Johan Bosman
Ina Boudier Bakkerlaan 59
3582 VR Utrecht
jgbosman@math.uu.nl

Lenny Taelman
Maurice Verdoncklaan 9
B9050, Gentbrugge, België
lenny.taelman@rug.ac.be

On sums of sums of values of Dirichlet characters

In de Universitaire Wiskunde Competitie (UWC) komt tweemaal per jaar een zogeheten ster-opgave voor. Voor een dergelijke opgave is bij de UWC-redactie geen oplossing bekend. Dit artikel is ontstaan naar aanleiding van de oplossing die Johan Bosman en Lenny Taelman hebben gevonden voor de ster-opgave van de UWC van vorig jaar (september 2000, pagina 303). Johan Bosman en Lenny Taelman zijn studenten wiskunde aan de universiteiten van respectievelijk Utrecht en Gent.

Using elementary techniques we prove relations between sums of Dirichlet characters. In the special case of Jacobi symbols these relations are closely related to the class number formula for quadratic number fields with negative discriminant. We then use these relations to prove problems from the Dutch-Flemish University Mathematics Competition.

Notations

Let n be a positive integer and let χ be a nontrivial Dirichlet character modulo n. We define the function $s : \mathbf{Z}_{>0} \to \mathbf{C}$ as follows:

$$s(m) = s_{n,\chi}(m) := \chi(0) + \cdots + \chi(m).$$

Note the following identity:

$$\sum_{x \in \mathbf{Z}/n\mathbf{Z}} \chi(x) = 0. \tag{1}$$

To see this, take $y \in \mathbf{Z}$ for which $\gcd(y,n) = 1$ and $\chi(y) \neq 1$, then $\sum_{x\in\mathbf{Z}/n\mathbf{Z}} \chi(x) = \sum_{x\in\mathbf{Z}/n\mathbf{Z}} \chi(xy) = \chi(y) \sum_{x\in\mathbf{Z}/n\mathbf{Z}} \chi(x)$ and (1) follows. We now see that $s(x+n) = s(x)$ for all $x \in \mathbf{Z}_{>0}$ so that s is well-defined as a map from $\mathbf{Z}/n\mathbf{Z}$ to $\mathbf{C}$ and hence s is well-defined on rational numbers $\frac{p}{q}$ with $\gcd(q,n) = 1$.

Rewriting (1) we have $\chi(0) + \cdots + \chi(x) + \chi(x+1) + \cdots + \chi(n-1) = 0$ or $s(x) + \chi(-1)s(n-x-1) = 0$, so we find the important identity

$$s(x) = -\chi(-1)s(-x-1). \tag{2}$$

Main Result

We shall prove the following result.

Theorem. *Let $m > 1$ be an integer coprime to n. Then the value of*

$$C = C(n,\chi) := \frac{\sum_{k=0}^{m-1} s(-\frac{k}{m})}{m - \overline{\chi(m)}}, \tag{3}$$

where $\overline{\chi(m)}$ denotes the complex conjugate of $\chi(m)$, does not depend on m.

Proof. Define $S := s(0) + \cdots + s(n-1)$. We will compute the expression $\sum_{l=0}^{n-1} \sum_{k=0}^{m-1} s(l - \frac{k}{m})$ in two ways.
First of all,

$$\sum_{l=0}^{n-1} \sum_{k=0}^{m-1} s\left(l - \frac{k}{m}\right) = mS, \tag{4}$$

since $l - \frac{k}{m}$ meets every residue class modulo n exactly m times. But on the other hand, $s(l - \frac{k}{m}) = s(-\frac{k}{m}) + \chi(1 - \frac{k}{m}) + \cdots + \chi(l - \frac{k}{m})$, so

$$\begin{aligned}
\sum_{l=0}^{n-1} \sum_{k=0}^{m-1} s\left(l - \frac{k}{m}\right) &= \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \left(s\left(-\frac{k}{m}\right) + \sum_{a=1}^{l} \chi\left(a - \frac{k}{m}\right)\right) \\
&= n \sum_{k=0}^{m-1} s\left(-\frac{k}{m}\right) + \sum_{l=0}^{n-1} \sum_{a=1}^{l} \sum_{k=0}^{m-1} \overline{\chi(m)}\chi(am-k) \\
&= n \sum_{k=0}^{m-1} s\left(-\frac{k}{m}\right) + \overline{\chi(m)} \sum_{l=0}^{n-1} s(ml) \\
&= n \sum_{k=0}^{m-1} s\left(-\frac{k}{m}\right) + \overline{\chi(m)}S.
\end{aligned}$$

Combining this with (4) we see that $n \sum_{k=0}^{m-1} s(-\frac{k}{m}) + \overline{\chi(m)}S = mS$ or, in other words,

$$\frac{\sum_{k=0}^{m-1} s(-\frac{k}{m})}{m - \overline{\chi(m)}} = \frac{S}{n},$$

which is clearly independent of m. □

Note that, in the special case where χ is the Jacobi symbol, and $n \equiv 3 (\mathrm{mod}\, 4)$ is square-free and not equal to 3, then $C(n,\chi)$ is the class number of $\mathbf{Q}(\sqrt{-n})$ (see for example the book *Zetafunktionen und quadratische Körper* by Don Zagier).

Corollary. *Let x be an element of $\mathbf{Z}/n\mathbf{Z}$. Then*

$$\frac{\sum_{k=0}^{m-1} s(\frac{x+k}{m}) - \overline{\chi(m)}s(x+m-1)}{m-\overline{\chi(m)}} = C(n,\chi), \tag{5}$$

where C is as in the theorem.

Proof. Note that after substitution $x=-m$ the expression (5) is clearly equal to (3) since we sum the same terms, but only in backward order. Now substitute $x=my$. We will prove by induction on y that the value of (5) is the same for all values of $y \in \mathbf{Z}/n\mathbf{Z}$. Above we showed this for $y=-1$, so let's get to the induction step. It is enough to show that

$$\sum_{k=0}^{m-1} s\left(y+1+\frac{k}{m}\right) - \sum_{k=0}^{m-1} s(y+\frac{k}{m})$$
$$= \overline{\chi(m)}\big(s(m(y+1)+m-1) - s(my+m-1)\big).$$

This is immediate, for denoting with LHS and RHS the lefthand and righthand side of this equality, respectively, we see that

$$\begin{aligned} LHS &= \sum_{k=0}^{m-1}\left(s\left(y+1+\frac{k}{m}\right) - s\left(y+\frac{k}{m}\right)\right) \\ &= \sum_{k=0}^{m-1} \chi\left(y+1+\frac{k}{m}\right) \\ &= \sum_{k=0}^{m-1} \overline{\chi(m)}\chi(my+m+k) = RHS. \end{aligned}$$

□

Examples

In this section we let n be a positive odd integer which is not a square and we let χ be the Jacobi symbol modulo n. In this case it is well-known that

$$\left(\frac{-1}{n}\right) = \begin{cases} 1 \text{ if } n \equiv 1 (\operatorname{mod} 4), \\ -1 \text{ if } n \equiv -1 (\operatorname{mod} 4), \end{cases}$$

and

$$\left(\frac{2}{n}\right) = \begin{cases} 1 \text{ if } n \equiv \pm 1 \ (\operatorname{mod} 8), \\ -1 \text{ if } n \equiv \pm 3 \ (\operatorname{mod} 8). \end{cases}$$

We will show by examples how we can use our theorem and its corollary together with (2) to prove identities involving sums of Jacobi symbols. The examples here are problems of the third edition of the Dutch-Flemish University Mathematics Competition 2000. Note that choosing $m=2$ in (3) we find that

$$C = \frac{s(\frac{n-1}{2})}{2-\left(\frac{2}{n}\right)}. \tag{6}$$

If $n \equiv 1 \ (\operatorname{mod} 4)$ then $\left(\frac{-1}{n}\right) = 1$. Using (2) we see that $s(\frac{n-1}{2}) = -s(\frac{n-1}{2})$ and therefore

$$C = 0 \quad \text{if } n \equiv 1 \ (\operatorname{mod} 4). \tag{7}$$

As a first example we will prove the following two identities:

$$\sum_{k=0}^{\frac{n-1}{2}} s(k) = \begin{cases} \dfrac{n+3}{6} s(\dfrac{n-1}{2}) \text{ if } n \equiv 3 \ (\operatorname{mod} 8), \\ \dfrac{n+1}{2} s(\dfrac{n-1}{2}) \text{ if } n \equiv 7 \ (\operatorname{mod} 8). \end{cases} \tag{8}$$

To see this, first note that in both cases $n \equiv -1 (\operatorname{mod} 4)$. Thus $\left(\frac{-1}{n}\right) = -1$, so that the following identity is a consequence of (2):

$$\sum_{k=0}^{\frac{n-1}{2}} s(k) = \frac{1}{2}\sum_{k=0}^{n-1} s(k) + \frac{1}{2}s\left(\frac{n-1}{2}\right).$$

If we substitute $m=n+1$ into (3), then we see that

$$\sum_{k=0}^{n} s(k) = nC = \frac{n}{2-\left(\frac{2}{n}\right)} s\left(\frac{n-1}{2}\right)$$

and (8) follows. The second example is

$$s\left(\frac{n-1}{4}\right) + s\left(\frac{3(n-1)}{8}\right) = s\left(\frac{n-1}{8}\right) \quad \text{if } n \equiv 1 \ (\operatorname{mod} 8). \tag{9}$$

To see this, first substitute $m=2$, $x=-\frac{7}{4}$ into (5) and consider (7) to derive

$$s\left(-\frac{3}{8}\right) + s\left(-\frac{7}{8}\right) - s\left(-\frac{3}{4}\right) = 0.$$

Using (2) we obtain from this that

$$s\left(-\frac{1}{4}\right) + s\left(-\frac{3}{8}\right) - s\left(-\frac{1}{8}\right) = 0,$$

which is equivalent with (9). The last example we show here is

$$s\left(\frac{n-3}{8}\right) + s\left(\frac{3n-1}{8}\right) = s\left(\frac{n-1}{2}\right) \quad \text{if } n \equiv 3 \ (\operatorname{mod} 8). \tag{10}$$

First we substitute $x=-\frac{3}{2}$, $m=2$ into (5) and we get:

$$\frac{s\left(-\frac{1}{4}\right) + s\left(-\frac{3}{4}\right) + s\left(-\frac{1}{2}\right)}{3} = C = \frac{s\left(-\frac{1}{2}\right)}{3}.$$

From (2) it follows that $s(-\frac{3}{4}) = s(-\frac{1}{4})$, so we see that

$$s\left(-\frac{3}{4}\right) = 0. \tag{11}$$

Next, we substitute $x=-\frac{7}{4}$, $m=2$ into (5) and we get:

$$s\left(-\frac{7}{8}\right) + s\left(-\frac{3}{8}\right) - s\left(-\frac{3}{4}\right) = s\left(-\frac{1}{2}\right).$$

Combining this with (11) and (2) we obtain that

$$s\left(-\frac{3}{8}\right) + s\left(-\frac{1}{8}\right) = s\left(-\frac{1}{2}\right),$$

which is equivalent with (10).

With the same tools one can also prove the following similar propositions:

$$s\left(\frac{n-5}{8}\right) + s\left(\frac{n-1}{4}\right) = s\left(\frac{3n-7}{8}\right) \quad \text{if } n \equiv 5 \ (\operatorname{mod} 8);$$
$$s\left(\frac{n-7}{8}\right) + s\left(\frac{3n-5}{8}\right) = 2s\left(\frac{n-3}{4}\right) \quad \text{if } n \equiv 7 \ (\operatorname{mod} 8);$$
$$s\left(\frac{n-3}{4}\right) = s\left(\frac{n-1}{2}\right) \quad \text{if } n \equiv 7 \ (\operatorname{mod} 8).$$