Algebraic Number Theory (MA3A6) - Problem Sheet 5

You should answer any **three** out of **five** questions. All questions have equal weight. Marks will also be awarded for clarity/presentation.

Deadline: **4pm** on **Thursday 24th February (week 7)**

1. Let K be a field. A subring $R \subseteq K$ is called a *valuation ring* of K if for each $x \in K^\times$ we have $x \in R$ or $x^{-1} \in R$.

   (a) Show that every valuation ring R of K is a local ring (a local ring is a ring that has exactly one maximal ideal). [Hint: prove that $R - R^\times$ is an ideal of R.]

   (b) Show that any valuation ring of K is integrally closed in K.

2. Let R be an integral domain whose field of fractions $K := \mathrm{Frac}(R)$ is a number field. The *ideal quotient* of fractional ideals I and J with $J \neq (0)$ is the fractional ideal $I : J$ defined by

$$I : J = \{\alpha \in K : \alpha J \subset I\}.$$

   Prove that the ideal quotient of fractional ideals satisfies the following properties:

   – $H : (IJ) = (H : I) : J$,

   – $(\bigcap_k I_k) : J = \bigcap_k (I_k : J)$,

   – $I : (\sum_k J_k) = \bigcap_k (I : J_k)$.

3. · Let I be an ideal in a commutative ring R and let $\sqrt{I} = \{r \in R \mid r^n \in I$ for some $n \in \mathbb{N}\}$. Show that $\sqrt{I}$ is an ideal. It is called the *radical* of I.

   · Suppose I is an ideal of a ring R. Show that if $\sqrt{I}$ is finitely generated, then for some integer N we have $\sqrt{I}^N \subseteq I$. Conclude that in a noetherian ring, the ideals I and J have the same radical if and only if there is some integer N with $I^N \subseteq J$ and $J^N \subseteq I$.

4. · Suppose $\mathfrak{p}$ and $\mathfrak{q}$ are distinct non-zero prime ideals in the ring of integers $\mathcal{O}_K$ of a number field K. Show that $\mathfrak{p} + \mathfrak{q} = \mathcal{O}_K$ and $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{p}\mathfrak{q}$.

   · Let I_1, I_2, I_3 and I_4 be non-zero ideals of an integral domain R. Show that if I_1I_2, I_2I_3 and I_3I_4 are all principal then I_1I_4 is also principal.

5. In $\mathbb{Z}[\sqrt{5}]$, find a non-zero ideal that is not invertible and a non-zero ideal that is not a product of prime ideals.