THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: April 2008

LOCAL FIELDS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) Let k be a local field with **discrete** valuation $|\quad|$. Let K/k be a totally ramified finite algebraic extension, and let π be a prime element for K. Show that $K = k(\pi)$ and that π is the root of an Eisenstein polynomial $f(X) \in k[X]$. **[10]**

b) Let $f(X) \in \mathbb{Z}_p[X]$ be a monic polynomial whose reduction in $\mathbb{F}_p[X]$ is irreducible. Let θ be a root of $f(X)$. Show that $\mathbb{Q}_p(\theta)/\mathbb{Q}_p$ is an unramified extension. **[7]**

c) Let $f(X) = X^4 + 2$ and let θ be a root of f. Show that $\mathbb{Q}_5(\theta)/\mathbb{Q}_5$ is unramified. **[8]**

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2. a) Suppose k is a complete non-archimedean field. Show that $\sum_{n=0}^{\infty} a_n$ converges if and only if $a_n \to 0$. **[6]**

b) Show that if p is a prime then the series $\sum_{n=1}^{\infty} \frac{p^n}{n}$ converges in $\mathbb{Q}_p$. **[7]**

c) Show that $|m!|_p \geq p^{-m/(p-1)}$ for positive integers m and primes p. Deduce that $\sum_{n=1}^{\infty} \frac{n!}{p^n}$ does not converge if p is odd. **[12]**

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3. a) State Strassman's theorem on the number of zeros of a power series (without proof). **[5]**

b) Let A, B be integers. Suppose that the polynomial $X^2 - AX - B$ has distinct roots α, β in $\mathbb{Z}_5$. Let u_m be the recurrence sequence given by

$$u_0 = 0,\ u_1 = 1, \qquad u_{m+2} = Au_{m+1} + Bu_m \quad (m \geq 0).$$

Show that $u_m = (\alpha^m - \beta^m)/(\alpha - \beta)$. **[10]**

c) Continuing with the notation of (b), suppose

$$\alpha \equiv \beta \equiv 1 \pmod 5.$$

Use Strassman's theorem to show that $u_m = 0$ only for $m = 0$. **[10]**

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4. a) Let k be complete with respect to a non-archimedean valuation $|\ \ |$. Let $\mathcal{O}$ be the ring of valuation integers and suppose $f(X) \in \mathcal{O}[X]$. Let $a_0 \in \mathcal{O}$ satisfy

$$|f(a_0)| < 1, \qquad |f'(a_0)| = 1.$$

Define the sequence

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}, \qquad n \geq 0.$$

Show that the sequence $\{a_n\}$ converges to some $a \in \mathcal{O}$ such that **[9]**

$$f(a) = 0, \qquad |a - a_0| < 1.$$

b) Continuing with the notation and assumptions of (a), show that if $b \in \mathcal{O}$ satisfies $f(b) = 0$ and $|b - a_0| < 1$ then $b = a$. **[7]**

c) Let p be a prime. Show that the polynomial $x^{p-1} - 1$ has precisely $p - 1$ roots in $\mathbb{Z}_p$. **[9]**

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5. a) Let k be a field. Define the terms **valuation**, **trivial valuation**, **archimedean valuation** and **non-archimedean valuation**. **[8]**

b) Let $C \geq 1$, and let $|\ \ |$ be a valuation on k satisfying the triangle inequality and $|e| \leq C$ for all e in the ring generated by 1 in k. Show that $|\ \ |$ is non-archimedean. **[8]**

c) Let $|\ \ |_1$, $|\ \ |_2$ be valuations on k and suppose the existence of an element $c \in k$ such that $|c|_1 < 1$, $|c|_2 > 1$. Let $b_1, b_2 \in k$ and let $\epsilon > 0$. Show the existence of an element $a \in k$ such that **[9]**

$$|a - b_1|_1 < \epsilon, \quad |a - b_2|_2 < \epsilon.$$

Hint: Examine the behaviour of the sequence $c^n/(1 + c^n)$ as $n \to \infty$ with respect to both valuations.
