Algebraic Geometry — Exercises 15 May 2007

The final assignment consists of all of the seven exercises below. Your grade for this assignment will count as 30% of your final grade.

1. Let A be a ring, let S be a multiplicative subset of A, and let M be an A-module

(a) Show that the $S^{-1}A$-modules $S^{-1}M$ and $S^{-1}A \otimes_A M$ are naturally isomorphic. *(Hint: Use the universal properties.)*

(b) Show that the functor which sends an A-module M to the $S^{-1}A$-module $S^{-1}M$ is exact, and conclude that $S^{-1}A$ is a flat A-module.

(c) Give an example where $S^{-1}A$ is not locally free as an A-module.

2. Let X be a locally ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module which is locally free of rank 1. Let $\mathcal{F}^\vee$ be the $\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, i.e.

$$\mathcal{F}^\vee(U) = \mathrm{Hom}_{\mathcal{O}_X|U}(\mathcal{F}|_U, \mathcal{O}_X|_U) \quad (U \subseteq X \text{ open}).$$

Show that the maps

$$\begin{aligned} \mathcal{F}^\vee(U) \times \mathcal{F}(U) &\longrightarrow \mathcal{O}_X(U) \\ (\phi, s) &\longmapsto \phi(s), \end{aligned}$$

for $U \subseteq X$ open, induce an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\sim} \mathcal{O}_X.$$

(Hint: Reduce to the case where $\mathcal{F}$ is free by replacing X by open subsets on which $\mathcal{F}$ is free; for $\mathcal{G}$ a presheaf on X and $U \subseteq X$ open, $(\mathcal{G}^+)|_U = (\mathcal{G}|_U)^+$.)

3. Let k be a field.

(a) Construct $X = \mathbf{P}^1_k \times_k \mathbf{P}^1_k$ by glueing four copies of $\mathbf{A}^2_k$: $\operatorname{Spec} k[x, y]$, $\operatorname{Spec} k[x^{-1}, y]$, etc.

(b) Show that $\operatorname{Pic}(X)$ is a free Abelian group of rank 2, generated by the classes of the line bundles $\mathcal{L}(\{\infty\} \times \mathbf{P}^1_k)$ and $\mathcal{L}(\mathbf{P}^1_k \times \{\infty\})$.

(c) Describe the intersection pairing on $\operatorname{Pic}(X) = \mathbf{Z}^2$.

4. Read Hartshorne, Chapter II, Theorem 7.1. Let k be a field.

(a) Let $\phi\colon \mathbf{P}^1_k \to \mathbf{P}^2_k$ be the morphism given by the invertible sheaf $\mathcal{L}(2 \cdot \infty)$ and the global sections 1, x, x^2. Describe ϕ explicitly and give an equation for its image. Show that ϕ is a closed immersion. *(Hint: For $i = 0, 1, 2$, let $X_i \subset \mathbf{P}^1_k$ and $U_i \subset \mathbf{P}^2_k$ be as in the proof of Theorem 7.1; apply Exercise II.2.18 to the morphisms $X_i \to U_i$.)*

(b) Let $\phi\colon \mathbf{P}^1_k \to \mathbf{P}^3_k$ be the morphism given by $\mathcal{L}(3 \cdot \infty)$, 1, x, x^2, x^3. Describe ϕ explicitly and give equations for its image (you need at least three equations). Show that ϕ is a closed immersion.

(c) Let $D = \mathbf{P}^1_k \times \{\infty\} + \{\infty\} \times \mathbf{P}^1_k$ on $\mathbf{P}^1_k \times_k \mathbf{P}^1_k$. Let $\phi\colon \mathbf{P}^1_k \times_k \mathbf{P}^1_k \to \mathbf{P}^3_k$ be the morphism given by $\mathcal{L}(D)$, 1, x, y, xy. Describe ϕ explicitly, give an equation for its image, and show that it is a closed immersion.

5. Read Hartshorne, Example II.7.1.1. Let k be a field and put $X = \mathbf{P}^1_k \times \mathbf{P}^1_k$. Identify $\operatorname{Pic}(X)$ with $\mathbf{Z}^2$ as in exercise 3.

(a) Show that any automorphism of X acts on $\operatorname{Pic}(X)$ as $(a,b) \mapsto (a,b)$ or $(a,b) \mapsto (b,a)$. *(Hint: the action preserves the intersection product.)*

(b) Determine $\operatorname{Aut}_k(X)$.

6. Let k be a field.

(a) Let $\phi : \mathbf{P}^1_k \to \mathbf{P}^2_k$ be the morphism given by a basis of

$$\{f \in \mathcal{L}(3\cdot\infty)(\mathbf{P}^1_k) : f(0) = f(1)\} \subset \mathcal{L}(3\cdot\infty)(\mathbf{P}^1_k).$$

Choose a basis f_0, f_1, f_2 for this space such that $v_\infty(f_0) = 0$, $v_\infty(f_1) = -2$, $v_\infty(f_2) = -3$, $f_1(0) = f_2(0) = 0$. Give an equation for the image.

(b) Replace the condition $f(0) = f(1)$ in part (a) by $f'(0) = 0$ and answer the same questions.

(c) Assume now that k has characteristic 2 and that there is an element $a \in k$ which is not a square. Let t be the closed point of $\mathbf{A}^1_k$ given by the maximal ideal $(x^2 - a)$ in $k[x]$. Let $\phi : \mathbf{P}^1_k \to \mathbf{P}^2_k$ be the morphism given by a basis of the space of sections $f \in \mathcal{L}(3\cdot\infty)(\mathbf{P}^1_k)$ whose image in $\kappa(t)$ lies in k. Choose a basis f_0, f_1, f_2 with f_1 and f_2 in $(x^2 - a)$. Give an equation for the image.

7. Let k be a field. For X a smooth k-scheme of dimension 2, the sheaf $\Omega^2_{X/k} = \bigwedge^2 \Omega^1(X/k)$ is an invertible $\mathcal{O}_X$-module such that for all closed points $x \in X$, the stalk $(\Omega^2_{X/k})_x$ is generated by $du \wedge dv$, where u and v are generators for the maximal ideal m_x. (See exercise II.5.16 for exterior powers of $\mathcal{O}_X$-modules).

(a) Compute the divisor of $dx \wedge dy$ on $\mathbf{P}^2_k$. Use the adjunction formula given in V.1.5 to compute the genus of a non-singular curve of degree d in $\mathbf{P}^2_k$ (see V.1.5.1).

(b) Do the same as in (a) but now for $\mathbf{P}^1_k \times \mathbf{P}^1_k$ and a curve of bidegree (a,b) (see V.1.5.2).