MA3A6
Algebraic Number Theory

Prerequisites: MA242 Algebra I, MA245 Algebra II, MA246 Number Theory. This course should be taken simultaneously with Galois Theory MA3D5 as there is some overlap between the two courses.

Contents: A Diophantine equation is a polynomial equation in several variables with integer coefficients and one desires the solutions in integers. An example is to ask for all the solutions in integers x, y of the equation

$$y^2 + 2 = x^3.$$

Euler solved this equation by factoring the left-hand side

$$(y + \sqrt{-2})(y - \sqrt{-2}) = x^3$$

and noting that, roughly speaking, the factors must be cubes. To make such arguments rigorous, we need to study factorisation in $\mathbb{Q}(\sqrt{-2})$ and similar fields. The field $\mathbb{Q}(\sqrt{-2})$ is an example of what is called a number field. This study of factorisation in number fields is what 'Algebraic Number Theory' is about. In this course we hope to cover the basic concepts of algebraic number theory (1–9 below); the applications to Diophantine equations will be interspersed throughout.

1. Factorisation—uniqueness and the failure of uniqueness.
2. Ideals. Prime ideals. Maximal ideals. Factorisation of ideals.
3. Dedekind domains.
4. Number fields. Norm. Trace.
5. Algebraic integers. Integral bases.
6. Quadratic and cyclotomic fields.
7. The class group.
8. Minkowski's Theorem.
9. Dirichelet's Units Theorem.
10. Applications to Diophantine Equations.

Books:

There are many algebraic number theory books. The easiest is:
Algebraic Number Theory and Fermat's Last Theorem, I. N. Stewart and D. O. Tall, third edition.

Those with more a more solid background in algebra might prefer:
Algebraic Theory of Numbers, Pierre Samuel.

Another good text that is available online is:
Algebraic Number Theory, J. Milne
`http://www.jmilne.org/math/index.html`.