Algebraic Geometry — Exercises 6 March 2007

1. *(2 points)* Hartshorne, Exercise II.1.22.

2. *(3 points)* Let k be an infinite field, let n be a non-negative integer, and let A be the polynomial ring $k[x_0, \ldots, x_n]$. For each $d \geq 0$, write A_d for the k-linear subspace of A consisting of the polynomials which are homogeneous of degree d. Then A is the direct sum of the A_d, i.e. any element $f \in A$ can be written in a unique way as a finite sum

$$f = \sum_{d \geq 0} f_d \quad \text{with } f_d \in A_d.$$

Consider the following action of the group $k^\times$ on the k-algebra A:

$$\begin{aligned} k^\times \times A &\longrightarrow A \\ (\lambda, f(x_0, \ldots, x_n)) &\longmapsto f(\lambda x_0, \ldots, \lambda x_n). \end{aligned}$$

An ideal $\mathfrak{a} \subseteq A$ is called $k^\times$*-invariant* if it is invariant under this action, i.e. if

$$f(\lambda x_0, \ldots, \lambda x_n) \in \mathfrak{a} \quad \text{for all } f \in \mathfrak{a} \text{ and } \lambda \in k^\times.$$

Show that the following are equivalent for every ideal $\mathfrak{a} \subseteq A$:

(1) $\mathfrak{a}$ is $k^\times$-invariant;

(2) for all $f \in \mathfrak{a}$ and all $d \geq 0$, the element $f_d \in A_d$ is in $\mathfrak{a}$;

(3) $\mathfrak{a} = \bigoplus_{d \geq 0} \mathrm{pr}_d\, \mathfrak{a}$, where $\mathrm{pr}_d \colon A \to A_d$ is the canonical projection;

(4) $\mathfrak{a} = \bigoplus_{d \geq 0} \mathfrak{a} \cap A_d$;

(5) $\mathfrak{a}$ is generated by homogeneous elements, i.e. by a subset of $\bigcup_{d \geq 0} A_d$.

(Note: You do not need to prove that (1) implies any of the other four; the implication (1)$\Longrightarrow$(2) was proved during the lecture. The properties (2)–(4) are still equivalent if A is an arbitrary graded ring (for the definition see Hartshorne, § I.2), and ideals satisfying them are called homogeneous ideals.*)*

3. *(3 points)* Hartshorne, Exercise I.2.11. *(Note: Varieties are required to be irreducible in Hartshorne's book.)*

4. *(3 points)* Hartshorne, Exercise I.2.14. *(Note: Varieties are irreducible. Since products of varieties have not been defined, the map $\mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^N$ is meant as a map of sets.)*

5. *(3 points)* Hartshorne, Exercise I.2.15 (a) and (b). *(Note: Varieties are irreducible. The map $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ is meant as a map of sets.)*