Algebraic Geometry – Exercises 8 May 2007

1. *(3 points)* Let k be an algebraically closed field of characteristic 2 and let n be a non-negative integer.

a) Show that $y^2 + y + x^{2n+1}$ is irreducible in $k[x, y]$.

b) Let C be the regular irreducible projective curve over k whose function field is equal to $k(x)[y]/(y^2 + y + x^{2n+1})$ and view $x \in K(C)$ as a morphism $x : C \to \mathbb{P}^1_k$. Describe $x : C \to \mathbb{P}^1_k$ by giving equations for the affine pieces $U = x^{-1}(\mathbb{P}^1_k - \{\infty\})$ and $V = x^{-1}(\mathbb{P}^1_k - \{0\})$. Show explicitly from these equations that C is non-singular.

c) Notation as in part b. Show that $x|_U : U \to \mathbb{A}^1_k$ is unramified. (Side remark: in characteristic zero this would be impossible: $\mathbb{A}^1_{\mathbb{C}}$ with its complex-analytic topology is simply connected, so no non-trivial unramified coverings exist).

d) Let P be the unique point in $C(k)$ where x has a pole. Compute the differential ramification index d_P and show that $g(C) = n$. Why does the Hurwitz formula using the ramification indices e_Q for $Q \in f^{-1}(P)$ fail here?

2. *(3 points)* In this exercise we shall give an explicit computation of what is called de Rham cohomology. Let k be a field of characteristic zero and let $a_1, \ldots, a_r \in k$ be distinct, where $a_1 = \infty$. Consider the sets $\Sigma := \{a_1, \ldots, a_r\} \subset \mathbb{P}^1(k)$ and $U := \mathbb{P}^1_k - \Sigma$.

a) Show that $\{dx\}$ is an $\mathcal{O}_{\mathbb{P}^1_k}(U)$-basis of $\Omega^1_{\mathbb{P}^1_k/k}(U)$.

b) Give a k-basis of $\mathcal{O}_{\mathbb{P}^1_k}(U)$.

c) Give a k-basis of

$$H^1_{\mathrm{dR}}(U) := \mathrm{Coker}\left(d : \mathcal{O}_{\mathbb{P}^1_k}(U) \to \Omega^1_{\mathbb{P}^1_k/k}(U)\right)$$

by giving representatives in $\Omega^1_{\mathbb{P}^1_k/k}(U)$.

3. *(3 points)* Let k be any field, $U := \mathbb{P}^1_k - \{\infty\}$, $V := \mathbb{P}^1_k - \{0\}$, $n \in \mathbb{Z}$. Define a map

$$\phi : \mathcal{L}(n \cdot \infty)(U) \oplus \mathcal{L}(n \cdot \infty)(V) \to \mathcal{L}(n \cdot \infty)(U \cap V)$$

by

$$(s, t) \mapsto s|_{U \cap V} - t|_{U \cap V}.$$

Give k-bases for $\mathrm{Ker}\, \phi$ and $\mathrm{Coker}\, \phi$. (Side remark: the vector spaces computed here are known as the 0-th and 1-st cohomology group of the sheaf $\mathcal{L}(n \cdot \infty)$ on $\mathbb{P}^1_k$.)

4. *(3 points)* Hartshorne, exercise IV.1.6.