Boundary Value Problems

Fourier transforms of partial derivative of a function:

1. $F_f\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2 F(u)$, where $F(u)$ is a fourier transform of u with respect to x.

2. $F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = s\cdot(u)_{x=0} - s^2 F_s(u)$ (sine transform)

3. $F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x}\right]_{x=0} - s^2 F_c(u)$ (cosine transform)

<u>Note</u>: If u at $x=0$ is given, take fourier sine transform, and if $\frac{\partial u}{\partial x}$ at $x=0$ is given, then use fourier cosine transform.

<u>Example</u>: Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, subject to the conditions

(i) $u=0$ when $x=0, t>0$

(ii) $u = \begin{cases} 1, & 0<x<1 \\ 0, & x\geq 1 \end{cases}$ when $t=0$

(iii) $u(x,t)$ is bounded.

<u>Sol^n</u>: In view of the initial conditions, we apply fourier sine transform $\int_0^\infty \frac{\partial u}{\partial t}\sin sx\,dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin sx\,dx$ [Fourier sine transform $F(s) = \int_0^\infty f(x)\sin sx\,dx$]

$$\Rightarrow \frac{\partial}{\partial t}\int_0^\infty u \sin sx\, dx = -s^2\bar{u}(s) + s\cdot u(0) \quad [\bar{u} \to \text{Fourier sine transform of } u]$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial t} = -s^2\bar{u} + 0 \quad [\because u = 0 \text{ when } x = 0]$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial t} + s^2\bar{u} = 0 \quad \cdots\cdots\cdots (1)$$

Let $\bar{u} = e^{mt}$ be the trial solⁿ of (1). Then from (1) we get

$$\frac{\partial}{\partial t}(e^{mt}) + s^2 e^{mt} = 0$$

$$\Rightarrow m e^{mt} + s^2 e^{mt} = 0$$

$$\Rightarrow e^{mt}(m + s^2) = 0$$

So, the auxiliary equation (A.E.) is $m + s^2 = 0$

$$\Rightarrow m = -s^2$$

$\therefore$ The solution is $\bar{u} = A e^{-s^2 t}$ $\cdots\cdots\cdots$ (2)

Again, we know $\bar{u} = \bar{u}(s,t) = \int_0^\infty u(x,t) \sin sx\, dx$

$$\therefore \bar{u} = \bar{u}(s,0) = \int_0^\infty u(x,0) \sin sx\, dx$$

$$= \int_0^1 1\cdot \sin sx\, dx$$

$$= \left[\frac{-\cos sx}{s}\right]_0^1 = \frac{1}{s}(-\cos s + 1)$$

$$= \frac{1-\cos s}{s} \quad \cdots\cdots\cdots (3)$$

Now, from (3) put the value of $\bar{u}(s,0)$ in (2) we get,

$$\frac{1-\cos s}{s} = A\cdot e^0 = A \quad (\text{at } t=0)$$

$\therefore$ From (2), $\bar{u} = \frac{1-\cos s}{s} e^{-s^2 t}$

Now from the inverse fourier sine transform, we get

$$u = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{1-\cos s}{s}\right) e^{-s^2 t} \sin sx \, ds. \quad (\text{Ans.})$$

<u>Note</u>: The inverse fourier sine transform of a function $F(s)$ is given by $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(s) \sin sx \, ds$.

<u>Exercise</u>: Solve: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ for $0 \leq x < \infty$, $t > 0$; given

the conditions (i) $u(x,0) = 0$ for $x \geqslant 0$

(ii) $\frac{\partial u}{\partial x}(0,t) = -a$ (constant)

(iii) $u(x,t)$ is bounded.