THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: April 2007

LOCAL FIELDS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

ANSWER 4 QUESTIONS.
If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. a) Let k be a field. Define the terms **valuation**, **trivial valuation**, **archimedean valuation** and **non-archimedean valuation**. **[8]**

b) Suppose that the valuation $|\ \ |$ on k satisfies $|1+b| \leq 2$ for all $b \in k$ satisfying $|b| \leq 1$. Show that the valuation satifies the triangle inequality. **[10]**

c) Let $|\ \ |_1$, $|\ \ |_2$ be valuations on k and suppose the existence of an element $c \in k$ such that $|c|_1 < 1$, $|c|_2 > 1$. (i.e. the two valuations are inequivalent). Let $b_1, b_2 \in k$. Show the existence of an element $a \in k$ such that

$$|a - b_1|_1 < 1, \quad |a - b_2|_2 < 1.$$

[7]

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2. a) Suppose k is a complete non-archimedean field. Show that $\sum_{n=0}^{\infty} a_n$ converges if and only if $a_n \to 0$. **[7]**

b) For which finite primes p does the series $\sum_{n=0}^{\infty} \left(\frac{15}{14}\right)^n$ converge in $\mathbb{Q}_p$? What is its limit in those fields? **[6]**

c) Show that if p is an odd prime then the series $\sum_{n=0}^{\infty} \frac{p^n}{n!}$ converges in $\mathbb{Q}_p$. (If you need any inequality involving factorials, you should prove it.) **[12]**

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3. a) Let k be complete with respect to a non-archimedean valuation $|\ \ |$. Let $\mathcal{O}$ be the ring of valuation integers and suppose $f(X) \in \mathcal{O}[X]$. Let $a_0 \in \mathcal{O}$ satisfy

$$|f(a_0)| < 1, \qquad |f'(a_0)| = 1.$$

A form of Hensel's Lemma says that there is a **unique** $a \in \mathcal{O}$ satisfying

$$f(a) = 0, \qquad |a - a_0| < 1.$$

Assuming the existence of a, prove its uniqueness. **[10]**

b) Use Hensel's Lemma to show that the equation $X^n = 7$ has a solution in $\mathbb{Z}_3$ for all integers $n \geq 1$, such that $3 \nmid n$. **[7]**

c) Show that the polynomial $X^4 - X^2 + X + 1$ has precisely two roots in $\mathbb{Z}_5$. **[8]**

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4. a) Let $|\ \ |$ be a discrete valuation on local field k. Let β be the root of an Eisenstein polynomial in $k[X]$. Show that $k(\beta)/k$ is completely ramified and β is a prime element of $k(\beta)$. **[10]**

b) Let $p \neq 2$. Show that $\mathbb{Q}_p(\zeta)$, where $\zeta^p = 1$, $\zeta \neq 1$, is completely ramified, and that $\lambda = \zeta - 1$ is a prime element. **[8]**

c) Let θ be a root of $X^3 - X - 1 \in \mathbb{Q}_3[X]$. Show that $\mathbb{Q}_3(\theta)$ is an unramified extension of $\mathbb{Q}_3$ of degree 3. **[7]**

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5. a) State Strassman's theorem on the number of zeros of a power series (without proof). **[5]**

b) Let A, B be integers. Suppose that the polynomial $X^2 - AX - B$ has two roots α, β in $\mathbb{Z}_7$, satisfying

$$\alpha \equiv 8 \pmod{7^2}, \qquad \beta \equiv 1 \pmod{7^2}.$$

Let u_m be the recurrence sequence given by

$$u_0 = 2,\ u_1 = A, \qquad u_{m+2} = Au_{m+1} + Bu_m \quad (m \geq 0).$$

Show that $u_m = \alpha^m + \beta^m$. Use Strassman's theorem to show that $u_m = 2$ only for $m = 0$. **[20]**

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