Eigenvalues and Eigenvectors

If A is an $n \times n$ matrix, then a non-zero vector v in R^n is called an eigenvector of A if Av is a scalar multiple of v, that is $Av = \lambda v$ ..... (1) for some scalar λ. The scalar λ is called an eigenvalue of A and v is said to be an eigenvector of A corresponding to λ.

Characteristic polynomial and characteristic equation:

To find the eigenvalue of $n \times n$ matrix A we write $Av = \lambda v$ as $Av = \lambda I v$, or equivalently, $(\lambda I - A)v = 0$ .... (2).

The matrix $\lambda I - A$, where I is the $n \times n$ identity matrix and λ is an indeterminate, is called the characteristic matrix of A.

For λ to be an eigenvalue of the matrix A, there must be a non-zero solution for the vector v of the equation (2) only if the rank of $\lambda I - A$ is less than its order, in which case its determinant is zero, that is, $|\lambda I - A| = 0$ ..... (3)

The determinant of the characteristic matrix $\lambda I - A$ is a polynomial in λ and is called the characteristic polynomial of A. Also equation (3) is called the characteristic equation of A, the scalars satisfying this equation are the eigenvalues of A.

Theorem: Any square matrix A and its transpose A^T have the same eigenvalues.

Ex.1. Find the eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

Soln: The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-2 & -1 & 0 \\ -3 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{bmatrix}$$

Now the determinant of $\lambda I - A$ is

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1 & 0 \\ -3 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix}$$

$$= (\lambda-2)\{(\lambda-2)(\lambda-4) - 0\} + 1\{-3(\lambda-4) - 0\} + 0$$

$$= (\lambda-2)^2(\lambda-4) - 3(\lambda-4)$$

$$= (\lambda-4)\{(\lambda-2)^2 - 3\}$$

Therefore the characteristic equation of A is

$$(\lambda-4)\{(\lambda-2)^2-3\}=0$$

$$\Rightarrow (\lambda-4)\{\lambda^2-4\lambda+4-3\}=0$$

$$\Rightarrow (\lambda-4)(\lambda^2-4\lambda+1)=0$$

$\therefore \lambda=4$ | $\lambda^2-4\lambda+1=0$

$$\therefore \lambda=\frac{4\pm\sqrt{16-4.1.1}}{2.1}$$

$$=\frac{4\pm2\sqrt{3}}{2}=2\pm\sqrt{3}$$

Hence the eigenvalues of A are

$\lambda_1=4$, $\lambda_2=2+\sqrt{3}$ and $\lambda_3=2-\sqrt{3}$. (Ans)

Ex.2: Find the eigenvalues of the matrix $A=\begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}$.

Sol^n: The characteristic matrix of A is

$$\lambda I-A=\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}-\begin{bmatrix}3 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}\lambda-3 & -2\\ 1 & \lambda\end{bmatrix}$$

Now the determinant of $\lambda I-A$ is

$$|\lambda I-A|=\begin{vmatrix}\lambda-3 & -2\\ 1 & \lambda\end{vmatrix}$$

$$=(\lambda-3).\lambda+2=\lambda^2-3\lambda+2$$

Therefore, the characteristic equation of A is

$\lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2 = 0$

$\Rightarrow \lambda(\lambda - 2) - 1(\lambda - 2) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 2) = 0$

$\therefore \lambda = 1, \lambda = 2$, which are the eigenvalues of A.

Ex. 3. Find the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

Solⁿ: The characteristic matrix of A is

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 4 \end{bmatrix}$$

Now the determinant of $\lambda I - A$ (the characteristic polynomial of A) is $|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 4) - 3$

Therefore, the characteristic equation of A is

$(\lambda - 2)(\lambda - 4) - 3 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 4\lambda + 8 - 3 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$

$\Rightarrow \lambda^2 - 5\lambda - \lambda + 5 = 0$

$\Rightarrow \lambda(\lambda - 5) - 1(\lambda - 5) = 0$

$\Rightarrow (\lambda - 1)(\lambda - 5) = 0$

$\therefore \lambda = 5, 1$, which are the eigenvalues of A.

Now by definition $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector of A corresponding to λ if and only if x is a non-trivial solution of

$(\lambda I - A)x = 0$, that is

$$\begin{bmatrix} \lambda-2 & -3 \\ -1 & \lambda-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{------ (1)}$$

If $\lambda = 5$, equation no. (1) becomes

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} 3x_1 - 3x_2 = 0 \\ -x_1 + x_2 = 0 \end{array}\right\} \Rightarrow x_1 - x_2 = 0$$

The system is in echelon form and consistent, since there are more unknowns than equation in echelon form the system has an infinite number of solutions.

Again, the equation begins with x_1 only, the other unknown x_2 is a free variable.

Let us take $x_2 = a$ (a is an arbitrary real number). $\therefore x_1 = a$. Therefore, the eigenvectors of A corresponding to the eigenvalue $\lambda = 5$ are non-zero vectors of the form $x = \begin{bmatrix} a \\ a \end{bmatrix}$.

In particular, let $a = 1$, then $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 5$.

If $\lambda = 1$, equation no. (1) becomes,

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -x_1 - 3x_2 = 0 \\ -x_1 - 3x_2 = 0 \end{aligned}\right\} \Rightarrow x_1 + 3x_2 = 0$$

The system is in echelon form and consistent, since there are more unknowns than equation in echelon form, the system has an infinite number of solutions. Again, the equation begins with x_1 only, the other unknown x_2 is a free variable.

Let us take $x_2 = b$ (b is an arbitrary real number). $\therefore x_1 = -3b$.

Therefore, the eigen vectors of A corresponding to the eigenvalue $\lambda = 1$, are the non-zero vectors of the form

$$x = \begin{bmatrix} -3b \\ b \end{bmatrix}.$$

In particular, let $b = 1$, then $x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 1$. (Ans)

Exercise: Find all eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$