Polynomial

Polynomial means many terms comes from poly means many and nomial means term. Each polynomial has three parts—constant, variables and exponents but never division by a variable. A polynomial P: f→ f is any type of function which can be written as:

$P(x) = a_0 + a_1x + a_2x^2 \ldots\ldots\ldots\ldots + a_{n-1}x^{n-1} + a_nx^n$ for all x ε F

Where, n is non negative integer which is known as exponent and a_i ε f known as constant. The highest power of x that appears is called the degree of p (deg(p)). The polynomial above has degree n (if an = 0 we wouldn't include it in the expression), if n = 3 then a polynomial is cubic, quadratic if n = 2, and linear if n = 1. We say a constant function $p(x) \equiv a_0$ has degree 0 then p is a monic polynomial.

Polynomial or not:

$$Polynomial: 3x^2 + \ 5x + 2 = 0$$

$$Not\ ploynomial: 3xy^{-2}\ ;\ {}^{x}/_{y} + 2$$

Evaluation of Polynomial:

Evaluation of Polynomials
The polynomial is a sum of n+1 terms and can be expressed as

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_0 + \sum_{i=1}^{n} a_i x^i = f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

Where a_i are real numbers representing the polynomial coefficients and x^n are the polynomial variables. The above polynomial is said to be of the n^{th} degree,

i.e. *deg(f(x)) = n*, where *n* represents the highest variable exponent.

e.g $f(x) = 3x^3 + 2x^2 - \ 5x + 2 = 0$ can be easily evaluated $f(x)$ using a loop, but this would require n(n+1)/2 multiplication and n additions.

Horner's Rule
We can write the polynomial f (x) using *Horner's rule* (also known as *nested multiplication*) as follows:

$$f(x) = ((\ldots\ldots((a_n x + a_{n-1})x + a_{n-2})x + \ldots\ldots\ldots\ldots + a_1)x + a_0)$$

The algorithm proposed by Horner's rule is based on evaluating the monomials formed above starting from the one in the inner-most parenthesis and move out to evaluate the monomials in the outer parenthesis.

Here the innermost expression $a_n\,x + a_{n-1}$ is evaluated first. The resulting value constitutes a multiplicand for the expression at the next level. The number of level or iteration required equals to n, the degree of polynomial. This approach needs a total of n additions and n multiplications.

Here is an example: $7x^4 + 2x^3 - 5x^2 + 4x - 3 = (((7x + 2)x - 5)x + 4)x - 3$.

The general iteration below gives $b_0 = p(x)$: $b_n = a_n$; $b_{j-1} = xb_j + a_{j-1}$, $j = n, n-1, \ldots, 2, 1$.

Algorithm: **Horner's Rule**

$p_n = a_n$

$p_{n-1} = p_n\,x + a_{n-1}$

. . .

. . .

. . .

$p_j = p_{j+1}\,x + a_j$

. . .

. . .

$p_1 = p_2\,x + a_1$

$f(x) = p_0 = p_1\,x + a_0$

The algorithm is executed following the below steps:

1. Set $j = n$
2. Let $b_j = a_j$
3. Let $b_{j-1} = a_{j-1} + b_j x_0$
4. Let $j = j - 1$
5. If $j \geq 0$ then go to step 3

Else End

Example: Evaluate the polynomial $f(x) = x^3 - 4x^2 + x + 6$ using Horner's rule at x = 2.

Solution: Here $n = 3$, $a_3 = 1$, $a_2 = -4$, $a_1 = 1$, $a0 = 6$

$p_3 = a_3 = 1$

$p_2 = 1 * 2 + (-4) = -2$

$p_1 = (-2) * 2 + 1 = -3$

$p_0 = (-3) * 2 + 6 = 0$

$f(2) = 0$

Deflation and Synthetic Division:

Polynomial of degree n can be expressed as:

P(x) = (x – x_r) q(x), where x_r is a root of the polynomial P(x), and q(x) is the quotient polynomial of degree n – 1. Synthetic division is a process of obtaining lower degree of q(x) by dividing P(x) by (x – x_r) once the x_r is known. The term synthetic is used because we find find q(x) actually without performing division. The idea of reducing degree of polynomial is defined as deflation.

Example:

Divide: $\frac{2x^3 \;\; - 5x^2 + 3x + 7}{x-2}$

To set up the problem, first, set the denominator equal to zero to find the number to put in the division box. Next, make sure the numerator is written in descending order and if any terms are missing you must use a zero to fill in the missing term, finally list only the coefficient in the division problem.

Step 1: Carry down the 2 that indicates the leading coefficient	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & & & \\ \hline & 2 & & & \end{array}$
Step 2: Multiply by the number on the left, and carry the result into the next column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & & \\ \hline & 2 & & & \end{array}$
Step 3: Add down the column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & & \\ \hline & 2 & -1 & & \end{array}$
Step 4: Multiply by the number on the left, and carry the result into the next column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & -2 & \\ \hline & 2 & -1 & & \end{array}$
Step 5: Add down the column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & -2 & \\ \hline & 2 & -1 & 1 & \end{array}$
Step 6: Multiply by the number on the left, and carry the result into the next column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & -2 & 2 \\ \hline & 2 & -1 & 1 & \end{array}$

Step 7: Add down the column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & -2 & 2 \\ \hline & 2 & -1 & 1 & 9 \end{array}$
Step 8: Write the final answer	$2x^2 - x + 1 + \dfrac{9}{x-2}$