Green's Theorem

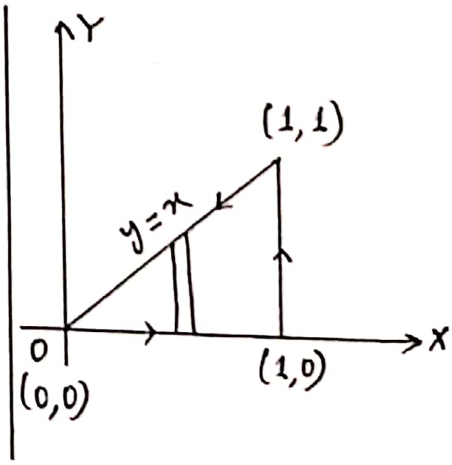
Green's Theorem (For a plane)

Statement: If $\Phi(x,y), \Psi(x,y), \frac{\partial \Phi}{\partial y}$ and $\frac{\partial \Psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve c in x-y plane, then

$$\oint_c (\Phi\, dx + \Psi\, dy) = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y} \right) dx\, dy$$

Example: Using Green's theorem evaluate $\int_c (x^2 y\, dx + x^2 dy)$ where c is the boundary described counter clockwise of the triangle with vertices $(0,0), (1,0), (1,1)$.

Sol^n: By Green's theorem



$$\oint_c (\Phi\, dx + \Psi\, dy) = \iint_R \left(\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y} \right) dx\, dy$$

$$\Rightarrow \int_c (x^2 y\, dx + x^2 dy) = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(x^2 y) \right] dx\, dy$$

Here, $\Phi = x^2 y$

$\Psi = x^2$

$$= \iint_R (2x - x^2)\, dx\, dy$$

(P.T.O.)

$$\Rightarrow \int_c (x^2y\,dx + x^2dy) = \int_{x=0}^{1} \int_{y=0}^{x} (2x - x^2)\,dx\,dy$$

$$= \int_{x=0}^{1} \left[2xy - x^2y\right]_0^x dx$$

$$= \int_{x=0}^{1} \left[(2x\cdot x - x^2\cdot x) - (2x\cdot 0 - x^2\cdot 0)\right] dx$$

$$= \int_{x=0}^{1} (2x^2 - x^3)\,dx$$

$$= 2\int_{x=0}^{1} x^2\,dx - \int_{x=0}^{1} x^3\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3}\left[(1)^3 - (0)^3\right] - \frac{1}{4}\left[(1)^4 - (0)^4\right]$$

$$= \frac{2}{3}\cdot 1 - \frac{1}{4}\cdot 1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12} \text{ (Ans.)}$$

Example: Verify Green's theorem in the plane for $\oint_c (xy+y^2)dx + x^2dy$, where c is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Solution: Given $y = x$ .... (1)

$y = x^2$ .... (2)

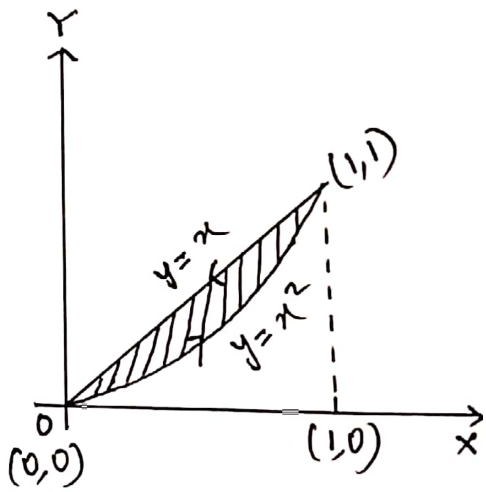
Put the value of y from (1) in (2) we get,

$$x = x^2$$
$$\Rightarrow x^2 - x = 0$$
$$\Rightarrow x(x-1) = 0 \quad \Rightarrow x = 0, 1$$

Now put the values of x in (1) we get $y = 0, 1$.

Hence $y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$.

The positive direction in traversing c is as shown in the adjacent diagram.



Given $\oint_c (xy + y^2)dx + x^2 dy$

Along $y = x^2$, the line integral equals,

$$\int_{x=0}^{1} \left[\left\{x \cdot (x^2) + (x^2)^2\right\} dx + x^2 \cdot d(x^2)\right]$$

$$= \int_0^1 \left[(x^3 + x^4)\,dx + x^2 \cdot 2x\,dx\right]$$

$$= \int_0^1 \left(x^3 dx + x^4 dx + 2x^3 dx\right)$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= 3\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4}\left[(1)^4-(0)^4\right]+\frac{1}{5}\left[(1)^5-(0)^5\right]$$

$$= \frac{3}{4}(1-0)+\frac{1}{5}(1-0) = \frac{3}{4}+\frac{1}{5} = \frac{15+4}{20} = \frac{19}{20}.$$

Now along $y=x$ from $(1,1)$ to $(0,0)$ the line integral equals,

$$\int_c \left[(xy+y^2)dx+x^2dy\right]$$

$$= \int_{x=1}^{0}\left[\left\{(x)(x)+(x)^2\right\}dx+x^2d(x)\right]$$

$$= \int_1^0\left[(x^2+x^2)dx+x^2dx\right]$$

$$= \int_1^0 3x^2dx = 3\left[\frac{x^3}{3}\right]_1^0 = \left[(0)^3-(1)^3\right] = (0-1) = -1.$$

Then the required line integral $= \frac{19}{20}-1 = \frac{19-20}{20} = -\frac{1}{20}.$

Again, $\iint_R \left(\frac{\partial\psi}{\partial x}-\frac{\partial\Phi}{\partial y}\right)dx\,dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2)-\frac{\partial}{\partial y}(xy+y^2)\right]dx\,dy$$

$$= \iint_R \left[2x-(x+2y)\right]dx\,dy$$

$$= \iint_R (x-2y)\,dx\,dy$$

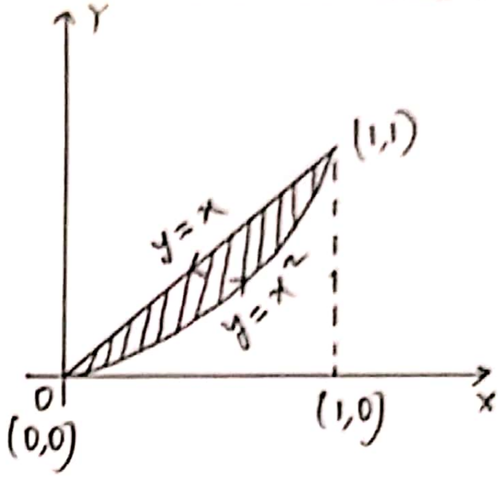
Green's theorem:

$$\oint_c (\Phi dx+\psi dy) = \iint_R\left(\frac{\partial\psi}{\partial x}-\frac{\partial\Phi}{\partial y}\right)dx\,dy$$

Given,

$$\oint_c \left[(xy+y^2)dx+x^2dy\right.$$

Here, $\Phi = xy+y^2$, and $\psi = x^2$



$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dx\, dy$$

$$= \int_{x=0}^{1} \left[xy - 2 \cdot \frac{y^2}{2} \right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[xy - y^2 \right]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[(x \cdot x - x^2) - (x \cdot x^2 - x^4) \right] dx$$

$$= \int_{x=0}^{1} \left[0 - x^3 + x^4 \right] dx$$

$$= \int_{0}^{1} (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$$= \left[\left(\frac{1}{5} - \frac{1}{4}\right) - \left(\frac{0}{5} - \frac{0}{4}\right) \right] = \frac{1}{5} - \frac{1}{4} - 0 = \frac{4-5}{20} = -\frac{1}{20}$$

Hence, $\oint_C (\Phi\, dx + \psi\, dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \Phi}{\partial y} \right) dx\, dy$

$$\Rightarrow -\frac{1}{20} = -\frac{1}{20}$$

So, that the Green's theorem is verified. ⊞

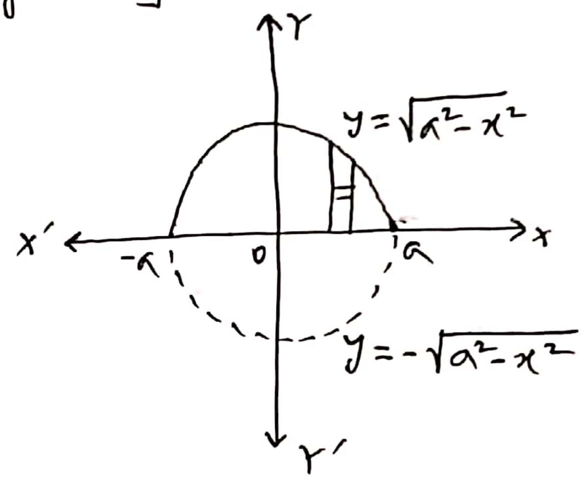
Exercise: Apply Green's theorem to evaluate

$\int_c [(2x^2-y^2)dx+(x^2+y^2)dy]$, where c is the boundary of the area enclosed by the x-axis and the upper half of circle $x^2+y^2=a^2$. [Ex. 82, Page: 433]

Hints: $x^2+y^2=a^2$

$\Rightarrow y^2=a^2-x^2$

$\therefore y=\pm\sqrt{a^2-x^2}$



Given, $\int_c [(2x^2-y^2)dx+(x^2+y^2)dy]$

Green's theorem,

$$\oint_c (\Phi\, dx+\psi\, dy)=\iint_R \left(\frac{\partial\psi}{\partial x}-\frac{\partial\Phi}{\partial y}\right)dx\,dy$$

Here, $\Phi=2x^2-y^2$, and $\psi=x^2+y^2$

$$=\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left[\frac{\partial}{\partial x}(x^2+y^2)-\frac{\partial}{\partial y}(2x^2-y^2)\right]dx\,dy$$

Exercise: Use Green's theorem to evaluate

$\int_c (x^2+xy)dx+(x^2+y^2)dy$ where c is the square formed by the lines $y=\pm1, x=\pm1$.