Algebraic Number Theory (MA3A6) - Problem Sheet 4

This sheet is not for submission, but you may ask for help or feedback if you wish.

1. Express $\mathbb{Q}(\sqrt{3}, \sqrt[3]{5})$ in the form $\mathbb{Q}(\theta)$.
2. Compute integral bases and discriminants of
   · $\mathbb{Q}(\sqrt{2}, \sqrt{3})$;
   · $\mathbb{Q}(\sqrt[4]{2})$.
3. Let R be an integral domain and I an invertible fractional ideal of R. Prove that I is finitely generated.
4. Let θ satisfy the equation $\theta^3 - \theta - 4 = 0$. Show that $\{1, \theta, \frac{1}{2}(\theta + \theta^2)\}$ is an integral basis of $\mathbb{Q}(\theta)$.
5. Find an example of a unique factorisation domain that is not a principal ideal domain.
6. Let d be a square-free integer, $d \not\equiv \pm 1 \pmod 9$. Prove that the ring of integers of $\mathbb{Q}(\sqrt[3]{d})$ is $\mathbb{Z}[\sqrt[3]{d}]$.
7. An integral domain B is said to be *integral* over a subring $A \subset B$ if every $b \in B$ is a zero of a monic polynomial in $A[X]$. Given inclusions of domains $A \subset B \subset C$, show that C is integral over A if and only if C is integral over B and B is integral over A.
8. Let $p \in \mathbb{Z}$ be a prime and $k \in \mathbb{Z}_{>0}$. Prove that the ring of integers of $\mathbb{Q}(\zeta_{p^k})$ is $\mathbb{Z}[\zeta_{p^k}]$.
9. Let R be a ring and $\mathfrak{a}$ an ideal that is contained in a finite union of prime ideals $\bigcup_i \mathfrak{p}_i$. Show that $\mathfrak{a}$ is contained in some $\mathfrak{p}_i$.
10. Let $\alpha_1, \ldots, \alpha_n$ be $\mathbb{Q}$-linearly independent algebraic integers in $\mathbb{Q}(\theta)$. If $\Delta[\alpha_1, \ldots, \alpha_n] = d$, the discriminant of $\mathbb{Q}(\theta)$, show that $\{\alpha_1, \ldots, \alpha_n\}$ is an integral basis for $\mathbb{Q}(\theta)$.