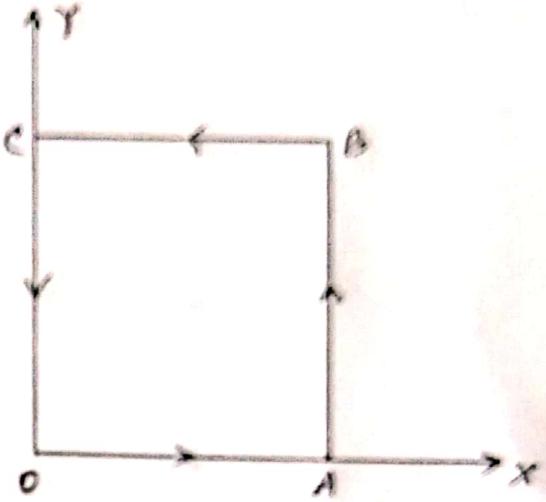
Example: Evaluate $\int_C \vec{F}\cdot\vec{dr}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane $z=0$ and bounded by the lines $x=0$, $y=0$, $x=a$ and $y=a$.



Sol^n: Here

$$\int_C \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CO} \vec{F}\cdot\vec{dr}$$

Again we know $\vec{r} = x\hat{i} + y\hat{j}$

$$\therefore \vec{dr} = dx\,\hat{i} + dy\,\hat{j}$$

Given, $\vec{F} = x^2\hat{i} + xy\hat{j}$

$$\therefore \vec{F}\cdot\vec{dr} = (x^2\hat{i} + xy\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$\Rightarrow \vec{F}\cdot\vec{dr} = x^2\,dx + xy\,dy \quad \cdots\cdots\cdots (1)$$

Now, on OA, $y=0$, $\therefore dy = 0$

So, from (1), $\vec{F}\cdot\vec{dr} = x^2\,dx$

$$\therefore \int_{OA} \vec{F}\cdot\vec{dr} = \int_{x=0}^{a} x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{1}{3}(a^3 - 0^3) = \frac{a^3}{3} \quad \cdots\cdots (2)$$

Again, on AB, $x = a$, $\therefore dx = 0$

So, from (1), $\vec{F}\cdot\vec{dr} = ay\,dy$

$$\therefore \int_{AB} \vec{F}\cdot\vec{dr} = \int_{y=0}^{a} ay\,dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a}{2}(a^2 - 0^2) = \frac{a^3}{2} \quad \cdots\cdots (3)$$

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Now on BC, $y = a$, $\therefore dy = 0$.

Then (1) becomes, $\vec{F} \cdot \vec{dr} = x^2 dx$.

$$\therefore \int_{BC} \vec{F} \cdot \vec{dr} = \int_{x=a}^{0} x^2 dx = \left[\frac{x^3}{3}\right]_a^0 = \frac{1}{3}(0^3 - a^3) = -\frac{a^3}{3} \quad \cdots\cdots (4)$$

Now on CO, $x = 0$, $\therefore dx = 0$.

Then from (1), $\vec{F} \cdot \vec{dr} = 0$

$$\therefore \int_{CO} \vec{F} \cdot \vec{dr} = 0 \quad \cdots\cdots\cdots\cdots (5)$$

Now, adding (2), (3), (4) and (5) we get,

$$\int_C \vec{F} \cdot \vec{dr} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$$

$$= \frac{a^3}{2}. \quad \text{(Ans)}.$$