Explicit Arithmetic for Modular Curves

Exercises III

(A) (Computational Exercise.) A point P on a curve C of genus g is called a **Weierstrass point** if there is a non-zero regular differential $\omega \in \Omega(C)$ such that $\mathrm{ord}_P(\omega) \geq g$. Determine all $N \leq 100$ such that the ∞ cusp of $X_0(N)$ is a Weierstrass point.

Hint: For a particular value of N, the magma command `Basis(CuspForms(N))` will give a basis of cusp forms of weight 2 and level N. This basis $f_1, f_2, \ldots, f_g$ is always given in "Echelon" form: i.e. $\mathrm{ord}_q(f_{i+1}) > \mathrm{ord}_q(f_i)$.

(B) To do this exercise you need to a little about how to calculate valutions at points. If this is unfamiliar, perhaps skip this exercise.

(i) Let

$$X \ : \ y^2 = a_{2g+2}x^{2g+2} + \cdots + a_0$$

be a curve of genus g where $a_{2g+2} \neq 0$. Let ∞_+ be one of the two points at infinity. Show that

$$\mathrm{ord}_{\infty_+}\left(\frac{dx}{y}\right) = g-1, \quad \mathrm{ord}_{\infty_+}\left(\frac{xdx}{y}\right) = g-2, \ldots, \mathrm{ord}_{\infty_+}\left(\frac{x^{g-1}dx}{y}\right) = 0.$$

(ii) Let

$$X \ : \ y^2 = a_{2g+1}x^{2g+1} + \cdots + a_0$$

be a curve of genus g (here necessarily $a_{2g+1} \neq 0$ otherwise the genus would be smaller than g). Let ∞ be the unique point at infinity. Show that

$$\mathrm{ord}_{\infty}\left(\frac{dx}{y}\right) = 2(g-1), \quad \mathrm{ord}_{\infty}\left(\frac{xdx}{y}\right) = 2(g-2), \ldots, \mathrm{ord}_{\infty}\left(\frac{x^{g-1}dx}{y}\right) = 0.$$

(C) A basis for $S_2(\Gamma_0(64))$ is

$$\begin{gathered} q - 3q^9 + O(q^{12}), \\ q^2 - 2q^{10} + O(q^{12}), \\ q^5 + O(q^{12}) \end{gathered}$$

Deduce (very very quickly) that $X_0(64)$ is not hyperelliptic. (Hint: Use exercises (A), (B)).

Fun Fact: $X_0(64)$ is actually the Fermat quartic $x^4 + y^4 = z^4$.