Example 2: Show that the vectors $(2,-1,4)$, $(3,6,2)$ and $(2,10,-4)$ are linearly independent.

Proof: Set a linear combination of the given vectors equal to the zero vector using unknown scalars x, y, z:

$$x(2,-1,4) + y(3,6,2) + z(2,10,-4) = (0,0,0)$$

$$\Rightarrow (2x,-x,4x) + (3y,6y,2y) + (2z,10z,-4z) = (0,0,0)$$

$$\Rightarrow (2x+3y+2z, -x+6y+10z, 4x+2y-4z) = (0,0,0)$$

Form a homogeneous system of linear equations equating the corresponding components:

$$\left.\begin{aligned} 2x+3y+2z &= 0 \\ -x+6y+10z &= 0 \\ 4x+2y-4z &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (1)$$

Reduce the system to echelon form by the elementary transformations. Interchange first and 2nd equations and get the equivalent system:

$$\sim \left.\begin{aligned} -x+6y+10z &= 0 \\ 2x+3y+2z &= 0 \\ 4x+2y-4z &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (2)$$

Now apply $L_1' = -L_1$ and $L_3' = \frac{L_3}{2}$, and get the equivalent system

$$\sim \left.\begin{aligned} x - 6y - 10z &= 0 \\ 2x + 3y + 2z &= 0 \\ 2x + y - 2z &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (3)$$

Now apply $L_2' = L_2 - 2L_1$ and $L_3' = L_3 - 2L_1$ and get

$$\sim \left.\begin{aligned} x - 6y - 10z &= 0 \\ 15y + 22z &= 0 \\ 13y + 18z &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (4)$$

Now apply $L_3' = L_3 - \frac{13}{15} L_2$ and get

$$\sim \left.\begin{aligned} x - 6y - 10z &= 0 \\ 15y + 22z &= 0 \\ -\frac{16}{15} z &= 0 \end{aligned}\right\} \cdots\cdots\cdots (5)$$

$$18 - 22 \times \frac{13}{15} = \frac{270 - 286}{15} = -\frac{16}{15}$$

Which is in echelon form.

In echelon form there are exactly three equations in three unknowns; hence the system has only the zero solution $x = 0$, $y = 0$ and $z = 0$. Accordingly, the vectors are linearly independent. [Proved]

Exercise: Examine whether the following sets of vectors are linearly dependent or independent:

(i) $\{(3,0,1,-1), (2,-1,0,1), (1,1,1,-2)\}$

(ii) $\{(1,-2,3), (5,6,-1), (3,2,1)\}$

(iii) $\{(1,0,2), (-1,1,0), (0,2,3)\}$

(iv) $\{(1,1,1), (0,1,1), (0,0,1)\}$

Gradient of a scalar point function:

Vector differential operator Del (∇):

The vector differential operator Del is denoted by ∇. It is defined as $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

The operator ∇ is also known as nabla.

The Gradient:

The gradient of a scalar point function $\Phi(x,y,z)$ is grad Φ or $\nabla\Phi$, and is defined as

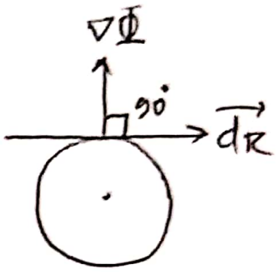
$$\nabla\Phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\Phi$$

$$= \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}$$

Note that $\nabla\Phi$ defines a vector field.

Normal and Directional Derivative

Normal: If $\Phi(x, y, z) = c$ represents a surface for a specific value of c, then $\nabla\Phi$ is a vector normal to the surface $\Phi(x, y, z) = c$. $\left[\text{where } \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]$



Note: $\vec{dr}$ is in the direction of tangent to the given surface $\Phi(x, y, z) = c$.

Scalar point function:

As for example, $\Phi(x, y, z) = x^2 + y^2 + z^2 = 16$

Vector point function:

As for example, $v(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$

Directional derivative of $\Phi(x, y, z)$ in the direction $\vec{d}$:

The directional derivative of a scalar point function $\Phi(x, y, z)$ in the direction of a vector $\vec{d}$ is equal to $\nabla\Phi \cdot \hat{d}$.

Physically, this is the rate of change of Φ at (x, y, z) in the direction $\hat{d}$.