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Tut-2] [Q.9]

$\rightarrow Q = x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$

$= [x_1\ x_2\ x_3] \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\left.\begin{aligned} x_1 &= y_1 + 2y_2 + 4y_3 \\ x_2 &= y_2 + 4y_3 \\ x_3 &= y_3 \end{aligned}\right\} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

i.e. Py

$Q = X^t A X$

$= (Py)^t A (Py)$

$= y^t P^t A P y$

$= y^t B y \qquad (B = P^t A P)$

$B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} 1 & -2 & 4 \\ -2 & 2 & 0 \\ 4 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$Q = y^t B y$

$= [y_1\ y_2\ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= y_1^2 - 2y_2^2 + 9y_3^2$

The given quadratic form $Q = x^t A x$ under the linear transformation $x = Py$ is transformed to $Q = y^t B y = y_1^2 - 2y_2^2 + 9y_3^2$

[Q.2]

$$Q = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x^t A x$$

$$A_{3\times3} = I_{3\times3} A_{3\times3} I_{3\times3}$$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1/3$

$R_3 \to R_3 - R_1/3$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + C_1/3$

$C_3 \to C_3 - C_1/3$

R_3 -

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2/7$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + \frac{C_2}{7}$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$$

$D = P^t A P$

$Q = Y^t D Y$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$Q = 6y_1^2 + 7/3\, y_2^2 + 16/7\, y_3^2$

Rank = no. of non-zero term (r) = 3

index = no. of positive term (p) = 3

Signature = diff of +ve & -ve term = 3

value class $r \to p = n$ posit

[Q.3]

$$Q = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_3x_1$$

$$= [x_1\ x_2\ x_3]\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= X^T A X$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 - C_1$

$C_3 \to C_3 + C_1$

~~$R_3 \to R_3 - 2R_2$~~

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = P^t A P$$

$Q = Y^t D Y$

$$= [y_1\ y_2\ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$Q = y_1^2 + y_2^2 - 2y_2^2$ ... value class and

The linear transform is $X = PY$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \begin{cases} x_1 = y_1 - y_2 + 3y_3 \\ x_2 = y_2 - 2y_3 \\ x_3 = y_3 \end{cases}$$

a) for Quadratic form Q to be +ve:
let

$y_1 = 2$, $y_2 = 1$, $y_3 = 1$ $\to$ $x_1 = 4$, $x_2 = -1$, $x_3 = 1$
$y_1 = 2$ $y_2 = 2$, $y_3 = 1$ $\to$ $x_1 = 3$, $x_2 = 0$, $x_3 = 1$

b) for Quadratic form Q to be -ve:
let

$y_1 = 0$, $y_2 = 0$, $y_3 = 1$ $\to$ $x_1 = 3$ $x_2 = -2$ $x_3 = 1$
$y_1 = 1$ $y_2 = 0$ $y_3 = 1$ $x_1 = 4$ $x_2 = -2$ $x_3 = 1$

[Q.5]

Find the singular value decomposition of $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

$\to$ $A_{m\times n} = U_{m\times n} \Sigma_{m\times n} V^T_{m\times n}$

$A_{2\times 3} = U_{2\times 2} \Sigma_{2\times 3} V^T_{3\times 3}$

$\Rightarrow$ $A^TA = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 6 \\ 0 & 5 & -3 \\ 6 & -3 & 9 \end{bmatrix}$

$\lambda^3 - (5+5+9)\lambda^2 + \left(\begin{vmatrix} 5 & -3 \\ -3 & 9 \end{vmatrix} + \begin{vmatrix} 5 & 6 \\ 6 & 9 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 0 & 5 \end{vmatrix} \right.$
$\left. \lambda - 0 \right) = 0$

$\lambda^3 - 19\lambda^2 + (36+9+25)\lambda = 0$

$\lambda^3 - 19\lambda^2 + 70\lambda = 0$ $\quad \lambda = 14, 5, 0$

$\sigma_1 = \sqrt{14}$; $\sigma_2 = \sqrt{5}$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$

a) when $\lambda = 14$

$[A - \lambda I]\lambda = 0$

$[A - 14I]x = 0$

$$\begin{bmatrix} -9 & 0 & 6 \\ 0 & -9 & -3 \\ 6 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Cramers Rule

$$\frac{x_1}{36} = \frac{-x_2}{18} = \frac{x_3}{54}$$

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$\|x_1\| = \sqrt{2^2 + (-1)^2 + 3^2}$

$$v_1 = \frac{1}{\|x_1\|} x_1$$

$$= \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

when $\lambda = 5$

$[A - \lambda I]x = 0$

$[A - 5I]x = 0$

$$\begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & -3 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{-9} = \frac{-x}{18} = \frac{x_3}{6}$$

$$x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\|x_2\| = \sqrt{1^2 + 2^2 + 0^2}$

$= \sqrt{5}$

$$v_2 = \frac{1}{\|x_2\|} x_2$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

when $\lambda = 0$

$[A - \lambda I]x = 0$

$[A]x = 0$

$$\begin{bmatrix} 5 & 0 & 6 \\ 0 & 5 & -3 \\ 6 & -3 & 9 \end{bmatrix}$$

$$\frac{x_1}{36} = \frac{-x}{18} = \frac{x_3}{-30}$$

$$x_3 = \begin{bmatrix} -6 \\ 3 \\ 5 \end{bmatrix}$$

$\|x_3\| = \sqrt{\ldots}$

$= \sqrt{70}$

$$v_3 = \frac{1}{\|x_3\|}$$

$$\frac{1}{\sqrt{70}} \begin{bmatrix} -6 \\ 3 \\ 5 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 2/\sqrt{14} \\ -1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \quad v_2 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} -6/\sqrt{70} \\ 3/\sqrt{70} \\ 5/\sqrt{70} \end{bmatrix}$$

$$V = [\, v_1 \;\; v_2 \;\; v_3 \,]$$

$$= \begin{bmatrix} 2/\sqrt{14} & 1/\sqrt{5} & -6/\sqrt{70} \\ -1/\sqrt{14} & 2/\sqrt{5} & 3/\sqrt{70} \\ 3/\sqrt{14} & 0 & 5/\sqrt{70} \end{bmatrix}$$

$$V^T = \begin{bmatrix} 2/\sqrt{14} & -1/\sqrt{14} & 3/\sqrt{14} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -6/\sqrt{70} & 3/\sqrt{70} & 5/\sqrt{70} \end{bmatrix}$$

$$U_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} -2/\sqrt{14} \\ -1/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$$

$$\Rightarrow U_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U_2 = \frac{1}{\sigma_2} A V_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \Rightarrow$$

$$U_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = [\, U_1 \;\; U_2 \,] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = U \Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{14} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{14} & -1/\sqrt{14} & 3/\sqrt{14} \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -6/\sqrt{70} & 3/\sqrt{70} & 5/\sqrt{70} \end{bmatrix}$$