Richardson Extrapolation for Differentiation:

Richardson Extrapolation is based on a model for error in numerical process and get better approximation of derivative.

Consider the central divide difference formula:

$$The\ first\ derivative\ f'(x) = \frac{f(x+h)-f(x-h)}{2h} + O(h^2)$$

$$\text{True Error } E_t = O(h^2)$$

So,

$$E_t = Ch^2$$

As we know,

True Error = True Value − Approximate Value

∴ True Value = Approximate Value + True Error

Therefore, True Value = (Approximate Value)$_h$ + Ch^2 …………………(1)

Similarly

True Value = (Approximate Value)$_{h/2}$ + $C(\frac{h}{2})^2$

= (Approximate Value)$_{h/2}$ + $C\frac{h^2}{4}$ ……………………………(2)

Equation (1) − equation (2) *4

− 3 True Value = (Approximate Value)$_h$ − 4(Approximate Value)$_{h/2}$

True Value = (Approximate Value)$_{h/2}$ + $\frac{(\text{Approximate Value})_{h/2} - (\text{Approximate Value})_h}{3}$ …….(3)

Equation 3 is Richardson Extrapolation formula for first derivative.

Example: Using the given data below, Richardson's extrapolation formula can provide better estimation.

x	− 0.5	− 0.25	0	0.25	0.5	0.75	1.0	1.25	1.5
$f(x) = e^x$	0.6065	0.7788	1.0000	1.2840	1.6487	2.1170	2.7183	3.4903	4.4817

Let us estimate $f'(x)$ at x = 0.5 and assume h = 0.5 and h=0.25. Using central difference formula,

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h}$$

$$f'(0.5) = \frac{f(0.5+0.5)-f(0.5-0.5)}{2(0.5)}$$

$$f'(0.5) = \frac{f(1)-f(0)}{2*0.5} = (2.7183 - 1.0000) = 1.7183$$

$$f'\left(\frac{h}{2} = 0.25\right) = \frac{f(0.75)-f(0.25)}{2*0.25} = (2.1170 - 1.2840) = 1.666$$

According to Richardson's extrapolation,

$$f'(x) = (\text{Approximate Value})_{h/2} + \frac{(\text{Approximate Value})_{\frac{h}{2}} - (\text{Approximate Value})_{h}}{3}$$

$$= (1.666) + \frac{(1.666)-(1.7183)}{3} = 1.6486$$

The correct answer is 1.6487. The result obtain from Richardson Extrapolation formula is much better than Central Difference Formula.

Romberg integration

The accuracy of Numerical Integration can be improved in two ways:

1. By increasing the number of subintervals (i.e. by decreasing 'h'): this decreases the magnitude of error terms. Here, the order of the method is fixed.
2. By using higher order methods: this eliminates the lower order error terms and the order of the method is varied and this method is known as variable order approach. It is implemented using Richardson Extrapolation formula.

The variable order method involves combining two estimates of a given order to obtain a third estimate of higher order. The method that incorporates this process to the trapezoidal rule is called Romberg Integration.

In order to achieve the Romberg Integration, we have the following steps:

1. Compute the integration with given value of 'h' by trapezoidal rule.
2. Each time, halve the value of 'h' and again compute the integral using composite trapezoidal rule.
3. Refine the above computed values using the relation given below:

$$I = I_2 + \frac{1}{3}(I_2 - I_1) = \frac{1}{3}(4I_2 - I_1)$$

Derivation of Romberg Integration

According to the Eular-Maclaruin formula, the error expansion of trapezoidal rule is:

$$\int_a^b f(x)dx - T(h) = a_2h^2 + a_4h^4 + a_6h^6 + \cdots \quad \text{.................} (4)$$

Here,

$T(h)$ *is the trapezoidal approximation with step size* h, *where* $h = \frac{(b-a)}{n}$ *and* n *is the number of inteval.*

Let's consider trapezoidal rule with no Richardson extrapolation being applied,

$$T(h, 0) = T(h)$$

Rewrite equation (4),

$$I = T(h, 0) + a_2h^2 + a_4h^4 + a_6h^6 + \cdots \quad \text{..........................}(5)$$

And for $h = \frac{(b-a)}{2n} = \frac{h}{2}$

$$I = T(h/2, 0) + a_2/4h^2 + a_4/16h^4 + a_6/32h^6 + \cdots \text{..............................} (6)$$

Multiplying equation 6 by 4 and subtracting by equation (5) we get,

$$I = \frac{4T\left(\frac{h}{2},0\right) - T(h,0)}{4-1} + b_4h^4 + b_6h^6 + \ldots.$$
$$= T\left(\frac{h}{2},1\right) + b_4h^4 + b_6h^6 + \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Where,

$$T\left(\frac{h}{2},1\right) = \frac{4T\left(\frac{h}{2},0\right) - T(h,0)}{3} = \frac{1}{3}[4I\left(\frac{h}{2}\right) - I(h)] = I(h,\frac{h}{2}),$$

is the corrected trapezoidal formula using Richardson extrapolation formula and truncation error = $O(h^4)$.

If we apply Richardson extrapolation technique in equation (7) and eliminate the error term h^4. Then we get,

$$I = \frac{16\ \left(\frac{h}{4},1\right) - T(\frac{h}{2},1)}{16-1} + c_6h^6 + \ldots.$$
$$= T\left(\frac{h}{4},2\right) + c_6h^6 + \cdots..$$

Where,

$$T\left(\frac{h}{4},2\right) = \frac{16T\left(\frac{h}{4},1\right) - T(\frac{h}{2},1)}{16-1} = I(h,\frac{h}{2},\frac{h}{4}),$$

is the second level estimation trapezoidal formula by applying second times Richardson's extrapolation.

Similarly third level correction of trapezoidal formula by applying third times Richardson's extrapolation is:

$$T\left(\frac{h}{8},3\right) = \frac{64T\left(\frac{h}{8},2\right) - T(\frac{h}{4},2)}{64-1} = I(h,\frac{h}{2},\frac{h}{4},\frac{h}{8})$$

After repeated use of Richardson's extrapolation the general form can be represented as:

$$T\left(\frac{h}{2^i},j\right) = \frac{4^jT\left(\frac{h}{2^i},\ j-1\right) - T(\frac{h}{2^{i-1}},\ j-1)}{4^j-1} \ldots\ldots\ldots..(*)$$

Where, i = 0,1,2,….denotes the depth of division of step size h and $j \leq i$ denotes the level of improvement.

Equation (*) is known as Romberg Integration.

The value of Romberg Integral in tabulated form:

$I(h)$			
	$I(h,\frac{h}{2})$		
$I(\frac{h}{2})$		$I(h,\frac{h}{2},\frac{h}{4})$	
	$I(\frac{h}{2},\frac{h}{4})$		$I(h,\frac{h}{2},\frac{h}{4},\frac{h}{8})$
$I(\frac{h}{4})$		$I(\frac{h}{2},\frac{h}{4},\frac{h}{8})$	
	$I(\frac{h}{4},\frac{h}{8})$		
$I(\frac{h}{8})$			

Where,

$$I\left(h,\frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right]$$
$$I\left(\frac{h}{2},\frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I(\frac{h}{2})\right]$$
$$\cdots$$
$$I\left(h,\frac{h}{2},\frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2},\frac{h}{4}\right) - I\left(h,\frac{h}{2}\right)\right]$$
$$I\left(\frac{h}{2},\frac{h}{4},\frac{h}{8}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4},\frac{h}{8}\right) - I\left(\frac{h}{2},\frac{h}{4}\right)\right]$$
$$I\left(h,\frac{h}{2},\frac{h}{4},\frac{h}{8}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2},\frac{h}{4},\frac{h}{8}\right) - I\left(h,\frac{h}{2},\frac{h}{4}\right)\right]$$

Example: Using Romberg's method compute $I = \int_0^{1.2} \frac{1}{1+x} dx$ correct to 4 decimal places.

Solution: Here

$$f(x) = \frac{1}{1+x}$$

We can take $h = 0.6, 0.3, 0.15$

i.e,

$h = 0.6, \frac{h}{2} = 0.3, \frac{h}{4} = 0.15$

x	0	0.15	0.30	0.45	0.60	0.75	0.90	1.05	1.20
$f(x)$	1	0.8695	0.7692	0.6896	0.6250	0.5714	0.5263	0.4878	0.4545

Using Trapezoidal Rule with $h = 0.6$

$I(h) = I(0.6) = \frac{0.6}{2}[(1 + 0.4545) + 2(0.6250)] = 0.8113$

$I\left(\frac{h}{2}\right) = I\left(\frac{0.6}{2}\right) = I(0.3) = \frac{0.3}{2}[(1 + 0.4545) + 2(0.7692 + 0.6250 + 0.5263)] = 0.7943$

$$I\left(\frac{h}{4}\right) = I\left(\frac{0.6}{4}\right) = I(0.15)$$
$$= \frac{0.15}{2}[(1 + 0.4545) + 2(0.8695 + 0.7692 + 0.6896 + 0.6250 + 0.5714 + 0.4878 + 0.5263)]$$
$$= 0.7899$$

Now, $I\left(h, \frac{h}{2}\right) = I(0.6, 0.3) = \frac{1}{3}[4 * I(0.3) - I(0.6)] = \frac{1}{3}[4 * 0.7943 - 0.8113] = 0.7886$

$$I\left(\frac{h}{2}, \frac{h}{4}\right) = I(0.3, 0.15) = \frac{1}{3}[4 * I(0.15) - I(0.3)] = \frac{1}{3}[4 * 0.7899 - 0.7943] = 0.7884$$

$$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = I(0.6, 0.3, 0.15) = \frac{1}{3}[4 * I(0.15,0.3) - I(0.3,0.6)] = \frac{1}{3}[4 * 0.7884 - 0.7886]$$
$$= 0.7883$$

0.8113		
	0.7886	
0.7943		
	0.7884	0.7883
0.7899		

$$\therefore \boldsymbol{I = \int_0^{1.2} \frac{1}{1 + x} dx = 0.7883}$$

References:

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