Algebraic Geometry — Exercises 24 April 2007

1. *(3 points)* Let $A \to B$ be a ring homomorphism, let $\Omega_{B/A}$ be the module of Kähler differentials of B over A, and let $d\colon B \to \Omega_{B/A}$ be the universal derivation. By definition, the canonical map

$$\begin{aligned} \mathrm{Hom}_B(\Omega_{B/A}, M) &\longrightarrow \mathrm{Der}_A(B, M) \\ h &\longmapsto h \circ d \end{aligned}$$

is an isomorphism for all B-modules M, where $\mathrm{Der}_A(B, M)$ is the B-module of A-derivations from B to M. Let $S \subseteq B$ be a multiplicative system; the composed map $A \to B \to S^{-1}B$ makes the localisation $S^{-1}B$ into an A-algebra.

(a) Show that for all $S^{-1}B$-modules M and all B-modules N, there is a canonical isomorphism of $S^{-1}B$-modules

$$\mathrm{Hom}_{S^{-1}B}(S^{-1}N, M) \xrightarrow{\sim} \mathrm{Hom}_B(N, M).$$

In particular, $\mathrm{Hom}_B(\Omega_{B/A}, M)$ and $\mathrm{Hom}_{S^{-1}B}(S^{-1}\Omega_{B/A}, M)$ are naturally isomorphic for all $S^{-1}B$-modules M.

(b) Show that for all $S^{-1}B$-modules M, there is a canonical isomorphism of $S^{-1}B$-modules

$$\mathrm{Der}_A(S^{-1}B, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M).$$

(c) Conclude that $\Omega_{S^{-1}B/A}$ is canonically isomorphic to $S^{-1}\Omega_{B/A}$.

For the following exercises, we consider an example treated in class. Let k be a field of characteristic different from 2, let d be a non-negative integer, and let

$$f = x^{2d+1} + f_{2d}x^{2d} + \cdots + f_0 \in k[x]$$

be a polynomial which has no double roots over an algebraic closure of k. Since f is not a square, the element $y^2 - f$ is irreducible in $k(x)[y]$ and the ring

$$K = k(x)[y]/(y^2 - f)$$

is a field. It was shown in class that there is a unique irreducible regular projective curve C over k which has K as its function field, and that C can be covered by the two affine open subsets

$$U = \mathrm{Spec}(k[x, y]/(y^2 - f)) \quad \text{and} \quad V = \mathrm{Spec}(k[u, v]/(v^2 - h)),$$

where

$$h = f_0u^{2d+2} + f_1u^{2d+1} + \cdots + f_{2d}u^2 + u.$$

The intersection of the two subsets is the affine scheme

$$\mathrm{Spec}(k[x, y, u, v]/(y^2 - f, ux - 1, x^{d+1}v - y))$$

seen as an open subset of both U and V via the morphisms induced by the obvious k-algebra homomorphisms. (In other words, the glueing data is given by the equations $u = x^{-1}$ and $v = x^{-d-1}y$.)

2. *(3 points)* Let $P \in C(k)$ be the unique point of C where x has a pole (i.e. the point $u = v = 0$ in terms of the coordinates on V). Compute a basis for the k-vector space $L(nP)$ for all $n \in \mathbf{Z}$. *(Hint: Find an expression for* $\mathrm{ord}_P(g_1(x) + yg_2(x))$ *in terms of the polynomials* g_1 *and* g_2*.)*

3. *(3 points)* Suppose $f_0 = 1$, and let Q be one of the two points where x has a zero. Compute a k-basis for $L(nQ)$ for all $n \in \mathbf{Z}$. *(Hint: First compute a* k*-basis for* $L(nQ + nR)$*, where* R *is the other point where* x *has a zero. Then show that the completion*

$$\widehat{\mathcal{O}}_{C,R} = \varprojlim_n \mathcal{O}_{C,R}/\mathfrak{m}_R^n$$

is isomorphic to the power series ring $k[[x]]$*. Finally, consider the natural map*

$$\phi\colon L(nQ + nR) \longrightarrow k((x)),$$

where the field of fractions of $\widehat{\mathcal{O}}_{C,R}$ *has been identified with the field* $k((x))$ *of Laurent series. Show that*

$$L(nQ) = \phi^{-1}(k[[x]])$$

and use this to determine a k*-basis for* $L(nQ)$*.)*

4. *(3 points)* Let E be the irreducible regular projective curve over the field $\mathbf{F}_3$ which has as its function field

$$K = \mathbf{F}_3(x)[y]/(y^2 - x^3 + x - 1).$$

(a) List the elements of $E(\mathbf{F}_3)$ (there are seven of them).

(b) Let $P \in E(\mathbf{F}_3)$ be the unique point where x has a pole. For every point $Q \in E(\mathbf{F}_3)$, write

$$Q^*\mathcal{L}(4P) = \mathcal{L}(4P)_Q/\mathfrak{m}_Q\mathcal{L}(4P)_Q.$$

Make the canonical $\mathbf{F}_3$-linear map

$$f\colon L(4P) \longrightarrow \bigoplus_{Q \in E(\mathbf{F}_3)} Q^*\mathcal{L}(4P)$$

explicit by giving a basis of both sides and the matrix of f with respect to these bases.

(Remark: In the language of coding theory, this map describes a linear error-correcting code over $\mathbf{F}_3$ *of dimension 4, length 7, and distance 3, which can correct one error.)*