Fourier Transforms

The fourier transform of a function $f(x)$ is given by $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx$

The inverse fourier transform of a function $F(s)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} ds$$

The fourier sine transform of a function $f(x)$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cdot \sin sx \, dx$$

The inverse fourier sine transform of a function $F(s)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \cdot \sin sx \, ds.$$

The fourier cosine transform of a function $f(x)$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cdot \cos sx \, dx$$

The inverse fourier cosine transform of a function $F(s)$ is

given by $f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \cdot \cos sx \, ds$

Ex Find the fourier transform of the function $f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$

Sol^n: The fourier transform of a function $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} \, dx$$

$x < a$
$-x < a$
$\Rightarrow x > -a$

$-a \quad 0 \quad a$

Substituting the value of $f(x)$, we get

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \cdot e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(is)} \left[e^{ias} - e^{-ias} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{s} \cdot \frac{e^{ias} - e^{-ias}}{2i}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2 \sin as}{s} = \sqrt{\frac{2}{\pi}} \, \frac{\sin as}{s} \cdot \text{(Ans)}$$

Ex. Find the fourier transform of the following function:

$$f(x) = \begin{cases} 1 - x^2 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

$\underline{Sol^n}$: Given $f(x) = \begin{cases} 1-x^2 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

or, $f(x) = \begin{cases} 1-x^2 & \text{if } -1 \le x \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$

The fourier transform of a function $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx \quad \text{-------------} \quad (1)$$

Substituting the values of $f(x)$ in (1), we get

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) e^{isx} dx$$

Using integrating by parts, we get $\left[[u \cdot v]_1 = uv_1 - u' \cdot v_2 + u'' \cdot v_3 \cdots \right]$

$$F(s) = \frac{1}{\sqrt{2\pi}} \left[(1-x^2) \cdot \frac{e^{isx}}{is} - (-2x) \cdot \frac{e^{isx}}{(is)^2} + (-2) \frac{e^{isx}}{(is)^3} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ 0 + (-2) \frac{e^{is}}{s^2} - (-2) \frac{e^{is}}{is^3} \right\} - \left\{ 0 + (2) \frac{e^{-is}}{s^2} - (-2) \frac{e^{-is}}{is^3} \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-2 \frac{e^{is}}{s^2} + 2 \frac{e^{is}}{is^3} - 2 \frac{e^{-is}}{s^2} - 2 \frac{e^{-is}}{is^3} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{s^2} (e^{is} + e^{-is}) + \frac{2}{is^3} (e^{is} - e^{-is}) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{s^2} \cdot (2 \cos s) + \frac{2}{is^3} (2i \sin s) \right] \quad \left[\because \cos x = \frac{e^{ix} + e^{-ix}}{2} \right.$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{4}{s^3} (-s \cos s + \sin s) \quad (\text{Ans}) \quad \left. \text{and } \sin x = \frac{e^{ix} - e^{-ix}}{2i} \right]$$