Fourier Integrals / Integral Transforms [P. 910, H.K. Dass]

List of Formulae of Fourier Integrals :

1. Fourier Integral for $f(x)$ is $f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,du\,dt$

2. Fourier Sine Integral for $f(x)$ is $f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\sin ut\,\sin ux\,du\,dt$

3. Fourier Cosine Integral for $f(x)$ is $f(x) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(t)\cos ut\cdot\cos ux\,du\,dt$

4. $\int e^{-ax}\cos bx\,dx = \frac{e^{-ax}}{a^2+b^2}(b\sin bx - a\cos bx)$

5. $\int e^{-ax}\sin bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\sin bx - b\cos bx)$

6. $[u\cdot v]_1 = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots$

where $[uv]_1 = \int uv\,dx$,

$v_1 = \int v\,dx$, $v_2 = \int v_1\,dx$ and so on.

$u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on.

Ex. 1 Find the Fourier sine integral for $f(x) = e^{-\beta x}$, hence show that $\frac{\pi}{2} e^{-\beta x} = \int_0^\infty \frac{\lambda \sin \lambda x}{\beta^2 + \lambda^2} d\lambda$.

Soln: The Fourier sine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \lambda x \, d\lambda \int_0^\infty f(t) \sin \lambda t \, dt \quad \cdots\cdots (1)$$

Putting the value of $f(x)$ in (1) we get

$$e^{-\beta x} = \frac{2}{\pi} \int_0^\infty \sin \lambda x \, d\lambda \int_0^\infty e^{-\beta t} \sin \lambda t \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \, d\lambda \left[\frac{e^{-\beta t}}{(\beta^2 + \lambda^2)} (-\beta \sin \lambda t - \lambda \cos \lambda t)^{*} \right]_0^\infty$$

$$= \frac{2}{\pi} \int_0^\infty \sin \lambda x \, d\lambda \left[0 + \frac{\lambda}{\beta^2 + \lambda^2} \right]$$

$$\therefore e^{-\beta x} = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{\beta^2 + \lambda^2} d\lambda$$

$$\Rightarrow \frac{\pi}{2} e^{-\beta x} = \int_0^\infty \frac{\lambda \sin \lambda x}{\beta^2 + \lambda^2} d\lambda. \quad \text{(Proved)}$$

$* \int e^{-ax} \sin bx \, dx = \frac{e^{-ax}}{a^2 + b^2} (-a \sin bx - b \cos bx)$

Ex. 2 Using Fourier cosine integral representation of an appropriate function, show that $\int_0^\infty \frac{\cos wx}{k^2 + w^2} dw = \frac{\pi e^{-kx}}{2k}$.

Soln: The Fourier cosine integral for $f(x)$ is

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos ux\, du \int_0^\infty f(t)\cos ut\, dt$$

Putting the value of $f(t)$ and replacing u by w we get

$$e^{-kx}=\frac{2}{\pi}\int_0^\infty \cos wx\, dw \int_0^\infty e^{-kt}\cos wt\, dt$$

$$=\frac{2}{\pi}\int_0^\infty \cos wx\, dw \cdot \left[\frac{e^{-kt}}{k^2+w^2}\left(-k\cos wt+w\sin wt\right)\right]_0^\infty$$

$$=\frac{2}{\pi}\int_0^\infty \cos wx\, dw\left[0+\frac{k}{k^2+w^2}\right]$$

$$\therefore e^{-kx}=\frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw$$

$$\therefore \int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw=\frac{\pi e^{-kx}}{2k}. \quad \text{(Proved)}$$

Ex. 3: Express the function $f(x)=\begin{cases}1 & \text{when } |x|\le 1\\ 0 & \text{when } |x|>1\end{cases}$ as a Fourier integral.

Hence evaluate $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda$.

Soln: The Fourier Integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cdot\cos\lambda(t-x)\, dt\, d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\int_{-1}^1 1\cdot\cos\lambda(t-x)\, dt\, d\lambda \quad (\text{since } f(t)=1).$$

$$=\frac{1}{\pi}\int_0^\infty\left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{-1}^1 d\lambda$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda(1-x)}{\lambda} + \frac{\sin\lambda(1+x)}{\lambda}\right] d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda \cdot \cos\lambda x}{\lambda}\, d\lambda \qquad [\because \sin(A+B)+\sin(A-B) = 2\cdot\sin A\cdot\cos B]$$

$$\therefore \int_0^\infty \frac{\sin\lambda\cdot\cos\lambda x}{\lambda}\, d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2} & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases} \quad \text{(Ans)}$$

<u>Exercise</u>: Express $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi \\ 0 & \text{for } x > \pi \end{cases}$ as a Fourier sine integral and hence evaluate $\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda} \sin\lambda x\, d\lambda$.