Stoke's Theorem

Mathematically, this theorem can be written as,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds, \text{ where } \hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

is a unit external normal to any surface ds and $ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$.

Example: Verify Stoke's theorem for $\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where s is the upper half surface of the sphere $x^2+y^2+z^2=1$ and c is its boundary. [Ex. 96, Page No. 443]

Solution: The boundary c of s is a circle in the xy-plane of radius one and center at the origin. Let $x=\cos t$, $y=\sin t$, $z=0$, $0 \le t < 2\pi$ be parametric equations of c.

$x=\cos t$, $y=\sin t$
$\therefore x^2+y^2 = \cos^2 t + \sin^2 t$
$\therefore x^2+y^2=1$
(circle)

Then,

$$\oint_c \vec{A}\cdot d\vec{r} = \oint_c \left[(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\right]\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \oint_c \left[(2x-y)dx - yz^2dy - y^2z\,dz\right]$$

$$= \int_{t=0}^{2\pi}\left[(2\cos t - \sin t)\,d(\cos t) - (\sin t)\cdot(0)^2 d(\sin t) - \sin^2 t\cdot 0\cdot d(0)\right]$$

$$= \int_{t=0}^{2\pi}\left[(2\cos t - \sin t)(-\sin t\,dt) - 0 - 0\right]$$

[P.T.O.]

$$\Rightarrow \oint_C \vec{A}\cdot d\vec{r} = \int_{t=0}^{2\pi} (-2\sin t\cdot \cos t + \sin^2 t)\,dt$$

$$= \int_{t=0}^{2\pi} (-\sin 2t + \sin^2 t)\,dt$$

$$= -\int_0^{2\pi} \sin 2t\,dt + \int_0^{2\pi} \sin^2 t\,dt$$

$$= -\int_0^{2\pi} \sin 2t\,dt + \frac{1}{2}\int_0^{2\pi} 2\sin^2 t\,dt$$

$$= -\int_0^{2\pi} \sin 2t\,dt + \frac{1}{2}\int_0^{2\pi} (1-\cos 2t)\,dt$$

$$= -\left[\frac{-\cos 2t}{2}\right]_0^{2\pi} + \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\cos 4\pi - \cos 0\right] + \frac{1}{2}\left[\left\{2\pi - \frac{\sin 4\pi}{2}\right\} - \left\{0 - \frac{\sin 0}{2}\right\}\right]$$

$$= \frac{1}{2}(1-1) + \frac{1}{2}\left[(2\pi - 0) - (0-0)\right]$$

$$= 0 + \pi = \pi$$

Now we have to evaluate the surface integral $\iint_S \text{curl}\vec{A}\cdot\hat{n}\,ds$,

where $\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$.

Now $\text{curl}\vec{A} = \nabla\times\vec{A}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left[(2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}\right]$$

(P.T.O.)

$$\Rightarrow \text{Curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x-y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right]$$

$$= \hat{i}\left[(-2yz + 2yz)\right] - \hat{j}(0-0) + \hat{k}(0-0+1)$$

$$= 0\hat{i} - 0\hat{j} + 1\cdot\hat{k} = \hat{k}.$$

Now put the value of curl $\vec{A}$ in the surface integral $\iint_S \text{curl}\,\vec{A}\cdot\hat{n}\,ds$

we get, $\iint_S \text{curl}\,\vec{A}\cdot\hat{n}\,ds$

$$= \iint_S \hat{k}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}} \quad \left[\because ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}\right]$$

$$= \iint_S dx\,dy$$

$x^2+y^2=1$

$$= \pi(1)^2 = \pi$$

So, $\oint_C \vec{A}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{A}\cdot\hat{n}\,ds$

$$\Rightarrow \pi = \pi$$

Hence the Stoke's theorem is verified. $\boxplus$

Exercise: Using Stoke's theorem evaluate

$\oint_C [(2x-y)dx - yz^2 dy - y^2 z\, dz]$ where C is the circle $x^2+y^2=1$, corresponding to the surface of sphere of unit radius.

[Ex. 86, Page No. 438]