Algebraic Number Theory (MA3A6) - Problem Sheet 6

This sheet is not for submission, but you may ask for help or feedback if you wish. You could also review questions from previous sheets.

1. In $\mathbb{Z}[\sqrt{-13}]$, define the ideals

$$\begin{aligned} \mathfrak{p} &= (2, 1+\sqrt{-13}), \\ \mathfrak{q} &= (7, 1+\sqrt{-13}), \\ \mathfrak{r} &= (7, 1-\sqrt{-13}). \end{aligned}$$

   Prove that these are maximal ideals, hence prime. Show that

$$\begin{aligned} \mathfrak{p}^2 &= (2), & \mathfrak{q}\mathfrak{r} &= (7), \\ \mathfrak{p}\mathfrak{q} &= (1+\sqrt{-13}), & \mathfrak{p}\mathfrak{r} &= (1-\sqrt{-13}). \end{aligned}$$

   Factorise the ideal (14) of of $\mathbb{Z}[\sqrt{-13}]$ into prime ideals and find two different factorisations of the element 14 into irreducible elements.

2. Prove that the ideals $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ in the previous exercise cannot be principal.

3. Find all ideals in $\mathbb{Z}[\sqrt{-13}]$ that contain the element 14.

4. Which of these rings are Dedekind domains? Explain your answer fully.

   (i) $\overline{\mathbb{Z}}$

   (ii) $\mathbb{Z}[x]$

   (iii) $\mathbb{Z}[\frac{1}{2}]$

   (iv) $\mathbb{C}[x]$

   (v) $\mathbb{C}[x, y]$

   (vi) $\mathbb{Z}[\sqrt{8}]$

   (vii) $\mathbb{Z}[i]$

   (viii) $\mathbb{Z}[\sqrt{17}]$

   (ix) $\mathbb{C}[x^2, x^3]$

   (x) $\mathbb{C}[x, y]/(x^2 + y^2)$

5. For each of these ideals, determine whether they are prime and whether they are maximal as ideals of $\mathbb{Z}[x]$ and as ideals of $\mathbb{Q}[x]$:

$$(x-7, 3), \quad (x^2-7), \quad (x^2-7, 3), \quad (x^2-7, 5).$$