Divergence or Gauss's theorem

Mathematically this theorem can be written as:

$$\iint_S \vec{F}\cdot\vec{ds} = \iiint_V \nabla\cdot\vec{F}\, dv$$

Note : Spherical co-ordinates (r, θ, Φ) :

$x = r \sin\theta \cdot \cos\Phi$,

$y = r \sin\theta \cdot \sin\Phi$,

$z = r\cos\theta$, where $r \geq 0$, $0 \leq \Phi < 2\pi$, $0 \leq \theta \leq \pi$

Here, $dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\Phi$.

Note : If $x = r\cos\theta$, $y = r\sin\theta$ (circle)

then, $dx\, dy = r\, dr\, d\theta$.

Again, $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2 \cdot 1$

$\therefore x^2 + y^2 = r^2$ (circle).

Example : Use divergence theorem to evaluate $\iint_S \vec{A}\cdot\vec{ds}$ where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and s is the surface of the sphere $x^2+y^2+z^2=a^2$.

Solution : The divergence theorem is

$$\iint_S \vec{A}\cdot\vec{ds} = \iiint_V \nabla\cdot\vec{A}\, dv$$

$$\Rightarrow \iint_S \vec{A}\cdot\vec{ds} = \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}\right) dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3)\right] dv$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)\, dx\, dy\, dz$$

$$\Rightarrow \iint_S \vec{A}\cdot\vec{ds} = 3\iiint_V (x^2+y^2+z^2)\, dx\, dy\, dz \text{ ---------- } (1)$$

Now put $x = r\sin\theta\cdot\cos\Phi$, $y = r\sin\theta\cdot\sin\Phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\Phi$ in (1) and get,

$$\iint_S \vec{A}\cdot\vec{ds} = 3\int_{r=0}^{a}\int_{\Phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(r^2\sin^2\theta\cos^2\Phi + r^2\sin^2\theta\cdot\sin^2\Phi + r^2\cos^2\theta\right) r^2\sin\theta\, dr\, d\theta\, d\Phi$$

$$= 3\int_{r=0}^{a}\int_{\Phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^4\sin\theta\left(\sin^2\theta\cdot\cos^2\Phi + \sin^2\theta\cdot\sin^2\Phi + \cos^2\theta\right) dr\, d\theta\, d\Phi$$

$$\Rightarrow \iint_S \vec{A}\cdot\vec{ds} = 3\int_{r=0}^{a}\int_{\Phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^4\sin\theta\left[\sin^2\theta\left(\cos^2\Phi+\sin^2\Phi\right)+\cos^2\theta\right]dr\,d\theta\,d\Phi$$

$$= 3\int_{r=0}^{a}\int_{\Phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^4\sin\theta\left(\sin^2\theta\cdot 1+\cos^2\theta\right)dr\,d\theta\,d\Phi$$

$$= 3\int_{r=0}^{a}\int_{\Phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^4\sin\theta\cdot 1\,dr\,d\theta\,d\Phi$$

$$= 3\int_{r=0}^{a}\int_{\Phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^4\sin\theta\,dr\,d\theta\,d\Phi$$

$$= 3\left[\frac{r^5}{5}\right]_0^a\cdot\left[\Phi\right]_0^{2\pi}\cdot\left[-\cos\theta\right]_0^{\pi}$$

$$= -\frac{3}{5}\left[(a)^5-(0)^5\right]\cdot(2\pi-0)\cdot(\cos\pi-\cos 0)$$

$$= -\frac{3}{5}(a^5-0)\cdot(2\pi)\cdot(-1-1)$$

$$= \frac{12}{5}\pi a^5. \text{ (Ans.)}$$

Example: Use Divergence theorem to evaluate $\iint_S \vec{F}\cdot\vec{ds}$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ and $z=3$.

Solution: The Divergence theorem is,

$$\iint_S \vec{F}\cdot\vec{ds} = \iiint_V \nabla\cdot\vec{F}\,dv$$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{ds} = \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right)dx\,dy\,dz$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right]dx\,dy\,dz$$

$$= \iiint_V (4 - 4y + 2z)\,dx\,dy\,dz$$

$$= \iint dx\,dy \int_{z=0}^{3} (4 - 4y + 2z)\,dz$$

$$= \iint dx\,dy \left[4z - 4yz + 2\cdot\frac{z^2}{2}\right]_0^3$$

$$= \iint dx\,dy \left[(4\cdot 3 - 4\cdot 3y + 3^2) - (0 - 0 + 0)\right]$$

$$\Rightarrow \iint_S \vec{F}\cdot\vec{ds} = \iint (21 - 12y)\,dx\,dy \;\text{-----------}\; (1)$$

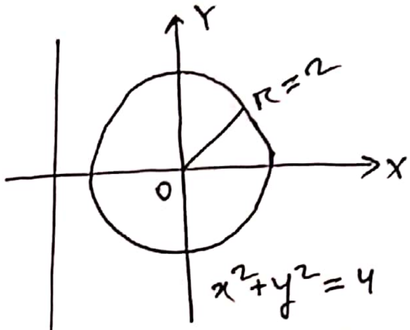

Fig. ①

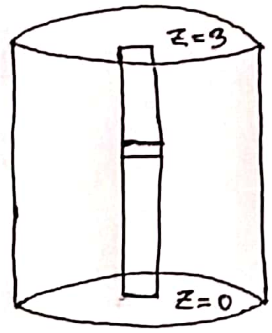

Fig. ②

Now let us put $x = r\cos\theta$, $y = r\sin\theta$ and $dx.dy = r\,dr\,d\theta$ in (1) and get,

$$\iint_S \vec{F}\cdot\vec{ds} = \iint (21-12r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (21-12r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} (21r-12r^2\sin\theta)\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[21\cdot\frac{r^2}{2} - 12\cdot\frac{r^3}{3}\sin\theta\right]_0^2 d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\left(\frac{21}{2}\cdot 2^2 - 4\cdot 2^3\sin\theta\right) - (0-0)\right] d\theta$$

$$= \int_{\theta=0}^{2\pi} (42-32\sin\theta)\, d\theta$$

$$= \left[42\theta + 32\cos\theta\right]_0^{2\pi}$$

$$= \left[(42\times 2\pi + 32\cos 2\pi) - (0+32\cos 0)\right]$$

$$= (84\pi + 32 - 32) = 84\pi. \quad \text{(Ans)}$$

<u>Exercise</u>: Evaluate $\iint_S \vec{F}\cdot\vec{ds}$ where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$. [Ex. 103, Page: 454]

<u>Exercise</u>: Ex. 115, Page: 459

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