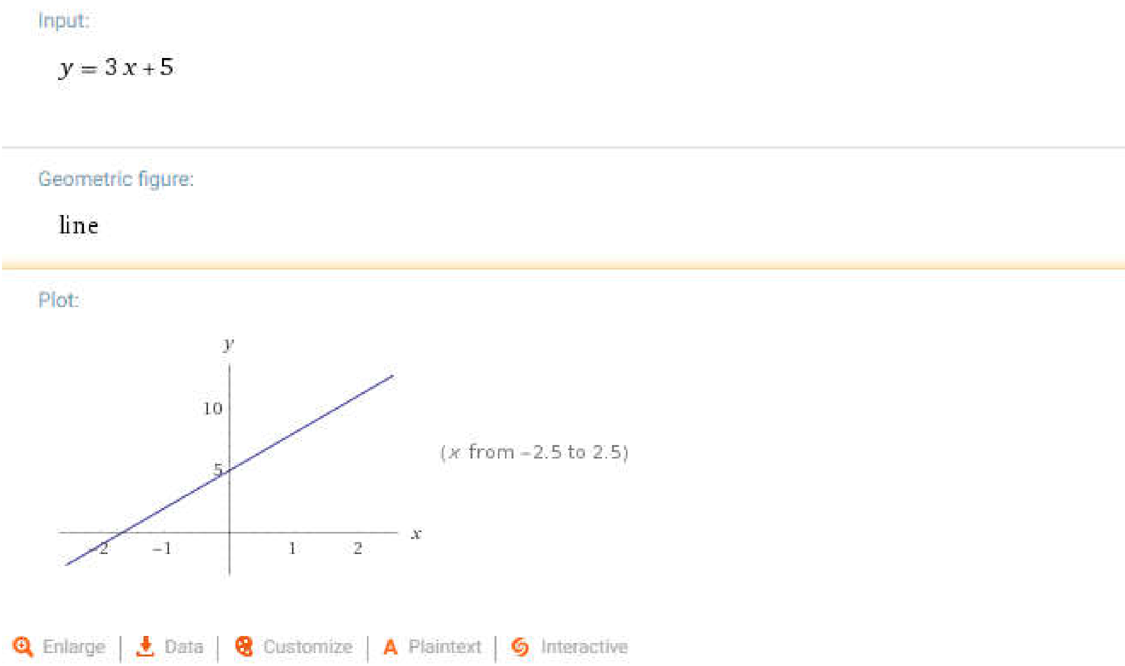
Roots of Non Linear Equation

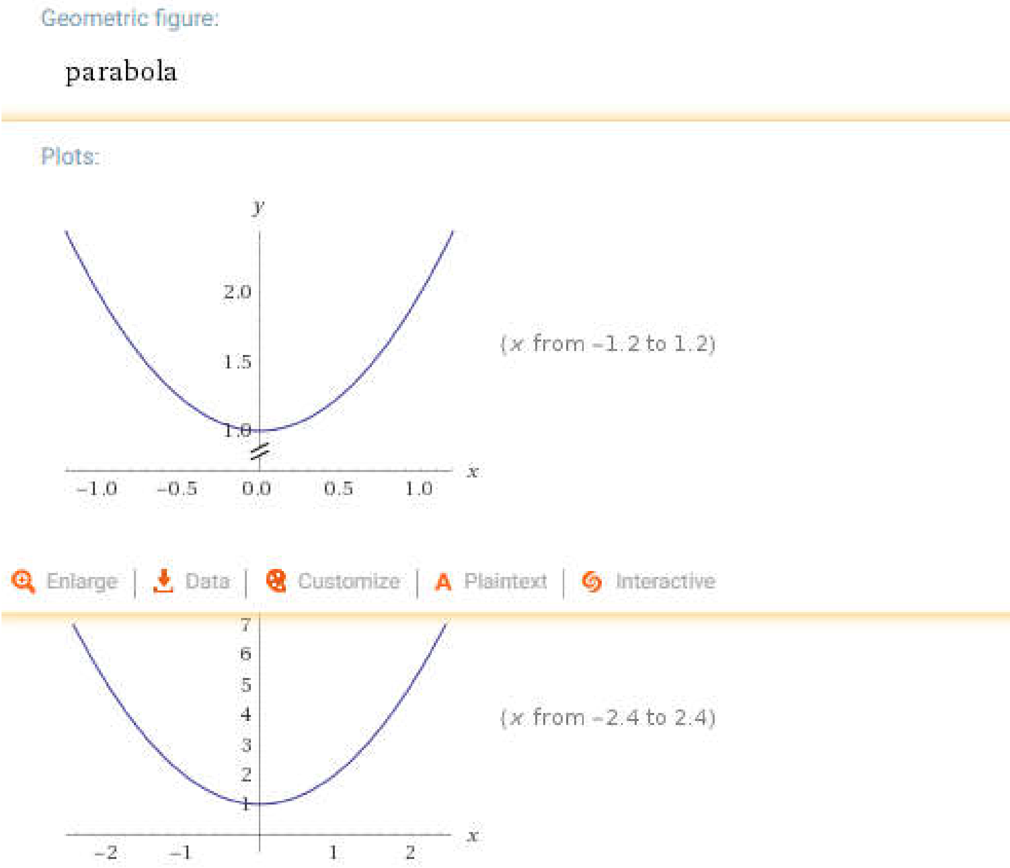
Mathematical models for a wide variety of problems in science and engineering can be formulated into equations of the form: $f(x) = 0$ ....(i), where x and f (x) may be real, complex or vector quantities. The values of x for which the equation (i) satisfy are called the *roots* of the equation.

Equation (i) may belong to one of the following types of equations:
i) Algebraic equation ii) Polynomial equation iii) Transcendental equations

- y = f(x) is a *linear* function, if the dependent variable y changes in direct proportion to the change in independent variable x. For example, y = 3x + 5 is a linear equation.



- y = f(x) is said to be *nonlinear*, if the response of the dependent variable y is not direct or exact proportion to the changes in the independent variable x. For example, y = x^2 + 1 is a non linear equation.



- ❖ Any function of one variable which does not graph as a straight line in two dimensions, or any function of two variables which does not graph as a plane in three dimensions, can be said as *nonlinear*.

Algebraic Equation

An equation of type $y = f(x)$ is said to be *algebraic* if it can be expressed in the form

$f_n y_n + f_{n-1} y_{n-1} + \ldots\ldots\ldots\ldots + f_1 y_1 + f_0 = 0$ ……………………………………………(ii), where f_i is an ith order polynomial in x. Equation (ii) can be written as general form as,

$f(x, y) = 0$

- ❑ Some examples are:

1. $3x + 5x - 2 = 0$ (linear)
2. $2x + 3xy - 25 = 0$ (non-linear)
3. $x^3 - xy + 3y^3 = 0$ (non-linear)

These equations have an infinite number of pairs of values and x and y which satisfy them.

Polynomial Equation

Polynomial equations are a simple class of algebraic equations that are represented as follows:

$a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_1 x + a_0 = 0$

This is called n^{th} degree polynomial and has n roots. The roots may be:

i) real and different
ii) real and repeated
iii) complex numbers

- Since a complex roots appear in pairs, if n is odd, then the polynomial has at least one real root.
- Some specific examples of polynomial equations are:

1. $5x^5 - x^3 + 3x^2 = 0$
2. $x^3 - 4x^2 + x + 6 = 0$

Transcendental Equations

A non-algebraic equation is called a *transcendental* equation. These include trigonometric, exponential and logarithmic functions.

- Examples of transcendental equation are:

1. $2 \sin x - x = 0$
2. $e^x \sin x - \log x = 0$

- A transcendental equation may have a finite or an infinite number of real roots or may not have real root at all.

Methods of Solution

There are a number of ways to find the roots of non linear equation such as:

1. Direct analytical methods
2. Graphical methods
3. Trial and error methods
4. Iterative methods

Direct analytical methods

In certain cases, roots can be found by using direct analytical methods. For example, we can easily calculate the roots of a quadratic equation: $ax^2 + bx + c = 0$ from the following equation:
$x = (- b \pm \sqrt{(b^2 - 4ac)}) / 2a$
However, there are equations that cannot be solved by analytical methods. For example, the simple transcendental equation $2 \sin x - x = 0$ cannot be solved analytically. Direct methods for solving non-linear equations do not exist except for certain simple cases.

Graphical methods

Graphical methods are useful when we are satisfied with approximate solution for a problem. This method involves plotting the given function and determining the points where it crosses x-axis. These points represent approximate values of the roots of the function.

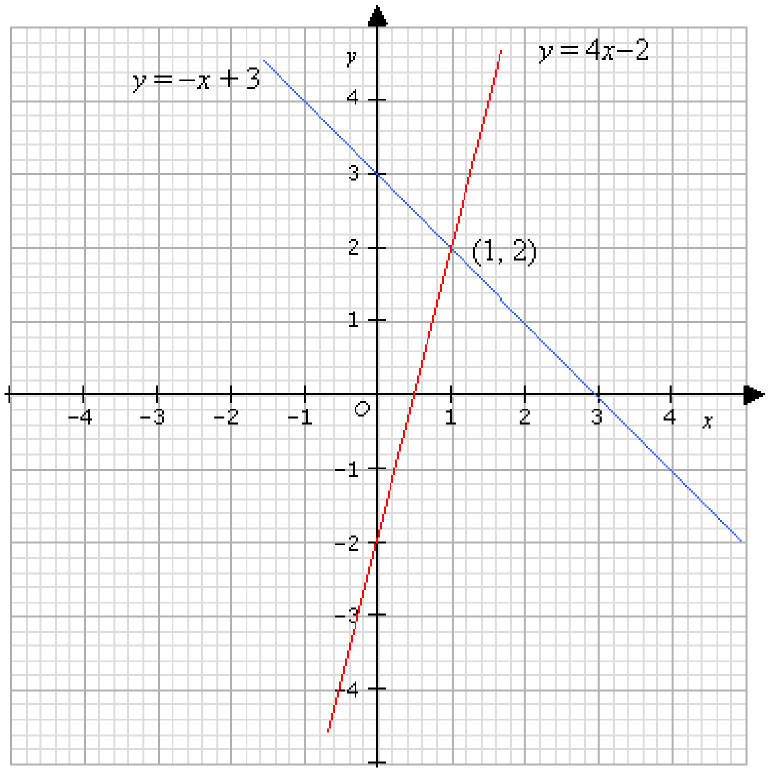
Example:

Using the graphical method, find the solution of the systems of equations

$y + x = 3$
$y = 4x - 2$

Solution:

Draw the two lines graphically and determine the point of intersection from the graph.
From the graph, the point of intersection is (1, 2)



Trial & Error methods
This method involves a series of guesses for x, each time evaluating the function to see whether it is close to zero. The value of x that causes the function value closer to zero is one of the approximate roots of the equation.

- Although graphical and trial error methods provide satisfactory approximations for many problem situations, they become cumbersome and time consuming. Moreover, the accuracy of the results is inadequate for the requirements of many engineering and scientific problems.

Iterative methods:
An iterative technique usually begin with an approximate value of the root, known as *initial guess*, which is then successively corrected iteration by iteration. The process of iteration stops when the desired level of accuracy is obtained.

- Iterative methods can be divided into two categories:

1. *Bracketing methods*: Bracketing methods (also known as *interpolation methods*) start with initial guesses that 'bracket' the root and then systematically reduce the width of the bracket until the solution is reached. Two popular method under this category are:
    - Bisection method
    - False position method
2. *Open end methods*: Open end methods (also known as *extrapolation methods*) use a single starting value or two values that do not necessarily bracket the root. The following iterative methods fall under this category:
    - Newton-Raphson method
    - Secant method.
    - Muller's method.
    - Methods of successive approximation (Fixed-point method)

Largest Possible Root

For a polynomial represented by: $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ the largest possible root is given by: $x = -\frac{a_{n-1}}{a_n}$.

- This value is taken as initial approximation when no other value is suggested by the knowledge of the problem at hand.

Search Bracket

Maximum absolute value of the root is,

$$|x_{\max}| = \sqrt{\{(\frac{a_{n-1}}{a_n})^2 - 2(\frac{a_{n-2}}{a_n})\}}$$

So, all real roots lie within the interval (- | x $_{max}$ |, | x $_{max}$ |).

Interval of the roots

All real roots x satisfy the inequality,

$$|x| \leq 1 + \frac{1}{a_n} \max \{ |a_0|, |a_1|, \ldots , |a_{n-1}| \}$$

where the 'max' denotes the maximum of the absolute values $|a_0|, |a_1|, \ldots , |a_{n-1}|$.

Example: Consider the polynomial equation: $2x^3 - 8x^2 + 12 = 0$. Estimate the possible initial guess values.

Solution: The largest possible root is x = - (-8 / 2) = 4. No root can be larger than the value 4.

All roots must satisfy the relation

$$|x_{\max}| <= \sqrt{\{(\frac{-8}{2})^2 - 2.\frac{0}{2}\}} = 4$$

Therefore, all real roots lie in the interval (-4, 4). We can use these two points as initial guesses for the bracketing methods and one of them as the open end methods.

Slopping Criterion

An iterative process must be terminated at some stage. We may use one (or combination) of the following tests, depending on the behavior of the function, to terminate the process:

1. $| x_{i+1} - x_i | \leq E_a$ (absolute error in x)

2. $\left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \leq E_r$ (relative error in x), $x \neq 0$

3. $| f(x_{i+1}) | \leq E$ (value of function at root)

4. $| f(x_{i+1}) - f(x_i) | \leq E$ (difference in function values)

5. $| f(x) | \leq F_{max}$ (large function value)

6. $| x_i | \leq XL$ (large value of x)

Here x_i represents the estimate of the root at i^{th} iteration and $f(x_i)$ is the value of the function at x_i.

- In cases where we do not know whether the process converges or not, we must have a limit on the number of iterations, like Iterations $\geq N$ (limit on iterations).