Rings and Modules
Example Sheet 1
Not for credit

(1) Let
$$\mathfrak{a} = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \quad : \quad a,\ c \in \mathbb{C} \right\}.$$
Show that $\mathfrak{a}$ is a left ideal of $M_2(\mathbb{C})$ but not a 2-sided ideal. Give a non-zero proper right ideal of $M_2(\mathbb{C})$.

(2) Let I be an ideal of the non-zero ring R (left, right or 2-sided). Show that I is proper if and only if $1 \notin I$. More generaly, show that I is proper if and only if $I \cap R^* = \emptyset$.

(3) Let R be an integral domain. Show that $R[X]^* = R^*$.

(4) Let R be a ring. An element $a \in R$ is called **nilpotent** if there is some positive integer n such that $a^n = 0$.
   (i) Show that if a is nilpotent, then $1+a$ is a unit. You might find the following identity useful
   $$(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1-x^n.$$
   (ii) Let p be a prime and $r \geq 2$. Show that $\overline{1}+\overline{p}X$ is a unit $(\mathbb{Z}/p^r\mathbb{Z})[X]$. Why doesn't this contradict (3)?

(5) Let R be a commutative ring, and n a positive integer. Define
$$\mathrm{GL}_n(R) = (M_n(R))^*.$$
Show that
$$\mathrm{GL}_n(R) = \{A \in M_n(R) \ : \ \det(A) \in R^*\}.$$
Hint: Remind yourself of the adjugate matrix. Note the operations available to you in a commutative ring are addition, subtraction and multiplication. You can assume standard properties of matrices that use only these operations.