Algebraic Number Theory (MA3A6) - Problem Sheet 3

You should answer any **three** out of **five** questions. All questions have equal weight. Marks will also be awarded for clarity/presentation.

Deadline: **5pm** on **Thursday 10th February (week 5)**

1. Let K, $L \subset \mathbb{C}$ be number fields.

   · Define KL to be the field generated by K and L, i.e. KL is the intersection of all subfields of $\mathbb{C}$ that contain both K and L. Prove that KL is a number field.

   · Likewise, define $\mathcal{O}_K \cdot \mathcal{O}_L$ to be the ring generated by $\mathcal{O}_K$ and $\mathcal{O}_L$, i.e. the intersection of all subrings of $\mathbb{C}$ that contain both $\mathcal{O}_K$ and $\mathcal{O}_L$. Prove that $\mathcal{O}_K \cdot \mathcal{O}_L$ is a full rank lattice in KL and a subring of $\mathcal{O}_{KL}$.

2. Do we necessarily have $\mathcal{O}_K \cdot \mathcal{O}_L = \mathcal{O}_{KL}$? Give a proof or a counterexample.

3. For $n \in \mathbb{Z}_{\geq 3}$, let P be a regular n-gon whose circumscribed circle has radius 1. Show that the product of the lengths of all the sides and diagonals of P is $\sqrt{n^n}$.

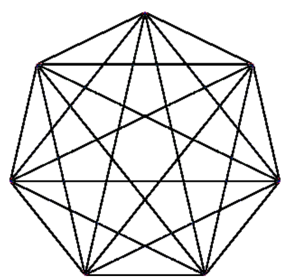

Figure 1: A heptagon and all of its diagonals

4. Let $f(x) = x^3 + x^2 + 2$ and suppose $f(\theta) = 0$. Put $K = \mathbb{Q}(\theta)$. Show that

   (i) $[K : \mathbb{Q}] = 3$;

   (ii) $\mathrm{Tr}(a + b\theta + c\theta^2) = 3a - b + c$ (for a, b and c in $\mathbb{Q}$);

   (iii) $\Delta(\mathbb{Z}[\theta])$ (i.e. $\Delta(1, \theta, \theta^2)) = -116$;

   (iv) $\mathbb{Z}[\theta] = \mathcal{O}_K$.

5. Let K be a number field and $\{x_1, \ldots, x_n\}$ a basis for K over $\mathbb{Q}$. Because the trace pairing is a non-degenerate bilinear form, we may define a dual basis $\{x_1^*, \ldots, x_n^*\}$ for K over $\mathbb{Q}$ satisfying
$$\mathrm{Tr}(x_i x_j^*) = \delta_{ij}$$
where δ_{ij} is the Kronecker delta symbol. Prove that
$$\Delta(x_1, \ldots, x_n) \cdot \Delta(x_1^*, \ldots, x_n^*) = 1.$$