Algebraic Number Theory (MA3A6) - Problem Sheet 7

You should answer any **three** out of **five** questions. All questions have equal weight. Marks will also be awarded for clarity/presentation.

Deadline: **4pm** on **Thursday 10th March (week 9)**

1. Let K be a number field with ring of integers $\mathcal{O}_K$. If $\alpha \in \mathcal{O}_K$ then $\mathrm{Tr}(\alpha\beta) \in \mathbb{Z}$ for all $\beta \in \mathcal{O}_K$; prove this. What about the converse: if $\alpha \in K$ and $\mathrm{Tr}(\alpha\beta) \in \mathbb{Z}$ for all $\beta \in \mathcal{O}_K$, is α necessarily in $\mathcal{O}_K$?

2. Let $\alpha \in \mathbb{C}$ be a root of $x^3 - x^2 + 2x + 1$ and consider $K = \mathbb{Q}(\alpha)$.
   (a) Compute $\mathrm{Tr}(\alpha)$ and $\mathrm{Tr}(\alpha^2)$.
   (b) Compute the discriminant of $\mathbb{Z}[\alpha]$ and prove that $\mathbb{Z}[\alpha] = \mathcal{O}_K$.
   (c) Factor the ideals (2), (3), (5) and (7) into prime ideals of $\mathcal{O}_K$. Which of these prime ideals are principal? What are their norms?

3. (a) Prove the Chinese Remainder Theorem for commutative rings: If I and J are ideals of a ring R with $I + J = R$, then $R/(I \cap J) \cong R/I \times R/J$.
   (b) If $P_1, \ldots, P_m$ are distinct maximal ideals of a ring R, prove that the natural map $R \to R/P_1 \times \cdots \times R/P_m$ is surjective. Prove that for all i, $P_i \setminus \cup_{j \neq i} P_j$ is non-empty.
   (c) An ideal I of a ring R is said to be *primary* if the following conditions hold: I is not the whole R and if $a, b \in R$, $ab \in I$, $a \notin I$, then there exists an integer $m \geq 1$ with $b^m \in I$.

      Show that every primary ideal of a Dedekind domain R is a power of a prime ideal. (Hint: use the factorisation of ideals into prime ideals).

4. Solve $x^2 + 56 = y^3$ in integers. You may use without proof that $h_{\mathbb{Q}(\sqrt{-14})} = 4$.

5. (a) Prove that if we have $[K : \mathbb{Q}] = 2$ then $K = \mathbb{Q}(\sqrt{d})$ for some square-free $d \in \mathbb{Z} \setminus \{1\}$. (We have used but never proved this in lectures).
   (b) Given $d \in \mathbb{Z}_{<0}$ square-free, prove that $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt{d})}$ is finite.