Minors and co-factors:

Let A be the n-square matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$

whose determinant is $|A|$. When from A the elements of its ith row and j-th column are removed, the determinant of the remaining $(n-1)$-square matrix is called the minor of the element a_{ij} and is denoted by M_{ij}. The signed minor, $(-1)^{i+j} M_{ij}$ is called the co-factor of a_{ij} and is denoted by α_{ij}.

Example: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

and $\alpha_{11} = (-1)^{1+1} M_{11} = M_{11}$

$$\alpha_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

$$\alpha_{13} = (-1)^{1+3} M_{13} = M_{13}$$

Then $|A| = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$

$$= a_{11}\alpha_{11} + a_{12}\alpha_{12} + a_{13}\alpha_{13}$$

Singular and non-singular matrices:

Let D be the determinant of the square matrix A, then if $D=0$, the matrix A is called the singular matrix and if $D \neq 0$, the matrix A is called the non-singular matrix.

As for example, $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a singular matrix, since $D_1 = |A| = 0$.

Again, $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is a non-singular matrix.

The adjoint of a square matrix:

Let $A = [a_{ij}]$ be an n-square matrix and α_{ij} be the co-factor of a_{ij}, then by definition

$$\text{adjoint } A = \text{adj } A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \cdots & \alpha_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n} & \alpha_{2n} & \cdots & \alpha_{nn} \end{bmatrix}.$$

or, adjugate matrix

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

$\therefore \alpha_{11} = 6,\ \alpha_{12} = -2,\ \alpha_{13} = -3,\ \alpha_{21} = 1,\ \alpha_{22} = -5,\ \alpha_{23} = 3,$

$\alpha_{31} = -5,\ \alpha_{32} = 4$ and $\alpha_{33} = -1$

and $\operatorname{adj} A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$ (Ans)

The inverse of a matrix:

If A and B are n-square matrices such that $AB = BA = I$, B is called the inverse of A, $(B = A^{-1})$ and A is called the inverse of B, $(A = B^{-1})$.

* An n-square matrix A has an inverse if and only if it is non-singular.

* If A is non-singular, then $AB = AC$ implies $B = C$.

Example 1: Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4-3 & -2+2 \\ 6-6 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Again $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Therefore A and B are invertible and are inverses of each other. That is $A^{-1} = B$ and $B^{-1} = A$.

■ Inverse from the adjoint / Process of finding the inverse of a square matrix:

Let the matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

Let D be the determinant of the matrix A. If $D=0$, the matrix A is singular and it has no inverse, if $D \neq 0$ the matrix A is non-singular and A^{-1} exists. Find the adjoint matrix adj A of the matrix A; then

$$A^{-1} = \frac{1}{D} \operatorname{adj} A = \frac{\operatorname{adj} A}{|A|}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{\alpha_{11}}{|A|} & \frac{\alpha_{21}}{|A|} & \cdots & \frac{\alpha_{n1}}{|A|} \\ \frac{\alpha_{12}}{|A|} & \frac{\alpha_{22}}{|A|} & \cdots & \frac{\alpha_{n2}}{|A|} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\alpha_{1n}}{|A|} & \frac{\alpha_{2n}}{|A|} & \cdots & \frac{\alpha_{nn}}{|A|} \end{bmatrix}$$

Example: From the previous example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

The adjoint of A is adj $A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$

The determinant of A, $|A| = D = 1(12-6) - 2(8-6) + 3(6-9)$

$= 6 - 4 - 9$

$= -7 \neq 0$

$$\therefore A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{5}{7} & -\frac{4}{7} \\ \frac{3}{7} & -\frac{3}{7} & \frac{1}{7} \end{bmatrix}$$