Algebraic Number Theory
Example Sheet 1

1. Which of the following are algebraic numbers and which are algebraic integers:

   $$23/7, \quad \sqrt{-5}, \quad \sqrt{2}+\sqrt{7}, \quad \exp(2\pi i/5), \quad \cos(2\pi/7).$$

   Write their minimal polynomials, conjugates, norms and traces.

2. Let $f(X) = X^3 - 2X - 2$.
   (i) Show that f is irreducible.
   (ii) Let θ be a root of f. Find the minimal polynomial for $\theta+1$.
   (iii) Find the minimal polynomial of θ^2.
   (**Hint for (iii):** You know that $\theta^3 - 2\theta = 2$. Now square both sides.)

3. Suppose d_1, d_2 are non-square integers. Show $\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})$ if and only if d_1/d_2 is a square of a rational number.

4. Use reduction modulo 5 to show that the polynomial $X^5 - X + 1$ is irreducible.

5. Let p be a prime number. Write $f(X) = X^{p-1} + X^{p-2} + \cdots + 1$. Prove that f is irreducible.
   (**Hint:** You know that $f(X) = (X^p - 1)/(X - 1)$. Now use Eisenstien to show that $f(X+1)$ is irreducible.)

6. You are given that the only integer solutions to the equation $u^3 - 2v^3 = 1$ are $(u, v) = (1, 0)$ and $(u, v) = (-1, -1)$. Use this fact to find the integer solutions to the Diophantine equation $x^2 - 1 = y^3$.

7. Show that $x^2 + 1 \not\equiv 0 \pmod 8$ for all integers x. Deduce that if x, y are integers satisfying $x^2 + 1 = y^3$ then x is even and y is odd. Use the fact that $\mathbb{Z}[i]$ is a unique factorisation domain to solve this Diophantine equation.
   (This is similar to the way we solved $4x^2 - 1 = y^3$ in the lectures. You will need a similar lemma. Make sure you justify every step.)