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1) ans

given

$$y = \sin(m \sin^{-1}x) \quad \text{--- (1)}$$

Diff. (1) wrt x, we get,

$$y_1 = \cos(m \sin^{-1}x) \cdot m \cdot \frac{1}{\sqrt{1-x^2}}$$

Sq. gives, $(1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1}x)$

$$= m^2[1-\sin^2(m \sin^{-1}x)]$$
$$= m^2[1-y^2]$$

or, $(1-x^2)y_1^2 - m^2 + m^2y^2 = 0$

Again, Diff. wrt x, we get,

$$(1-x^2)2y_1y_2 + (-2x)y_1^2 + 2m^2yy_1 = 0$$
$$2y_1[(1-x^2)y_2 - xy_1 + m^2y] = 0$$
$$\therefore (1-x^2)y_2 - xy_1 + m^2y = 0 \quad \text{--- (11)}$$

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Diff. (11) wrt x, up to 'n' times using Leibnitz theorem,

$\Rightarrow [(1-x^2)y_{n+1} + n y_{n+1}(-2x) + \frac{n(n-1)}{2} y_n(-2)] - (x y_{n+1} + n y_n] + m^2 y_n = 0$

$\Rightarrow (1-x^2)y_{n+2} - 2nx y_{n+1} - n^2 y_n + n y_n - x y_{n+1} - n y_n + m^2 y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (m^2-n^2)y_n = 0$

Proved

2) Soln

Statement: If a function $f(x)$ is
a) continuous in $[a,b]$
b) differentiable in (a,b)

then there exist at least one point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof: Let us define a function
$F(x) = f(x)[b-a] - [f(b) - f(a)]x$ —— (1)

which is continuous in (a,b) and differentiable in (a,b).

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Put $x = a$ in eqn ① we get,

$$f(a) = f(a)[b-a] - [f(b) - f(a)]\, a$$

$$= bf(a) - af(a) - af(b) + af(a)$$

$$= bf(a) - af(b)$$

Put $x = b$ in eqn ①, we get

$$f(b) = f(b)[b-a] - [f(b) - f(a)]\, b$$

$$= bf(b) - af(b) - bf(b) + bf(a)$$

$$= bf(a) - af(b)$$

$$\therefore f(a) = f(b)$$

Then, by rolles theorem there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$

$$f'(x) = f'(x)[b-a] - [f(b) - f(a)]$$

$$f'(c) = f'(c)[b-a] - [f(b) - f(a)]$$

$$0 = f'(c)[b-a] - [f(b) - f(a)]$$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$ proved

3) $\lim_{x \to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

Soln given

$\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

since, $\lim_{x \to 0} (1+x)^{1/x} = e$

So, $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}(1+x)^{1/x}}{1}$

$= \lim_{x \to 0} \frac{(1+x)^{1/x} \{x - (1+x)\log(1+x)\}}{x^2(1+x)}$

$= e \cdot \lim_{x \to 0} \frac{x - (1+x)\log(1+x)}{x^2(1+x)}$ $\left(\frac{0}{0} \text{ form}\right)$

$= e \cdot \lim_{x \to 0} \frac{1 - (1+x)\frac{1}{(1+x)} - \log(1+x)}{2x + 3x^2}$

$= e \cdot \lim_{x \to 0} \frac{-\log(1+x)}{(2x + 3x^2)}$ $\left(\frac{0}{0} \text{ form}\right)$

$$= e \cdot \lim_{x\to 0} \frac{\frac{-1}{1+x}}{2+6x} = -e \cdot \frac{1}{2} = -\frac{e}{2}$$

$$\therefore \lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = -\frac{e}{2}$$

4) Soln.

$$(x+y)^2(x+2y+2) = x+9y-2$$

Here the eqn of given curve is,

$$(x+y)^2(x+2y+2) = x+9y-2$$

or, $(x+y)^2(x+2y) + 2(x+y)^2 = x+9y-2$

or, $(x^2+2xy+y^2)(x+2y) + 2(x^2+2xy+y^2) = x+9y-2$

or, $x^3+2y^3+4x^2y + 5xy^2+2x^2+2y^2+4xy-9y+2=0.$

which is 3^{rd} degree eqn. in x and y, so it has at most three asymptotes. Since x^3 and y^3 are both present, there are no horizontal and vertical asymptotes.

For the oblique asymptotes,

let $y = mx + c$ be an asymptote,

for, this, put $x=1$, $y=m$ in 3^{rd}, 2^{nd} and 1^{st} degree terms of the given curve then,

$\phi_3(m) = 1 + 2m^3 + 4m + 5m^2$

$\phi_2(m) = 2 + 2m^2 + 4m$

$\phi_1(m) = -1 - 9m$

$\phi_3'(m) = 6m^2 + 4 + 10m$

$\phi_3''(m) = 12m + 10$

$\phi_2'(m) = 4m + 4$

for the value of m, put $(\phi_3(m)) = 0$

$2m^3 + 5m^2 + 4m + 1 = 0$

or, $2m^3 + 2m^2 + 3m^2 + 3m + m + 1 = 0$

or, $(m+1)(2m^2 + 3m + 1) = 0$

or, $(m+1)(m+1)(2m+1) = 0$

$\therefore m = -1, -2, -1/2$

Also, $c = \dfrac{-\phi_2(m)}{\phi_3'(m)} = \dfrac{-2 + 2m^3 + 4m}{6m^2 + 4 + 10m}$

when, $m = -\frac{1}{2}$, $c = -1$.

Here; $m = -1$ is repeated root and $\phi_3'(-1) = 0$

So, using

$\dfrac{c^2}{2!}\phi_3''(m) + \dfrac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$

or, $\frac{c^2}{2!}(12m+10) + c(4m+4) + (-1-9m) = 0$

when, $m = -1$

$\frac{c^2}{2!} \times -2 + 0 + 8 = 0$

or, $-c^2 + 8 = 0$

$\therefore c = \pm 2\sqrt{2}$

Hence the asymptotes are

$x + 2y + 2 = 0$, $x + y \pm 2\sqrt{2} = 0$.

5) Soln.

Here, $r^2 = a^2 \cos 2\theta$ ——— (1)

Differentiating,

$$2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

or, $\frac{d\theta}{dr} = -\frac{r}{a^2 \sin 2\theta}$

or, $r\frac{d\theta}{dr} = -\frac{r^2}{a^2 \sin 2\theta} = -\frac{a^2}{a^2}\frac{\cos 2\theta}{\sin 2\theta}$

or, $\tan \phi = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$

$\therefore \phi = \frac{\pi}{2} + 2\theta$ ——— (ii)

we have,

$p = r \sin \phi$

Using (ii)

$p = r \sin\left(\frac{\pi}{2} + 2\theta\right)$

or, $p = r \cos 2\theta$

Using (1),

$p = r \cdot \frac{r^2}{a^2}$

$\therefore r^3 = a^2 p$ is the req. ~~eq~~ pedal equation.

6) Soln

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \frac{\pi^2}{4}$$

By using property of definite integral,

$$I = \int_0^{\pi} \frac{(\pi - x)\, \sin(\pi - x)}{1+\cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\, \sin x}{1+\cos^2 x}$$

$$= \int_0^{\pi} \frac{\pi . \sin x}{1+\cos^2 x} - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}$$

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}$$

Let $\cos x = t$

$dt = -\sin x \, dx$

When, $x = 0, \ t = 1$

$x = \pi, \ t = -1$

now,

$$2I = -\pi \int_{1}^{-1} \frac{1}{1+t^2}\, dt$$

$$= -\pi \left(\tan^{-1} t\right)_{1}^{-1}$$

$$= -\pi \left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= -\pi \left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$2I = \frac{2\pi^2}{4}$$

$$\therefore \quad I = \frac{\pi^2}{4} \quad \text{ans}$$

7) $\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{\pi}{3}$

Soln

given integral is, $\int_0^1 \frac{dx}{(1-x^6)^{1/6}}$

Put, $x^6 = t$,

$6x^5\, dx = dt$, $dx = \frac{dt}{6t^{5/6}}$

When $x=0$, $t=0$, when, $x=1$, $t=1$

So, $\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{1}{6}\int_0^1 \frac{dt}{(1-t)^{1/6}\, t^{5/6}}$

$= \frac{1}{6}\int_0^1 t^{-5/6}(1-t)^{-1/6}\, dt$

$= \frac{1}{6}\int_0^1 t^{\frac{1}{6}-1}(1-t)^{\frac{5}{6}-1}\, dt$

$= \frac{1}{6}\, B\left(\frac{1}{6}, \frac{5}{6}\right)$

$$= \frac{1}{6} \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}+\frac{5}{6}\right)} = \frac{1}{6} \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(1-\frac{1}{6}\right)}{\Gamma(1)}$$

$$= \frac{1}{6} \frac{\pi}{\sin \pi/6} = \frac{1}{6} \frac{\pi}{1/2}$$

$$= \pi/3.$$

$$\therefore \int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \pi/3 \quad \text{proved}$$

8) Soln

Here

$$I = \int_0^\infty \frac{e^{-ax}\sin x}{x}\, dx$$

diff. wrt a,

$$\frac{dI}{da} = \frac{d}{da}\int_0^\infty \frac{e^{-ax}\sin x}{x}\, dx$$

$$= \int_0^\infty -\frac{\sin x}{x} \cdot \frac{de^{-ax}}{d\,ax} \cdot \frac{dax}{dx}$$

$$= -\int_0^\infty \frac{\sin x}{x} \cdot x\, e^{-ax} \cdot dx$$

$$dI = \frac{-1}{a^2+1^2} \cdot da$$

Integrating,

$$\int dI = -\int \frac{da}{a^2+1^2}$$

$$I = -\tan^{-1}(a) + c \quad \text{—— (1)}$$

Put $a = \infty$, $I = 0$

$$I = -\tan^{-1}(\infty) + C$$

$$0 = -\frac{\pi}{2} + C$$

$$\therefore C = \frac{\pi}{2}$$

Putting in eqn (1),

$$I = -\tan^{-1}(a) + \frac{\pi}{2}$$

$$= \cot^{-1}(a)$$

$$\therefore \int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx = \cot^{-1}(a)$$

$$= \int_0^\infty -\frac{\sin x}{x} \cdot d e^{-ax}$$

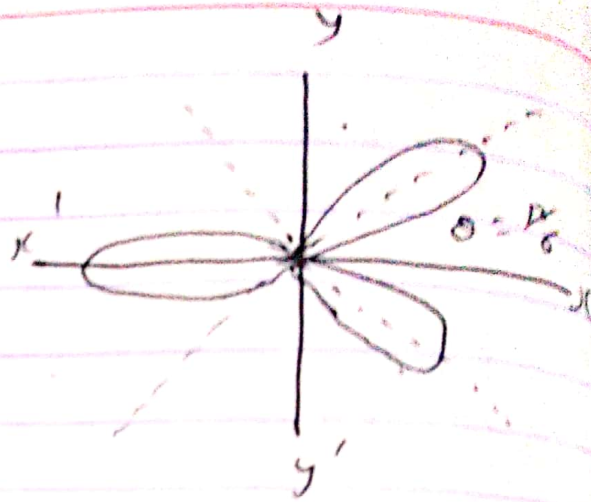
Put $a = 0$ in both sides,

$$\int_0^\infty \frac{\sin x}{x} \, dx = \cot^{-1}(0) = \frac{\pi}{2}$$ Proved

9) <u>Given</u>

$r = a\sin 3\theta$

It is symmetrical and 3 is odd so three equal loops are formed.



Hence,

Required Area, $A = 3\int_0^{\pi/6} 2\left(\frac{1}{2}r^2 d\theta\right)$

$= 3\int_0^{\pi/6} a^2 \sin^2 3\theta \, d\theta$

$= 3a^2 \int_0^{\pi/6} \frac{1-\cos^2 6\theta}{2} \, d\theta$

$= \frac{3a^2}{2}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/6}$

$= \frac{3a^2}{2} \times \frac{\pi}{6}$

$= \frac{\pi a^2}{4}$ sq. units

10) Sol^n

$$3x^2+3y^2+2xy=2 \quad \text{—— (1)}$$

Since the axes be turned through angle $\theta = 45^\circ$, then put,

$$x = x\cos\theta - y\sin\theta$$
$$= x\cos 45 - y\sin 45$$
$$= x\frac{1}{\sqrt{2}} - y\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x-y)$$

and,

$$y = y\cos\theta + x\sin\theta$$
$$= y\cos 45 + x\sin 45$$
$$= y\frac{1}{\sqrt{2}} + x\frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(x+y)$$

Putting this value in (1) we get,

$$\frac{3}{2}(x-y)^2 + \frac{3}{2}(x+y^2) + 2\frac{1}{\sqrt{2}}(x-y)\frac{1}{\sqrt{2}}(x+y) = 2$$

$$3(x^2-2xy+y^2) + 3(x^2+2xy+y^2 + 2x^2-2y^2 = 4$$

or, $3x^2+3y^2+3x^2+3y^2+2x^2-2y^2=4$

or, $8x^2+4y^2=4$

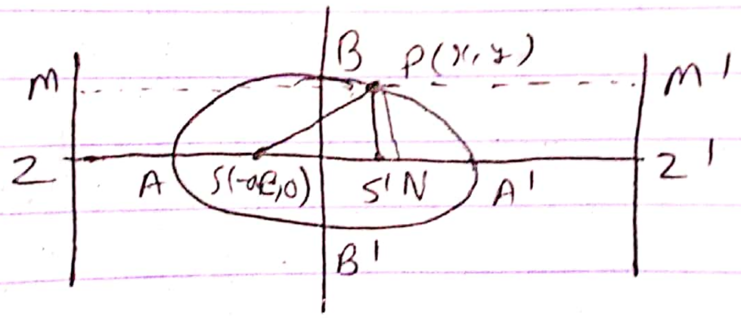
$\therefore 2x^2+y^2=1$

Hence the transferred eqn is,

$2x^2+y^2=1$ ans

11) Soln

Let S be fixed point and ZM be the fixed straight line on an ellipse. and Draw SZ perpendicular to ZM.



By Defn of ellipse,

let A be a point on the locus such that

$SA = eAZ$ —— (i)

Also, the the point A' be on the locus such that $SA' = eA'Z$ —— (ii)

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Let C be the middle point AA' and $AA' = 2a$ then

$$CA = CA' = a$$

on addition (1) and (ii)

$$SA + SA' = e(AZ + A'Z)$$
$$\text{or, } AA' = [e[(CZ - CA) + CA' + CZ]$$
$$\text{or, } 2a = e[(CZ - a + a + CZ)]$$

$$\text{or, } 2a = 2eCZ$$
$$\therefore CZ = \frac{a}{e}$$

on Subtraction (1) and (ii),

$$SA - SA' = e(AZ - A'Z)$$
$$\text{or, } (CS + CA) = e(AZ - A'Z).$$
$$\text{or, } (CS + CA) - (CA' - CS) = eAA'$$
$$\text{or, } 2CS = 2ae$$
$$\therefore CS = ae$$

Let C be the origin, CA be along the axis of x, a line through C perpendicular to ACA' be the y-axis, so that the coordinates of foci S and S' are $S(-ae, 0)$ and $S'(ae, 0)$.

Let $P(x, y)$ be any point on the ellipse. Draw PN perpendicular to x-axis and PM perpendicular to the directrix such that

$$CN = x, \quad PN = y$$

By defn of the ellipse

$$SP = e\,PM.$$

$$\text{or, } SP^2 = e^2 PM^2 = e^2 (ZN)^2$$

$$= e^2 (CZ + CN)^2$$

$$\text{or, } (x + ae)^2 + (y - 0)^2 = e^2 \left(\frac{a}{e} + x\right)^2 = (a + ex)^2$$

$$\text{or, } x^2 + 2aex + a^2e^2 + y^2 = a^2 + 2aex + e^2x^2$$

$$\text{or, } x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\text{or, } \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\text{where, } b^2 = a^2(1 - e^2).$$

This is the required standard equation of ellipse.

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12)

given $r = \frac{10 \operatorname{cosec}\theta}{2\operatorname{cosec}\theta + 3}$

now,

$$r = \frac{10 \operatorname{cosec}\theta}{2\operatorname{cosec}\theta + 3}$$

$$= \frac{\frac{10}{\sin\theta}}{\frac{2}{\sin\theta} + 3}$$

$$= \frac{10}{2 + 3\sin\theta}$$

$$= \frac{10}{2\left(1 + \frac{3}{2}\sin\theta\right)}$$

$$\therefore r = \frac{5}{1 + \frac{3}{2}\sin\theta} \quad \text{—— (1)}$$

comparing eqn (1) with $r = \frac{de}{1 + e\sin\theta}$

$e = \frac{3}{2}$, $de = 5$

or, $d = 5 \times \frac{2}{3}$

$\therefore d = \frac{10}{3}$

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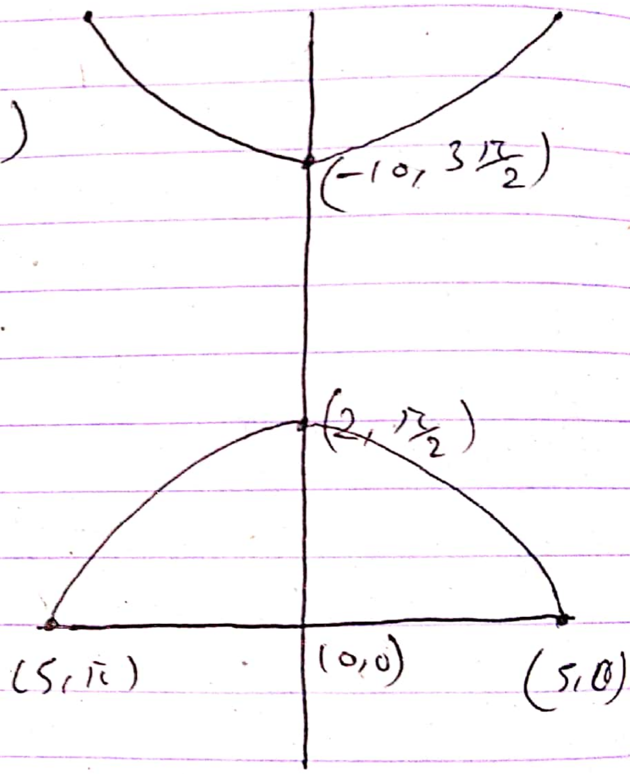
Since,

$e > 1$; we know the conic is hyperbola.

θ	0°	$\pi/2$	π	$3\pi/2$
r (r, θ)	5 $(5, 0)$	2 $(2, \frac{\pi}{2})$	5 $(5, \pi)$	-10 $(-10, \frac{3\pi}{2})$

now, focus $= (0,0), (12, \frac{\pi}{2})$

vertex $= (2, \frac{\pi}{2}), (10, \frac{\pi}{2})$



eqn of directrix

$y = r \sin\theta$

$y = \frac{10}{3} \sin 90$

$y = \frac{10}{3}$

or, $y = r \sin\theta$

$y = \frac{20}{3} \sin 90$

$y = \frac{20}{3}$

13) Here

$$x\frac{dy}{dx} + y\log y = xye^{x}$$

Here, $x\frac{dy}{dx} + y\log y = xye^{x}$

Dividing by xy,

$$y^{-1}\frac{dy}{dx} + \frac{\log y}{x} = e^{x}$$

Put $\log y = v$, $y^{-1}\frac{dy}{dx} = \frac{dv}{dx}$

So, the eqn reduces to,

$$\frac{dv}{dx} + \frac{v}{x} = e^{x}$$

This is linear eqn form, $P = \frac{1}{x}$, $Q = e^{x}$

$$IF = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$\therefore$ its general soln is;

$$v \times IF = \int Q \times (IF)\,dx$$

$$vx = \int x e^{x}\,dx + C$$

$\therefore$ $x\log y = xe^{x} - e^{x} + C$ is the req. soln

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14) $y - 3px + ayp^2 = 0$

$$x = \frac{y}{3p} + \frac{ayp}{3} \quad \text{——} ①$$

which is solvable for x, diff. wrt y,

$$\frac{dx}{dy} = \frac{1}{3} \frac{p - y\frac{dp}{dy}}{p^2} + \frac{1}{3} ay \frac{dp}{dy} + \frac{1}{3} ap$$

or, $\frac{1}{p} = \frac{1}{3p} + \frac{ap}{3} + \frac{dp}{dy}\left(\frac{ay}{3} - \frac{y}{3p^2}\right)$

or, $\frac{1}{p} - \frac{1}{3p} - \frac{ap}{3} = \frac{dp}{dy}\left(\frac{ay}{3} - \frac{y}{3p^2}\right)$

or, $\left(\frac{3 - 1 - ap^2}{3p}\right) = \frac{dp}{dy} \frac{ayp^2 - y}{3p^2}$

or, $\frac{(2-ap^2)}{3p} = \frac{dp}{dy} \frac{y(ap^2-1)}{3p^2}$

or, $\frac{dy}{y} = \frac{y(ap^2-1)}{p(2-ap^2)}$

or, $\frac{dy}{y} = \left(\frac{1}{2p} - \frac{ap}{2(2ap^2)}\right) dp$

or, $\frac{2dy}{y} = \left[\frac{1}{p} - \frac{ap}{(2-ap^2)}\right] dp$

or, $4\frac{dy}{y} = \left[\frac{2}{p} - \frac{2ap}{(2-ap^2)}\right] dp$

or, $4\log y = 2\log p + \log(2-ap^2) + \log C$

or, $\log y^4 = \log p^2(2-ap^2)c$

or, $y^4 = p^2(2-ap^2)c$ —— (ii)

Since p cannot be easily eliminated from (i) and (ii), so (i) and (ii) give the required soln.

15) $(D^2+3D+2)y = e^{2x}\sin x$

its auxilary eqn is,

$$m^2+3m+2=0$$
$$(m+1)(m+2)=0$$
$$\therefore m=-1,-2$$

$$\therefore CF = C_1e^{-x}+C_2e^{-2x}$$

now PI $= \dfrac{1}{D^2+3D+2} e^{2x}\sin x$

$$= e^{2x}\frac{1}{(-1)^2+3D+2}\sin x$$

$$= e^{2x}\frac{1}{3D+1}\sin x$$

$$= e^{2x}\frac{3D-1}{(3D)^2-1}\sin x$$

$$= -\frac{1}{10} e^{2x} \cdot (3D-1) \sin x$$

$$= -\frac{1}{10} e^{2x} (3\cos x - \sin x)$$

Thus, $y = CF + PI$

$$= C_1 e^{-x} + C_2 e^{-2x} - \frac{1}{10} e^{2x} (3\cos x - \sin x)$$

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16) $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$

This is homogeneous ~~linear~~ differential eqn.

Put, $x = e^z$ or, $z = \log x$

$D = \frac{d}{dz}$

$x \cdot \frac{dy}{dx} = Dy$ and $x^2 \frac{d^2y}{dx^2} = D(D-1)y$

Substituting value in eqn ①, it becomes,

$[D(D-1) + 4D + 2]y = e^{e^z}$

$D^2 + 3D + 2\, y = e^{e^z}$

$\therefore$ its auxilary eqn is,

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$\therefore m = -1, -2$

$\therefore CF = C_1 e^{-x} + C_2 e^{-2x}$

$= \frac{C_1}{x} + \frac{C_2}{x^2}$

$$PI = \frac{1}{D^2+3D+2} e^{e^z}$$

$$= \frac{1}{(D+1)(D+2)} e^{e^z}$$

$$= \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$$

$$= \frac{1}{(D+1)} e^{e^z} - \frac{1}{D+2} e^{e^z}$$

$$= e^{-z} \int e^{e^z} . e^z \, dz - e^{-2z} \int e^{e^z} . e^{2z} \, dz$$

Put, $e^z = t$, so, $e^z dz = dt$

$$PI = e^{-z} \int e^t dt - e^{-2z} \int e^t . t . dt$$

$$= e^{-z} e^t - e^{-2z}(te^t - e^t)$$

$$= e^{-z} e^{e^z} - e^{-2z} . e^{e^z} .$$

$$= e^{-2z} e^{e^z}$$

$$= x^{-2} e^x$$

$$= \frac{e^x}{x^2}$$

∴ The complete solution is,

$$y = CF + PI$$

$$= \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2} \quad \text{ans}$$