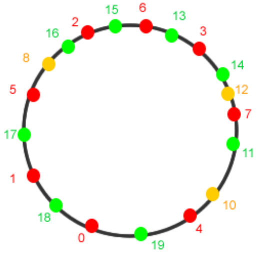
- He gave a constructive proof by showing one explicit graceful labeling for each such n.

An example



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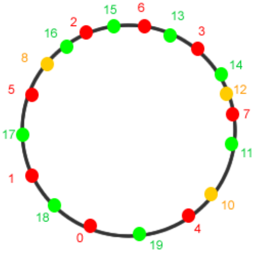

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- In any graceful labeling of C_n exactly one of the vertex labels from $\{0, 1, 2, ..., n\}$ is missing.

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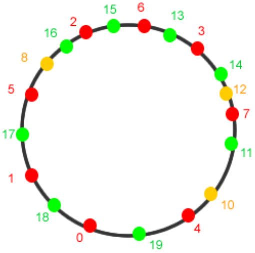

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- In any graceful labeling of C_n exactly one of the vertex labels from $\{0, 1, 2, ..., n\}$ is missing.
- We call this missing value m. In the above example, $m = 9$.

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- Barrientos showed that all hairy cycles are graceful.

An Algorithm

- We have developed an algorithm to generate all possible graceful labelings of C_n when n is 0 or 3 $(mod\ 4)$.

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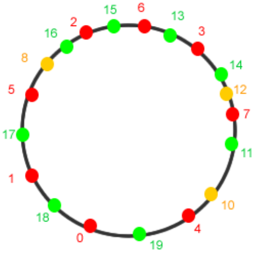
- We have developed an algorithm to generate all possible graceful labelings of C_n when n is 0 or 3 $(mod\ 4)$.
- We discovered several more graceful labelings and observed that they all satisfy interesting properties.

An example



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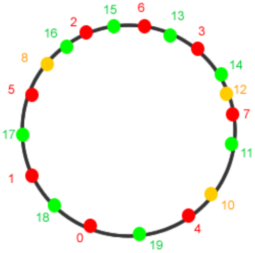

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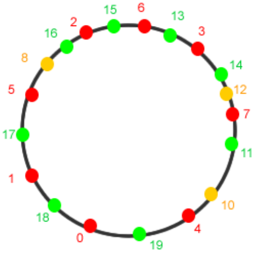

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Graceful Labeling data

	n												
		3	4	7	8	11	12	15	16	19	20	23	24
Number		2	2	12	24	208	492	7,764	20,464	424,784	1,204,540	33,492,078	107,399,400
	0												
	1	1	1										
	2	1		3	3								
	3		1	3	6	26	26						
	4			3	6	42	80	299	299				
	5			3	6	36	80	789	1,476	5,932	5,932		
	6				3	36	120	1,301	3,190	22,210	39,692	162,634	162,634
	7					42	80	1,493	3,494	49,714	104,688	787,218	1,393,740
	8					26	80	1,493	3,646	61,758	162,606	2,218,596	4,813,618
m	9						26	1,301	3,494	72,778	191,238	3,690,788	9,785,048
	10							789	3,190	72,778	196,228	4,633,029	13,567,488
	11							299	1,476	61,758	191,238	5,252,774	15,837,020
	12								299	49,714	162,606	5,253,774	16,280,304
	13									22,210	104,688	4,633,029	15,837,020
	14									5,932	39,692	3,690,788	13,567,488
	15										5,932	2,218,596	9,785,048
	16											787,218	4,813,618
	17											162,634	1,393,740
	18												162,634
	19												
	20												

Algorithm - Details

Level n	0 n			
Level n-1	n-1 0 n		0 n 1	
Level n-2	1 n-1 0 n	n-1 0 n 2	n-2 0 n 1	0 n 1 n-1
Level n-3	...			
...	...			
Level n-k	...			

Possible Differences

Edge Label	n		n-1		n-2		n-3	
Adjacent Vertex Label	n	0	n-1	0	n-2	0	n-3	0
			n	1	n-1	1	n-2	1
					n	2	n-1	2
							n	3

General Step

Given the following sublabeling S at step k:

$$a_2 \;\; a_3 \;\; a_4 \;\; \ldots \;\; 0 \;\; n \;\; \ldots \;\; a_{k-1} \;\; a_k$$

Where next edge label is e
Find two labels a and b such that

- $|\mathrm{a} - \mathrm{b}| = e$
- a, b $\notin S$
- $0 \leq$ a, b $\leq n$

General Step - 2

$$\text{a b } X_1 \ a_2 \ a_3 \ a_4 \ \ldots \ 0 \ n \ \ldots \ a_{k-1} \ a_k$$
$$\text{b a } X_2 \ a_2 \ a_3 \ a_4 \ \ldots \ 0 \ n \ \ldots \ a_{k-1} \ a_k$$

Where X_1 and X_2 are sublabelings between b a_2 and a a_2 respectively.

Description of Algorithm

- $|\text{a} - a_2| = e$ or $|\text{a} - a_i| = e$ or $|\text{a} - a_k| = e$
  - a a_2 a_3 a_4 … 0 n … a_{k-1} a_k
  - a_2 a_3 a_4 … a_i a X_1 … 0 n … a_{k-1} a_k
  - a_2 a_3 a_4 … X_1 a a_i … 0 n … a_{k-1} a_k
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  - a_2 a_3 a_4 ... X_1 a a_i ... 0 n ... a_{k-1} a_k
  - a_2 a_3 a_4 ... 0 n ... a_{k-1} a_k a
- $|$ a - b $| = e$
  - a b X_1 a_2 a_3 a_4 ... 0 n ... a_{k-1} a_k
  - b a X_1 a_2 a_3 a_4 ... 0 n ... a_{k-1} a_k
  - a_2 a_3 a_4 ...X_1 a b X_2 ... 0 n ... a_{k-1} a_k
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- $|$ a - b $| = e$
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  - a_2 a_3 a_4 ...X_1 b a X_2 ... 0 n ... a_{k-1} a_k
- $|a_i - a_{i+1}| = e$ where a_i, $a_{i+1} \in$ (... a_i X_1 a_{i+i} ...)
  - a_2 a_3 a_4 ... a_i X_1 a_{i+1} ... 0 n ... a_{k-1} a_k $\Rightarrow$ a_2 a_3 a_4 ... a_i a_{i+1} ... 0 n ... a_{k-1} a_k

Sample Output

8 — 0 8
7 — 7 0 8
6 — 1 7 0 8
5 — 6 1 7 0 8
4 — 2 6 1 7 0 8
3 — 5 2 6 1 7 0 8
2 — 3 5 2 6 1 7 0 8
3 — 2 6 1 7 0 8 5
* 2 — 4 2 6 1 7 0 8 5 [1] $\longrightarrow$ 3 H(#3;21)={6 7 8}; L(#3;3)={0 1 2}; I(#2;9)={4 5}
* 2 — 2 6 1 7 0 8 5 3 [2] $\longrightarrow$ 4 H(#3;21)={6 7 8}; L(#3;3)={0 1 2}; I(#2;8)={3 5}
4 — 6 1 7 0 8 4
3 — 3 6 1 7 0 8 4
* 2 — 5 3 6 1 7 0 8 4 [3] $\longrightarrow$ 2 H(#4;26)={5 6 7 8}; L(#4;8)={0 1 3 4}; I(#0;0)={}
* 2 — 3 6 1 7 0 8 4 2 [4] $\longrightarrow$ 5 H(#3;21)={6 7 8}; L(#3;3)={0 1 2}; I(#2;7)={3 4}
3 — 5 2 -2 6 1 7 0 8 4
2 — 3 5 2 -2 6 1 7 0 8 4
3 — 2 5 -2 6 1 7 0 8 4
* 2 — 6 1 7 0 8 4 2 5 [5] $\longrightarrow$ 3 H(#3;21)={6 7 8}; L(#3;3)={0 1 2}; I(#2;9)={4 5}
2 — 2 5 3 -2 6 1 7 0 8 4
5 — 1 7 0 8 3
4 — 5 1 7 0 8 3
3 — 2 5 1 7 0 8 3
* 2 — 4 2 5 1 7 0 8 3 [6] $\longrightarrow$ 6 H(#4;24)={4 5 7 8}; L(#4;6)={0 1 2 3}; I(#0;0)={}
2 — 6 4 -2 2 5 1 7 0 8 3
2 — 4 6 -2 2 5 1 7 0 8 3
...

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- Note that $S_n = \{0, n\}$ and $S_1 = f$

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- Example: $f =< 4, 15, 0, 16, 2, 11, 3, 13, 1, 14, 7, 9, 12, 6, 10, 5 >$ $S_{13} =< 15, 0, 16, 2 >< 1, 14 >$
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- Example: $f =< 4, 15, 0, 16, 2, 11, 3, 13, 1, 14, 7, 9, 12, 6, 10, 5 >$ $S_{13} =< 15, 0, 16, 2 >< 1, 14 >$
- Note that sub labeling S_k is a set of paths
- The proof of the correctness of the algorithm is carried out using induction on the edge label $k = n, n-1, ..., 2, 1$

Proof of correctness

- **Base Case:** $S_n = \{0, n\}$ which is trivially true for any f
- Now at $n-1$ we have two alternatives
  - $S_{n-1} =< n-1, 0, n >$
  - $S_{n-1} =< 0, n, 1 >$
- **Induction Hypothesis:** The algorithm achieves S_{k+1}
- We will now show that the algorithm achieves S_k

Proof of correctness

- By definition S_k produces edge label n through k. Suppose $|a - b| = k$

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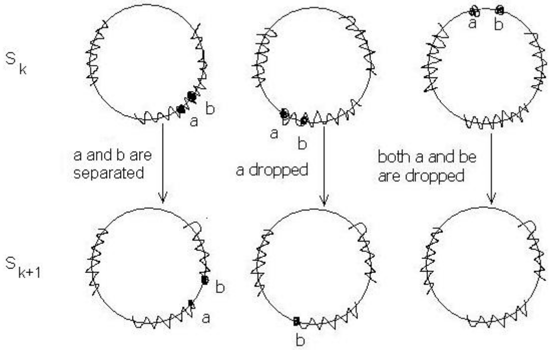
- There are three possibilities



Algorithm for Paths(1)

- We have modified our cycle algorithm for paths and obtained graceful labelings of paths up to order 16.
- Abrham and Kotzig (1990) showed that the number of graceful labelings of paths grows exponentially.

Algorithm for Paths(2)

- Aldred et al (2003) showed that for large n, the number of graceful labelings of P_n is at least $\left(\frac{5}{3}\right)^n$.

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n	#	$\left(\frac{5}{3}\right)^n$	**Ratio**
6	16	21	0.75
7	20	36	0.56
8	60	60	1.01
9	148	99	1.49
10	324	165	1.96
11	664	276	2.41
12	1600	459	3.48
13	4956	766	6.47
14	12796	1276	10.03
15	27960	2127	13.15
16	71596	3545	20.20

Comparison with Other Cycle Algorithms

n	Performance (s)	
	E & A [1]	Ours
8	< 0.01	< 0.01
15	$< .65$	< 0.01
20	< 105.32	< 0.01
55	N/A	< 0.03
72	N/A	< 0.04
112	N/A	< 0.15

[1] [E & A] Eshghi and Azimi (2003)

Graceful Labeling data

		n											
		3	4	7	8	11	12	15	16	19	20	23	24
Number		2	2	12	24	208	492	7,764	20,464	424,784	1,204,540	33,492,078	107,399,400
	0												
	1	1	1										
	2	1		3	3								
	3		1	3	6	26	26						
	4			3	6	42	80	299	299				
	5			3	6	36	80	789	1,476	5,932	5,932		
	6				3	36	120	1,301	3,190	22,210	39,692	162,634	162,634
	7					42	80	1,493	3,494	49,714	104,688	787,218	1,393,740
	8					26	80	1,493	3,646	61,758	162,606	2,218,596	4,813,618
m	9						26	1,301	3,494	72,778	191,238	3,690,788	9,785,048
	10							789	3,190	72,778	196,228	4,633,029	13,567,488
	11							299	1,476	61,758	191,238	5,252,774	15,837,020
	12								299	49,714	162,606	5,253,774	16,280,304
	13									22,210	104,688	4,633,029	15,837,020
	14									5,932	39,692	3,690,788	13,567,488
	15										5,932	2,218,596	9,785,048
	16											787,218	4,813,618
	17											162,634	1,393,740
	18												162,634
	19												
	20												

Graceful Cycles

- J. Bagga, A. Heinz, M. Majumder, *An Algorithm for Graceful Labelings of cycles*, Congressus Numerantium **186** (2007), pp. 57-63.
- J. Bagga, A. Heinz, M. Majumder, *Properties of Graceful Labelings of cycles*, Congressus Numerantium **188** (2007), pp. 109-115.

Graceful Paths and Hairy Cycles

- C. Barrientos, *Graceful graphs with pendant edges*, Australasian J. of Comb. **33** (2005), pp. 99-107.

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- All hairy cycles are graceful.
- Hairy cycle: A unicyclic graph in which the deletion of any edge in the cycle yields a caterpillar.

Sun graphs

- Sun graph: The vertices of degree at least two induce a cycle.

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- Corona $G_1 \odot G_2$ (Frucht and Harary, 1970)
- Corona $C_n \odot mK_1$
- Algorithm to generate graceful labelings for the $1 - Sun$ $(m = 1)$.

Graceful Labeling Algorithm for 1-Sun

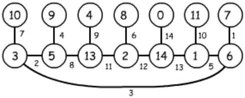
Given SUN_n a 1-Sun
$c_1, c_2, ..., c_{n/2}$ the vertex labels in the cycle
$r_1, r_2, ..., r_{n/2}$ the vertex labels in the rays such as ray r_i is attached to c_i.
A given a labeling $f =< c_1, c_2, ..., c_{n/2}, r_1, r_2, ..., r_{n/2} >$ of SUN_n can be considered an ordered sequence of labels.

A *sublabeling* is an ordered union of disjoint subsequences of f.
If f is graceful, there are n sublabelings S_k of f, where S_k generates edge labels $k, k+1, ..., n$.
This sublabeling S_k is the ordered union of sublabelings in SUN_n containing edges with labels k through n.

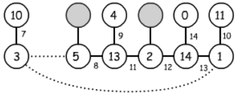
Graceful Labeling Algorithm for 1-Sun

For instance, given the graceful labeling $f =< 3, 5, 13, 2, 14, 1, 6, 10, 9, 4, 8, 0, 11, 7 >$ of SUN_{14}, we observe that $S_7 =< 3, 10 >< 5, 13, 2, 14, 1, \Phi, 4, \Phi, 0, 11 >$ and therefore, S_7 is the ordered union of two sublabelings. We also observe that for any graceful labeling $f = S_1$.

f



S_7



General Step

For a given level l, denote the previous sublabeling S_{l+1} by S_{l+1} by $< s_1 >< s_2 > ... < s_p >$.

Now we wish to add two labels v_i and v_j such that $|v_i - v_j| = l$.

General Step - Cases

There are three cases to consider.

(i) $v_i \in S_{l+1}$ *and* $v_j \in S_{l+1}$. In this case, two subsequences are merged into a single subsequence.

(ii) $v_i \notin S_{l+1}$ *and* $v_j \notin S_{l+1}$. In this case, $< v_i, v_j >$ is added between $< s_{r-1} >$ and $< s_r >$, for every $2 \leq r \leq p$. The computation splits into several branches for all these cases.

(iii) $v_i \notin S_{l+1}$ *and* $v_j \in S_{l+1}$. In this case, if v_j is an end label of some s_r we then obtain S_l be adding v_i to that end. Also, if the ray adjacent to v_i is free, another computation branch is created by placing v_j as a ray of v_i.

If none of these cases occur at a given branch, then that computation branch dies.

General Step - Case (i)

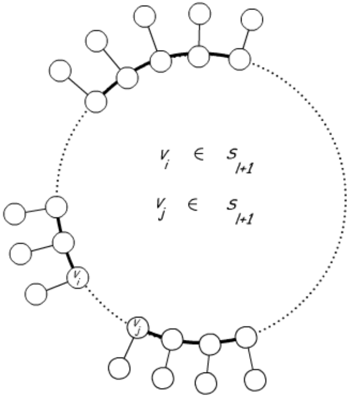


Figure: Case (i)

General Step - Case (ii)

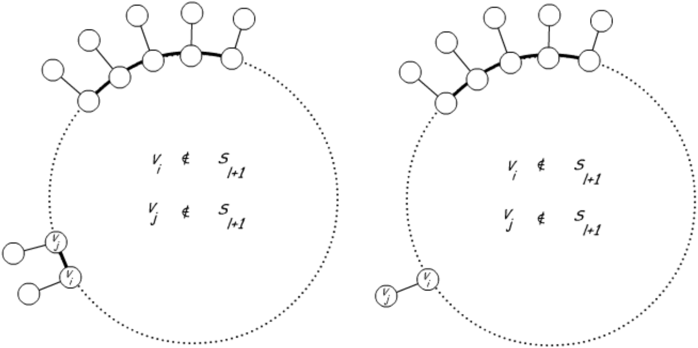


Figure: Case (ii)

General Step - Case (iii)

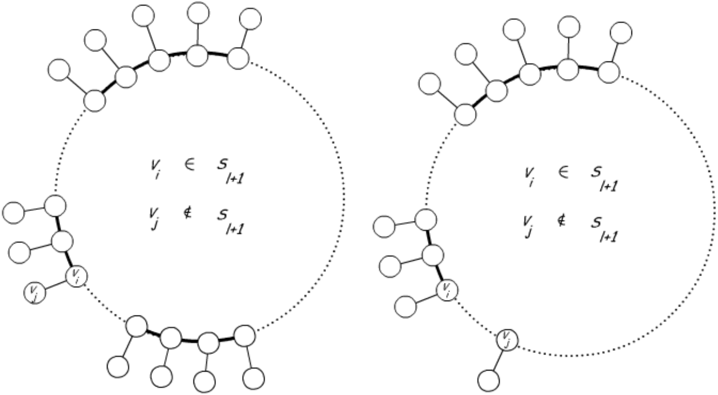


Figure: Case (iii)

Execution Branches

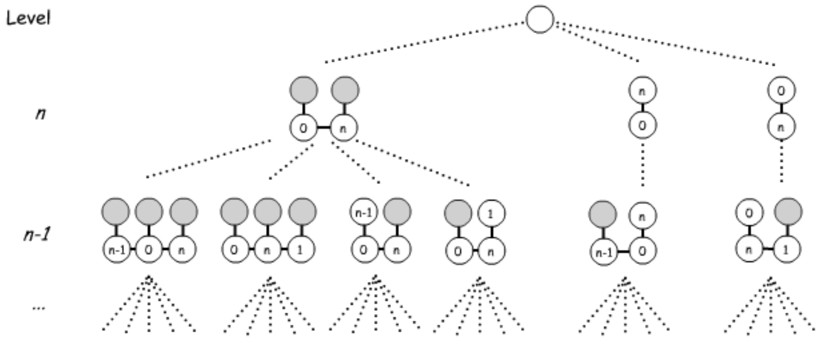


Figure: Execution branches

Proof of Correctness

Theorem

Suppose $f =< x_1, x_2, ..., x_n >$ *is a graceful labeling of* SUN_n*.*
The algorithm achieves f *exactly once.*

Proof of Correctness

(i) $v_i \in S_{l+1}$ *and* $v_j \in S_{l+1}$.

(ii) $v_i \notin S_{l+1}$ *and* $v_j \notin S_{l+1}$.

(iii) $v_i \notin S_{l+1}$ *and* $v_j \in S_{l+1}$.

A Graceful Cycle