Algebraic Geometry — Exercises 10 April 2007

These exercises are about various properties of schemes and morphisms of schemes. The definitions are given in the exercises and in Hartshorne, § I.1 (dimension of a topological space) and § II.3 (other definitions).

1. *(3 points)* Hartshorne, Exercise I.1.10. (Note that this exercise is from Chapter I.)

2. *(3 points)* Determine the dimension of $\mathbf{A}^1_{\mathbf{Z}}$. *(Hint: Consider intersections of prime ideals of* $\mathbf{Z}[x]$ *with* $\mathbf{Z}$*.)*

3. *(3 points)* Hartshorne, Exercise II.2.3.

4. *(3 points)* Hartshorne, Exercise II.3.5. *(Hint: Reduce to the case* $f = \operatorname{Spec}\phi$*, where* $\phi\colon A \to B$ *is a ring homomorphism making* B *into a finite* A*-algebra. For every prime ideal* $\mathfrak{p} \subset A$*, let* $k(\mathfrak{p})$ *denote the residue class field of the local ring* $A_{\mathfrak{p}}$ *(which is also the field of fractions of* $A/\mathfrak{p}$*), and let* $\overline{k(\mathfrak{p})}$ *be an algebraic closure. Furthermore, let* $B_{\mathfrak{p}}$ *denote the localisation of* B *at* $\mathfrak{p}$ *as an* A*-module. To prove (a), show that there is a surjective map*

$$\operatorname{Hom}_{k(\mathfrak{p})\text{-}\mathbf{Alg}}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}, \overline{k(\mathfrak{p})}) \longrightarrow \{\mathfrak{q} \in \operatorname{Spec} B \mid \phi^{-1}\mathfrak{q} = \mathfrak{p}\}.$$

For (b), reduce to proving the statement that if ϕ *is injective, then* $\operatorname{Spec}\phi$ *is surjective. Show that in this case* $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ *is injective for every prime ideal* $\mathfrak{p} \subset A$*. Then use Nakayama's lemma (see e.g. Lang,* Algebra*, Ch. X, Lemma 4.1) to prove that there exists a prime ideal of* $B_{\mathfrak{p}}$ *containing* $\mathfrak{p}B_{\mathfrak{p}}$*.)*

5. *(2 points)* Hartshorne, Exercise II.3.6.