Algebraic Geometry – Exercises 27 March 2007

Definition. Let X be a scheme over a scheme S. For any S-scheme T we define the set $X(T)$ of T-*valued points* as

$$X(T) := \mathrm{Hom}_{\mathrm{Sch}_S}(T, X).$$

In the special case $T = \mathrm{Spec}(A)$ we also use the notation $X(A)$ for $X(T)$ and speak of A-valued points.

1. *(3 points)* Hartshorne, exercise II.2.18.

2. *(2 points)* Hartshorne, exercise II.2.19.

3. *(3 points)* Determine the largerst integer n such that there are morphisms $f_1, \ldots, f_n$ from $\mathrm{Spec}(\mathbb{Z})$ to $\mathbb{A}^1_{\mathbb{Z}}$ whose images are mutually disjoint.

4. *(3 points)* Let $f_1, \ldots, f_m$ be elements of $\mathbb{Z}[x_1, \ldots, x_n]$ and consider

$$X = \mathrm{Spec}\left(\mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)\right).$$

Prove that for any ring A one can identify $X(A)$ with

$$\{(a_1, \ldots, a_n) \in A^n : f_i(a_1, \ldots, a_n) = 0 \text{ for all } i\}.$$

This clarifies the terminology of A-valued points.

5. *(3 points)* If X is a scheme over a scheme S then a morphism of S-schemes $T \to T'$ induces a map $X(T') \to X(T)$. In particular there is a natural map

$$\mathbb{P}^1_{\mathbb{Z}}(\mathbb{Z}) \to \mathbb{P}^1_{\mathbb{Z}}(\mathbb{Q}).$$

Show that this map is a bijection.