Algebraic Geometry – Exercises 27 February 2007

1. *(3 points)* For $n \geq 0$ we denote by $C^\infty_{\mathbb{R}^n}$ the sheaf of C^∞-functions on open subsets of $\mathbb{R}^n$. Show that for $n, m \geq 0$ the map

$$\mathrm{Hom}_{\mathrm{LRS}_\mathbb{R}}\left((\mathbb{R}^n, C^\infty_{\mathbb{R}^n}), (\mathbb{R}^m, C^\infty_{\mathbb{R}^m})\right) \to C^\infty_{\mathbb{R}^n}(\mathbb{R}^n)^m$$

given by

$$(f, \phi) \mapsto (\phi(\mathbb{R}^m)(x_1), \ldots, \phi(\mathbb{R}^m)(x_m))$$

is a bijection.

2. *(2 points)* Let k be an algebraically closed field. Determine $\mathrm{Aut}_{\mathrm{LRS}_k}(\mathbb{A}^1_k)$.

3. *(2 points)* Let k be an algebraically closed field. Let f_1 and f_2 be squarefree elements of $k[x]$, monic and of the same degree d. Put $Y_1 = Z(f_1)$ and $Y_2 = Z(f_2)$.

a) Show that Y_1 and Y_2 are isomorphic as varieties.

b) For which values of d is there always an automorphism σ of $\mathbb{A}^1_k$ such that $\sigma Y_1 = Y_2$ holds?

4. *(3 points)* Let A be a ring (commutative with 1 as usual) and let $S \subset A$ be a subset closed under multiplication and containing 1. Define an equivalence relation $\sim$ on $A \times S$ as follows:

$$(a, s) \sim (b, t) \quad \Leftrightarrow \quad \exists u \in S : u(at - bs) = 0.$$

Denote $(A \times S)/{\sim}$ by $S^{-1}A$.

a) Show that $S^{-1}A$ carries naturally the structure of a ring together with a homomorphism $A \to S^{-1}A$ that satisfies the following universal property: for any ring B, any homomorphism $\phi : A \to B$ with $\phi(S) \subset B^*$ factors in a unique way as

$$A \to S^{-1}A \to B.$$

b) Prove that $S^{-1}A$ is isomorphic to

$$A[\{x_s : s \in S\}]/(\{sx_s - 1 : s \in S\}).$$

c) For $f \in A$, consider $S = \{1, f, f^2, \ldots\}$ and denote $S^{-1}A$ by A_f. Find a necessary and sufficient condition for f in order that A_f be the zero ring.

d) Prove that in any ring, the intersection of all the prime ideals is equal to the nilradical.

e) For $\mathfrak{p}$ a prime ideal of A consider $S = A - \mathfrak{p}$ and denote $S^{-1}A$ by $A_\mathfrak{p}$. Prove that $A_\mathfrak{p}$ is a local ring.

5. *(3 points)* Let $(X, \mathcal{O}_X)$ be a locally ringed space. For $x \in X$, define $k(x)$ to be the residue field of the local ring $\mathcal{O}_{X,x}$. Take $f \in \mathcal{O}_X(X)$ and define

$$D(f) := \{x \in X : \overline{f}_x \neq 0 \text{ in } k(x)\}.$$

Prove that $D(f)$ is an open subset of X and that $f|_{D(f)}$ is a unit in $\mathcal{O}_X(D(f))$.