ON FACTORIALS EXPRESSIBLE AS SUMS OF AT MOST THREE FIBONACCI NUMBERS

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Abstract. In this paper, we determine all the factorials which are a sum of at most three Fibonacci numbers.

1. introduction

Diophantine equations involving factorials have a long history. For example, in 1876 Brocard [6] asked for the integral solutions of $n!+1=x^2$; this was asked again (apparently independently) by Ramanujan [17] in 1913. The Brocard–Ramanujan equation is still an unsolved problem today; see D25 in Guy's book [13]. Other Diophantine equations involving factorials have proved more tractable. For example, Erdős and Obláth [9] showed that the equation $x^p+y^p=n!$ has no solutions with x, y coprime and $p>2$. Many have considered equations of the form $P(x)=n!$ where P is a polynomial; the best results so far appear to be those of Berend and Harmse [1] who show that there are only finitely many solutions if P has an irreducible factor of relatively large degree.

Diophantine equations involving Fibonacci numbers have been no less popular as documented Guy's book [13, D25] and in the historical sections of [8] and [7]. Moreover, there have been several papers attacking Diophantine equations that involve both factorials and Fibonacci numbers. For example, in [12] it is shown that if k is fixed, then there are only finitely many positive integers n such that

$$F_n = m_1! + m_2! + \cdots + m_k!$$

holds for some positive integers $m_1, \ldots, m_k$ and all solutions of the above equation with $k \le 2$ have been determined. It is conjectured in [12] that if $m_1 < m_2 < \cdots < m_k$ holds in the above equation, then k itself must be bounded. Some results on this problem can be found in [3]. In [16], it was shown that the largest solution of the Diophantine equation

$$F_{n_1} F_{n_2} \cdots F_{n_k} = m_1! m_2! \cdots m_\ell!$$

with $1 \le n_1 < \cdots < n_k$ and $2 \le m_1 \le \cdots \le m_\ell$ is $F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12} = 11!$.

In this paper we prove the following result.

Theorem 1. *Let $(F_m)_{m\ge 0}$ be the Fibonacci sequence given by $F_0=0,\ F_1=1$ and $F_{m+2}=F_{m+1}+F_m$ for all $m\ge 0$. The only factorials expressible as the sum of at*

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most three Fibonacci numbers are

$$0! = 1! = F_1 = F_2, \qquad 2! = F_1 + F_1 = F_2 + F_1 = F_2 + F_2 = F_3,$$
$$3! = F_4 + F_4 = F_5 + F_1 = F_5 + F_2 = F_3 + F_3 + F_3 = F_4 + F_3 + F_1 = F_4 + F_3 + F_2,$$
$$4! = F_8 + F_4 = F_6 + F_6 + F_6 = F_7 + F_6 + F_4 = F_8 + F_3 + F_1 = F_8 + F_3 + F_2,$$
$$6! = F_{15} + F_{10} + F_{10} = F_{15} + F_{11} + F_8.$$

It is not hard to show that every positive integer N has a representation, called the *Zeckendorf decomposition*, of the form $N = F_{n_1} + F_{n_2} + \cdots + F_{n_s}$, where $n_i - n_{i+1} \geq 2$ and that, up to identifying F_1 with F_2, this representation is unique. Our problem is therefore related to the Zeckendorf decomposition of factorials. Denote by $Z(N)$ the number s of Fibonacci numbers appearing in the Zeckendorf decomposition of N.

Conjecture. *$Z(n!)$ tends to infinity with n.*

We are unable to prove our conjecture, but our Theorem 1 determines all positive integers n such that $Z(n!) \leq 3$.

It is appropriate to point out some analogous results to our conjecture appearing in [15]. Let $b \geq 2$ be a positive integer. For a positive integer N let $s_b(N)$ be the sum of the base b digits of n. In [15], it is shown that the inequality $s_b(n!) \gg \log n$ holds for all positive integers n, where the implied constant depends on b. Thus, the complexity of representing $n!$ in base b grows as n tends to infinity. The method of proof is elementary and it is based on the observation that $n!$ is a multiple of $b^m - 1$ for all $m = 1, 2, \ldots, \lfloor \log n / \log b \rfloor$.

Particular Diophantine equations of the form

$$a_1^{x_1} + \cdots + a_k^{x_k} = n!,$$

where $a_1, \ldots, a_k$ are given positive integers and $x_1, \ldots, x_k, n$ are nonnegative integer unknowns have been studied in the recent paper [11]. For example, all the solutions of the Diophantine equation

$$2^{x_1} + 3^{x_2} + 5^{x_3} + 7^{x_4} + 11^{x_5} = n! \tag{1}$$

have $n \leq 6$.

For the purpose of the present paper, as $F_0 = 0$, it suffices to determine all solutions to the following Diophantine equation

$$F_x + F_y + F_z = n!, \qquad x, y, z \geq 0, \quad n \geq 1. \tag{2}$$

Before doing this, we explain very briefly why our method for solving this equation is far more complicated that the method for solving (1) in [11]. To solve (1), all we have to do is to find a positive integer M such that the congruence

$$2^{x_1} + 3^{x_2} + 5^{x_3} + 7^{x_4} + 11^{x_5} \equiv 0 \pmod{M}$$

has no solutions. Once this is done, we know that for any solution to (1), $M \nmid n!$, giving a bound on n. This elementary idea cannot be used for (2); for example, $F_0 + F_{-2} + F_1 = 0$, and so the congruence $F_x + F_y + F_z \equiv 0 \pmod{M}$ has solutions for all M.

Our strategy for (2) is as follows. Let $(L_m)_{m \geq 0}$ be the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$ and $L_{m+2} = L_{m+1} + L_m$ for all $m \geq 0$. Let $m \geq 6$ be an even integer such that $L_{m/2} \leq n$. We compute the first few terms of an expansion of F_x as an '$F_{m/2}$–adic' power series, in a way that is very similar to Strassman's Theorem

[10, pages 59–73], except that we do not require $F_{m/2}$ to be prime. From this, we deduce congruence conditions modulo m and modulo $F_{m/2}$ on the unknowns x, y, z in (2); the idea here is reminiscent of Skolem's method [10, pages 228–231], [4, pages 290–300]. We use the Chebyshev θ-function to combine the information obtained from all even $m \geq 6$ with $L_{m/2} \leq n$. For n very large, this shows that x, y, z are too large compared to n for the equation (2) to hold, and so gives a bound on n. Our initial bound obtained this way is $n \leq L_{501} < 5.045 \times 10^{104}$. An iterative argument, using the same information derived from the 'Strassman' expansion, is applied 50 times to reduce the bound to $n \leq L_{37} = 54,018,521$. The proof is completed using a sieving argument.

2. Inequalities

In this section, we gather some inequalities that will be useful later.

Lemma 2.1. *For all integers $n \geq 2$,*

$$\log(n!) < (n+1/2)\log n - n + 1 \leq n \log n. \tag{3}$$

Proof. By Stirlings' formula,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\lambda_n},$$

where

$$\frac{1}{12n+1} < \lambda_n < \frac{1}{12n}.$$

Hence,

$$\log(n!) \leq (n+1/2)\log n - n + \frac{1}{2}\log(2\pi) + \lambda_n.$$

But $\log(2\pi)/2 + \lambda_n \leq \log(2\pi)/2 + 1/24 < 1$, leading to the first inequality in the statement of the lemma. To obtain the second, we need to show that

$$n \geq \frac{\log n}{2} + 1$$

holds for all $n \geq 2$. This in fact true for $n = 2$, and so true for all real $n \geq 2$ since

$$\frac{d}{dt}\left(t - \frac{\log t}{2} - 1\right) = 1 - \frac{1}{2t} > 0 \qquad \text{for all } t \geq 2.$$

□

We write $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ for the two roots of the characteristic equation $\lambda^2 - \lambda - 1 = 0$ of the Fibonacci sequence. It is well-known that the *Binet formula*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{holds for all } n \geq 0.$$

We will find it convenient to extend the Fibonacci sequence to negative subscripts either using Binet's formula directly, or by defining $F_{n-2} = F_n - F_{n-1}$ for all $n = 1, 0, -1, \ldots$. The Binet formula for the Lucas numbers is

$$L_n = \alpha^n + \beta^n \qquad \text{for all } n \geq 0.$$

As with the Fibonacci numbers, we will sometimes make use of negative subscripted Lucas numbers. It is easy to see that if $n \geq 0$, then $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$.

Here are a few inequalities involving the Fibonacci and Lucas numbers.

Lemma 2.2. *Any solution to the Diophantine equation* (2) *with* $n \geq 3$ *satisfies*

$$\max(x, y, z) \leq \frac{n \log n}{\log \alpha} - 1\,.$$

Proof. From $F_x \leq n!$ and the Binet formula, we obtain that $\alpha^x \leq \sqrt{5}n! + 1$. Hence,

$$\begin{aligned} x \log \alpha &\leq \log(n!) + \log(\sqrt{5}) + \log(1 + 1/\sqrt{5}n!) \\ &\leq (n + 1/2)\log n - n + 1 + \log(\sqrt{5}) + \log(1 + 1/6\sqrt{5}). \end{aligned}$$

The lemma follows from the inequality

$$n \geq \frac{\log n}{2} + 1 + \log \alpha + \log(\sqrt{5}) + \log(1 + 1/6\sqrt{5}),$$

which is easily established for $n \geq 3$ by modifying the argument at the end of the proof of Lemma 2.1. □

Lemma 2.3. *Let* $y \geq 1$. *If* $x \geq 2.079 \log y + 2.441$, *then* $F_x \geq y$. *Moreover, if* $y \geq 200$ *and* $x \leq 2.076 \log y$, *then* $L_x \leq y$.

Proof. For the first part, note that $2.079 \log \alpha = 1.0004\ldots$ to four decimal places. Thus,

$$\begin{aligned} x \log \alpha &\geq \log y + 2.441 \log \alpha \\ &\geq \log \sqrt{5}y + \log(1 + 1/\sqrt{5}) \\ &\geq \log \sqrt{5}y + \log(1 + 1/(\sqrt{5}y)). \end{aligned}$$

Hence, $\alpha^x \geq \sqrt{5}y + 1$, giving $F_x \geq y$. For the second part, observe that $2.076 \log \alpha = 0.998996\ldots$ to six decimal places. So, $x \log \alpha \leq 0.999 \log y$, and therefore

$$L_x \leq \alpha^x + 1 \leq y^{0.999} + 1 \leq y \left(\frac{1}{200^{0.001}} + \frac{1}{200} \right) < y.$$

□

Lemma 2.4. *For* $x \geq 0$, *we have* $\log(1 + x) \leq x$. *For* $0 \leq x \leq 1/2$, *we have* $\log(1 - x) \geq -2x$.

2.1. **The Chebyshev Function.** We shall need some estimates involving the Chebyshev function

$$\theta(x) = \sum_{p \leq x} \log p,$$

where the sum is taken over all primes $\leq x$. It is well-known that $\theta(x)/x \to 1$ as $x \to \infty$. Here, we need lower estimates for this ratio for small values of x.

Proposition 2.5. *For all real* $x \geq 1$, $\theta(x) \leq 1.001102x$. *Moreover,*

$$\begin{array}{ll} \text{if } 10 \leq x \leq 20, & \text{then } \theta(x)/x \geq 0.4861 \\ \text{if } 20 \leq x \leq 30, & \text{then } \theta(x)/x \geq 0.6628 \\ \text{if } 30 \leq x \leq 40, & \text{then } \theta(x)/x \geq 0.7033 \\ \text{if } 40 \leq x \leq 50, & \text{then } \theta(x)/x \geq 0.7228 \\ \text{if } 50 \leq x \leq 500, & \text{then } \theta(x)/x \geq 0.7615 \\ \text{if } 500 \leq x \leq 1000, & \text{then } \theta(x)/x \geq 0.9194 \\ \text{if } x \geq 1000, & \text{then } \theta(x)/x \geq 0.9456. \end{array}$$

Proof. Theorem 6 of [18] gives $\theta(x) < 1.001102x$ if $0 < x$, and $\theta(x) \geq 0.998684x$ if $x \geq 1319007$. To obtain the lower bounds claimed by the proposition, we used a simple `MAGMA` script to determine the infima of $\theta(x)/x$ in the finite ranges above as well as in the range $1000 \leq x \leq 2 \times 10^6$. Note that over the interval $[p, p']$, where p, p' are primes, the infimum of $\theta(x)/x$ is $\theta(p)/p'$. $\square$

3. Elementary Lemmas

We shall also need the following elementary properties of the Fibonacci and Lucas numbers. Properties (4), (5) and (6) are well-known (see for example [14]), and can be proved immediately using the Binet formulas for the Fibonacci and Lucas numbers. For integers n,

$$F_{2n} = F_n L_n, \quad L_{2n} = 5F_n^2 + 2(-1)^n, \quad L_n^2 = 5F_n^2 + 4(-1)^n; \tag{4}$$

For all pairs of integers m and n,

$$2F_{m+n} = F_m L_n + F_n L_m, \qquad 2L_{m+n} = L_m L_n + 5F_m F_n; \tag{5}$$

If m and n have the same parity, then

$$F_m + F_n = \begin{cases} F_{(m+n)/2} L_{(m-n)/2} & \text{if } m \equiv n \pmod 4, \\ F_{(m-n)/2} L_{(m+n)/2} & \text{if } m \not\equiv n \pmod 4. \end{cases} \tag{6}$$

Lemma 3.1. *Let m be a non-zero integer. Then*

$$F_{\lambda m}^2 \equiv (-1)^{(\lambda+1)m} \lambda^2 F_m^2 \pmod{F_m^4}.$$

Moreover, if m is even, then

$$F_{\lambda m} \equiv (-1)^{m(\lambda+1)/2} \lambda F_m \pmod{F_{m/2}^3}.$$

Proof. Define

$$H_\lambda = \left(\frac{F_{\lambda m}}{F_m}\right)^2 = \frac{\alpha^{2\lambda m} + \beta^{2\lambda m} - 2(-1)^{\lambda m}}{(\alpha^m - \beta^m)^2}.$$

This is a ternary recurrence sequence with characteristic polynomial

$$(X - \alpha^{2m})(X - \beta^{2m})(X - (-1)^m) = (X^2 - L_{2m}X + 1)(X - (-1)^m).$$

However, from (4) we have $L_{2m} \equiv 2(-1)^m \pmod{F_m^2}$. Hence,

$$H_{\lambda+3} \equiv 3(-1)^m H_{\lambda+2} - 3H_{\lambda+2} + (-1)^m H_\lambda \pmod{F_m^2}.$$

Moreover, $H_0 = 0$, $H_1 = 1$, and $H_2 = L_m^2 \equiv 4(-1)^m \pmod{F_m^2}$, again by (4). An easy induction proves shows that

$$H_\lambda \equiv (-1)^{(\lambda+1)m} \lambda^2 \pmod{F_m^2},$$

and multiplying by F_m^2 completes the proof the first part of the lemma.

The proof of the second part is similar, but easier, using the binary recurrence sequence of general term $G_\lambda = F_{\lambda m}/F_m$. $\square$

Lemma 3.2. *Let m be a non-zero even integer. Then*

$$F_{x_0+2\lambda m} \equiv F_{x_0} \pmod{F_m}, \tag{7}$$

and

$$2F_{x_0+\lambda m} \equiv 2(-1)^{m\lambda/2} F_{x_0} + 2(-1)^{m(\lambda+1)/2} F_m L_{x_0} \lambda + 5F_m^2 F_{x_0} \lambda^2 \pmod{F_{m/2}^3}. \tag{8}$$

Proof. Using (5), we see that

$$\begin{aligned} 2F_{x_0+2\lambda m} &= F_{x_0}L_{2\lambda m} + L_{x_0}F_{2\lambda m} \\ &= F_{x_0}(2+5F_{\lambda m}^2) + L_{x_0}F_{2\lambda m}. \end{aligned}$$

However, F_m divides $F_{\lambda m}$ and $F_{2\lambda m}$. If $2 \nmid F_m$, then (7) follows. Suppose now that $2 \mid F_m$. Then $2 \mid L_m$, and we note that $2F_m$ divides both $F_{\lambda m}^2$ and $L_m F_m = F_{2m} \mid F_{2\lambda m}$. Hence, $2F_{x_0+2\lambda m} \equiv 2F_{x_0} \pmod{2F_m}$. This completes the proof of (7).

We now drop the requirement that 2 divides F_m and we move on to prove (8). To this aim, we combine (4) and (5) with Lemma 3.1 to obtain

$$\begin{aligned} 2F_{x_0+\lambda m} &= F_{x_0}L_{\lambda m} + L_{x_0}F_{\lambda m} \\ &= F_{x_0}(2(-1)^{m/2} + 5F_{\lambda m}^2) + L_{x_0}F_{\lambda m} \\ &\equiv F_{x_0}(2(-1)^{m/2} + 5F_m^2\lambda^2) + (-1)^{m(\lambda+1)/2}F_m L_{x_0}\lambda \pmod{F_{m/2}^3}, \end{aligned}$$

which gives (8). □

4. Some Congruences

The following two results are useful in applying the 'Strassman procedure' alluded to in the introduction.

Lemma 4.1. *Let $m \geq 4$ be even and $-m < x_0, y_0 \leq m$ be integers such that x_0 is odd, y_0 is even and $F_{x_0} + F_{y_0} \equiv 0 \pmod{F_m}$. Then $(x_0, y_0) \in \{(\pm(m-1), m-2), (\pm 1, 2)\}$.*

Proof. Since $x_0, y_0 \in (-m, m]$ and $m \geq 4$, it follows that if either one of F_{x_0} or F_{y_0} is negative, then it is $< F_m$ in absolute value. Thus, $F_{x_0}+F_{y_0} \in \{-F_m, 0, F_m, 2F_m\}$.

The case $F_{x_0} + F_{y_0} = 2F_m$ is impossible since $F_{x_0} \leq F_m$, $F_{y_0} \leq F_m$ and at least one of these two inequalities is strict because x_0 is not equal to y_0.

Assume that $F_{x_0} + F_{y_0} = F_m$. Since both x_0 and y_0 are at most m, it follows that $F_{x_0} \geq 0$ and $F_{y_0} \geq 0$. Further, both these inequalities are in fact strict since equality is achieved only in the case $y_0 = 0$ leading to $F_{x_0} = F_m$, so $x_0 = m$, which in turn is impossible because x_0 must be odd. Thus, both F_{x_0} and F_{y_0} are positive and $< F_m$. If $\max\{F_{x_0}, F_{y_0}\} \leq F_{m-2}$, then

$$F_{m-3} + 2F_{m-2} = F_m = F_{x_0} + F_{y_0} < 2F_{m-2},$$

leading to $F_{m-3} < 0$, which is impossible. So, one of F_{x_0} or F_{y_0} equals F_{m-1}, and therefore the other one is $F_m - F_{m-1} = F_{m-2}$. By parity arguments, we get that $F_{y_0} = F_{m-2}$ and $F_{x_0} = F_{m-1}$; so, $(x_0, y_0) = (\pm(m-1), m-2)$.

Assume that $F_{x_0} + F_{y_0} = 0$. Then $F_{x_0} = -F_{y_0} = F_{-y_0}$, since y_0 is even. Since x_0 is odd, F_{x_0} is positive, so $y_0 < 0$. If $y_0 = -2$, we then get $F_{x_0} = F_2 = 1$, leading to $x_0 = \pm 1$. This gives us the possibilities $(x_0, y_0) = (\pm 1, -2)$. Finally, if $y_0 \leq -4$, then $x_0 = \pm y_0$, which is false since x_0 must be odd.

Assume that $F_{x_0} + F_{y_0} = -F_m$. Then $F_m \geq |F_{y_0}| = |-F_m - F_{x_0}| = F_m + F_{x_0} > F_m$, again because $F_{x_0} > 0$, but this last inequality is false. This completes the proof of this lemma. □

Lemma 4.2. *Let $m \geq 6$ be even and let $-m < x_0, y_0, z_0 \leq m$ be integers satisfying $F_{x_0} + F_{y_0} + F_{z_0} \equiv 0 \pmod{F_m}$. Then*

(i) *either all three x_0, y_0, z_0 are even and their sum is a multiple of m, or*

(ii) *up to symmetries,* (x_0, y_0, z_0) *equals* $(a, \pm(a+1), -a-2)$ *for some even integer* a*, or*
(iii) *up to symmetries,* (x_0, y_0, z_0) *equals one of* $(\pm 3, -2, -2)$*,* $(\pm 1, \pm 3, -4)$*,* $(m, \pm 1, -2)$*,* $(m, \pm(m-1), m-2)$*,* $(\pm(m-1), \pm(m-3), m-4)$*,* $(m-2, m-2, \pm(m-3))$*,* $(0, \pm(m-1), m-2)$*.*

Proof. Assume that one of $x_0, y_0, z_0 \in \{0, m\}$. Say $x_0 \in \{0, m\}$. Then $F_{y_0} + F_{z_0} \equiv 0 \pmod{F_m}$. If y_0 and z_0 have distinct parities, then, by Lemma 4.1, we get that, up to symmetries,

$$(x_0, y_0, z_0) \in \{(m, \pm(m-1), m-2), (m, \pm 1, -2)\}.$$

Assume now that y_0 and z_0 have the same parities. If $y_0 \in \{0, m\}$, then $F_m \mid F_{z_0}$, so that $z_0 \in \{0, m\}$ and $x_0 + y_0 + z_0 \equiv 0 \pmod{m}$. Assume now that none of y_0, z_0 is in $\{0, m\}$. If they are even, then $|F_{y_0}| \le F_{m-2}$ and $|F_{z_0}| \le F_{m-2}$, therefore $|F_{y_0} + F_{z_0}| \le 2F_{m-2} < F_m$, so $F_{y_0} + F_{z_0} = 0$, leading to $F_{y_0} = -F_{z_0} = F_{-z_0}$. Since both y_0 and z_0 are even, we get that $y_0 = -z_0$, so indeed all three numbers $x_0,\ y_0,\ z_0$ are even and $x_0 + y_0 + z_0 \equiv 0 \pmod{m}$. If y_0 and z_0 are odd, then we may replace y_0 and z_0 by their absolute values and note that F_{y_0} and F_{z_0} are both positive. Assume that $y_0 \ge z_0$. If $y_0 \le m-3$, then $F_{y_0} + F_{z_0} \le 2F_{m-3} < F_m$, which is impossible. If $y_0 = m-1$, then $F_{z_0} = F_m - F_{m-1} = F_{m-2}$, leading to $z_0 = m-2$ (because $m-2 \ge 4$), which is false since z_0 is odd.

From now on, we assume that $x_0, y_0, z_0 \in (-m, m)$ and that none of them is zero. Then $|F_{x_0} + F_{y_0} + F_{z_0}| \le 3F_{m-1} < 2F_m$, so that $F_{x_0} + F_{y_0} + F_{z_0} \in \{0, \pm F_m\}$. Assume that $F_{x_0} + F_{y_0} + F_{z_0} = 0$. Since none of these numbers is zero, it follows that at least one of them is negative. Say z_0 is such that F_{z_0} is negative and has the largest absolute value (among the negative numbers from the set $\{F_{x_0}, F_{y_0}, F_{z_0}\}$). Then z_0 is even and negative and $F_{x_0} + F_{y_0} = F_{-z_0}$. Assume first that F_{x_0} and F_{y_0} are positive. Then $|y_0| < |z_0|$. If $|y_0| = |z_0| - 1 = -z_0 - 1$, we then get that $F_{x_0} = F_{-z_0-2}$. If $|z_0| \ge 6$, then $x_0 = -z_0 - 2$. Putting $x_0 = a$, we get that $x_0 = a,\ y_0 = \pm(a+1),\ z_0 = -a-2$. Thus, we got the possibility

$$(x_0, y_0, z_0) = (a, \pm(a+1), -a-2), \qquad \text{for some even } a.$$

If $z_0 = -4$, then $|y_0| = 3$ and $F_{x_0} = F_2$, therefore $x_0 \in \{\pm 1, 2\}$. The case $x_0 = 2$ is part of the previous parametric family with $a = 2$, while for $x_0 = \pm 1$, we get the possibilities

$$(x_0, y_0, z_0) = (\pm 1, \pm 3, -4).$$

Assume now still that $F_{x_0} + F_{y_0} = F_{-z_0}$ but that one of F_{x_0} and F_{y_0} is negative. Say $F_{y_0} < 0$. Then y_0 is even and negative. Thus, $F_{x_0} = F_{|y_0|} + F_{|z_0|}$. If $|z_0| = 2$, then $|y_0| = 2$ and we get that $F_{x_0} = 2$, so $x_0 = \pm 3$. Thus, we got the possibility

$$(x_0, y_0, z_0) = (\pm 3, -2, 2).$$

If $|z_0| \ge 4$, then either $F_{|y_0|} + F_{|z_0|} = 2F_{|z_0|} \in (F_{|z_0|+1}, F_{|z_0|+2})$, so this is not a Fibonacci number, or $|y_0| \le |z_0| - 2$, therefore $F_{|z_0|} < F_{|z_0|} + F_{|y_0|} \le F_{|z_0|} + F_{|z_0|-2} < F_{|z_0|+1}$, so $F_{|z_0|} + F_{|y_0|}$ is not a Fibonacci number either.

Now assume that $F_{x_0} + F_{y_0} + F_{z_0} = F_m$. If $\max\{F_{x_0}, F_{y_0}, F_{z_0}\} \le F_{m-3}$, we then get that $F_{x_0} + F_{y_0} + F_{z_0} \le 3F_{m-3} < F_m$, which is impossible. So, let us assume that $F_{x_0} \in \{F_{m-1}, F_{m-2}\}$ and that $F_{x_0} \ge F_{y_0} \ge F_{z_0}$. If $F_{x_0} = F_{m-1}$, then $F_{y_0} + F_{z_0} = F_{m-2}$. If $F_{y_0} = F_{m-1}$ also, then $F_{z_0} = F_{m-2} - F_{m-1} = -F_{m-3}$, which is impossible since $m-3$ is odd. Clearly, $F_{y_0} \ne F_{m-2}$, since the contrary leads to $F_{z_0} = 0$. If $F_{y_0} = F_{m-3}$, then $F_{z_0} = F_{m-2} - F_{m-3} = F_{m-4}$. Thus, we have obtained the possibilities

$$(x_0, y_0, z_0) = (\pm(m-1), \pm(m-3), m-4).$$

If $F_{y_0} \le F_{m-4}$, then $F_{z_0} = F_{m-2} - F_{y_0} \ge F_{m-3} > F_{m-4} \ge F_{y_0}$, which is impossible. This takes care of the case when $F_{x_0} = F_{m-1}$. Assume now that $F_{x_0} = F_{m-2}$. Then $F_{y_0} + F_{z_0} = F_{m-1}$. If $F_{y_0} \le F_{m-3}$, then $F_{y_0} + F_{z_0} \le 2F_{m-3} < F_{m-1}$, which is a contradiction. Thus, $F_{y_0} = F_{m-2}$, giving $F_{z_0} = F_{m-3}$. Hence,

$$(x_0, y_0, z_0) = (m-2, m-2, \pm(m-3)).$$

Now assume that $F_{x_0} + F_{y_0} + F_{z_0} = -F_m = F_{-m}$. If at most two of the Fibonacci numbers involved were negative, then they were in absolute value $\le F_{m-2}$ (because their indices are even). Thus, $F_m = |F_{x_0} + F_{y_0} + F_{z_0}| < 2F_{m-2}$, which is false. Consequently, all three Fibonacci numbers are negative, so all their indices are negative and even. Changing (x_0, y_0, z_0) to $(-x_0, -y_0, -z_0) = (|x_0|, |y_0|, |z_0|)$, we get a solution to

$$F_{|x_0|} + F_{|y_0|} + F_{|z_0|} = F_m.$$

If at most two of the Fibonacci numbers involved are $\le F_{m-2}$, then

$$F_m = F_{|x_0|} + F_{|y_0|} + F_{|z_0|} \le 2F_{m-2} + F_{m-4} < F_m,$$

which is impossible, while if all three of them are F_{m-2}, we then get $3F_{m-2} = F_m$, which is also impossible for $m \ge 6$ since the left hand side is in fact larger than the right hand side.

This completes the proof of Lemma 4.2. □

5. Skolem's method

In this section—using an argument similar to Skolem's method—we deduce that if x, y, z, n is a solution of the Diophantine equation (2), then certain linear forms in x, y, z are multiples of m or $F_{m/2}$ for all even integers $m \ge 6$ such that F_m is not too large with respect to n. Throughout this section, we study the equation

$$F_x + F_y + F_z = n!$$

in non-negative integers x, y, z with $n \ge 7$. From now on, we make the following convention. If precisely two of the unknowns x, y, z are even, then we shall suppose that these are x and z. If exactly one of them is even, we shall suppose that it is x.

Lemma 5.1. *Let x, y, z, n are as above. Let $m \ge 6$ be an even integer such that both F_m and $2F_{m/2}^2$ divide $n!$ (a sufficient condition for both of these to divide $n!$ is $L_{m/2} \le n$). Then*

(a) *Not all of x, y, z are odd.*
(b) *If x, y, z are all even, m divides $x + y + z$.*
(c) *If x is even and y, z are odd, m divides $x + 4$.*
(d) *If x, z are both even and y is odd, either m divides $x + z + 2$, or $F_{m/2}$ divides $3x \pm 4y + 3z$. Moreover, this latter expression is non-zero.*

Proof. Let $m \ge 6$ be an even integer. First, we prove the observation that $L_{m/2} \le n$ implies that both F_m and $2F_{m/2}^2$ divide $n!$. Thus, suppose that $L_{m/2} \le n$. Then $F_{m/2} < L_{m/2}$ and $F_m = F_{m/2}L_{m/2}$. Hence, F_m divides $n!$. Clearly, $2F_{m/2}^2$ divides $n!$ for $m = 6$. Suppose $m \ge 8$, so $F_{m/2} < \frac{1}{2}L_{m/2}$. Thus, $2F_{m/2}^2$ divides $n!$. This proves the observation.

From now on, we drop the condition that $L_{m/2} \le n$, but assume that both F_m and $2F_{m/2}^2$ divide $n!$. Write

$$x = x_0 + 2\lambda m, \qquad y = y_0 + 2\mu m, \qquad z = z_0 + 2\epsilon m,$$

where $-m < x_0, y_0, z_0 \le m$. By (7), $F_{x_0} + F_{y_0} + F_{z_0} \equiv 0 \pmod{F_m}$. Lemma 4.2 gives a number of possibilities for x_0, y_0, z_0. Clearly x, y, z have the respectively the same parities as x_0, y_0, z_0. By examining the possibilities in Lemma 4.2, we see that x, y, z are not all odd, and that if x, y, z are all even then m divides $x+y+z$.

Suppose that two of x, y, z are odd and one is even. By our convention above, x must be even. Then, from Lemma 4.2, we see that (x, y, z) is congruent modulo m to one of

$$(-4, \pm 1, \pm 3), (-4, \pm 3, \pm 1),$$

showing that m divides $x + 4$.

There now only remains the case in which precisely two of x, y, z are even and one is odd. By our convention, x, z are even and y is odd. From Lemma 4.2, we see that (x, y, z) is congruent modulo $2m$ to one of

$$(a, \pm(a+1), -a-2),\ (-a-2, \pm(a+1), a),\ (m, \pm(m-1), m-2),\ (m-2, \pm(m-1), m),$$
$$(0, \pm(m-1), m-2),\ (m-2, \pm(m-1), 0),\ (m, \pm 1, -2),\ (-2, \pm 1, m);$$

or to one of $(-2, \pm 3, -2)$, $(m-2, \pm(m-3), m-2)$. In all but the last two cases, m divides $x + z + 2$.

It remains to consider the case where (x, y, z) is congruent modulo $2m$ to one of $(-2, \pm 3, -2)$, $(m-2, \pm(m-3), m-2)$. Note here that $x \equiv z \pmod 4$, and $(x, y, z) \equiv (-2, \pm 3, -2) \pmod m$. We now write

$$x = -2 + \lambda_1 m, \qquad y = \pm 3 + \mu_1 m, \qquad z = -2 + \epsilon_1 m.$$

Observe that λ_1, μ_1 and ϵ_1 have the same parity. Moreover, $F_{-2} + F_{\pm 3} + F_{-2} = 0$. It follows from (8) that

$$2n! \equiv \pm 2F_m(L_{-2}\lambda_1 + L_{\pm 3}\mu_1 + L_{-2}\epsilon_1) + 5F_m^2(F_{-2}\lambda_1^2 + F_{\pm 3}\mu_1^2 + F_{-2}\epsilon_1^2) \pmod{F_{m/2}^3}.$$

Here, our observation that λ_1, μ_1 and ϵ_1 have the same parity is crucial. We now consider two subcases. The first is $3 \nmid m$. This means that $F_{m/2}$, $L_{m/2}$ are odd and coprime. Recall that $2F_{m/2}^2$ divides $n!$. From $F_m = F_{m/2}L_{m/2}$ and the coprimality of $F_{m/2}$ and $L_{m/2}$, we obtain that $F_{m/2}$ divides $L_{-2}\lambda_1 + L_{\pm 3}\mu_1 + L_{-2}\epsilon_1 = 3\lambda_1 \pm 4\mu_1 + 3\epsilon_1$. Thus, $3x \pm 4y + 3z = m(3\lambda_1 \pm 4\mu_1 + 3\epsilon_1)$ is divisible by $F_{m/2}$. The second subcase is $3 \mid m$. Hence, $F_{m/2}$ and $L_{m/2}$ are both even, and their greatest common divisor is 2. We now obtain that $\frac{1}{2}F_{m/2}$ divides $3\lambda_1 \pm 4\mu_1 + 3\epsilon_1$. But m is even. Thus, $3x \pm 4y + 3z$ is divisible by $F_{m/2}$ in this case as well.

All that remains to show is that the expression $3x \pm 4y + 3z$ does not vanish. This is clearly true for $3x + 4y + 3z$. Suppose that $3x - 4y + 3z = 0$. Recall our observation above that $x \equiv z \pmod 4$. Then $y = 3(x+z)/4$, and using (6), we get

$$n! = F_x + F_z + F_{3(x+z)/4} = F_{(x+z)/2}L_{(x-z)/2} + F_{3(x+z)/4}.$$

The right-hand side is divisible by $F_{(x+z)/4}$, and so this divides $n!$. If $(x+z)/4 \le 12$ then we can list all the solutions. Hence, suppose that $(x+z)/4 > 12$. By the Primitive Divisor Theorem [2], $F_{(x+z)/4}$ has some prime divisor p such that $p \equiv \pm 1 \pmod{(x+z)/4}$. But $p \mid n!$, which gives that $p \le n$. Thus,

$$(x+z)/4 \le n + 1,$$

and so $x \le 4n+4$, $y \le 3n+3$, and $z \le 4n+4$. However, $F_{\max\{x,y,z\}} \ge n!/3$, giving a contradiction for $n \ge 7$. $\square$

6. Bounds on n when x is even and y, z have the same parity

6.1. **Case I: x, y, z are all even.** Let us suppose that $n \ge 200$. In this case, we know from Lemma 5.1 that all even $m \ge 6$ with $L_{m/2} \le n$ satisfy $m \mid (x+y+z)$.

Let p run through the integers

$$3 \le p \le 2.076 \log n.$$

By Lemma 2.3, $L_p \le n$, and so $2p$ divides $x+y+z$. Thus, by Lemma 2.2,

$$\theta(2.076 \log n) \le \log(x+y+z) \le \log\left(\frac{3n\log n}{\log\alpha}\right).$$

The first bound that we prove for n is $n \le L_{31}$. Suppose that $n \ge L_{31}+1$. Then $2.076 \log n > 30$, and so, by Proposition 2.5,

$$\theta(2.076 \log n) \ge 0.7033 \times 2.076 \log n > 1.46 \log n.$$

Hence,

$$1.46 \log n < \log\left(\frac{3n\log n}{\log\alpha}\right) = \log n + \log\log n + \log(3/\log\alpha).$$

This is impossible for $n \ge L_{31}+1$. Thus, $n \le L_{31}$.

Hence,

$$x+y+z \le \frac{3n\log n}{\log\alpha} \le 279962456.$$

Suppose that $n \ge L_{19}$. Then $x+y+z$ is divisible by $2p$ for all integers $3 \le p \le 19$. However,

$$\mathrm{lcm}(6, 8, \ldots, 38) = 465585120 > 279962456 \ge x+y+z,$$

giving a contradiction. Thus, $n \le L_{19}$. Repeating the argument shows that $n \le L_{13}$ and finally that $n \le L_{11} = 199$. This contradicts our initial assumption that $n \ge 200$, and so $n \le 199$.

6.2. **Case II: x is even and y and z are odd.** In this case, we know from Lemma 5.1 that all even $m \ge 6$ with $L_{m/2} \le n$ satisfy $m \mid (x+4)$. A similar argument to the one above now shows that $n \le 199$.

7. Bound for n when x, z are even, y odd

7.1. **An initial bound.** Suppose that $n \ge 200$. Let $0 < \gamma < 1$ be a real number to be chosen later. Let p be a prime satisfying

$$2.079\gamma \log n + 2.441 \le p \le 2.076 \log n.$$

By Lemma 2.3, we have that $L_p \le n$ and $F_p \ge n^\gamma$. We know, by Lemma 5.1 applied to $m = 2p$, that either $2p$ divides $x+z+2$, or F_p divides one of the (non-zero) expressions $3x \pm 4y + 3z$.

From Lemma 2.2,

$$x+z+2 \le \frac{2n\log n}{\log\alpha}, \qquad |3x \pm 4y + 3z| \le \frac{10n\log n}{\log\alpha}.$$

Suppose that k is a positive integer satisfying

$$n^{\gamma(k+1)} > \frac{10n\log n}{\log\alpha}. \tag{9}$$

Then at most k of the numbers F_p for the primes p in the given range divide $3x+4y+3z$, and at most another k of these divide $3x-4y+3z$. Note that here we are making use of the fact that the F_p are coprime as p runs through the primes; this is a consequence of the well-known property $\gcd(F_u, F_v) = F_{\gcd(u,v)}$ for all integers u and v.

It follows that for all but at most $2k$ primes p in the range above, $2p$ divides $x+z+2$. Hence,

$$\begin{aligned}(10)\quad \theta(2.076\log n) &- \theta(2.079\gamma\log n + 2.441) - 2k\log(2.076\log n)\\ &\le \log(x+z+2) \le \log\left(\frac{2n\log n}{\log\alpha}\right) \le \log n + \log\log n + 1.425.\end{aligned}$$

Now suppose that k and γ are fixed and that n is very large. Recall that for large x, $\theta(x) = x + o(x)$ as $x \to \infty$. Thus, the above inequalities give

$$(1.076 - 2.079\gamma)\log n \le o(\log n) \qquad \text{as } n \to \infty,$$

showing that n must be in fact bounded provided that γ is small enough. We use this idea to obtain an explicit bound for n.

We first show that $n \le L_{501}$. So, suppose that $n \ge L_{501}+1$. We let $k=2$ and $\gamma = 0.35$. It is easy to show that (9) holds. Moreover,

$$2.076\log n \ge 500,$$

and so, by Proposition 2.5, we have

$$\theta(2.076\log n) \ge 0.9194 \times 2.076\log n > 1.908\log n,$$

and

$$\theta(2.079\gamma\log n + 2.441) \le 1.001102(2.079\gamma\log n + 2.441) < 0.729\log n + 2.444.$$

Equation (10) gives

$$0.188\log n \le 5\log\log n + 6.791.$$

This is impossible for $n \ge L_{501}+1$. Hence, $n \le L_{501} < 5.045 \times 10^{104}$.

7.2. **A recursive procedure for reducing the bound.** We now give an iterative argument which will be used repeatedly to reduce the above bound. Our argument is reminiscent to that given at the end of Section 6, but substantially more complicated. Write

$$\mathcal{E} = x+z+2, \qquad \mathcal{F} = |(3x+4y+3z)(3x-4y+3z)|.$$

For a positive integer $b \ge 2$ we put

$$C_{\mathcal{E},b} = \frac{2L_b\log L_b}{\log\alpha}, \quad C_{\mathcal{F},b} = \frac{60L_b^2(\log L_b)^2}{(\log\alpha)^2}.$$

Lemma 7.1. *If $n \le L_b$, then $\mathcal{E} \le C_{\mathcal{E},b}$ and $\mathcal{F} \le C_{\mathcal{F},b}$.*

Proof. This follows from Lemma 2.2. ∎

For positive integers u and a with $2 \le u \le a$ define

$$H_a(u) = \operatorname{lcm}\{F_v \;:\; 3 \le v \le a, \quad u \mid v\}.$$

For $2 \le u_1 < u_2 < \cdots < u_n \le a$, define

$$H_a(u_1, \dots, u_n) = \operatorname{lcm}(H_a(u_1), \dots, H_a(u_n)).$$

Lemma 7.2. *Suppose that $n \ge L_a$ and that $2 \le u \le a$. If $u \nmid \mathcal{E}$, then $H_a(u) \mid \mathcal{F}$. If $2 \le u_1 < u_2 < \cdots < u_n \le a$ and all $u_i \nmid \mathcal{E}$, then $H_a(u_1, \dots, u_n)$ divides $\mathcal{F}$.*

Proof. Let v be an integer satisfying $3 \le v \le a$ and $u \mid v$. Write $m = 2v$. Then m is an even integer satisfying $m \ge 6$ and $L_{m/2} \le L_a \le n$. By Lemma 5.1, either m divides $\mathcal{E}$, or $F_v = F_{m/2}$ divides $\mathcal{F}$. But $u \mid v \mid m$ and $u \nmid \mathcal{E}$. Thus, $m \nmid \mathcal{E}$, and so F_v divides $\mathcal{F}$ as required. □

Now let $2 \le a \le b$. Put

$$\mathcal{P}_a = \{u \;:\; 2 \le u \le a, \quad u \text{ is a prime power}\}.$$

We define a sequence of subsets of the powerset of $\mathcal{P}_a$ as follows

$$S_0(a,b) = \{\emptyset\}, \quad S_1(a,b) = \{\{u\} \;:\; 2 \le u \le a, \quad H_{u,a} \le C_{\mathcal{F},b}\},$$

and for $k \ge 1$, we define $S_{k+1}(a,b)$ to be the set of $k+1$–tuples $\{u_1, \dots, u_{k+1}\}$ satisfying:

- $2 \le u_1 < u_2 < \cdots < u_{k+1} \le a$,
- $\{u_1, \dots, u_k\} \in S_k(a,b)$,
- $\{u_{k+1}\} \in S_1(a,b)$,
- $H_a(u_1, \dots, u_{k+1}) \le C_{\mathcal{F},b}$.

We put

$$S(a,b) = \bigcup_{k \ge 0} S_k(a,b).$$

Lemma 7.3. *Let $2 \le a \le b$, and suppose that $L_a \le n \le L_b$. Let $V = \{u \in \mathcal{P}_a : u \mid \mathcal{E}\}$. Then $\mathcal{P}_a \backslash V \in S(a,b)$.*

Proof. Write $\mathcal{P}_a \backslash V = \{u_1, \dots, u_j\}$, where $u_1 < \cdots < u_j$. No u_i divides $\mathcal{E}$, and so, by Lemma 7.2, $H_a(u_1, \dots, u_j)$ divides $\mathcal{F}$. By Lemma 7.1, we have $H_a(u_1, \dots, u_j) \le C_{\mathcal{F},b}$. Clearly, for each $k \le j-1$, $H_a(u_1, \dots, u_{k-1}) \le C_{\mathcal{F},b}$ and $H_a(u_k) \le C_{\mathcal{F},b}$. This shows inductively that $\{u_1, \dots, u_k\} \in S_k(a,b)$ for $k = 1, \dots, j$. Thus, $\mathcal{P}_a \backslash V \in S_j(a,b) \subseteq S(a,b)$. □

Lemma 7.4. *Let $2 \le a \le b$. Suppose that for each $U \in S(a,b)$, we have*

$$\operatorname{lcm}(\mathcal{P}_a \backslash U) > C_{\mathcal{E},b}.$$

Then there is no solution to the Diophantine equation (2) with x, z even, y odd, and $L_a \le n \le L_b$.

Proof. Suppose that $L_a \le n \le L_b$. Let $V = \{u \in \mathcal{P}_a : u \mid \mathcal{E}\}$, and $U = \mathcal{P}_a \backslash V$. By Lemma 7.3, $U \in S(a,b)$. Moreover,

$$\operatorname{lcm}(\mathcal{P}_a \backslash U) = \operatorname{lcm}(V) \mid \mathcal{E}.$$

However, by Lemma 7.1, $\mathcal{E} \le C_{\mathcal{E},b}$. This gives a contradiction. □

We have shown previously that $n \le L_{501}$. We shall apply Lemma 7.4 to repeatedly reduce this bound on n. First, we let $a = 490$ and $b = 501$. We used a simple `MAGMA` script to compute $\mathcal{P}_a$ and $S_k(a, b)$. We found that $\mathcal{P}_a$ has 112 elements, $S_1(a, b)$ has 84 elements, $S_2(a, b)$ has 2565 elements, $S_3(a, b)$ has 8609 elements, $S_4(a, b)$ has 16 elements, and $S_k(a, b) = \emptyset$ for $k \ge 5$. Altogether, $S(a, b)$ has 11275 elements. We check the criterion of Lemma 7.4 and find that it holds for all $U \in S(a, b)$. Thus, there are no solutions to (2) with x, z even, y odd, and $L_{490} \le n \le L_{501}$. This shows that $n \le L_{490}$. Repeating the above argument another 50 times shows that $n \le L_{37} = 54,018,521$.

8. The Final Sieve

We know from the previous three sections that all solutions of the Diophantine equation (2) satisfy $n \le 54,018,521$. In this section, we shall determine all solutions to (2) with $n \le 6 \times 10^7$, and thus complete the proof of Theorem 1.

Lemma 8.1. *In* (2)*, suppose* $x \ge y \ge z$. *Then*

$$\frac{\log n!}{\log \alpha} + C_1 \le x \le \frac{\log n!}{\log \alpha} + C_2, \tag{11}$$

where

$$C_1 = \frac{\log(\sqrt{5}/3) - 1/\sqrt{5}}{\log \alpha}, \qquad C_2 = \frac{\log(\sqrt{5}) + 1/(6\sqrt{5})}{\log \alpha}.$$

Proof. The lemma is easily checked for $n \le 2$, so suppose $n \ge 3$. Clearly,

$$\frac{n!}{3} \le F_x \le n!,$$

and so

$$\frac{\sqrt{5}n!}{3} - 1 \le \alpha^x \le \sqrt{5}n! + 1.$$

Taking logarithms, we find that

$$x \log \alpha \le \log n! + \log \sqrt{5} + \log(1 + 1/(n!\sqrt{5})) \le \log n! + \log \sqrt{5} + 1/(6\sqrt{5}),$$

by using Lemma 2.4 and the fact that $n \ge 3$. Moreover,

$$x \log \alpha \ge \log n! + \log(\sqrt{5}/3) + \log(1 - 3/(n!\sqrt{5})) \ge \log n! + \log(\sqrt{5}/3) - 1/\sqrt{5},$$

again using Lemma 2.4 and $n \ge 3$. This completes the proof. □

For the purpose of searching for all solutions to (2), we may without loss of generality suppose that $x \ge y \ge z$. The last lemma above gives, for each n, an interval containing at most three integers x.

Now let

$$l_1 = F_{43} = 433494437, \qquad l_2 = F_{47} = 2971215073.$$

Both these Fibonacci numbers are prime, and they have been chosen because the period of the Fibonacci sequence modulo each l_i is particularly small; the periods are respectively 172 and 188, respectively. Let

$$T_1 = \{F_u + F_v \pmod{l_1} \ : \ 0 \le u \le 171\},$$

and

$$T_2 = \{F_u + F_v \pmod{l_2} \ : \ 0 \le u \le 187\}.$$

Using MAGMA, we find that T_1 has 2821 elements and T_2 has 3453 elements. Thus,

$$\#T_1/l_1 \approx 6.5 \times 10^{-6}, \qquad \#T_2/l_2 \approx 1.16 \times 10^{-6}.$$

Now our MAGMA program for determining the solutions of (2) with $n \le 6 \times 10^7$ is as follows. For each n, we need to compute three quantities. The first is $\log(n!)/\log(\alpha)$, the second is $n! \pmod{l_1}$, and the third is $n! \pmod{l_2}$. Knowing these for $n = k-1$ quickly give these for $n = k$. For each n, we determine the integers x in the interval (11). For each x, we compute $F_x \pmod{l_1}$ and $F_x \pmod{l_2}$. If $n! - F_x$ modulo l_1 does not belong to T_1, or $n! - F_x$ modulo l_2 does not belong to T_2, then we know that there is no solution to the equation (2) with the given values of n and x. Computing $F_x \pmod{l_i}$ can be done in $O(\log x) = O(\log n)$ steps as it involves only computing α^x modulo l_i, and so it is very fast. Our script took less than 6 hours to run on a dual core 3.00 GHz Opteron and produced only the following pairs of values of (x, n) for which $n! - F_x$ belongs to T_i modulo l_i $(i = 1, 2)$:

$$(0,1), (1,1), (0,2), (1,2), (2,2), (3,2), (3,3), (4,3), (5,3),$$
$$(6,4), (7,4), (8,4), (9,5), (11,5), (14,6), (15,6).$$

From this, we easily recover our list of solutions in Theorem 1.

Note that the probability of a random integer belonging modulo l_1 to T_1 and modulo l_2 to T_2 is less than 10^{-11}. Since the possibilities for (x, n) are most $3 \times 6 \times 10^7 < 2 \cdot 10^8$, it is not at all surprising that our sieve found only pairs of (x, n) for which there are solutions (n, x, y, z) to (2).

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