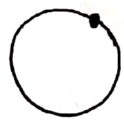
# Vector Integration

## Line Integral [Page: 421]

Line integral $= \int_C \vec{F}\cdot d\vec{r}$, where $C$ is the curve.

**Note:** If $\vec{F}$ represents the variable force acting on a particle along arc $AB$, then the total work done $= \int_A^B \vec{F}\cdot d\vec{r}$

**Note:** When the path of integration is a closed curve then notation of integration is $\oint_C$ in place of $\int_C$.



$\Rightarrow$ closed curve

**Example 1:** If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displaces a particle in the $xy$-plane from $(0,0)$ to $(1,4)$ along a curve $y = 4x^2$. Find the work done. [Page: 421]

**Solution:** Work done $= \int_C \vec{F}\cdot d\vec{r}$

$\vec{r} = x\hat{i} + y\hat{j}$

$\therefore d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$= \int_C \left(2x^2y\,\hat{i} + 3xy\,\hat{j}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$= \int_C \left(2x^2y\,dx + 3xy\,dy\right).$$

Given $y = 4x^2$

$\therefore dy = 8x\,dx$

$$= \int_{x=0}^{1} \left[2x^2\cdot(4x^2)\,dx + 3x\,(4x^2)\,8x\,dx\right]$$

$$= \int_{x=0}^{1} \left(8x^4 dx + 96x^4 dx\right)$$

$$= 104 \int_{x=0}^{1} x^4 dx$$

$$= 104 \left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{104}{5}\left[(1)^5 - (0)^5\right] = \frac{104}{5}(1-0) = \frac{104}{5}. \text{ (Ans)}$$

Example 2: A vector field is given by $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$. Evaluate $\int_c \vec{F}\cdot\vec{dr}$ along the path c which is $x = 2t,\ y = t,\ z = t^3$ from $t = 0$ to $t = 1$. [Ex. 67, Page. 422]

Solution: Here $\int_c \vec{F}\cdot\vec{dr}$

$$= \int_c \left[(2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}\right]\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right)$$

$$\left[\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$\therefore \int_c \vec{F}\cdot\vec{dr} = \int_c \left[(2y+3)dx + xz\,dy + (yz-x)dz\right] \text{ ---------- } (1)$$

Now the path c given $x = 2t$ | $y = t$ | $z = t^3$

$\therefore \frac{dx}{dt} = 2$ | $\therefore \frac{dy}{dt} = 1$ | $\therefore \frac{dz}{dt} = 3t^2$

$\Rightarrow dx = 2dt$ | $\Rightarrow dy = dt$ | $\Rightarrow dz = 3t^2 dt$

Then equation (1) becomes,

$$\int_c \vec{F}\cdot\vec{dr} = \int_{t=0}^{1}\left[(2t+3)(2dt) + (2t)(t^3)(dt) + (t\cdot t^3 - 2t)(3t^2 dt)\right]$$

(P.T.O.)

$$\Rightarrow \int_c \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} \left[(4t+6)dt + 2t^4 dt + (3t^6 - 6t^3)dt\right]$$

$$= \int_{t=0}^{1} (4t+6+2t^4+3t^6-6t^3)\,dt$$

$$= \left[4\cdot\frac{t^2}{2} + 6t + 2\cdot\frac{t^5}{5} + 3\cdot\frac{t^7}{7} - 6\cdot\frac{t^4}{4}\right]_0^1$$

$$= \left[\left\{2\cdot(1)^2 + 6\cdot 1 + \frac{2}{5}(1)^5 + \frac{3}{7}(1)^7 - \frac{3}{2}(1)^4\right\} - \left\{2\cdot(0)^2 + 6\cdot 0 + \frac{2}{5}(0)^5 + \frac{3}{7}(0)^7 - \frac{3}{2}(0)^4\right\}\right]$$

$$= \left[2+6+\frac{2}{5}+\frac{3}{7}-\frac{3}{2}\right] - \left[0+0+0+0-0\right]$$

$$= 7.328 \text{ (Ans)}$$

<u>Exercise</u>: If $\vec{A} = (3x^2+6y)\hat{i} - 14yz\,\hat{j} + 20xz^2\hat{k}$, evaluate the line integral $\int_c \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $c$ which is $x=t, y=t^2, z=t^3$. [Ex. 69, Page: 423].

<u>Hints</u>: $\int_c \vec{A}\cdot d\vec{r} = \int_c (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz \;\cdots\cdots\; (1)$

If $x=t, y=t^2, z=t^3$ then points $(0,0,0)$ and $(1,1,1)$ correspond to $t=0$ and $t=1$ respectively. Then from (1),

$$\int_c \vec{A}\cdot d\vec{r} = \int_{t=0}^{1} \left[(3t^2+6t^2)\,d(t) - 14(t^2)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)\right].$$