Fourier series in complex form:

Fourier series in complex form of a function $f(x)$ of period $2l$ is

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{-\frac{in\pi x}{l}}, \text{ where}$$

$$c_0 = \frac{a_0}{2} = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$c_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-\frac{in\pi x}{l}}\,dx, \quad \text{and} \quad c_{-n} = \frac{1}{2l}\int_0^{2l} f(x)\, e^{\frac{in\pi x}{l}}\,dx$$

$$c_n \downarrow \frac{1}{2}(a_n - ib_n) \qquad c_{-n} \downarrow \frac{1}{2}(a_n + ib_n)$$

Example: Obtain the complex form of the Fourier series of the function $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ 1, & 0 \le x \le \pi \end{cases}$.

Sol^n: Here, $\pi - (-\pi) = 2\pi$

$\therefore 2l = 2\pi \quad \therefore l = \pi$.

So, for the given function, the fourier series in complex form is $f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx}$ .... (1), where

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\cdot dx + \int_0^{\pi} 1\cdot dx\right] = \frac{1}{2\pi}\left[0 + [x]_0^{\pi}\right]$$

$$= \frac{1}{2\pi}[\pi - 0] = \frac{1}{2}.$$

Now, $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$

$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} 0 \cdot e^{-inx}\, dx + \int_{0}^{\pi} 1 \cdot e^{-inx}\, dx \right]$

$= \frac{1}{2\pi} \int_{0}^{\pi} e^{-inx}\, dx$

$= \frac{1}{2\pi} \left[\frac{e^{-inx}}{-in} \right]_0^{\pi}$

$= -\frac{1}{2\pi n i} \left[e^{-in\pi} - e^{0} \right]$

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

$e^{-in\pi} = \cos n\pi - i \sin n\pi$

$\because \sin n\pi = 0$

$\therefore \cos n\pi = (-1)^n$

$= -\frac{1}{2\pi n i} \left[\cos n\pi - i \sin n\pi - 1 \right]$

$= -\frac{1}{2\pi n i} \left[(-1)^n - 1 \right]$

$= \begin{cases} \frac{1}{in\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$

Again, $c_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx$

$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} 0 \cdot e^{inx}\, dx + \int_{0}^{\pi} 1 \cdot e^{inx}\, dx \right]$

$= \frac{1}{2\pi} \left[0 + \left[\frac{e^{inx}}{in} \right]_0^{\pi} \right]$

$= \frac{1}{2\pi n i} \left[e^{in\pi} - e^{0} \right]$

$$= \frac{1}{2\pi n i}\left[\cos n\pi + i\sin n\pi - 1\right]$$

$$= \frac{1}{2\pi n i}\left[(-1)^n - 1\right] = \begin{cases} -\frac{1}{in\pi}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases}$$

Hence, from $f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx}$, we get

$$f(x) = \frac{1}{2} + \left[\frac{1}{i\pi}e^{ix} + \frac{1}{3i\pi}e^{3ix} + \frac{1}{5i\pi}e^{5ix} + \cdots\right] +$$

$$\left[\frac{(-1)}{i\pi}e^{-ix} + \frac{(-1)}{3i\pi}e^{-3ix} + \frac{(-1)}{5i\pi}e^{-5ix} + \cdots\right]$$

$$= \frac{1}{2} + \frac{1}{i\pi}\left[\frac{e^{ix}}{1} + \frac{e^{3ix}}{3} + \frac{e^{5ix}}{5} + \cdots\right] -$$

$$\frac{1}{i\pi}\left[\frac{e^{-ix}}{1} + \frac{e^{-3ix}}{3} + \frac{e^{-5ix}}{5} + \cdots\right].$$ ~~(Ans)~~

which is the required complex form of the fourier series of the given function. (Ans)