Additional exercise 1.

Let X be the set $\{-1, 0, 1\}$ with the following topology:

$$\mathrm{Open}(X) = \big\{\emptyset, \{0\}, \{-1, 0\}, \{0, 1\}, X\big\}.$$

a) Show that a presheaf on X naturally corresponds to a commutative diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B \arrow[dr] & & C \arrow[dl] \\ & D & \end{tikzcd} \tag{1}$$

of abelian groups.

b) Determine the stalks of $\mathcal{F}$ at all the points of X in terms of the diagram (1).

c) Let $\mathcal{F}$ be a presheaf on X. Which properties should the diagram (1) associated to $\mathcal{F}$ satisfy in order that $\mathcal{F}$ be a sheaf? Determine also the sheafification $\mathcal{F}^+$ of $\mathcal{F}$ in terms of (1).

d) Consider the subpace $Y = \{-1, 1\}$ of X, with the natural inclusion $\iota : Y \hookrightarrow X$. Give a surjective morphism

$$\phi : \mathbb{Z}_X \to \iota_* \mathbb{Z}_Y$$

such that there is an open $U \subset X$ with $\phi(U)$ not surjective. Here, for X a topological space, $\mathbb{Z}_X$ denotes the constant sheaf with values in $\mathbb{Z}$ on X.