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GRACEFUL TREE LABELING

(Ročníkový projekt)

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Chapter 1

Introduction

This work is focused on known results connected with graceful labelings of trees. Graceful labeling is assignment of distinct labels from $\{0, \dots, |E|\}$ to vertices of graph, where edges are labeled by difference of absolute values of adjacent vertices and every label from $\{1, 2, \dots, |E|\}$ is used exactly once as an edge label. It is still not known, if every tree has a graceful labeling. This was conjectured by Rosa at International Symposium in Rome in July 1966. Since that time, there was lot of attempts to prove (and few to disprove) this conjecture, but there are still just very few classes of trees, whose are proved to be graceful.

Original motivation for graceful tree conjecture was, that if this claim is true, then it implies true of Ringel's conjecture (posed on Graph theory symposium in Smolenice, 1963) about decompositions of complete graph, which states that every $(m+1)$ -order tree decomposes K_{2m+1} . This conjecture is also still unsolved. It should be noted that for solving this conjecture, we don't necessarily need gracefully label every tree, it will be sufficient, if every tree has ρ -labeling or σ -labeling. Both of this labelings would be defined in next chapter and both of them were introduced by Rosa in the same paper as Graceful labeling conjecture.

From other known applications of graceful labelings should be noted: Radar pulse codes, convolution codes or resolving ambiguities in X-Ray crystallography. More information about possible applications could be found in Golomb's work [4].

Another results about gracefulness could be used in topology. Aldred, J. Širáň and M. Širáň showed in [3] lower bound of number of graceful labelings of path P_n is at least $(\frac{5}{3})^n$ and noted, that further improved this lower

bounds may lead to improved bounds on the number of vertex-transitive triangulations of complete graphs.

This paper is divided to 6 chapters. In next chapter we will formally define notions used in this branch of graph theory including bipartite and locally bipartite labelings and various range relaxed labelings. Chapter number 3 is devoted to progress that was made in classification of classes which are known to be graceful. In chapter 4 we will show 2 interesting properties of trees – k -rotatability and k -centred gracefulness, with aim at $k = 0$. This property is crucial in progress with graceful tree conjecture, because it was very soon shown that not all trees allow to label arbitrary vertex with 0 and that makes a lots of problems by doing constructions for graceful labeling trees. Chapter 5 is about constructions of larger graceful trees from smaller ones, this was long time main way of trying to prove GTC but none of known construction helps us to label some larger or not symmetric class of trees. Last chapter is devoted to some interesting problems which are either open or unexplored and it is hoped that their solving would lead to better understanding of gracefulness on trees.

Main sources used by writing this paper were works from Frank van Bussel [23] [22], where in the first one is very large part devoted to results made in GTL and second is shorter version of his results. From other used materials is very interesting survey made by Michelle Edwards and Lea Howard [8], which is aimed mainly at graceful labeling of trees, but including some interesting variations like harmonious labeling etc. For reader interested in labelings in general could be very helpful Dynamic Survey of Graph Labelings from Joseph A. Gallian [9] – this is really comprehensive survey with 180 pages and almost 800 cited papers. Other works related to concrete results are mentioned in text.

Chapter 2

Definitions and preliminaries

Definition 2.1. Graceful labeling of graph $G = (V, E)$ is labeling $f : V \rightarrow \{0 \dots |E|\}$ inducing edge labeling g defined by $g(uv) = |f(u) - f(v)|$ such that:

- $\forall u, v \in V, u \neq v, f(u) \neq f(v)$
- g is bijection from $E(G)$ to $\{1 \dots |E|\}$

If graph G has graceful labeling, we say that G itself is graceful. Graceful labeling is sometimes called β -valuation

Conjecture 2.1. (Graceful labeling conjecture) Every tree has a graceful labeling.

It is easy to see that if labeling f is graceful, then also labeling $h(v) = |E| - f(v)$ is graceful. Such labeling is called *complementary labeling*. First condition trivially holds. And for each edge $f(uv) = |u - v| = |v - u| = ||E| - |E| + v - u| = |(|E| - u) - (|E| - v)| = h(uv)$, so complementary labeling also preserves the second condition. This implies that every nontrivial graph has even number of graceful labelings.

Definition 2.2. Bipartite labeling is graceful labeling with further property, that there exists $x \in 0 \dots |E|$ such that for arbitrary edge uv either $f(u) \leq x < f(v)$ or $f(v) \leq x < f(u)$ holds.

Tree which admits bipartite labeling is sometimes called *balanced, interlaced* or α -valuation.

Bipartite labeling allows us to do other operation to generate new graceful labelings.

Definition 2.3. Let f be bipartite labeling of graph G with bipartition (A, B) with k such that there exists such vertex v that $v \in B, f(v) = k$ and $u \in A, f(u) = k - 1$. We call g the reverse labeling of f if g is defined by:

$$g(x) = \begin{cases} k - 1 - f(v), v \in A \\ |E| + k - f(v), v \in B \end{cases}$$

If f is bipartite labeling, then also g is. So if we have any bipartite labeling f with value k as a smallest label from bipartition with higher values of labels, then we could label with zero not only vertex $|E|$, but also k or $k - 1$ – just by using complementary and reverse operations.

Positive on bipartite labelings is that we have more information about vertices labels – we know that in one bipartition will be smaller labels than in second one, and we even know that edge label 1 will be between vertices labeled k and $k - 1$, where k is value of smallest label of bipartition with larger values. Another positive is, that we could scale them by adding a constant to bipartition with higher values, what is used by some constructions which generate larger graceful trees from smaller ones.

Negative is, that not all trees admit bipartite labeling, so using any labeling technique using this property wouldn't lead to successive labeling of any tree or solving graceful labeling conjecture. But there is some hope, that knowing more about bipartite labeling would lead to better understanding of graceful labeling in general.

Definition 2.4. Locally bipartite labeling of graph G with bipartitions (A, B) is graceful labeling with further property, that for arbitrary edge uv such that $u \in A$ and $v \in B$ holds $f(u) < f(v)$.

Locally bipartite labelings share some properties with bipartite labelings, but not all of them. We could for example scale them almost as easy as bipartite labelings. But positive difference is, that we still don't know any tree, that isn't locally bipartite. This labelings were examined by van Bussel in [22], one of computer verified result is, that there isn't any non-locally bipartite tree in all trees with order smaller than 20.

There are more kinds of near graceful labelings, that don't fulfill conditions of gracefulnes, but are still interesting enough and often used to prove statements which we cannot prove for graceful labeling or to show near gracefulnes of graphs which aren't graceful. We will define some of them in next few paragraphs.

Definition 2.5. σ -valuation is labeling, where the induced edge labeling must satisfy the condition of being bijection from $E(G)$ to $\{1 \dots |E|\}$, but the vertex range is relaxed to $\{0, \dots, 2|E|\}$

Definition 2.6. ρ -valuation is labeling, where the induced edge labeling is relaxed to $\{1, \dots, 2|E|\}$, under the condition that either label i or label $2|E| + 1 - i$ is used, but not both and the vertex range is relaxed to $\{0, \dots, 2|E|\}$.

Rosa introduced in 1988 another class called *nearly graceful labelings*, where the range of the vertices and edges is allowed to go up to $|E| + 1$, but with restriction that $|E|$ and $|E| + 1$ cannot both be an edge label. Year later Moulton introduced a modification of such labeling, where $|E|$ cannot be used as a vertex label, he called such labeling *almost-graceful*.

Slater and Maheo with Thullier independently introduced *k-graceful* labeling scheme

Definition 2.7. *k-graceful labeling* f is labeling such that

- f is injection into $\{0, 1, \dots, |E| + k - 1\}$
- edge labels are shifted to $\{k, \dots, |E| + k - 1\}$

Every graph with bipartite graceful labeling is *k-graceful* for all k , since we can add any constant we want to labels from partition with higher label values as we have done by generating graceful labelings from two graphs, where at least one of them is bipartite. But there are graphs which are *k-graceful* for all k and do not have a bipartite graceful labeling.

Definition 2.8. (k,d) -graceful labeling f is labeling such that

- f vertex labels are in range $\{0, 1, \dots, (|E| - 1)d + k\}$
- edge labels take values $\{k, k + d, k + 2d, \dots, (|E| - 1)d\}$

Theorem 2.1. • Every *k-graceful* graph is (kd, d) -graceful.

- Every connected (kd, d) -graceful graph is *k-graceful*.
- If graph is (k, d) -graceful and not bipartite then $k \leq (m - 2)d$

Rosa and Širáň [19] introduced the term *gracesize*. In this relaxation they no longer ask for existence of graceful labeling, but they are interested in question of bounds. So they are interested in the maximum number of distinct edge labels.

Definition 2.9. Gracesize $gs(G)$ of graph G is defined as

$$gs(G) = \max |g(E(G))|$$

over all vertex labelings $f : V(G) \rightarrow \{0, \dots, |E(G)|\}$, where g is the induced edge labeling $g(uv) = |f(u) - f(v)|$. α -size is defined similarly, except in that, we are confined to bipartite labelings of G .

Theorem 2.2. For every tree T with m edges $\alpha(T)$, the α -size of T , satisfies:

$$\alpha(T) \geq \frac{5}{7}(m + 1)$$

This value is naturally also a lower bound on $gs(T)$, so we know that all trees are at least $\frac{5}{7}$ graceful.

As we know, there are trees without bipartite graceful labelings, so there are trees with α -size strictly less than $|E|$. An upper bound on the alpha size of an arbitrary tree is:

Theorem 2.3. Define $\alpha(m)$ as the minimum of all $\alpha(T)$ over all trees with m edges. We then have

$$\alpha(m) \leq \frac{5m + 9}{6}$$

This is obtained through an α -optimal labeling of the $\frac{m}{2}$ -comet. It is believed that comets are the worst behaved trees in terms of bipartite graceful labelings, so Rosa and Širáň conjectured that $\frac{5}{6}m$ is asymptotically correct value of $\alpha(m)$. This was later verified by Bonnington and Širáň [5] for all trees on $n \leq 12$ vertices and maximum degree 3.

Chapter 3

Classes of trees known to be graceful

Definition 3.1. Caterpillar tree T is tree which consists of path P_n and vertices not on P_n , each joined to exactly one vertex on P_n .

Theorem 3.1. Every caterpillar is graceful.

Proof. (due to Rosa [18]) There is a algorithm for labeling which could be used for all caterpillars. For labeling a special case of caterpillars – paths – one could label vertices on this path by alternatively using largest and smallest label possible. Similar approach is applicable for any caterpillar. One has to take a vertex which is beginning of longest path in the tree and start labeling vertices along this path. For every vertex on this path one has to use largest(smallest) label possible, then label all adjacent vertices not lying on that path with consecutive available labels of opposite size and move to next vertex, which would be labeled with smallest(largest) label possible. \square

Maheo in 1980 defined that graph is strongly graceful, if it possesses a bipartite graceful labeling with the extra condition that for every vertex v the labels of all incident edges form a sequence of consecutive integers. Today is strongly graceful labeling denoting to other labeling, so this paragraph is the only place where we will use term strongly graceful in this sense, in all other parts of this work we will use it as it was defined by Broersma and Hoede. As we have seen labeling from proof of last theorem is strongly graceful, so every caterpillar is strongly graceful. Bodendiek and Schumacher have shown, that caterpillars are the only trees with this property.

Definition 3.2. Let T be a caterpillar and let $P_n = v_1, v_2, \dots, v_n$ be a longest path in T . We would call vertices v_1 and v_n tail and head and other vertices of path P_n feet. If every foot has the same degree $t + 2$, T is called a t -toed caterpillar.

Chung and Hwang used in their work [13] modification of construction from previous theorem to show next property:

Theorem 3.2. Every t -toed caterpillar is zero-rotatable.

Rosa showed in [17] other interesting property for paths:

Theorem 3.3. There exists a bipartite labeling of any path in which any vertex may be labeled 0 if and only if the vertex is not the central vertex of P_5 .

Definition 3.3. A lobster T is a tree consisting of path P_n and vertices not on P_n at distance at most two from a vertex in P_n .

Even though we could get a caterpillar from any lobster simply by deleting all its leaves, there is very little known about gracefulness of class of lobsters, up to some special cases. Wang et al. 1994 [24] proved the following:

Theorem 3.4. Let $T_i, i = 1, 2, \dots, 2n+1$ be stars where the degrees of central vertices are either all positive even numbers or all positive odd numbers. For positive integers a_0, \dots, a_m with sum n , select any $2a_0 + 1, 2a_1, \dots, 2a_m$ stars and identify a leaf of each with the $m + 1$ vertices of path v_0, \dots, v_m . The lobster obtained in this way is graceful.

Caro et al. [7] proved the following:

Theorem 3.5. All lobsters have a ρ -labeling.

Huang, Kotzig and Rosa studied gracefulness of trees with at most 4 leaves and trees not admitting bipartite labeling in 1982 paper [?], we will present here some of their results:

Definition 3.4. Let T be a tree and v a vertex of T . A branch vertex of T is a vertex of degree at least 3 in T . A v -endpath of T is a path P from v to a leaf of T such that each internal vertex of P has a degree two in T . A spider $S(a_1, \dots, a_r)$ is a tree with exactly one branch vertex v and v -endpaths of lengths $1 \leq a_1 \leq \dots \leq a_r$ where $r = \deg v$. These v -endpaths will be called legs.

Theorem 3.6. • *The tree $S(p, q, r)$ with 3 leaves has a bipartite labeling if and only if $(p, q, r) \neq (2, 2, 2)$*

- *Every tree $S(p, q, r)$ with 3 leaves has a graceful labeling.*

Trees with exactly four leaves could be divided to two groups. First could be described by using spider notation as $S(p, q, r, s)$. Second group are trees with two branch vertices u and v , both with degree 3. We would describe this trees by using notation $(p, q; r; s, t)$, where the numbers refer to lengths of u -endpaths, distance between u and v and lengths of v -endpaths respectively.

Theorem 3.7. • *If at least two of p, q, r, s do not equal 2, then there exists a bipartite labeling of spider $S(p, q, r, s)$.*

- *Every tree $S(p, q, r, s)$ has a graceful labeling.*
- *Every tree $S(p, q; r; s, t)$ has a graceful labeling.*

Corollary 3.1. *All trees with at most 4 leaves are graceful.*

Theorem 3.8. *Let T_{rs} denote to tree of diameter three on $r + s + 2$ vertices with two vertices of degree $s + 1$ and $r + 1$ adjacent to each other and $r + s$ leaves. Let P_{rs} be the tree of diameter six obtained by replacing each edge in T_{rs} with a path of length two. Then next claims holds.*

- *The tree P_{rs} has a bipartite labeling if and only if $|r - s| \leq 1$.*
- *Every tree P_{rs} has a graceful labeling.*

Huang, Kotzig and Rosa proved also next theorem:

Theorem 3.9. *Let T be a tree all of whose vertices are of odd degree. Let T^* be obtained from T by replacing every edge of T by a path of length two. If $|V(T)| \equiv 0 \pmod{4}$, then T^* does not have a bipartite labeling.*

Theorem 3.10. *Let T be a tree with diameter 4 and T is not a caterpillar nor path, then T has no bipartite labeling.*

Theorem 3.11. *Let $T(q_1, q_2, q_3; s)$ be the tree rooted at its centre x , where x is adjacent to s leaves and to 3 vertices having degrees $q_i + 1$, where each of these are adjacent only to x and leaves of T . In this case, the following holds:*

For any $q_1, q_2, q_3 \geq 1$ there exists a graceful labeling f of $T(q_1, q_2, q_3; s)$ with $f(x) = 0$

Kotzig [14] showed, that we are able to guarantee gracefulness by next two modifications of any tree.

Theorem 3.12. • *If a leaf of a long-enough path is joined any leaf of an arbitrary tree, the resulting tree is graceful.*

- *If a long-enough path replaces an arbitrary edge in an arbitrary tree, the resulting tree is graceful.*

Theorem 3.13. *Every tree of diameter 5 is graceful*

Sketch of proof of this theorem could be found in survey work from Edwards and Howard [8]. Original proof is in Hrniar and Haviar 2001 [11].

Definition 3.5. *A bamboo tree is a rooted tree consisting of branches of equal length, the leaves of which are identified with leaves of stars of stars of equal size.*

Theorem 3.14. *All bamboo trees are graceful.*

Proof could be found in Sekar 2002 [21].

Definition 3.6. *An olive tree T_k is a rooted tree consisting of k branches, the i th branch is a path of length i . This is in spider notation spider $S(1, \dots, k)$.*

Theorem 3.15. *All olive trees are graceful.*

Proof. (sketch, due to Abhyankar, Bhat-Nayak [1]) The tree T_k is labeled according to parity of k by labeling branch vertex first, with $q = (n+1)(2n+1)$ for tree T_{2n+1} or with label $q = n$ for T_{2n} . In second step are labeled vertices adjacent to the branch vertex according to parity. In final step are labeled remaining vertices such that sum of any two adjacent vertices is either $q-1$ or q for first case, or q or $q+1$ for second case. \square

Definition 3.7. *Symmetrical tree T is a rooted tree in which every level contains vertices of the same degree.*

Theorem 3.16. *Every symmetrical tree is graceful.*

This theorem was proven by Stanton and Zarnke [20] with modified version of construction from theorem 5.1.

Theorem 3.17. *All trees with number of vertices less or equal 27 are graceful.*

This was proved by Aldred and McKay [2] in 1998. They used algorithm, which begins with random permutation of labels $\{0, 1, \dots, |E|\}$ and switches such pairs of vertex labels which increased cardinality of the edge labels set. If there wasn't any such pair, then they used other initial permutation. They used some other heuristics to obtain graceful labeling very fast. Time consumption for one tree needed to be very low, because the number of various trees is growing very fast – for example there are 751,065,460 trees of order 27.

Chapter 4

k-rotatability

Definition 4.1. *Labeling f of tree T is k -centred if for centre vertex v , $f(v) = k$. If tree T has odd diameter then either endpoint of the centre edge is acceptable.*

Definition 4.2. *The tree T is k -rotatable (or k -ubiquitously graceful) if for every vertex v there exists a graceful labeling of T such that $f(v) = k$. If tree T is k -rotatable for all $k \in \{0 \dots |E|\}$ then T is ubiquitously graceful.*

We could similarly define k -edge-centred labelings and k -edge-rotatable labelings.

Non-zero-rotatability seems to be the major stumbling block in the attempt to prove graceful labeling conjecture. If any vertex of any tree could be labeled 0 under a graceful labeling, it would be possible to generate graceful labelings for arbitrary tree by using simple inductive step – adding new vertex to previously labeled subtree, label the new vertex it with $|E|$ and connect it with vertex with label 0. Similar problems to non-zero-rotatability arise when trying to solve this problem with other approaches. Smallest non-zero-rotatable tree is tree which could be obtained from $P_5 = v_1, v_2, v_3, v_4, v_5$ by adding an edge v_2v_6 . Vertex which couldn't be labeled zero in this graph is then v_3 .

Frank van Bussel made some progress in specification of trees, which are zero-rotatable and zero-centred, we will show major results of his work in next few paragraphs. We will define some new notation introduced by van Bussel.

Definition 4.3. *Branch A of diameter 4 tree T is a vertex v adjacent to*

centre vertex of T with any number of leaves adjacent to v . $|A|$ is defined as number of leaves adjacent to v .

Theorem 4.1. *Let T be a tree of diameter 4 having 2 branches A and B , with $|A| \geq |B|$. T has a zero-centred graceful labeling if and only if there exists integers x and r such that*

$$|A| = (|B| + 2 - x)(r - 1) - x$$

where

$$2 \leq r \leq \frac{|E(T)|}{2}$$

$$0 \leq x \leq \min(r - 1, |B|)$$

x is even if r is odd; its parity is not constrained otherwise.

Definition 4.4. *Class of diameter 4 trees with centre degree 2 that fail the conditions imposed by previous theorem will be denoted \mathcal{D} . Class \mathcal{D}' is class of trees which are obtained by identifying an end vertex of an arbitrary path with the centre of a tree in \mathcal{D} .*

Theorem 4.2. *Let T be a tree with diameter less or equal 4. If $T \notin \mathcal{D}$ then T has a zero-centred graceful labeling.*

Theorem 4.3. *Let T be a tree with diameter less or equal 4:*

- *If T is not in \mathcal{D}' , it is zero-rotatable.*
- *If T is in \mathcal{D}' , it can be gracefully labeled with 0 assigned to any vertex but one (up to automorphism).*

Proofs of these theorems are too long to reproduce here, reader can found them in [22] and [23].

Because of an empirical evidence, van Bussel postulates next two conjectures.

Conjecture 4.1. *The class \mathcal{D} contains all trees which are not zero-centred graceful.*

Conjecture 4.2. *The class \mathcal{D}' contains all trees which are not zero-rotatable.*

Non-zero-rotatability surprisingly seems to be very rare. For example there are just 14 non-zero-rotatable trees of order 14 and 7 non-zero-centred trees of order 18 (and no such tree in class of trees with order 17).

Chapter 5

Constructions of larger graceful trees

One possible way of constructing larger graceful trees from smaller is by using various tree products.

Definition 5.1. *One of basic tree product could be defined as follows – we have trees S and T , where $|E(S)| = n_S$ and $|E(T)| = n_T$, we distinguish vertex $v \in T$ and we obtain product of S and T by making n_S copies of T and identifying every vertex v_i of S with vertex v in one of these trees.*

This construction could be gracefully labeled as was shown by Stanton and Zarnke [20] in following theorem:

Theorem 5.1. *Let S and T be trees with n_S and n_T vertices respectively, and let v^* be a distinguished vertex in T . Let S' denote the tree obtained by replacing each distinct vertex u_i of S with a copy of T rooted v^* . The individual copy of T replacing u_i will be denoted as $T_i, i = 1, \dots, n_S$, with the vertex v of T copied to the vertex v_i of T_i . If S and T both possess graceful labelings f_S and f_T respectively, then the tree S' has a graceful labeling f' given by:*

$$f'(v_i) = \begin{cases} (n_S - f_S(u_i) + 1)n_T + f_T(v), & \text{dist}(v, v^*) \text{ in } T \text{ is odd} \\ f_S(u_i)n_T + f_T(v), & \text{dist}(v, v^*) \text{ in } T \text{ is even} \end{cases}$$

Modification of this construction allows us to create a graceful labeling for tree product of trees S and T , where we replace all vertices of S but one,

with a copy of T rooted at v^* . We could label such tree if S and T fulfill condition, that they both have graceful labelings f_S and f_T respectively, such that f_S assigns 0 to u' . By using this modified labeling, we are able to find graceful labeling for symmetrical trees defined in 3.7

Next construction is called *canonical amalgamation construction* and allows us to generate larger graceful trees by connecting two graceful trees:

Theorem 5.2. *Let T_1 and T_2 be vertex disjoint trees with distinguished vertices v_1 and v_2 respectively, let (A, B) , $v_1 \in A$, be the bipartition of T_1 and let $T_1 \circ T_2$ denote the amalgamation of T_1 and T_2 at (v_1, v_2) . Assume T_1 has a bipartite graceful labelling f_1 with $f_1(v_1) = 0$ and that T_2 has a graceful labeling f_2 with $f_2(v_2) = 0$. Let g_1 denote to reverse labeling to f_1 . We then have a graceful labeling f of $T_1 \circ T_2$ defined by:*

$$f(v) = \begin{cases} g_1(v), & v \in A \subset V(T_1) \\ f_2(v) + |A| - 1, & v \in V(T_2) \\ g_1(v) + |V(T_2) - 1|. & v \in B \subset V(T_1) \end{cases}$$

The labels on vertices from T_2 are just shifted by a constant, so the edge labels on T_2 remain unchanged. Edge labels on T_1 are stretched, the reverse labeling is used in order to have the same value on vertices v_1 and v_2 .

If both trees are bipartite, than the resultant tree is also bipartite.

This construction could be used to form larger tree from two trees with disjoint vertices by using it first with one of these trees and K_1 and then by adding the second tree to this new tree.

Chapter 6

Possible fields of study

Kotzig attempts in his work [15] to characterize the relationship between gracefully labeled trees of same size. He made a construction of super-graph G whose vertices are gracefully labeled trees and edge between two vertices $t_1, t_2 \in G$ exists if tree represented by t_2 could be obtained by changing the endpoints of some edge in tree represented by t_1 . The edges in G are labeled by the value of exchanged edge. Although Kotzig didn't prove that we could get any gracefully labeled tree from another graceful tree of same size with series of edge exchanges, he was able to show that a graceful tree could always exchange one edge of value 1 for another with the result remaining a graceful tree.

Broersma and Hoede [6] proposed variation of graceful labeling called strongly graceful defined as follows:

Definition 6.1. *If T is a tree on m vertices which has a perfect matching M , then the labeling f is strongly graceful if:*

- f is graceful
- for every edge $uv \in M$, $f(u) + f(v) = m$

They showed that Graceful tree conjecture is equivalent to the conjecture, that every tree with a perfect matching has strongly graceful labeling. They also showed several simple transformations and a method of creating larger strongly graceful trees out of smaller ones. They were mainly interested in two types of trees – *spiketrees* and *contrees*, where spiketree is a tree with

perfect matching obtained from an arbitrary smaller tree by adding a pendant edge to each vertex and a contree is a tree obtained from larger tree with perfect matching M by contracting every edge in M . Authors admit that that new problem is smaller in scope but much harder. But the potential of this approach still wasn't fully explored and the more predictable structure of this assumptions, make some interesting recursive constructions possible.

Following conjecture comes from Gvozdiak's PhD thesis [10]. It is of concern of graceful labeling of path P_n . He is using notation, that if $P_n = v_0, v_1, \dots, v_n$, f is graceful labeling and $f(0) = a, f(n) = b$, then such labeling is denoted as $(a, b; n)$

Conjecture 6.1. *An $(a, b; n)$ -graceful labeling exists if and only if the integers a, b, n satisfy this conditions:*

- $b - a$ has the same parity as $1 + 2 + \dots + n = n(n + 1)/2$
- $0 < |b - a| \leq n/2 \leq a + b \leq 3n/2$

Both conditions are necessary, but it is still not proved if they are also sufficient. Computed experiments showed, that this conjecture holds at least for all paths with $n < 22$.

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