Do-little LU Decomposition method

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Do-little LU Decomposition method

- The system of linear equation can be expressed in the matrix form as:

$$Ax = b\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Gauss Elimination method is inefficient when equation with same coefficient [A], but with different right hand side constant {B}

- LU Factorization:
- In LU Factorization method, the coefficient matrix A of a system of linear equations can be factorized or decomposed into two triangular matrices L and U such that:

$$A = LU\ \ldots\ldots\ldots(2).$$

- Rearrange equation (1): $Ax - b = 0\ldots\ldots\ldots\ldots\ldots\ldots(3)$

Do-little LU Decomposition method

- Suppose equation (3) could be expressed as upper triangular system:

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow [U]\{x\} - \{d\} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

- Consider Lower triangular matrix with 1's on diagonal….

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

- Multiply equation (4) and (5)
- [L]{[U]{x} – {d}} = [A]{x} – {b} ……………………………..(6)
- From equation 6:
  - [L][U] = [A] ……………………………………………………(7)
  - And [L] {d} = {b} ………………………………………………(8)

Do-little LU Decomposition method

- Steps of LU decomposition Methods:

(a)Factorization

[A] {x} = {B}

[U] [L]

[L] {d} = {b}

{d}

(b) Forward Substitution

[U] {x} = {d}

{x}

(c) Back Substitution

Do-little LU Decomposition method

- LU factorization Step: [A] is decomposed or factored in Lower[L] and Upper [U] triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- Forward Substitution Step: [L] {d} = {b} is used to generate an intermediate vector {d}

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b \end{bmatrix}$$

- Backward Substitution Step: The result from forward substitution is used substitute [U]{X} = {d} to solve {x}

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Do-little LU Decomposition method

- Multiply [L]and [U] we get:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{12}+l_{32}u_{23}+u_{33} \end{bmatrix} \text{..........(9)}$$

From equation (9) we can find the entries of L and U

$\boxed{u_{11} = a_{11}}$ $\boxed{u_{12} = a_{12}}$ $\boxed{u_{13} = a_{13}}$

$\boxed{l_{21} = \frac{a_{21}}{u_{11}}}$ $\boxed{u_{22} = a_{22} - l_{21}u_{12}}$ $\boxed{u_{23} = a_{23} - l_{21}u_{13}}$

$\boxed{l_{31} = \frac{a_{31}}{u_{11}}}$ $\boxed{l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}}}$ $\boxed{u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}}$

Do-little LU Decomposition method

- [L]and [U] can be:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{u_{11}} & 1 & 0 \\ \frac{a_{31}}{u_{11}} & \frac{a_{32} - l_{31}u_{12}}{u_{22}} & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - l_{21}u_{12} & a_{23} - l_{21}u_{13} \\ 0 & 0 & a_{33} - l_{31}u_{13} - l_{32}u_{23} \end{bmatrix}$$

- Now we get {d} from forward substitution
- Solve {x} from backward substitution

Example:

- A = $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$ X = $\begin{matrix} x1 \\ x2 \\ x3 \end{matrix}$ B = $\begin{matrix} b1 \\ b2 \\ b3 \end{matrix}$

Suppose we have system of equation AX = B

We will find the matrix L and U where A = LU

L = $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ U = $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

- Multiplying L and U LU and setting the answer equal to A gives:

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

Now we use this to find the entries in L and U. Consider 1st Row:

$\boxed{u_{11} = 1}$ $\boxed{u_{12} = 2}$ $\boxed{u_{13} = 4}$

- Now consider the 2nd row

$l_{21}u_{11}$ = 3 | l_{21}*1 = 3 | $\boxed{l_{21} = 3}$

$l_{21}u_{12} + u_{22}$= 8 | 3*2 +u_{22} = 8 | $\boxed{u_{22} = 2}$

$l_{21}u_{13} + u_{23}$= 14 | 3*4 +u_{23} = 14 | $\boxed{u_{23} = 2}$

Notice how, at each step, the equation being considered has only one unknown in it, and other quantities that we have already found. This pattern continues on the last row

- Now consider the last row

$l_{31}u_{11}$ = 2	l_{31}*1 = 2	l_{31} = 2
$l_{31}u_{12} + l_{32}u_{22}$ = 6	2*2 + l_{32}*2 = 6	l_{32}= 1
$l_{31}u_{13} + l_{32}u_{23} + u_{33}$ = 13	(2*4)+(1*2) +u_{33} = 13	u_{33} = 3

- We have show that:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

And this is an LU decomposition of A

Example

- Solve the following system using LU decomposition.

$x_1 + x_2 + x_3 = 1$

$4x_1 + 3x_2 - x_3 = 6$

$3x_1 + 5x_2 + 3x_3 = 4$

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \qquad x = \begin{matrix} x1 \\ x2 \\ x3 \end{matrix} \qquad b = \begin{matrix} 1 \\ 6 \\ 4 \end{matrix}$$

Example

- Step 1: Factorization Step:

[A] = [L][U]

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

- Now we use this to find the entries in L and U.
- Consider 1st Row:

$u_{11} = 1$ $\quad u_{12} = 1$ $\quad u_{13} = 1$

Example

- Now consider the 2nd row

$l_{21}u_{11}$ = 4	l_{21}*1 = 4	l_{21} = 4
$l_{21}u_{12}+u_{22}$= 3	4*1 +u_{22} = 3	u_{22} = -1
$l_{21}u_{13}+u_{23}$= -1	4*1 +u_{23} = -1	u_{23}= -5

- Now consider the last row

$l_{31}u_{11}$ = 3	l_{31}*1 = 3	l_{31} = 3
$l_{31}u_{12}+l_{32}u_{22}$ = 5	3*1 + l_{32}*(-1) = 5	l_{32}= -2
$l_{31}u_{13}+l_{32}u_{23}+u_{33}$ = 3	(1*3)+(-5*-2) +u_{33} = 3	u_{33} = -10

Example

- Now we have:

L = $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$

Forward Substitution Step: [L] {d} = {b} is used to generate an intermediate vector {d}

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

- applying matrix multiplication:
  - $\boldsymbol{d_1}$**=1**
  - $4d_1 + d_2$ +0 = 6

    $\boldsymbol{d_2}$ **= 2**
  - $3d_1 - 2d_2 + d_3 = 4$

    $\boldsymbol{d_3}$**= 5**

Example

- Backward Substitution Step: The result from forward substitution is used substitute [U]{X} = {d} to solve {x}

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

- applying matrix multiplication we can solve the system:
  - $-10x_3=5$

    $x_3 = -\frac{1}{2}$
  - $-x_2 - 5x_3 = 2$

    $x_2 = \frac{1}{2}$
  - $x_1 + x_2 + x_3 = 1$

    $x_1=1$

Solution of the system:
$x_1=1$; $x_2 = \frac{1}{2}$; $x_3 = -\frac{1}{2}$

LU Factorization using Gauss Elimination

$3x_1 + 2x_2 + x_3 = 10$

$2x_1 + 3x_2 + 2x_3 = 14$

$x_1 + 2x_2 + 3x_3 = 14$

Solution:

$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ $\quad x = \begin{matrix} x1 \\ x2 \\ x3 \end{matrix}$ $\quad b = \begin{matrix} 10 \\ 14 \\ 14 \end{matrix}$

- We know, [A] = [L][U]
- From Gauss Elimination method we get:

$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

LU Factorization using Gauss Elimination

$R1$: $3x_1 + 2x_2 + x_3$ = 10

R2: $2x_1 + 3x_2 + 2x_3$ = 14

R3: $x_1 + 2x_2 + 3x_3$ = 14

Eliminate x_1 from R2 and R3

R2->R2+R1*(-3/2)

R3->R3+R1*(-1/3)

Modified System:

$R1$: $3x_1 + 2x_2 + x_3$ = 10

R2': $5/3x_2 + 4/3x_3$ = 22/3

R3': $4/3x_2 + 8/3x_3$ = 32/3

LU Factorization using Gauss Elimination

$R1$: $3x_1 + 2x_2 + x_3$ = 10

R2': $5/3x_2 + 4/3x_3$ = 22/3

R3': $4/3x_2 + 8/3x_3$ = 32/3

Eliminate x_2 from R3'

R3'->R3'+R2'*(-4/5)

Modified System:

$R1$: $3x_1 + 2x_2 + x_3$ = 10

R2': $5/3x_2 + 4/3x_3$ = 22/3

R3'': $24/15x_3$ = 72/15

LU Factorization using Gauss Elimination

- Now we get L and U from Gauss Elimination:

- U = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 24/15 \end{bmatrix}$

- L = $\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix}$; $l_{21} = \frac{a_{21}}{a_{11}}$ $l_{31} = \frac{a_{31}}{a_{11}}$ and $l_{32} = \frac{a'_{32}}{a'_{22}}$

LU Factorization using Gauss Elimination

[L] {d} = {b} is used to generate an intermediate vector {d}

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

- applying matrix multiplication:
  - $\boldsymbol{d_1}$**=10**
  - $2/3d_1 + d_2 = 14$

    $\boldsymbol{d_2}$ **= 22/3**
  - $1/3d_1 + 4/5d_2 + d_3 = 14$

    $\boldsymbol{d_3}$**= 72/15**

LU Factorization using Gauss Elimination

- [U]{X} = {d} used to solve {x}

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 24/15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 72/15 \end{bmatrix}$$

- applying matrix multiplication:
  - x_3**=3**
  - $5/3x_2 + 4/3x_3 = 22/3$

    x_2 **= 2**
  - $3x_1 + 2x_2 + x_3 = 10$

    x_1**= 1**

Solution of the system:
$x_1=1$; $x_2=2$; $x_3=3$