Finding Roots

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1 Convergence Rate for Secant Method

Below we derive the convergence rate for the secant method. We start with our scheme

$$x^{(n+1)} = x^{(n)} - f(x^{(n)}) \frac{x^{(n)} - x^{(n-1)}}{f(x^{(n)}) - f(x^{(n-1)})}$$

We now substitute $x^{(n)} = \epsilon^{(n)} + r$, where $\epsilon^{(n)}$ is the error of the nth iteration and r is the real root.

$$\epsilon^{(n+1)} + r = \epsilon^{(n)} + r - f(\epsilon^{(n)} + r) \frac{\epsilon^{(n)} + r - \epsilon^{(n-1)} - r}{f(\epsilon^{(n)} + r) - f(\epsilon^{(n-1)} + r)}$$

$$\epsilon^{(n+1)} = \epsilon^{(n)} - f(\epsilon^{(n)} + r) \frac{\epsilon^{(n)} - \epsilon^{(n-1)}}{f(\epsilon^{(n)} + r) - f(\epsilon^{(n-1)} + r)}$$

We can expand the function f in terms of its Taylor expansion about $x = r$, and use the fact that $f(r) = 0$ to simplify.

$$f(\epsilon^{(n)} + r) \approx \epsilon^{(n)} f'(r) + \frac{{\epsilon^{(n)}}^2}{2} f''(r)$$

$$f(\epsilon^{(n-1)} + r) \approx \epsilon^{(n-1)} f'(r) + \frac{{\epsilon^{(n-1)}}^2}{2} f''(r)$$

Plugging this back into our equation gives

$$\epsilon^{(n+1)} = \epsilon^{(n)} - (\epsilon^{(n)} f'(r) + \frac{{\epsilon^{(n)}}^2}{2} f''(r)) \frac{\epsilon^{(n)} - \epsilon^{(n-1)}}{\epsilon^{(n)} f'(r) + \frac{{\epsilon^{(n-1)}}^2}{f}''(r) - \epsilon^{(n-1)} f'(r) + \frac{\epsilon^{(n-1)2}}{2} f''(r)}$$

$$= \epsilon^{(n)} - \frac{\epsilon^{(n)}(1 + \epsilon^{(n)} \frac{f''(r)}{2f'(r)})}{1 + (\epsilon^{(n)} + \epsilon^{(n-1)}) \frac{f''(r)}{2f'(r)}}$$

$$= \epsilon^{(n)} - \epsilon^{(n)}(1 + \epsilon^{(n)} \frac{f''(r)}{2f'(r)})[1 - (\epsilon^{(n)} + \epsilon^{(n-1)}) \frac{f''(r)}{2f'(r)}]$$

$$= \epsilon^{(n)} - \epsilon^{(n)} + {\epsilon^{(n)}}^2 \frac{f''(r)}{2f'(r)} + \epsilon^{(n)}\epsilon^{(n-1)} \frac{f''(r)}{2f'(r)} - {\epsilon^{(n)}}^2 \frac{f''(r)}{2f'(r)} + {\epsilon^{(n)}}^3 \frac{f''(r)}{2f'(r)}^2 + {\epsilon^{(n)}}^2 \epsilon^{(n-1)} \frac{f''(r)}{2f'(r)}^2$$

$$\approx \epsilon^{(n)}\epsilon^{(n-1)} \frac{f''(r)}{2f'(r)}$$

Now we assume that $\epsilon^{(n+1)}$ is of the form $\epsilon^{(n+1)} = C{\epsilon^{(n)}}^R \; \forall n$, where C and R are constants independent of n. This gives us the following

$$\epsilon^{(n+1)} = C{\epsilon^{(n)}}^R \approx \epsilon^{(n)}\epsilon^{(n-1)} \frac{f''(r)}{2f'(r)}$$

$${\epsilon^{(n)}}^{R-1} = \frac{f''(r)}{2Cf'(r)} \epsilon^{(n-1)}$$

$$\epsilon^{(n)} = \frac{f''(r)}{2Cf'(r)}^{\frac{1}{R-1}} {\epsilon^{(n-1)}}^{\frac{1}{R-1}}$$

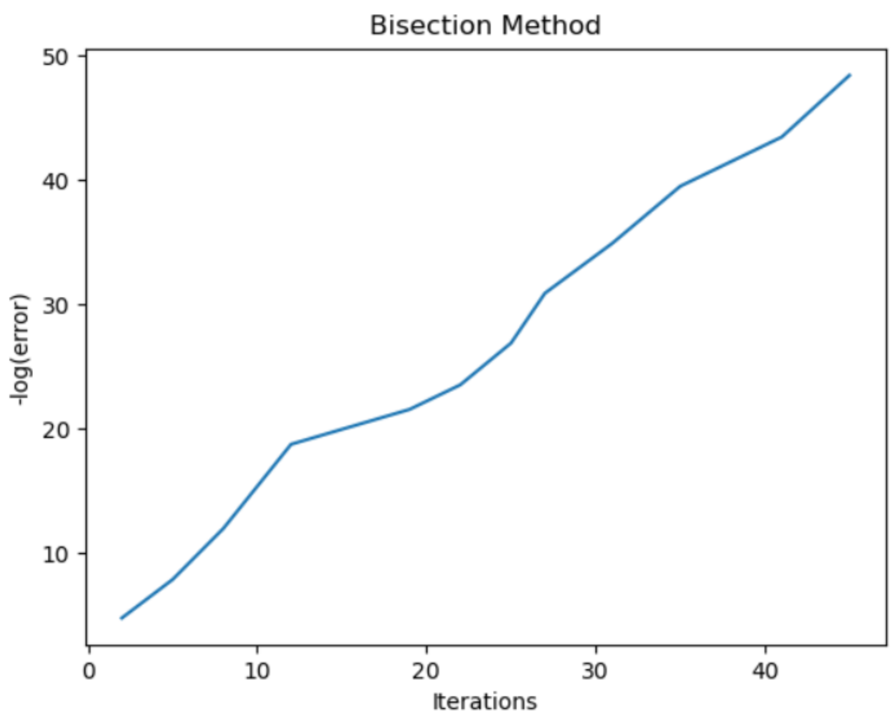
This gives us $C = (\frac{f''(r)}{2Cf'(r)})^{\frac{1}{R-1}}$ and $R = \frac{1}{R-1}$. Solving for R and C gives us the following

$$R = \frac{1+\sqrt{5}}{2} = \phi$$

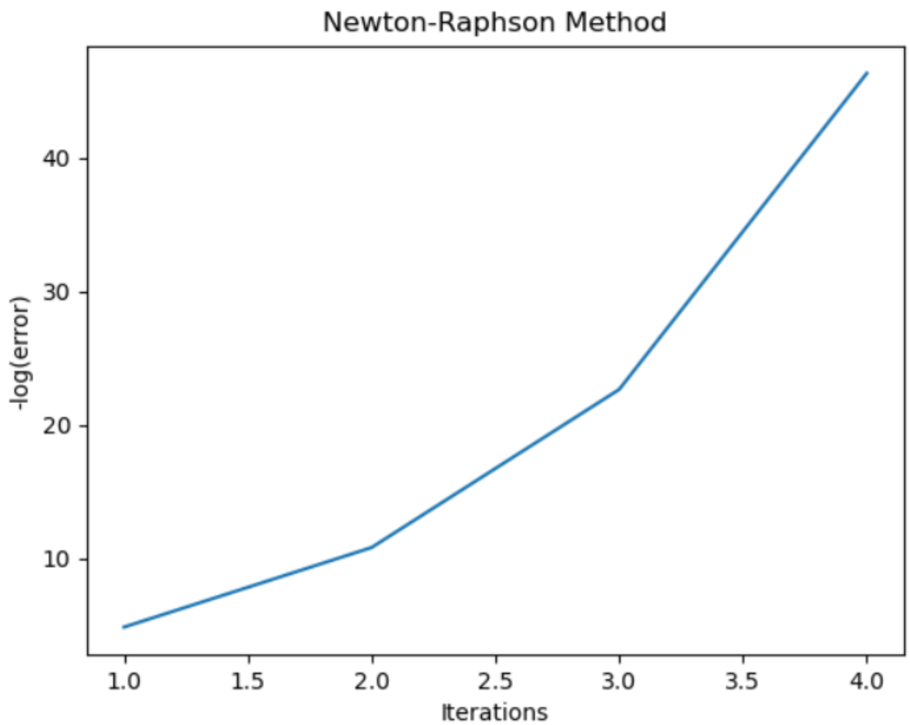
$$C = (\frac{f''(r)}{2f'(r)})^{\frac{1}{R}}$$

2 Visualization of Convergence Rates

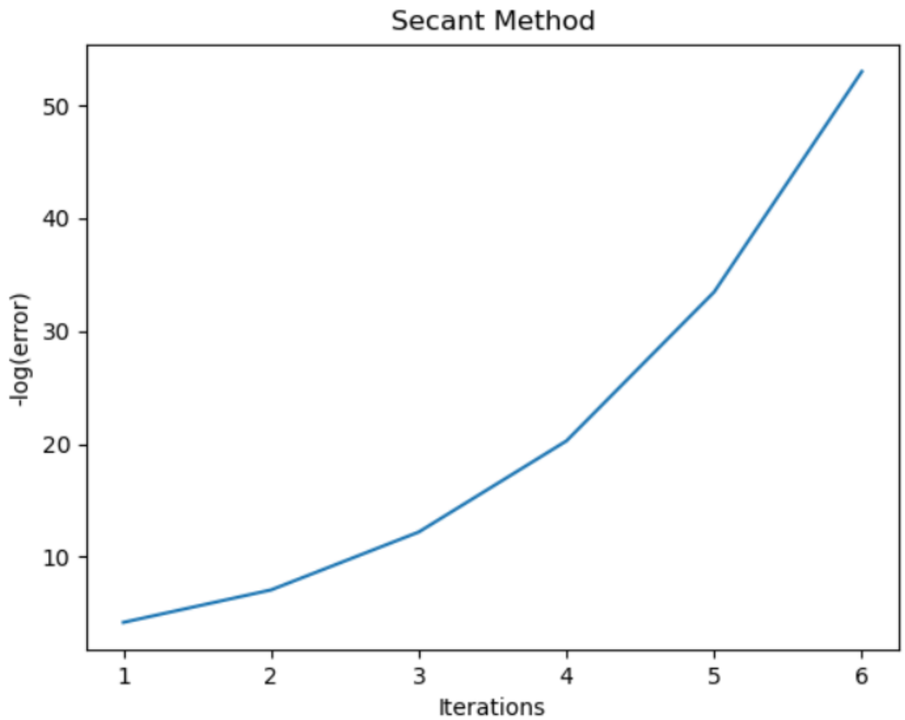
Attached in the Git repository is file "RootFinding.py" which implements the three root finding methods: the bisection method, the Newton-Raphson method, and the secant method. Since the bisection method only provided a bracket for the root, a modified version of the bisection method was implented which returns a float representing the determined root rather than a tuple representing the final bracket.
Below are the plots for the visualization of the convergence rates of the three methods. Plotted is $-log_2(|r - \frac{\pi}{4}|)$ vs N, where r is the determined root of the method, $\frac{pi}{4}$ is the root of the equation $f_{\frac{\sqrt{2}}{2}}(x) = sin(x) - \frac{\sqrt{2}}{2}$ which the methods are implemented on, and N is the number of iterations corresponding to the value of $-log_2(|r - \frac{\pi}{4}|)$.



Linear relationship between N and $-log_2(|error|)$ as seen in figure.



Quadratic relationship between N and $-log_2(|error|)$ as seen in figure.



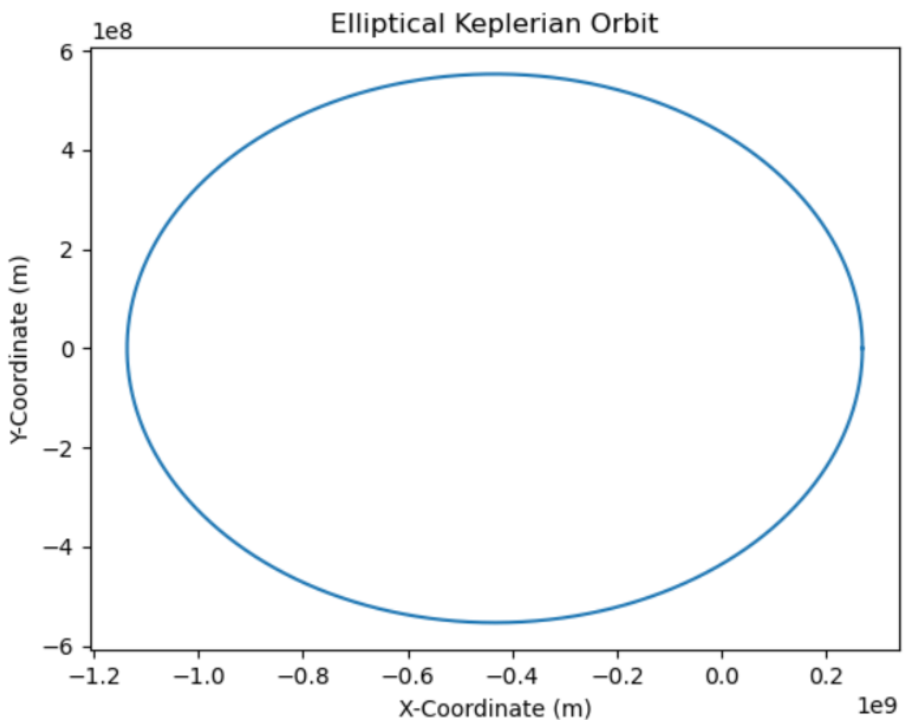
$-log_2(|r - \frac{\pi}{4}|) \approx CN^{\phi}$ for some constant C and $\phi = \frac{1+\sqrt{5}}{2}$ as seen in figure.

3 Elliptical Keplerian Orbits

Elliptical orbit of two body system was computed using the Newton-Raphson method to compute the roots of ξ which correspond to the time dependent equation below. Then, using this numerically determined value of ξ the x and y values at that specified time can be computed from the equations below.

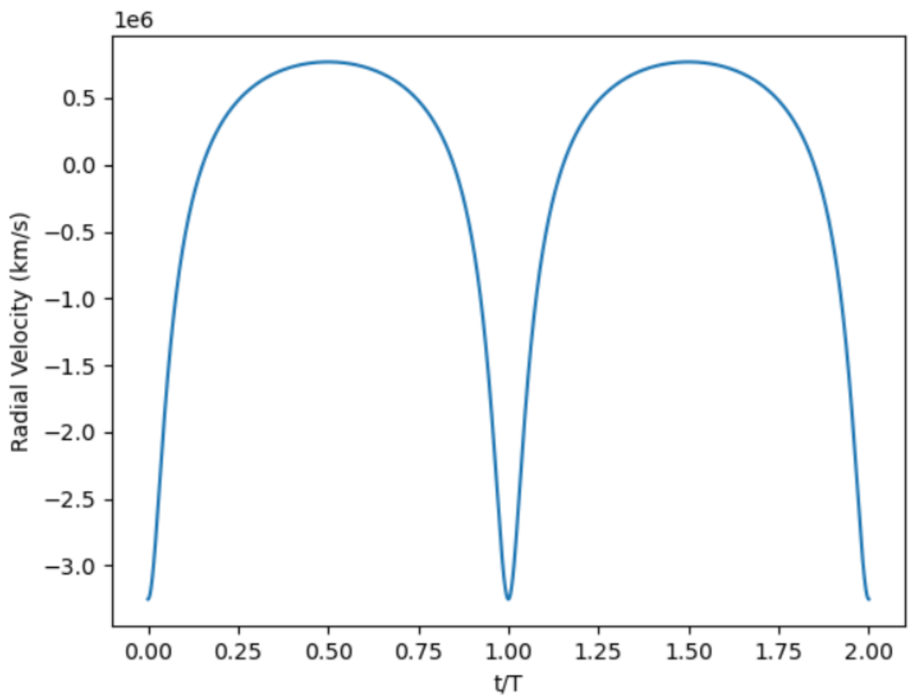
$$\frac{T}{2\pi}(\xi - esin\xi) - t* = 0x = a(cos\xi - e); y = a\sqrt{1-e^2}sin\xi$$

With values $e = 0.617139$, $T = 27906.98161s$, and $a = a = 2.34186sxc$, where $c = 3\text{x}10^8\frac{m}{s}$.



4 Radial Velocity Plot

We see if our calculated orbit agrees qualitatively with Figure 3 provided in the assignment. We does by plotting the radial velocity ($\frac{km}{s}$) versus time ($\frac{t}{T}$) and seeing if it agrees qualitatively with the figure. The value of ϕ determined which provides a the most accurate plot is $\phi = \frac{\pi}{2}$. Below is the plotted radial velocity over two periods.



This plot qualitatively agrees with Figure 3, except for the arbitrary phase shift associated with Figure 3.