Algebraic Number Theory (MA3A6) - Problem Sheet 1

You should answer any **three** out of **five** questions. All questions have equal weight. Marks will also be awarded for clarity/presentation.

Deadline: **5pm** on **Thursday 27th January (week 3)**

1. Find all integer solutions to

   · $x^2 + 1 = y^7$ (You may use the fact that $\mathbb{Z}[i]$ is a UFD)

   · $x^2 + 8 = y^3$ (You may use the fact that $\mathbb{Z}[\sqrt{-2}]$ is a UFD)

2. Prove that the only integer solutions of $x^2 + 19 = y^3$ are $(\pm 18, 7)$. (You may use the fact that $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a UFD)

3. Let $n > 2$ be an integer and put $d = n^2 - 2$.
   Prove that $n^2 - 1 + n\sqrt{d}$ is a unit of $\mathbb{Z}[\sqrt{d}]$. Is it necessarily a fundamental unit? Provide a proof or a counterexample.

4. Let d be an integer, not a square and put $\alpha := a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ and $u := N(\alpha)$. Let x_n, y_n for $n \in \mathbb{Z}_{\geq 0}$ be defined by $\alpha^n = x_n + y_n\sqrt{d}$.

   (a) Show that x_n and y_n satisfy the recursions:

   $$\begin{aligned} x_0 = 1,\ x_1 = a,\ x_{n+2} &= 2ax_{n+1} - ux_n \\ y_0 = 0,\ y_1 = b,\ y_{n+2} &= 2ay_{n+1} - uy_n \end{aligned}$$

   (b) Give explicit formulae for x_n and y_n in terms of a, b, d, u and n.

5. Let d be an integer, not a square and ϵ_d a fundamental unit of $\mathbb{Z}[\sqrt{d}]$.

   (a) Show that for $n \in \mathbb{Z}_{>1}$ there is an $m \in \mathbb{Z}_{>0}$ for which ϵ_d^m is a fundamental unit of $\mathbb{Z}[n\sqrt{d}]$.

   (b) Prove $m < n^2$. [Hint: work in $(\mathbb{Z}[\sqrt{d}]/n\mathbb{Z}[\sqrt{d}])^\times$ and consider powers of ϵ_d.]