TCC Course
Arithmetic of Curves of Higher Genus
Example Sheet 2

Students taking this course for credit, please answer the homework questions and mail your solutions to me: Samir Siksek, Mathematics Institute, University of Warwick, Coventry CV4 7AL. Feel free to contact me by email if you have any questions: `samirsiksek@yahoo.com`

(1) Let C be a smooth projective curve over a field k.

(i) Show that $\operatorname{div} : k(C)^* \to \operatorname{Div}(C)$ is a homomorphism.

(ii) Suppose $\operatorname{ord}_P(f) \neq \operatorname{ord}_P(g)$, where $P \in C(\overline{k})$ and f, $g \in k(C)$. Show that

$$\operatorname{ord}_P(f+g) = \min\{\operatorname{ord}_P(f), \operatorname{ord}_P(g)\}.$$

(iii) Let $t \in \overline{k}(C)$ be a uniformizer at $P \in C(\overline{k})$. Show that any function of the form

$$s = \frac{a_1 t + a_2 t^2 + \cdots a_m t^m}{b_0 + b_1 t + \cdots b_n t^n}$$

with a_1, $b_0 \neq 0$ is also a uniformizer at P.

(2) Let $C \subset \mathbb{P}^2$ be the smooth projective curve with affine patch $y^2 = x^3 - x$ over field k of characteristic 0. Determine $\operatorname{div}(y)$ and $\operatorname{div}(dy)$.

(3) Let C be the affine curve $y^2 = x^4 + 1$. Show that $\deg(\operatorname{div}(f)) = 0$ need not hold for non-constant $f \in k(C)^*$. Show that there are non-constant $f \in k(C)^*$ with no poles.

(4) Let k be a field of characteristic $\neq 2$. Suppose a, b, $c \in k^*$ and let $C \subset \mathbb{P}^2$ be the curve

$$C : ax^4 + by^4 + cz^4 = 0.$$

(i) Show that C is non-singular of genus of C is 3.

(ii) Write down a positive k-rational divisor D_0 of degree 4.

(iii) Suppose C has a k-rational divisor D_1 of odd degree. Show that C has a k-rational divisor D of degree 1. (**Hint:** Take D to be a suitable linear combination of D_0 and D_1.)

(iv) Continuing with the assumption of (iii), show that either $C(k) \neq \emptyset$, or that there is a positive k-rational divisor of degree 3. (**Hint:** You need to consider two cases, according to whether $l(K - 3D)$ is positive or zero.)