Fourier Series of the function defined in two or more sub-ranges:

Example: Find the fourier series of the following function:

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ +1 & \text{for } \frac{\pi}{2} < x < \pi . \end{cases}$$

Sol^n: Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$ (1)

Here $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$ $\quad$ $| \pi - (-\pi) = 2\pi$

$$= \frac{1}{\pi} \int_{-\pi}^{-\pi/2} (-1)\, dx + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 0 \cdot dx + \frac{1}{\pi} \int_{\pi/2}^{\pi} (1)\, dx$$

$$= -\frac{1}{\pi} \left[x \right]_{-\pi}^{-\pi/2} + 0 + \frac{1}{\pi} \left[x \right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi} \left[-\frac{\pi}{2} - (-\pi) \right] + \frac{1}{\pi} \left[\pi - \frac{\pi}{2} \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - \pi \right] + \frac{1}{\pi} \left[\pi - \frac{\pi}{2} \right] = \frac{1}{\pi} \left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2} \right] = 0 .$$

Now, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi} \int_{-\pi}^{-\pi/2} (-1) \cos nx\, dx + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} (0) \cos nx\, dx + \frac{1}{\pi} \int_{\pi/2}^{\pi} (1) \cos nx\, dx$$

$$= -\frac{1}{\pi} \left[\frac{\sin nx}{n} \right]_{-\pi}^{-\pi/2} + 0 + \frac{1}{\pi} \left[\frac{\sin nx}{n} \right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi} \left[-\frac{1}{n} \sin \frac{n\pi}{2} + \frac{1}{n} \sin n\pi \right] + \frac{1}{\pi} \left[\frac{1}{n} \sin n\pi - \frac{1}{n} \sin \frac{n\pi}{2} \right]$$

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$$= -\frac{1}{\pi}\left[-\frac{1}{n}\sin\frac{n\pi}{2} + \frac{1}{n}\cdot 0\right] + \frac{1}{\pi}\left[\frac{1}{n}\cdot 0 - \frac{1}{n}\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{n\pi}\sin\frac{n\pi}{2} - \frac{1}{n\pi}\sin\frac{n\pi}{2} = 0.$$

Again, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx\, dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nx\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\sin nx\, dx$$

$$= -\frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} + 0 + \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\pi}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] + \left[\cos\frac{n\pi}{2} - \cos n\pi\right]\cdot\frac{1}{n\pi}$$

$$= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$\therefore b_1 = \frac{2}{\pi}\left[\cos\frac{\pi}{2} - \cos\pi\right] = \frac{2}{\pi}\left[0+1\right] = \frac{2}{\pi}$

$\therefore b_2 = \frac{2}{2\pi}\left[\cos\pi - \cos 2\pi\right] = \frac{1}{\pi}(-1-1) = -\frac{2}{\pi}$

Similarly, $b_3 = \frac{2}{3\pi}, \ldots\ldots$

Now putting the values of a_0, a_n, b_n in (1) we get,

$$f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \ldots\ldots\right]$$

(Ans.)

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Exercise : Find the Fourier series for the periodic function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}.$$

Even function :

A function $f(x)$ is said to be even (or, symmetric) function if

$$f(-x) = f(x).$$

The area under such a curve from $-\pi$ to π is double the area from 0 to π.

$$\therefore \int_{-\pi}^{\pi} f(x)\,dx = 2\int_{0}^{\pi} f(x)\,dx$$

Odd function :

A function $f(x)$ is called odd (or skew symmetric) function if

$$f(-x) = -f(x).$$

The area under the curve from $-\pi$ to π is zero. That is,

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

Expansion of an even function :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

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$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = 0$ [As $\sin nx$ is an odd function so

$f(x)\cdot \sin nx$ is also an odd function.]

Expansion of an odd function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx\, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cdot \sin nx\, dx.$$

Example: Find the Fourier series expansion of the periodic function of period 2π: $f(x) = x^2$, $-\pi \le x \le \pi$.

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$

Solⁿ: Given $f(x) = x^2$, $-\pi \le x \le \pi$.

This is an even function. $\therefore b_n = 0$.

$$\therefore a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\, dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}\left[\pi^3 - 0\right] = \frac{2}{3}\pi^2.$$

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Now, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\cdot\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right) - 0\right]_0^{\pi}$$

$$\left[\because [u\cdot v]_1 = u\cdot v_1 - u' v_2 + u'' v_3 - u''' v_4 + \cdots\cdots\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi \cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right\} - \{0+0-0\}\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} - 0\right] = \frac{4(-1)^n}{n^2}.$$

Now substituting the values of a_0, a_n and b_n in the Fourier series:

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + a_4\cos 4x + \cdots\cdots +$$

$$b_1 \sin x + b_2 \sin 2x + \cdots\cdots\cdots,$$ we get

$$\Rightarrow x^2 = \frac{1}{2}\cdot\frac{2\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \frac{4}{4^2}\cos 4x - \cdots\cdots$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\cdots\right]$$

On putting $x = 0$, we get, $0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots\right]$

$$\Rightarrow 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots\right] = \frac{\pi^2}{3}$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots = \frac{\pi^2}{12}.$$ (Ans)

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The graph of $f(x) = x^2, -\pi \le x \le \pi$ is given below:

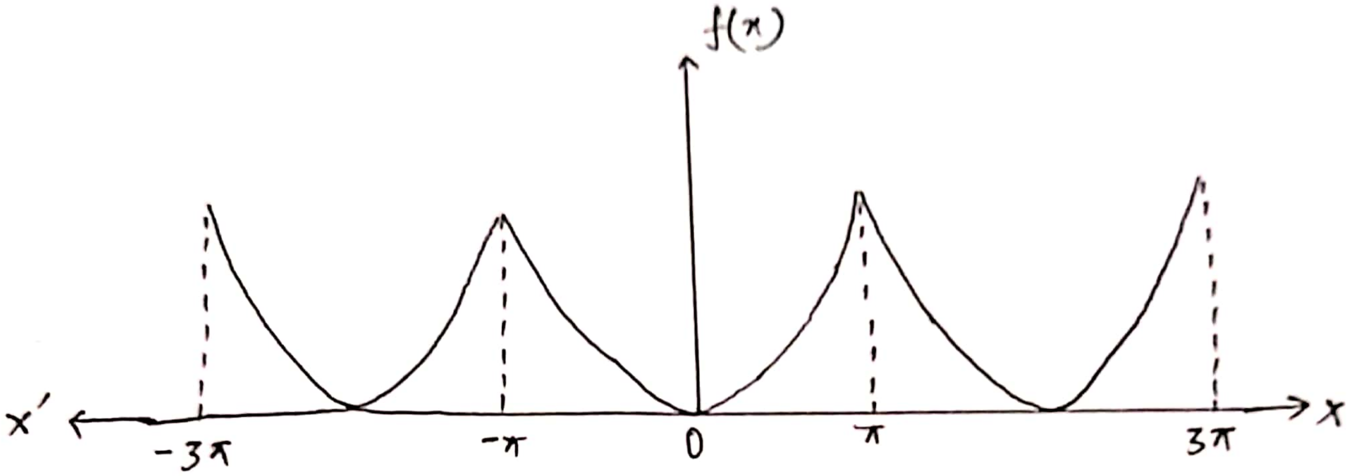

Fig. ①

Exercise: Obtain a Fourier expression for

$$f(x) = x^3 \text{ for } -\pi < x < \pi.$$

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