

Finding Roots

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April 2020

1 Convergence Rate for Secant Method

Below we derive the convergence rate for the secant method. We start with our scheme

$$x^{(n+1)} = x^{(n)} - f(x^{(n)}) \frac{x^{(n)} - x^{(n-1)}}{f(x^{(n)}) - f(x^{(n-1)})}$$

We now substitute $x^{(n)} = \epsilon^{(n)} + r$, where $\epsilon^{(n)}$ is the error of the nth iteration and r is the real root.

$$\epsilon^{(n+1)} + r = \epsilon^{(n)} + r - f(\epsilon^{(n)} + r) \frac{\epsilon^{(n)} + r - \epsilon^{(n-1)} - r}{f(\epsilon^{(n)} + r) - f(\epsilon^{(n-1)} + r)}$$

$$\epsilon^{(n+1)} = \epsilon^{(n)} - f(\epsilon^{(n)} + r) \frac{\epsilon^{(n)} - \epsilon^{(n-1)}}{f(\epsilon^{(n)} + r) - f(\epsilon^{(n-1)} + r)}$$

We can expand the function f in terms of its Taylor expansion about $x = r$, and use the fact that $f(r) = 0$ to simplify.

$$f(\epsilon^{(n)} + r) \approx \epsilon^{(n)} f'(r) + \frac{\epsilon^{(n)2}}{2} f''(r)$$

$$f(\epsilon^{(n-1)} + r) \approx \epsilon^{(n-1)} f'(r) + \frac{\epsilon^{(n-1)2}}{2} f''(r)$$

Plugging this back into our equation gives

$$\begin{aligned} \epsilon^{(n+1)} &= \epsilon^{(n)} - \left(\epsilon^{(n)} f'(r) + \frac{\epsilon^{(n)2}}{2} f''(r) \right) \frac{\epsilon^{(n)} - \epsilon^{(n-1)}}{\epsilon^{(n)} f'(r) + \frac{\epsilon^{(n-1)2}}{2} f''(r) - \epsilon^{(n-1)} f'(r) + \frac{\epsilon^{(n-1)2}}{2} f''(r)} \\ &= \epsilon^{(n)} - \frac{\epsilon^{(n)} \left(1 + \epsilon^{(n)} \frac{f''(r)}{2f'(r)} \right)}{1 + (\epsilon^{(n)} + \epsilon^{(n-1)}) \frac{f''(r)}{2f'(r)}} \\ &= \epsilon^{(n)} - \epsilon^{(n)} \left(1 + \epsilon^{(n)} \frac{f''(r)}{2f'(r)} \right) \left[1 - (\epsilon^{(n)} + \epsilon^{(n-1)}) \frac{f''(r)}{2f'(r)} \right] \end{aligned}$$

$$\begin{aligned}
&= \epsilon^{(n)} - \epsilon^{(n)} + \epsilon^{(n)2} \frac{f''(r)}{2f'(r)} + \epsilon^{(n)} \epsilon^{(n-1)} \frac{f''(r)}{2f'(r)} - \epsilon^{(n)2} \frac{f''(r)}{2f'(r)} + \epsilon^{(n)3} \frac{f''(r)^2}{2f'(r)} + \epsilon^{(n)2} \epsilon^{(n-1)} \frac{f''(r)^2}{2f'(r)} \\
&\approx \epsilon^{(n)} \epsilon^{(n-1)} \frac{f''(r)}{2f'(r)}
\end{aligned}$$

Now we assume that $\epsilon^{(n+1)}$ is of the form $\epsilon^{(n+1)} = C\epsilon^{(n)R} \forall n$, where C and R are constants independent of n . This gives us the following

$$\epsilon^{(n+1)} = C\epsilon^{(n)R} \approx \epsilon^{(n)} \epsilon^{(n-1)} \frac{f''(r)}{2f'(r)}$$

$$\epsilon^{(n)R-1} = \frac{f''(r)}{2Cf'(r)} \epsilon^{(n-1)}$$

$$\epsilon^{(n)} = \frac{f''(r)^{\frac{1}{R-1}}}{2Cf'(r)} \epsilon^{(n-1)^{\frac{1}{R-1}}}$$

This gives us $C = \left(\frac{f''(r)}{2Cf'(r)}\right)^{\frac{1}{R-1}}$ and $R = \frac{1}{R-1}$. Solving for R and C gives us the following

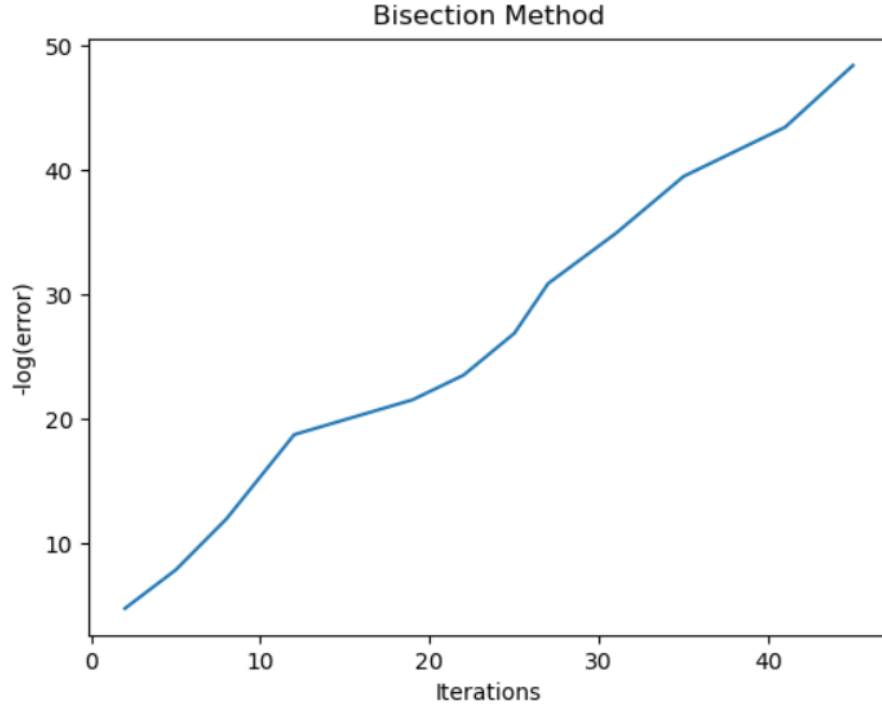
$$R = \frac{1 + \sqrt{5}}{2} = \phi$$

$$C = \left(\frac{f''(r)}{2f'(r)}\right)^{\frac{1}{R}}$$

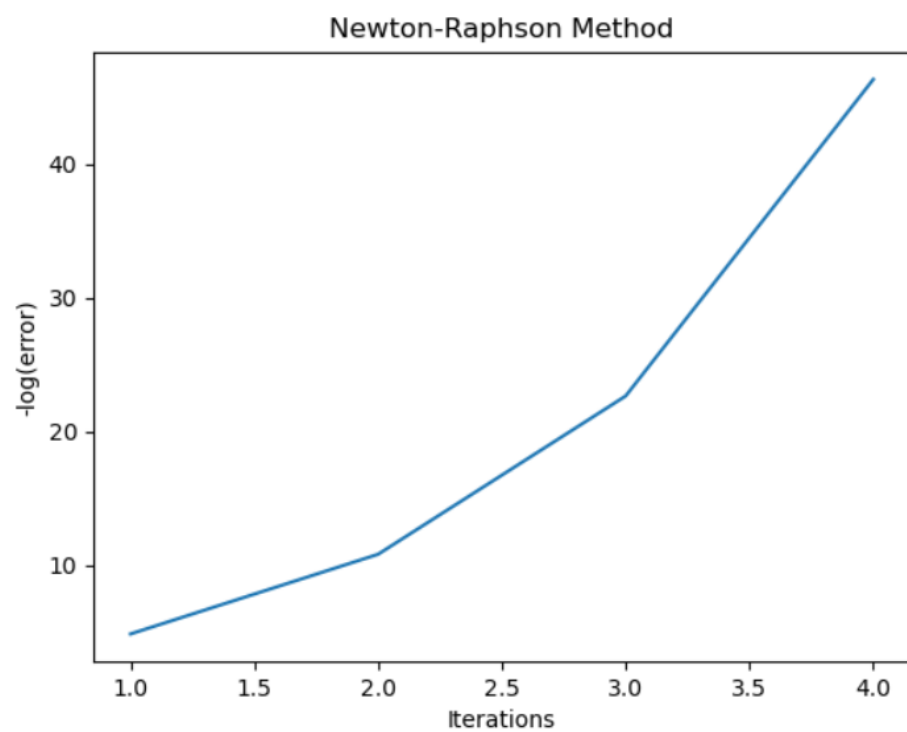
2 Visualization of Convergence Rates

Attached in the Git repository is file "RootFinding.py" which implements the three root finding methods: the bisection method, the Newton-Raphson method, and the secant method. Since the bisection method only provided a bracket for the root, a modified version of the bisection method was implented which returns a float representing the determined root rather than a tuple representing the final bracket.

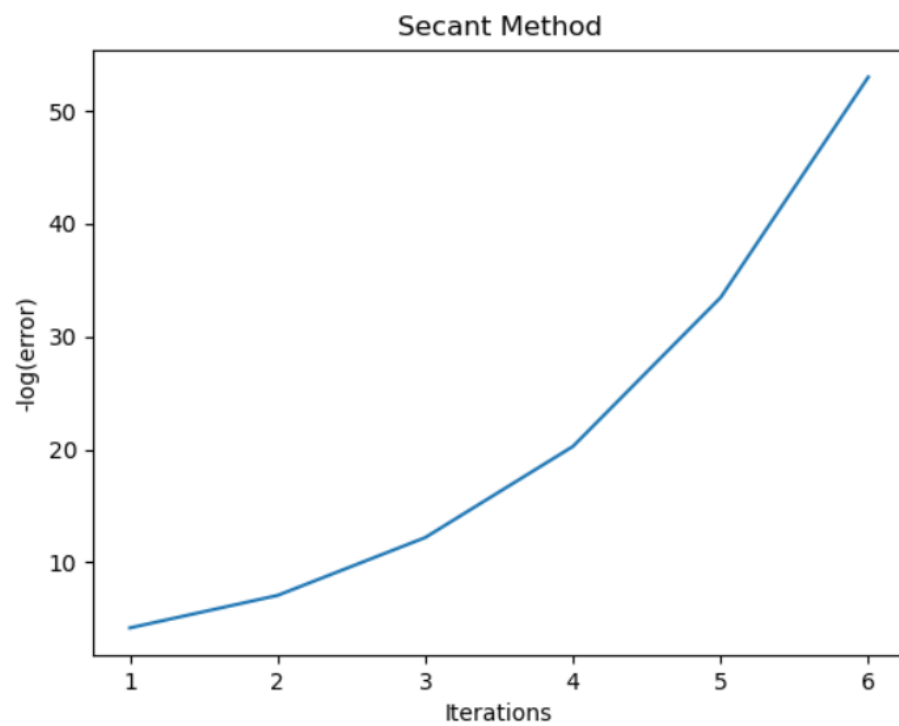
Below are the plots for the visualization of the convergence rates of the three methods. Plotted is $-\log_2(|r - \frac{\pi}{4}|)$ vs N , where r is the determined root of the method, $\frac{\pi}{4}$ is the root of the equation $f_{\frac{\sqrt{2}}{2}}(x) = \sin(x) - \frac{\sqrt{2}}{2}$ which the methods are implemented on, and N is the number of iterations corresponding to the value of $-\log_2(|r - \frac{\pi}{4}|)$.



Linear relationship between N and $-\log_2(|\text{error}|)$ as seen in figure.



Quadratic relationship between N and $-\log_2(|error|)$ as seen in figure.



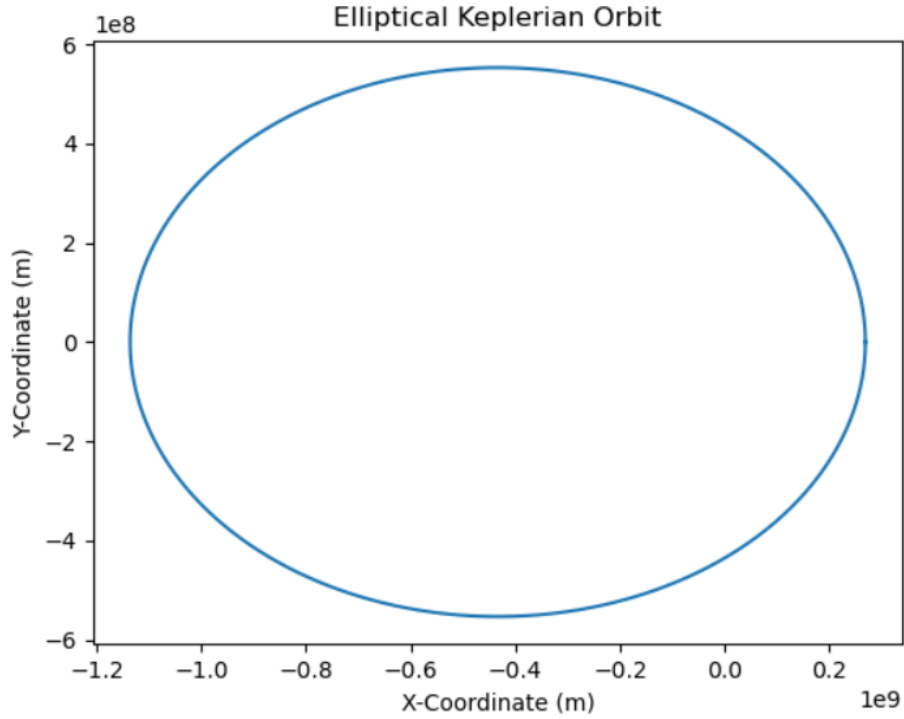
$-\log_2(|r - \frac{\pi}{4}|) \approx CN^\phi$ for some constant C and $\phi = \frac{1+\sqrt{5}}{2}$ as seen in figure.

3 Elliptical Keplerian Orbits

Elliptical orbit of two body system was computed using the Newton-Raphson method to compute the roots of ξ which correspond to the time dependent equation below. Then, using this numerically determined value of ξ the x and y values at that specified time can be computed from the equations below.

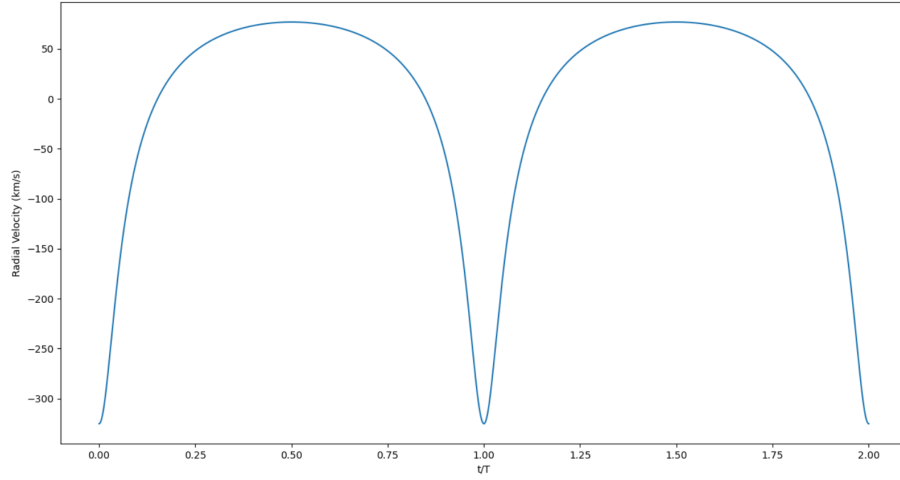
$$\frac{T}{2\pi}(\xi - e \sin \xi) - t^* = 0 \quad x = a(\cos \xi - e); y = a\sqrt{1 - e^2} \sin \xi$$

With values $e = 0.617139$, $T = 27906.98161s$, and $a = 2.34186 \text{ sxc}$, where $c = 3 \times 10^8 \frac{m}{s}$.



4 Radial Velocity Plot

We see if our calculated orbit agrees qualitatively with Figure 3 provided in the assignment. We do this by plotting the radial velocity ($\frac{km}{s}$) versus time ($\frac{t}{T}$) and seeing if it agrees qualitatively with the figure. The value of ϕ determined which provides the most accurate plot is $\phi = \frac{\pi}{2}$. Below is the plotted radial velocity over two periods.



This plot qualitatively agrees with Figure 3, except for the arbitrary phase shift associated with Figure 3.