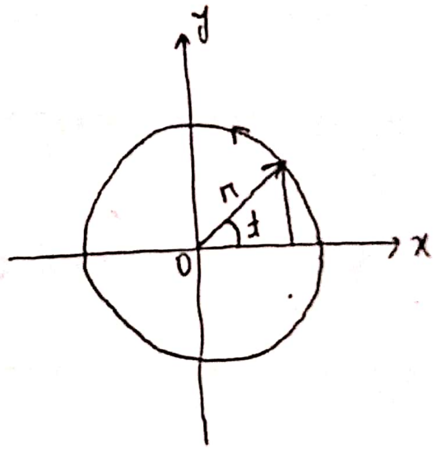
Ex: Find the work done in moving a particle once around a circle C in the xy plane, if the circle has centre at the origin and radius 3 and if the force field is given by

$$\vec{F} = (2x-y+z)\hat{i} + (x+y-z^2)\hat{j} + (3x-2y+4z)\hat{k}.$$

Soln: In the plane $z=0$, $\vec{F} = (2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$ so that the work done is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left[(2x-y)\hat{i} + (x+y)\hat{j} + (3x-2y)\hat{k}\right]\cdot\left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$= \int_C \left[(2x-y)\,dx + (x+y)\,dy\right]$$



Choose the parametric equations of the circle as $x = 3\cos t$, $y = 3\sin t$ where t varies from 0 to 2π. Then the line integral equals

$\vec{r} = x\hat{i} + y\hat{j}$
$= 3\cos t\,\hat{i} + 3\sin t\,\hat{j}$

$$\int_{t=0}^{2\pi} \left[2(3\cos t) - 3\sin t\right]\left[-3\sin t\right]dt + \left[3\cos t + 3\sin t\right]\left[3\cos t\right]dt$$

$$= \int_0^{2\pi} \left[-18\cos t\cdot\sin t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cdot\cos t\right]dt$$

$$= \int_0^{2\pi} (9 - 9\sin t\cdot\cos t)\,dt$$

$$= \int_0^{2\pi} \left(9 - \frac{9}{2}\sin 2t\right) dt$$

$$= \left[9t + \frac{9}{2}\cdot\frac{\cos 2t}{2}\right]_0^{2\pi} = \left[\left(18\pi + \frac{9}{4}\cos 4\pi\right) - \left(0 + \frac{9}{4}\right)\right]$$

$$= 18\pi$$

In traversing C we have chosen the counterclockwise direction indicated in the adjoining figure. We call this the positive direction, or say that C has been traversed in the positive sense. If C were traversed in the clockwise (negative) direction the value of the integral would be -18π.

(Ans)

Ex: If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve in the xy plane, $y = 2x^2$, from $(0,0)$ to $(1,2)$

Soln: Since the integration is performed in the xy plane $(z=0)$, we can take $\vec{r} = x\hat{i} + y\hat{j}$. Then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (3xy\hat{i} - y^2\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_C (3xy\,dx - y^2\,dy)$$

Now substitute $y = 2x^2$ directly, where x goes from 0 to 1. Then

$$\int_C \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} \{3x(2x^2)\,dx - (2x^2)^2\,d(2x^2)\}$$

$$= \int_{x=0}^{1} \{6x^3\,dx - 4x^4\cdot 2\cdot 2x\,dx\}$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= 6\left[\frac{x^4}{4}\right]_0^1 - 16\left[\frac{x^6}{6}\right]_0^1$$

$$= \frac{3}{2}(1-0) - \frac{8}{3}(1-0)$$

$$= \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = -\frac{7}{6} \text{ (Ans)}$$