Algebraic Number Theory (MA3A6) - Problem Sheet 2

This sheet is not for submission, but you may ask for help or feedback if you wish.

1. Is $\frac{3+2\sqrt{6}}{1-\sqrt{6}}$ an algebraic integer?

2. Given a ring R (commutative and with 1) and an ideal $I \subseteq R$, show that

   · R/I is a field $\Leftrightarrow$ I is maximal

   · R/I is an integral domain $\Leftrightarrow$ I is prime.

3. Let α be an algebraic number. Show that $n\alpha$ is an algebraic integer for some $n \in \mathbb{Z}_{>0}$.

4. Factorise 24 and $5 + 3\sqrt{-7}$ as product of irreducible elements;

   (a) in $\mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$,

   (b) in $\mathbb{Z}\left[\sqrt{-7}\right]$.

5. Consider $\mathbb{Q}(\alpha)$ where α is a root of $\alpha^3 + \alpha^2 + \alpha + 2 = 0$.

   Express
   $$(\alpha^2 + \alpha + 1)(\alpha^2 + \alpha) \text{ and } (\alpha - 1)^{-1}$$
   in the form $a\alpha^2 + b\alpha + c$ with $a, b, c \in \mathbb{Q}$.

6. Show that $\sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$ of degree 4.

7. Let p and q be coprime square-free integers. Show that the minimal polynomial of $\sqrt{p}$ over $\mathbb{Q}(\sqrt{q})$ is $X^2 - p$.

8. Let F be an extension of a field K with basis $\{\gamma_1, \dots, \gamma_n\}$ over K. Show that for any $x \in F^*$, $\{x\gamma_1, \dots, x\gamma_n\}$ is also a K-basis for F.

9. Let M be an $n \times n$ matric over a field k. Assume $\mathrm{tr}(MX) = 0$ for all $n \times n$ matrices X over k. Show that $M = 0$.

10. Let L be a free module over $\mathbb{Z}$ with basis $e_1, \dots, e_n$. Let M be a free sub-module of the same rank, with basis $u_1, \dots, u_n$. Let $u_i = \sum c_{ij} e_j$. Show that the index $(L : M)$ is given by the determinant
    $$(L : M) = |\det(c_{ij})|.$$