TCC Course
Arithmetic of Curves of Higher Genus
Example Sheet 1

Students taking this course for credit, please answer the homework questions and mail your solutions to me: Samir Siksek, Mathematics Institute, University of Warwick, Coventry CV4 7AL. Feel free to contact me by email if you have any questions: `samirsiksek@yahoo.com`

(1) Show that
$$H^0\left(\mathrm{Gal}(\overline{k}/k), \mathbb{P}^n(\overline{k})\right) = \mathbb{P}^n(k).$$
Big Hint: In Silverman's book (page 20) it is suggested that you use Hilbert's Theorem 90. This is actually a good way of learning how H90 works. However, you can do this exercise by taking a point in $\mathbb{P}^n(\overline{k})$ fixed by $\mathrm{Gal}(\overline{k}/k)$ and scaling so one of the coordinates is 1.

(2) Let $V \subset \mathbb{A}^n$ be given a single non-constant irreducible polynomial
$$V : f(x_1, \ldots, x_n).$$
Show that V has dimension $n-1$.

(3) Let
$$C_0 : y^2 = x^3 + x.$$
Define $\phi_0 : C_0 \dashrightarrow \mathbb{A}^1$ by $\phi(x,y) = x/y$. Observe that ϕ_0 is not a morphism. Now let $C \subset \mathbb{P}^2$ be the projective closure of C_0. Show that ϕ_0 extends to a morphism $\phi : C \to \mathbb{P}^1$.

(4) Let V be a projective k-variety given by a single equation; say $V : F(x_0, \ldots, x_n) = 0$. Here F must be a homogeneous polynomial.

(i) Show that the singular locus is given by
$$\frac{\partial F}{\partial x_0} = \cdots = \frac{\partial F}{\partial x_n} = F = 0.$$
(ii) if $\mathrm{char}(k)$ does not divide d, show that the singular locus is the set of solutions to
$$\frac{\partial F}{\partial x_0} = \cdots = \frac{\partial F}{\partial x_n} = 0.$$
Hint for (ii): google 'Euler's Homogeneous Function Theorem'.