Iterative Solution for System of Linear Equations

Raqeebir Rab

System of Linear Equations: Iterative Methods

- Assume that we are given set of n equations:

$$[A]\{x\} = \{b\}$$

- If the diagonal elements are all non zero then 1st equation solved for x_1, 2nd equation for x_2 and 3rd equation for x_3

Therefore,

$$x_1^j = \frac{b_1 - a_{12}\, x_2^{j-1} - a_{13}\, x_3^{j-1}}{a_{11}}$$

$$x_2^j = \frac{b_2 - a_{21}\, x_1^{j-1} - a_{23}\, x_3^{j-1}}{a_{22}}$$

$$x_3^j = \frac{b_3 - a_{31}\, x_1^{j-1} - a_{32}\, x_2^{j-1}}{a_{33}}$$

Here,

j-->present iteration

$j-1$→ previous iteration

System of Linear Equations: Iterative Methods

Process:

- Start Condition:
  - To start the solution process initial guess made for x's; the simple approach is to assume that they all are zero.
- End condition:
  - The process will continue until the solution converges close to true value.
  - Convergence criterion for all i: $\varepsilon_{a,i} = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| \times 100\% \leq \varepsilon_s$
- Two types of Iterative methods:
  - Jacobi Method
  - Gauss Seidel Methods

Jacobi Method

$x_1 = \frac{b_1}{a_{11}}$
$x_2 = \frac{b_2}{a_{22}}$
$x_3 = \frac{b_3}{a_{33}}$

First Iteration:

x_1=x_2 = x_3=0

$$x_1 = \frac{b_1 - a_{12}\,x_2^{j-1} - a_{13}\,x_3^{j-1}}{a_{11}}$$
$$x_2 = \frac{b_2 - a_{21}\,x_1^{j-1} - a_{23}\,x_3^{j-1}}{a_{22}}$$
$$x_3 = \frac{b_3 - a_{31}\,x_1^{j-1} - a_{32}\,x_2^{j-1}}{a_{33}}$$

Second Iteration

$$x_1 = \frac{b_1 - a_{12}\,x_2^{j-1} - a_{13}\,x_3^{j-1}}{a_{11}}$$
$$x_2 = \frac{b_2 - a_{21}\,x_1^{j-1} - a_{23}\,x_3^{j-1}}{a_{22}}$$
$$x_3 = \frac{b_3 - a_{31}\,x_1^{j-1} - a_{32}\,x_2^{j-1}}{a_{33}}$$

Third Iteration

$$x_1 = \frac{b_1 - a_{12}\,x_2^{j-1} - a_{13}\,x_3^{j-1}}{a_{11}}$$
$$x_2 = \frac{b_2 - a_{21}\,x_1^{j-1} - a_{23}\,x_3^{j-1}}{a_{22}}$$
$$x_3 = \frac{b_3 - a_{31}\,x_1^{j-1} - a_{32}\,x_2^{j-1}}{a_{33}}$$

Value of x_1, x_2 and x_3 calculated from 1st iteration used to calculate new value of x_1, x_2, x_3

Gauss Seidal Method

First Iteration:

$Let, x_2 = x_3 = 0$

$$x_1 = \frac{b_1 - a_{12}\,x_2^{j-1} - a_{13}\,x_3^{j-1}}{a_{11}}$$

Calculate $x_1 = \frac{b_1}{a_{11}}$

$$x_2 = \frac{b_2 - a_{21}\,x_1^{j-1} - a_{23}\,x_3^{j-1}}{a_{22}}$$

New value of x_1 used to calculate x_2

$$x_3 = \frac{b_3 - a_{31}\,x_1^{j-1} - a_{32}\,x_2^{j-1}}{a_{33}}$$

New value of x_1 and x_2 used to calculate x_3

Second Iteration

$$x_1 = \frac{b_1 - a_{12}\,x_2^{j-1} - a_{13}\,x_3^{j-1}}{a_{11}}$$

New value of x_3 and x_2 calculated from 1st iteration used to calculate x_1

$$x_2 = \frac{b_2 - a_{21}\,x_1^{j-1} - a_{23}\,x_3^{j-1}}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}\,x_1^{j-1} - a_{32}\,x_2^{j-1}}{a_{33}}$$

Convergence Condition

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

- i.e. the absolute value of the diagonal coefficient in each of the equations must be larger than the sum of the absolute value of the other coefficient in the equations, such system is known as diagonally dominant.
- The criterion is sufficient but not necessary for convergence.

Example of Gauss-Seidel Methods

Problem statement:

Consider the system of linear equations:

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$

$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$

$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$

Suppose the solution for the system is $\{x\}^T = \{3,-2.5,7\}$

Solve the system using Gauss Siedel Method.

Example of Gauss-Seidel Methods

$$x_1^j = \frac{b_1 - a_{12}\, x_2^{j-1} - a_{13}\, x_3^{j-1}}{a_{11}}$$

$$x_2^j = \frac{b_2 - a_{21}\, x_1^{j-1} - a_{23}\, x_3^{j-1}}{a_{22}}$$

$$x_3^j = \frac{b_3 - a_{31}\, x_1^{j-1} - a_{32}\, x_2^{j-1}}{a_{33}}$$

Rearranging the equations:

$$x_1^j = \frac{7.85 + 0.1x_2^{j-1} + 0.2x_3^{j-1}}{3}$$

$$x_2^j = \frac{-19.3 - 0.1\, x_1^{j-1} + 0.3x_3^{j-1}}{7}$$

$$x_3^j = \frac{71.4 - 0.3x_1^{j-1} - 0.2x_2^{j-1}}{10}$$

Initial Guess:

$x_1^0 = x_2^0 = x_3^0 = 0$

Example of Gauss-Seidel Methods

1^{st} Iteration; j=1

$$x_1^1 = \frac{7.85 + 0.1x_2^0 + 0.2x_3^0}{3}$$

$$x_2^1 = \frac{-19.3 - 0.1\, x_1^0 + 0.3x_3^0}{7}$$

$$x_3^1 = \frac{71.4 - 0.3x_1^0 - 0.2x_2^0}{10}$$

$$x_1^1 = \frac{7.85 + 0.1(0) + 0.2(0)}{3} = 2.61667$$

$$x_2^1 = \frac{-19.3 - 0.1\,(2.61667\,) + 0.3(0)}{7} = -2.794524$$

$$x_3^1 = \frac{71.4 - 0.3(2.61667)\ - 0.2(-2.794524)}{10} = 7.0005610$$

Example of Gauss-Seidel Methods

2nd Iteration; j=2

$$x_1^2 = \frac{7.85+ 0.1x_2^1+0.2x_3^1}{3}$$

$$x_2^2 = \frac{-1\ \ .3 - 0.1\ x_1^1+0.3x_3^1}{7}$$

$$x_3^2 = \frac{71.4 - 0.3x_1^1 - 0.2x_2^1}{10}$$

$$x_1^2 = \frac{7.85+0.1(-2.794524\ )+0.2(7.0005610\ )}{3} = 2.990557$$

$$x_2^2 = \frac{-19.3 - 0.1\ (2.990557\ )+0.3(7.0005610\ )}{7} = \text{-`}2.499625$$

$$x_3^2 = \frac{71.4 - 0.3(2.990557\ )\ - 0.2 - \text{`}2.499625\ )}{10} = 7.000291$$

Example of Gauss-Seidel Methods

3rd Iteration; j=3

x_1^3 = 3.0004

x_2^3 = -2.5

x_3^2 = 6.9997

4th Iteration; j=4

x_1^3 = 3.0003

x_2^3 = -2.5

x_3^2 = 6.9997

Additional Iteration will be applied to improve the result.

Example of Gauss-Seidel Methods

- If true value is unknown in that case we have calculate error at each iteration until the result is met at least the specified tolerance level ε_s

Error for x_1

- $\varepsilon_{a,1} = \left|\frac{2.990557-2.61667}{2.990557}\right| \times 100\% = 12.5\%$

Error for x_2

- $\varepsilon_{a,2} = \left|\frac{2.499625 - 2.794524}{2.499625}\right| \times 100\% = 11.8\%$

Error for x_3

- $\varepsilon_{a,3} = \left|\frac{7.000291 - 7.0005610}{7.000291}\right| \times 100\% = 0.076\%$