CAUCHY'S THEOREM. THE CAUCHY-GOURSAT THEOREM

Let $f(z)$ be analytic in a region $\mathcal{R}$ and on its boundary C. Then

$$\oint_C f(z)\,dz = 0 \tag{4.9}$$

This fundamental theorem, often called *Cauchy's integral theorem* or briefly *Cauchy's theorem*, is valid for both simply- and multiply-connected regions. It was first proved by use of Green's theorem with the added restriction that $f'(z)$ be continuous in $\mathcal{R}$ [see Problem 4.11]. However, *Goursat* gave a proof which removed this restriction. For this reason the theorem is sometimes called the *Cauchy–Goursat theorem* [see Problems 4.13–4.16] when one desires to emphasize the removal of this restriction.

3i

Theorem 4.4 Let $f(z)$ be analytic in a region bounded by two simple closed curves C and C_1 [where C_1 lies inside C as in Fig. 4.5] and on these curves. Then

C

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz \tag{4.16}$$

where C and C_1 are both traversed in the positive sense relative to their interiors [counterclockwise in Fig. 4.5].

The result shows that if we wish to integrate $f(z)$ along curve C, we can equivalently replace C by any curve C_1 so long as $f(z)$ is analytic in the region between C and C_1 as in Fig. 4.6.

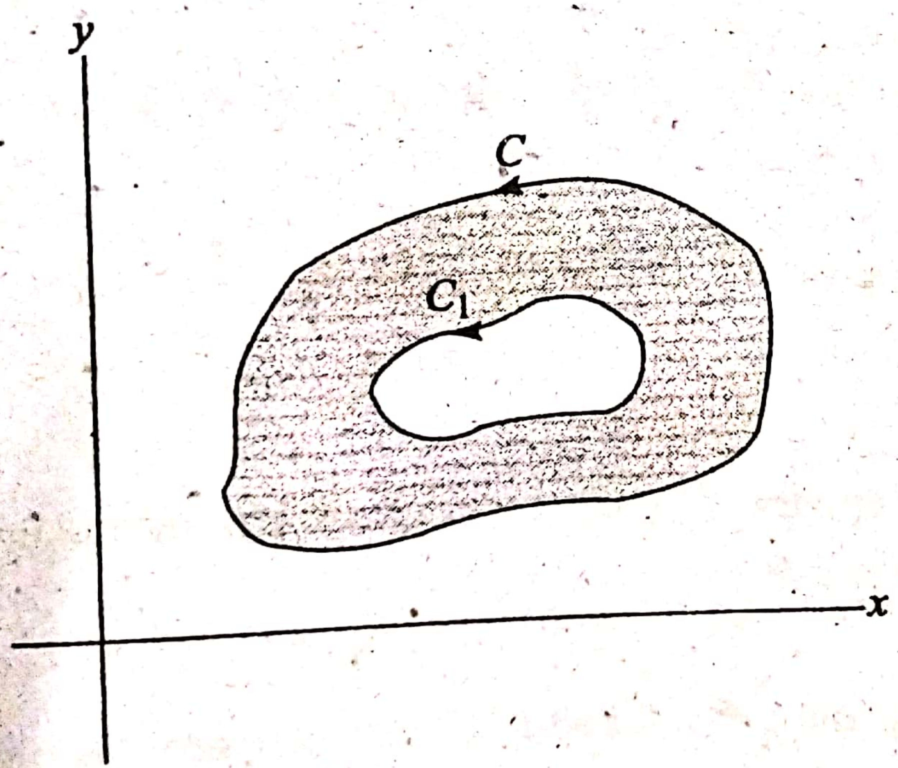


Fig. 4.5

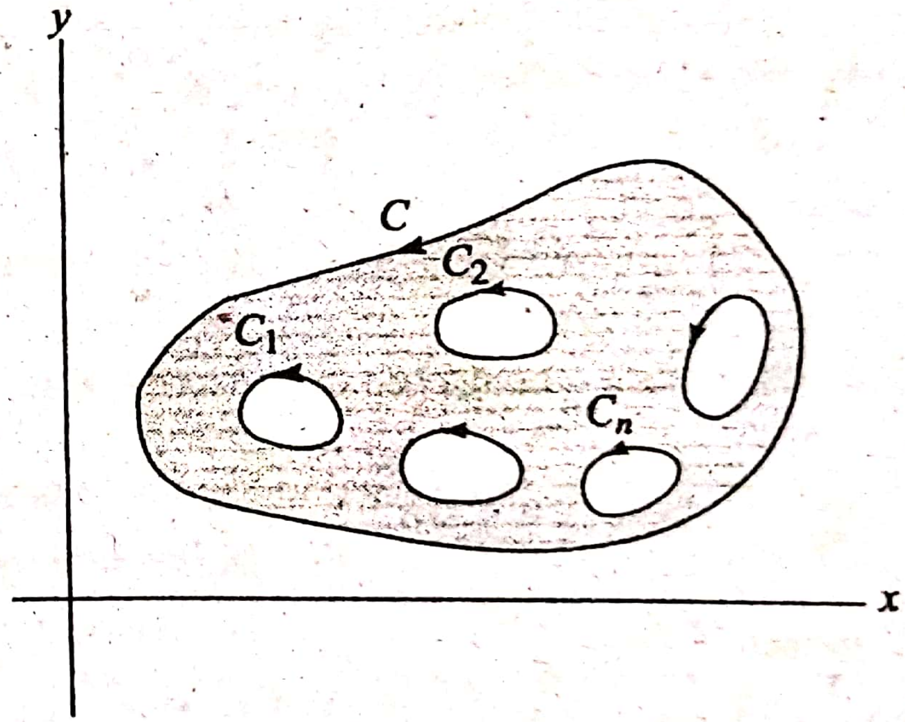
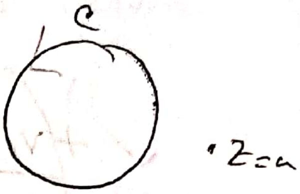


Fig. 4.6

Evaluate $\oint_C \frac{dz}{z-a}$, where c is any simple closed curve and $z=a$ is (i) outside c ii) inside c.

S.S- P- 4.20

Also See Kuddus P-177 Example-177

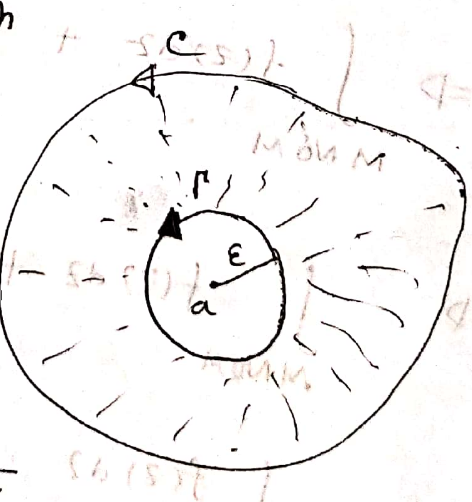


Solution:

i) If a is outside c, $f(z)=\frac{1}{z-a}$ is analytic inside and on c. Then by Cauchy's theorem

$$\oint_C f(z)\,dz = 0 \cdot \oint_C \frac{1}{z-a}\,dz = 0$$

ii) Suppose 'a' is inside c. Let Γ be a circle of radius ϵ with center/centre at $z=a$ where Γ lies inside c. Then we have



$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} \quad -(1)$$

Now on Γ, $|z-a| = \epsilon$; $0 \le \theta \le 2\pi$

or, $z-a = \epsilon e^{i\theta}$; $0 \le \theta \le 2\pi$

or, $z = a+\epsilon e^{i\theta}$; $0 \le \theta \le 2\pi$

Thus, since $dz = i\epsilon e^{i\theta}d\theta$, the right side of (1) becomes,

$$\int_0^{2\pi} \frac{i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}}\, d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

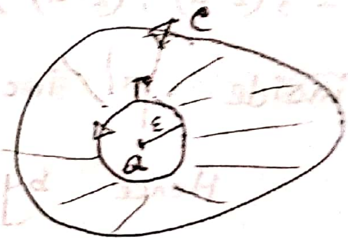
Therefore ① $\Rightarrow$ $\oint_C \frac{dz}{z-a} = 2\pi i.$

* Show that

$$\oint_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & \text{if } n=1 \\ 0 & \text{if } n=2,3,4,\dots \\ 0 & \text{if } n=0,-1,-2,-3,\dots \end{cases}$$

where C is a simple closed curve bounding a region having $z=a$ as interior point.

Sol^n: Let Γ be a circle of radius ε with centre at $z=a$ where Γ lies inside C. Then we have,



$$\oint_C \frac{dz}{(z-a)^n} = \oint_\Gamma \frac{dz}{(z-a)^n} \quad \text{——} ①$$

Now on Γ we have

$|z-a| = \varepsilon$

or $z-a = \varepsilon e^{i\theta}$

$\therefore$ $z = a+\varepsilon e^{i\theta}$, $0 \le \theta \le 2\pi$

$\therefore$ $dz = i\varepsilon e^{i\theta}\, d\theta$

Then ① $\Rightarrow$

$$\oint_C \frac{dz}{(z-a)^n} = \oint_\Gamma \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\, d\theta}{\varepsilon^n e^{i\theta n}}$$

$$= \frac{i}{\varepsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta}\, d\theta \quad \text{——} ②.$$

First Part: Now putting $n=1$ in ② $\Rightarrow$

$$\oint_C \frac{dz}{(z-a)} = \frac{i}{1}\int_0^{2\pi} d\theta = 2\pi i.$$

Second part:

$$② \Rightarrow \oint_C \frac{dz}{(z-a)^n} = \frac{i}{r^{n-1}} \left[\frac{e^{(1-n)i\theta}}{(1-n)i} \right]_0^{2\pi}$$

$$= \frac{1}{r^{n-1}(1-n)} \left[e^{2(1-n)\pi i} - e^0 \right]$$

$$= \frac{1}{r^{n-1}(1-n)} [1-1] = 0,$$

if $n = 2, 3, 4, \ldots$

Third part:

If $n = 0, -1, -2, \ldots$ then the integrand $(z-a)^{-n}$ becomes $1, (z-a), (z-a)^2, \ldots$ and they are all analytic everywhere inside Γ including $z = a$.

Hence by Caucy - Goursat theorem we have

$$\oint_C \frac{dz}{(z-a)^n} = 0 \quad \text{if} \quad n = 0, -1, -2, -3, \ldots$$

Cauchy's Integral Formulae and Related Theorems

CAUCHY'S INTEGRAL FORMULAE

If $f(z)$ be analytic inside and on a simple closed curve C and a is any point inside C [Fig. 5.1]. Then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{5.1}$$

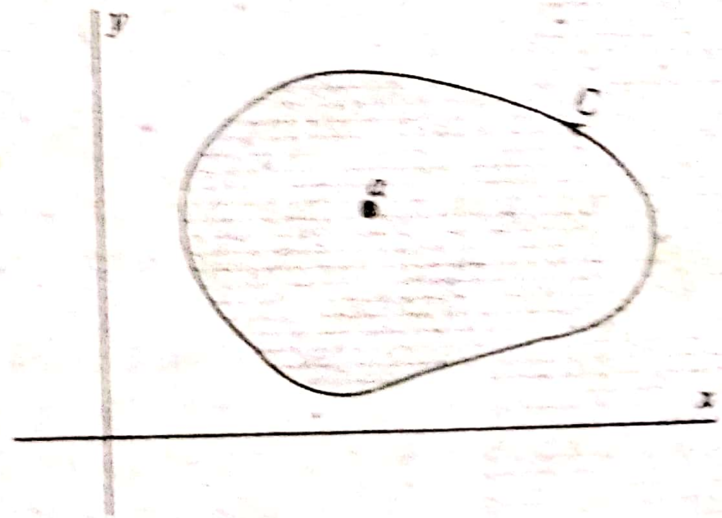
where C is traversed in the positive (counterclockwise) sense.

Also, the nth derivative of $f(z)$ at $z = a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad n = 1, 2, 3, \ldots \tag{5.2}$$

The result (5.1) can be considered a special case of (5.2) with $n = 0$ if we define $0! = 1$.

The results (5.1) and (5.2) are called *Cauchy's integral formulae* and are quite remarkable because they show that if a function $f(z)$ is known *on* the simple closed curve C, then the values of the function and all its derivatives can be found at all points *inside* C. Thus if a function of a complex variable has a first derivative, i.e. is analytic, in a simply-connected region $\mathcal{R}$, all its higher derivatives exist in $\mathcal{R}$. This is not necessarily true for functions of real variables.



Chirchill 29-04-15.

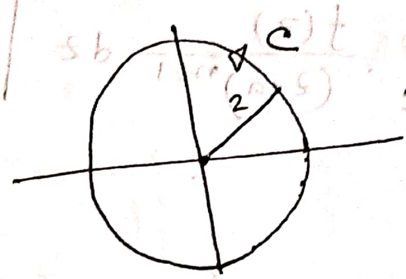
C → P-157

Evaluate the integral

Jn-2011 $\oint_C \frac{z\,dz}{(9-z^2)(z+i)}$ | $\oint_C \frac{z}{z^2+9}dz$ where C is the circle $|z-2i|=4$ Incourse

where C is the positively oriented circle $|z|=2$.

Solution:



Cauchy Integral Theorem:

$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$

$f(z)$ is analytic

$\Rightarrow \int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$

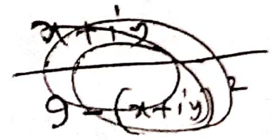
Let

$f(z)=\frac{z}{9-z^2}$. Then $f(z)$ is analytic within and on C. Also $z=-i$ lies inside the circle C. Then from Cauchy's integral formula

$$\oint_C \frac{f(z)}{z+i}dz = 2\pi i\, f(-i)$$

$$= 2\pi i \cdot \frac{-i}{10}$$

$$= \frac{\pi}{5}$$



$\frac{-i}{9-(i)^2}$

$= \frac{-i}{10}$

Chirchill Page-163 1.(d),(e) 2 (a)&(b) (4)

Do this

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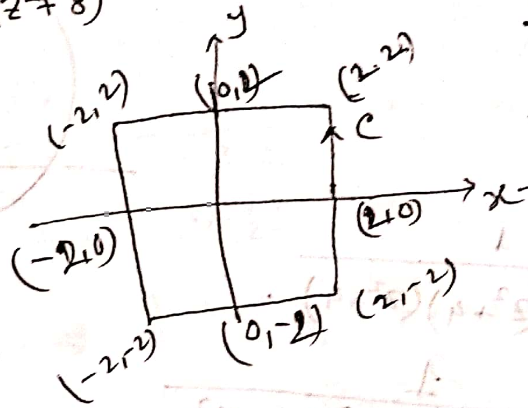
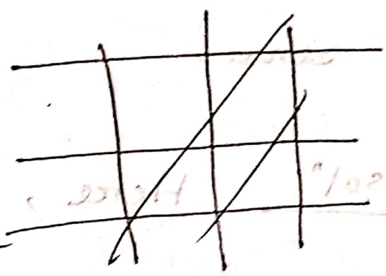
Exercise: Let C denote the positively oriented boundary of the square whose sides lies along the lines $x = \pm 2$, $y = \pm 2$.

Evaluate:

$$\oint_C \frac{\cos z}{z(z^2+8)}\, dz$$





$z^2 = -8$

$z^2 = 8i^2$

$z = \pm 2\sqrt{2}i$

$I = \int \frac{\cos x}{x(x^2+1)}\, dx$, $I = ?$

Soln:

Let $f(z) = \frac{\cos z}{z^2+8}$ then $f(z)$ is analytic within and on C. Also $z = 0$ lies inside C.

Then from Cauchy's Integral formula

$$\oint_C \frac{\cos z}{z(z^2+8)}\, dz = \oint_C \frac{f(z)}{z-0}\, dz = \oint_C \frac{f(z)}{z-0}\, dz$$

$$= 2\pi i\, f(0)$$

$$= 2\pi i \cdot \frac{1}{8}$$

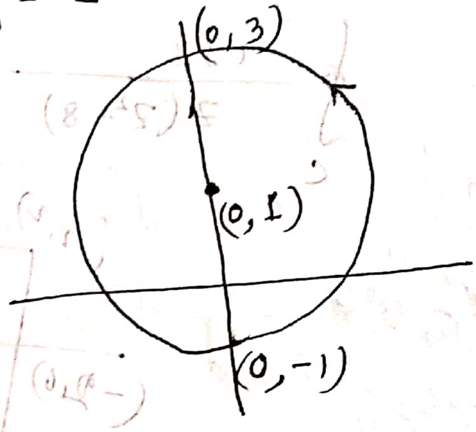
$$= \frac{1}{4}\pi i.$$

? [Cauchy Integral formula for Real field

[4] Evaluate the Integral   C-163 - Exercise

$$\oint_C \frac{1}{(z^2+4)^2}\, dz$$

where C is the circle $|z-i| = 2$ in the positive sense.



Sol^n: Here,

$$\frac{1}{(z^2+4)^2} = \frac{1}{(z^2+4)(z^2+4)}$$

$$= \frac{1}{(z+2i)^2(z-2i)^2}$$

$$(z^2+4)^2 = (z^2 - (2i)^2)^2 = \{(z+2i)(z-2i)\}^2$$

So, Let

$f(z) = \dfrac{1}{(z+2i)^2}$ Then $f(z)$ is analytic within and on C. Also, $z = +2i$ ~~in a point~~ lies inside C. Then,

$$\oint_C \frac{1}{(z^2+4)^2}\, dz = \oint_C \frac{f(z)}{(z-2i)^2}\, dz = 2\pi i \cdot f'(2i)$$

Now $f(z) = (z+2i)^{-2}$

$\therefore f'(z) = (-2)(z+2i)^{-3}$

$$\therefore f'(2i) = \frac{(-2)}{(2i+2i)^3} = \frac{-2}{(4i)^3} = \frac{-2}{64 i^3} = \frac{-2}{64 i^2 \cdot i} = \frac{2}{64 i} = \frac{1}{32 i}$$

$$= 2\pi i \cdot \frac{1}{32 i} = \frac{\pi}{16}$$

Ans

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Cauchy Integral formula related প্রশ্ন নং 275 270

Evaluate the Integral

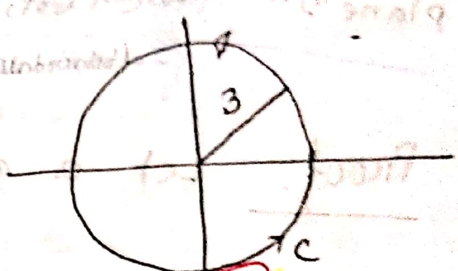
S.S p-5.6

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)\cancel{(z+1)}(z-2)}\, dz$$

where C is the circle $|z| = 3$ in the positive sense.

? Cauchy integral formula of multiply-connected region

if C is the boundary of a region (unbounded)



Sol^n:

Here,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\Rightarrow 1 = A(z-2) + B(z-1)$$

if $z=2$ then
$1 = A.0 + B(2-1)$
$\boxed{1 = B}$

if $z=1$ then
$1 = A(1-2) + B(0)$
$1 = -A$
$\therefore \boxed{A = -1}$

pending

Cauchy Integral formula related প্রশ্ন
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$$\therefore \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}.$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)(-1)}{z-1}\, dz + \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}\, dz$$

By Cauchy's integral formula with $a = 1$, and $a = 2$ respectively, we have

Exercise
S.S.
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P-134 (30-35, 38, 39)

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz = 2\pi i f(1)$$
$$= 2\pi i \{\sin \pi (1)^2 + \cos \pi (1)^2\}$$
$$= 2\pi i (\sin \pi + \cos \pi)$$
$$= 2\pi i (0 - 1) = -2\pi i$$

$-(-2\pi i)$

$$\oint \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz = 2\pi i f(2)$$
$$= 2\pi i \{\sin \pi (2)^2 + \cos \pi (2)^2\}$$
$$= 2\pi i (\sin 4\pi + \cos 4\pi)$$
$$= 2\pi i (0 + 1) = 2\pi i.$$

Since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then the required integral has the value

$$-(-2\pi i) + (2\pi i)$$
$$= 4\pi i.$$