A Survey of Probability Concepts

Chapter 5

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GOALS

- Define probability.
- Describe the classical, empirical, and subjective approaches to probability.
- Explain the terms experiment, event, outcome, permutations, and combinations.
- Define the terms conditional probability and joint probability.
- Calculate probabilities using the rules of addition and rules of multiplication.
- Apply a tree diagram to organize and compute probabilities.
- Calculate a probability using Bayes’ theorem.

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Definitions

A probability is a measure of the likelihood that an event in the future will happen. It it can only assume a value between 0 and 1.

- A value near zero means the event is not likely to happen. A value near one means it is likely.
- There are three ways of assigning probability:
  - classical,
  - empirical, and
  - subjective.

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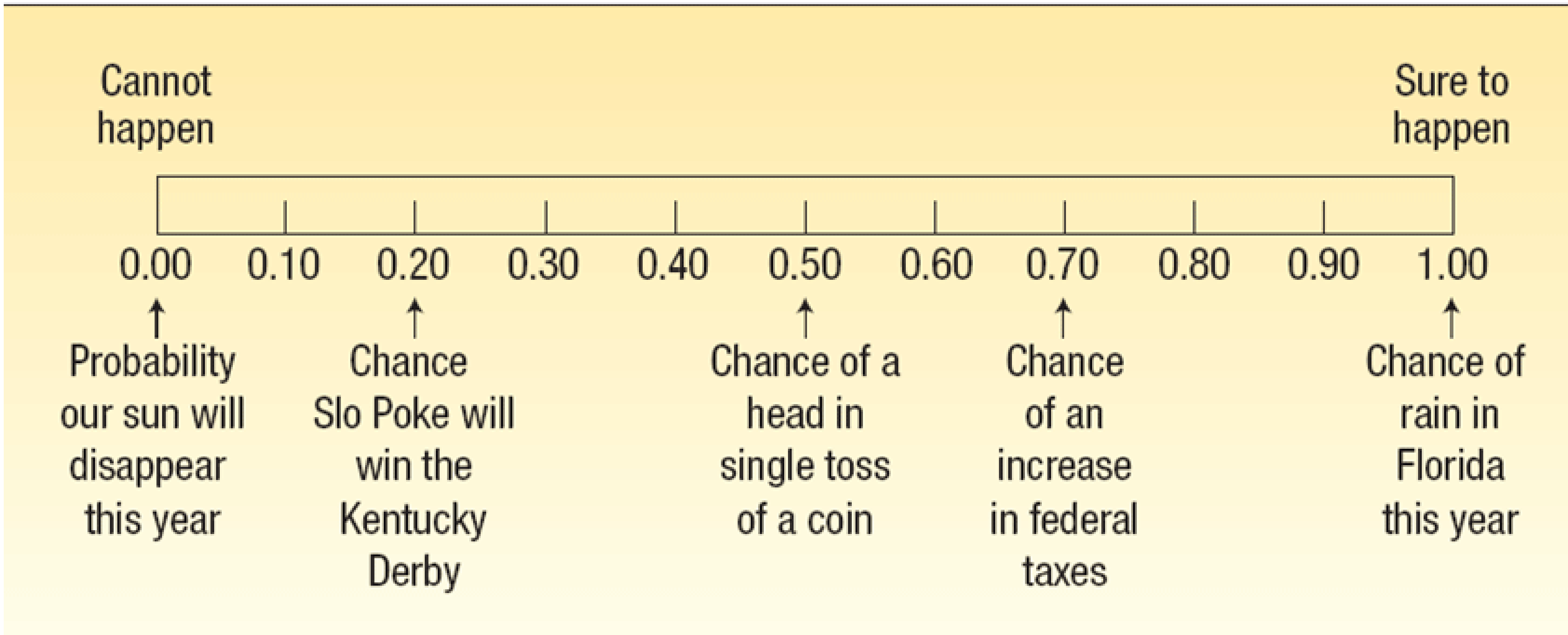
Probability Examples



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Definitions *continued*

- An experiment is the observation of some activity or the act of taking some measurement.
- An outcome is the particular result of an experiment.
- An event is the collection of one or more outcomes of an experiment.

Experiments, Events and Outcomes

		
Experiment	Roll a die	Count the number of members of the board of directors for Fortune 500 companies who are over 60 years of age
All possible outcomes	Observe a 1 Observe a 2 Observe a 3 Observe a 4 Observe a 5 Observe a 6	None are over 60 One is over 60 Two are over 60 ... 29 are over 60 ... ... 48 are over 60 ...
Some possible events	Observe an even number Observe a number greater than 4 Observe a number 3 or less	More than 13 are over 60 Fewer than 20 are over 60

Assigning Probabilities

Three approaches to assigning probabilities

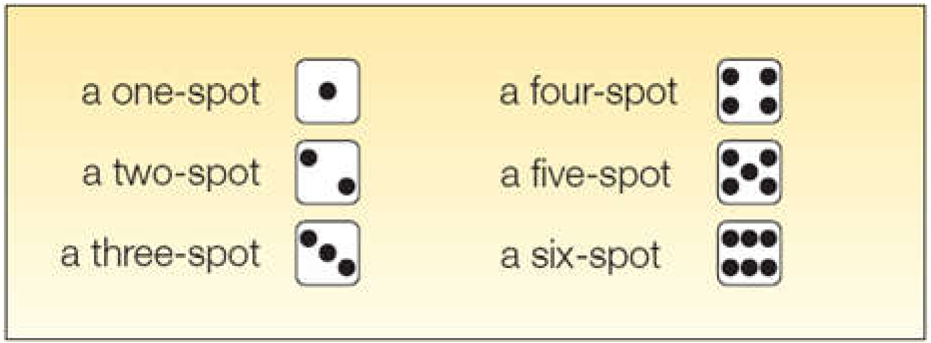
- Classical
- Empirical
- Subjective

Classical Probability

CLASSICAL PROBABILITY $$\text{Probability of an event} = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}} \qquad [5\text{-}1]$$

Consider an experiment of rolling a six-sided die. What is the probability of the event "an even number of spots appear face up"?

The possible outcomes are:



There are three "favorable" outcomes (a two, a four, and a six) in the collection of six equally likely possible outcomes.

Mutually Exclusive Events

- Events are mutually exclusive if the occurrence of any one event means that none of the others can occur at the same time.
- Events are independent if the occurrence of one event does not affect the occurrence of another.

Collectively Exhaustive Events

- Events are collectively exhaustive if at least one of the events must occur when an experiment is conducted.

Empirical Probability

EMPIRICAL PROBABILITY The probability of an event happening is the fraction of the time similar events happened in the past.

The empirical approach to probability is based on what is called the law of large numbers. The key to establishing probabilities empirically is that more observations will provide a more accurate estimate of the probability.

LAW OF LARGE NUMBERS Over a large number of trials the empirical probability of an event will approach its true probability.

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Law of Large Numbers

Suppose we toss a fair coin. The result of each toss is either a head or a tail. If we toss the coin a great number of times, the probability of the outcome of heads will approach .5. The following table reports the results of an experiment of flipping a fair coin 1, 10, 50, 100, 500, 1,000 and 10,000 times and then computing the relative frequency of heads

Number of Trials	Number of Heads	Relative Frequency of Heads
1	0	.00
10	3	.30
50	26	.52
100	52	.52
500	236	.472
1,000	494	.494
10,000	5,027	.5027

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Empirical Probability - Example

On February 1, 2003, the Space Shuttle Columbia exploded. This was the second disaster in 113 space missions for NASA. On the basis of this information, what is the probability that a future mission is successfully completed?

$$\text{Probability of a successful flight} = \frac{\text{Number of successful flights}}{\text{Total number of flights}}$$

$$= \frac{111}{113} = 0.98$$

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Subjective Probability - Example

SUBJECTIVE CONCEPT OF PROBABILITY The likelihood (probability) of a particular event happening that is assigned by an individual based on whatever information is available.

- If there is little or no past experience or information on which to base a probability, it may be arrived at subjectively.
- Illustrations of subjective probability are:
  1. Estimating the likelihood the New England Patriots will play in the Super Bowl next year.
  2. Estimating the likelihood you will be married before the age of 30.
  3. Estimating the likelihood the U.S. budget deficit will be reduced by half in the next 10 years.

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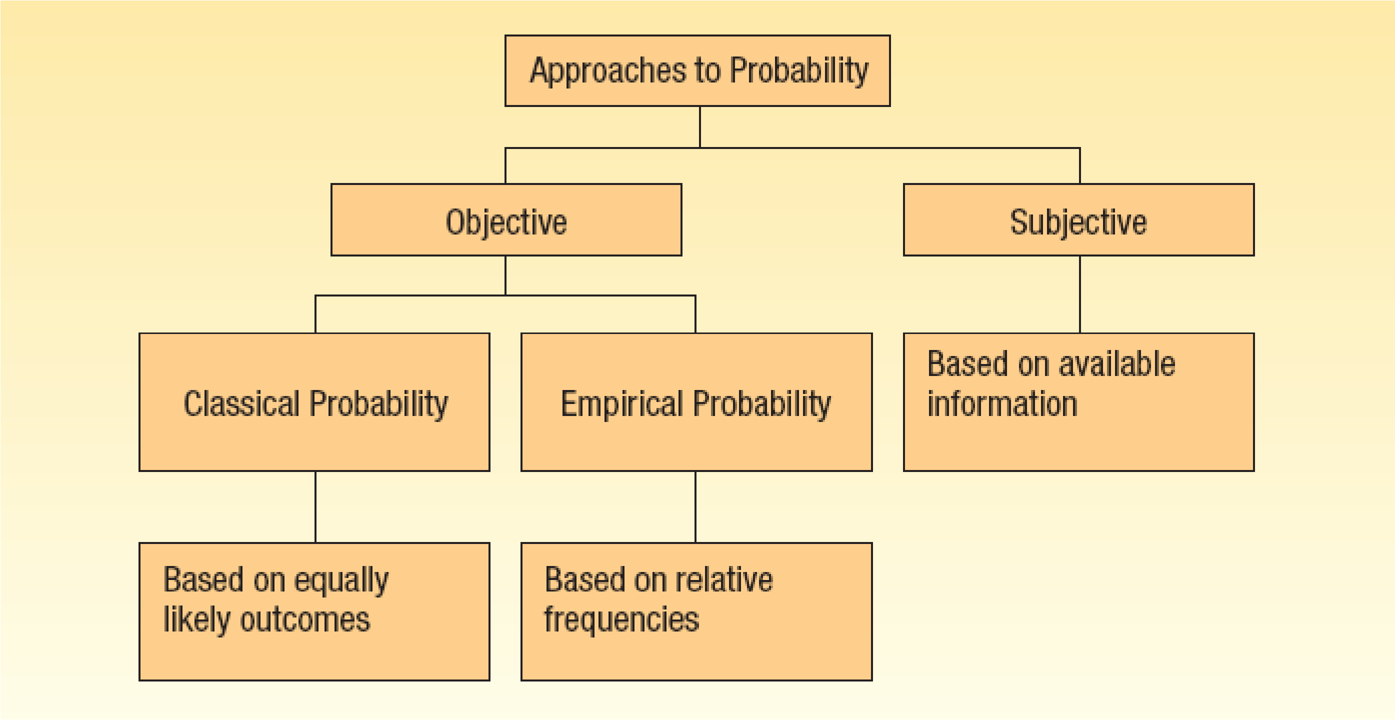
Summary of Types of Probability

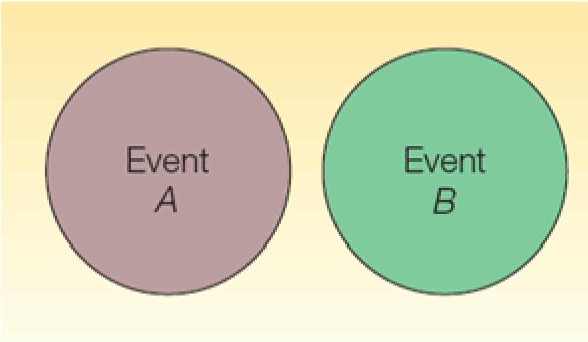


Rules for Computing Probabilities

Rules of Addition

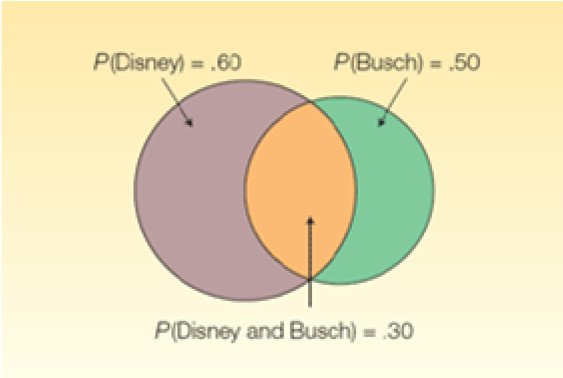
- Special Rule of Addition - If two events *A* and *B* are mutually exclusive, the probability of one *or* the other event's occurring equals the sum of their probabilities.

  P(A or B) = P(A) + P(B)



- The General Rule of Addition - If A and B are two events that are not mutually exclusive, then P(A or B) is given by the following formula:

  P(A or B) = P(A) + P(B) - P(A and B)



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Addition Rule - Example

What is the probability that a card chosen at random from a standard deck of cards will be either a king or a heart?

Card	Probability		Explanation
King	$P(A)$	$= 4/52$	4 kings in a deck of 52 cards
Heart	$P(B)$	$= 13/52$	13 hearts in a deck of 52 cards
King of Hearts	$P(A \text{ and } B)$	$= 1/52$	1 king of hearts in a deck of 52 cards

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$= 4/52 + 13/52 - 1/52$

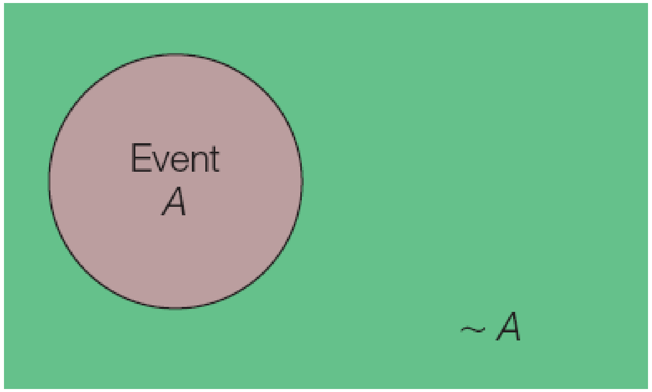
$= 16/52$, or .3077

The Complement Rule

The complement rule is used to determine the probability of an event occurring by subtracting the probability of the event *not* occurring from 1.

$$P(A) + P(\sim A) = 1$$

or $$P(A) = 1 - P(\sim A).$$



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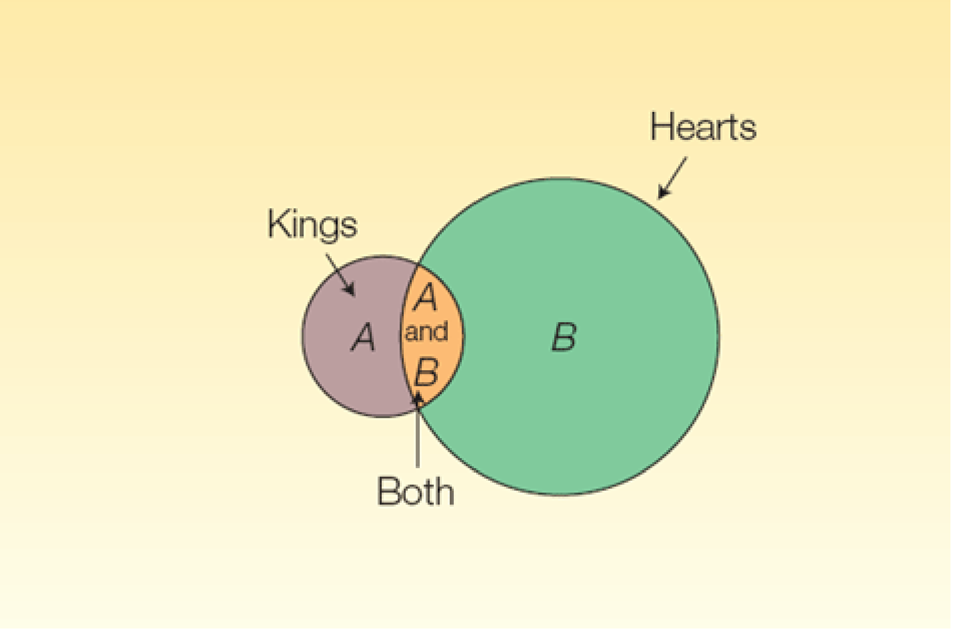
Joint Probability – Venn Diagram

JOINT PROBABILITY A probability that measures the likelihood two or more events will happen concurrently.



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Special Rule of Multiplication

- The special rule of multiplication requires that two events A and B are *independent*.
- Two events A and B are independent if the occurrence of one has no effect on the probability of the occurrence of the other.
- This rule is written: $P(A \text{ and } B) = P(A)P(B)$

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Multiplication Rule-Example

A survey by the American Automobile association (AAA) revealed 60 percent of its members made airline reservations last year. Two members are selected at random. What is the probability both made airline reservations last year?

Solution:

The probability the first member made an airline reservation last year is .60, written as $P(R_1) = .60$

The probability that the second member selected made a reservation is also .60, so $P(R_2) = .60$.

Since the number of AAA members is very large, you may assume that R_1 and R_2 are independent.

$$P(R_1 \text{ and } R_2) = P(R_1)P(R_2) = (.60)(.60) = .36$$

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Conditional Probability

A conditional probability is the probability of a particular event occurring, given that another event has occurred.

The probability of the event A given that the event B has occurred is written $P(A|B)$.

General Multiplication Rule

The general rule of multiplication is used to find the joint probability that two events will occur.

Use the general rule of multiplication to find the joint probability of two events when the events are not independent.

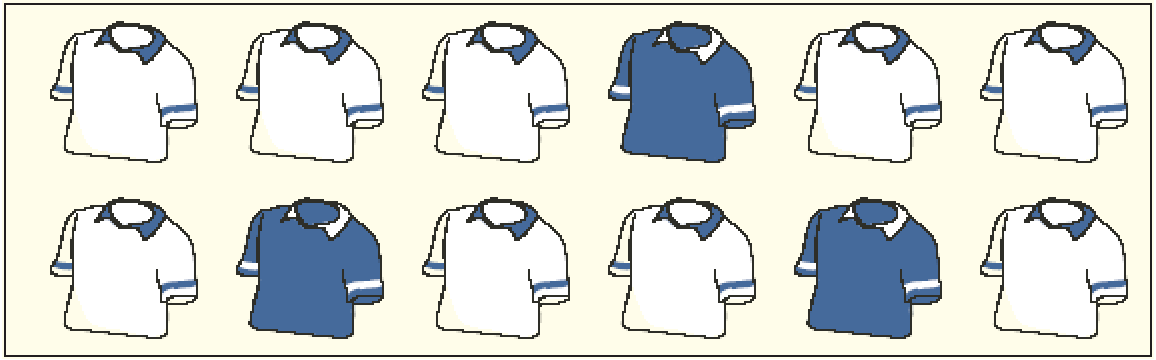
It states that for two events, *A* and *B,* the joint probability that both events will happen is found by multiplying the probability that event *A* will happen by the conditional probability of event *B* occurring given that *A* has occurred.

GENERAL RULE OF MULTIPLICATION $$P(A \text{ and } B) = P(A)P(B|A) \quad [5\text{-}6]$$

General Multiplication Rule - Example

A golfer has 12 golf shirts in his closet. Suppose 9 of these shirts are white and the others blue. He gets dressed in the dark, so he just grabs a shirt and puts it on. He plays golf two days in a row and does not do laundry.

What is the likelihood both shirts selected are white?



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General Multiplication Rule - Example

- The event that the first shirt selected is white is W_1. The probability is $P(W_1) = 9/12$
- The event that the second shirt selected is also white is identified as W_2. The conditional probability that the second shirt selected is white, given that the first shirt selected is also white, is $P(W_2 \mid W_1) = 8/11$.
- To determine the probability of 2 white shirts being selected we use formula: P(AB) = P(A) P(B|A)
- $P(W_1 \text{ and } W_2) = P(W_1)P(W_2 \mid W_1) = (9/12)(8/11) = 0.55$

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Contingency Tables

A CONTINGENCY TABLE is a table used to classify sample observations according to two or more identifiable characteristics

E.g. A survey of 150 adults classified each as to gender and the number of movies attended last month. Each respondent is classified according to two criteria—the number of movies attended and gender.

Movies Attended	Gender: Men	Gender: Women	Total
0	20	40	60
1	40	30	70
2 or more	10	10	20
Total	70	80	150

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Contingency Tables - Example

A sample of executives were surveyed about their loyalty to their company. One of the questions was, "If you were given an offer by another company equal to or slightly better than your present position, would you remain with the company or take the other position?" The responses of the 200 executives in the survey were cross-classified with their length of service with the company.

	Length of Service				
Loyalty	Less than 1 Year, B_1	1–5 Years, B_2	6–10 Years, B_3	More than 10 Years, B_4	Total
Would remain, A_1	10	30	5	75	120
Would not remain, A_2	25	15	10	30	80
	35	45	15	105	200

What is the probability of randomly selecting an executive who is loyal to the company (would remain) and who has more than 10 years of service?

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Contingency Tables - Example

Event A_1 happens if a randomly selected executive will remain with the company despite an equal or slightly better offer from another company. Since there are 120 executives out of the 200 in the survey who would remain with the company

$P(A_1)$ = 120/200, or .60.

Event B_4 happens if a randomly selected executive has more than 10 years of service with the company. Thus, P(B4| A1) is the conditional probability that an executive with more than 10 years of service would remain with the company. Of the 120 executives who would remain 75 have more than 10 years of service, so P(B4| A1) = 75/120.

$$P(A_1 \text{ and } B_4) = P(A_1)P(B_4|A_1) = \left(\frac{120}{200}\right)\left(\frac{75}{120}\right) = \frac{9{,}000}{24{,}000} = .375$$

Tree Diagrams

A tree diagram is useful for portraying conditional and joint probabilities. It is particularly useful for analyzing business decisions involving several stages.

A tree diagram is a graph that is helpful in organizing calculations that involve several stages. Each segment in the tree is one stage of the problem. The branches of a tree diagram are weighted by probabilities.

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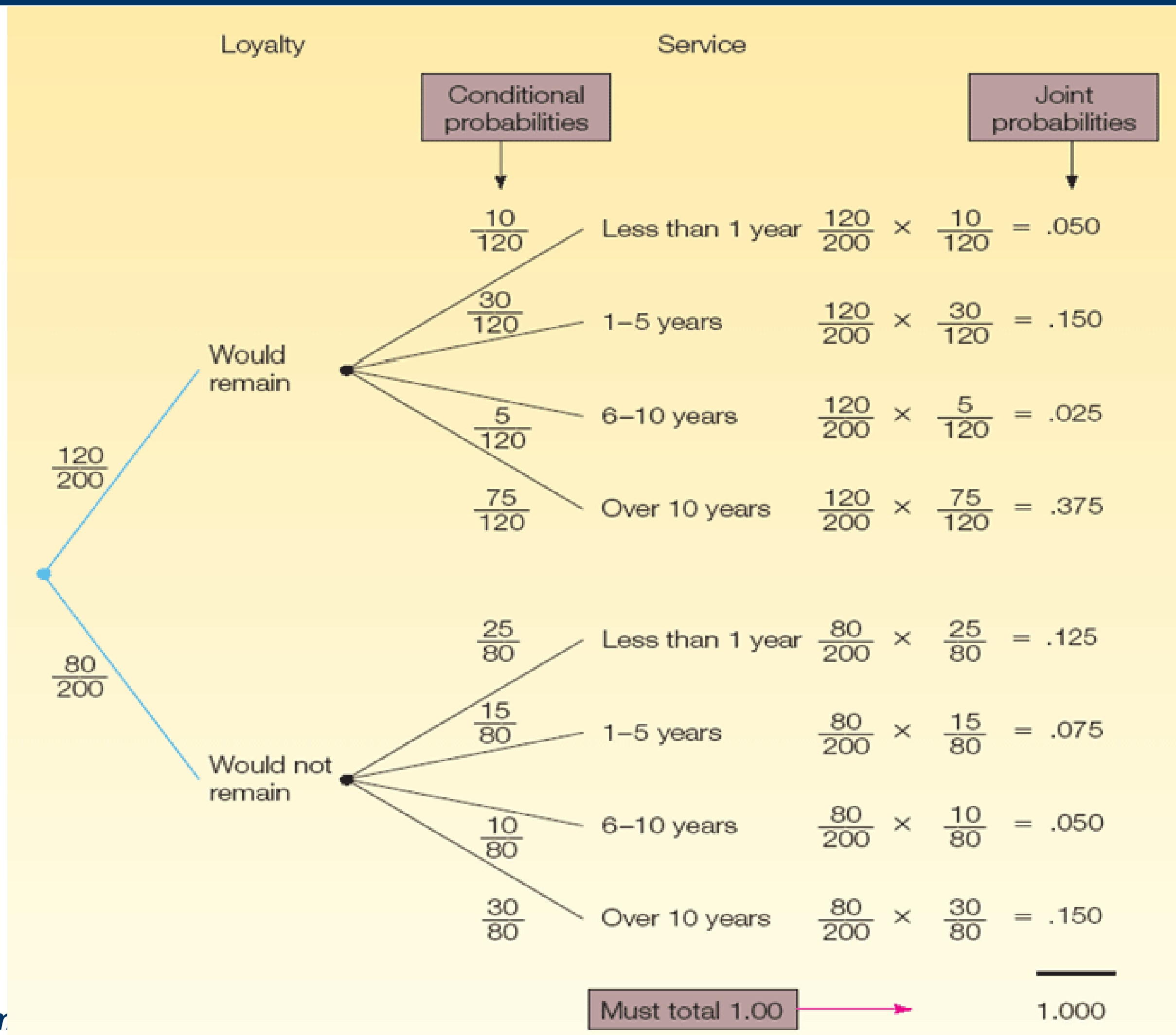
Loyalty
Service
Conditional probabilities
Joint probabilities
120/200
Would remain
10/120
Less than 1 year 120/200 × 10/120 = .050
30/120
1–5 years 120/200 × 30/120 = .150
5/120
6–10 years 120/200 × 5/120 = .025
75/120
Over 10 years 120/200 × 75/120 = .375
80/200
Would not remain
25/80
Less than 1 year 80/200 × 25/80 = .125
15/80
1–5 years 80/200 × 15/80 = .075
10/80
6–10 years 80/200 × 10/80 = .050
30/80
Over 10 years 80/200 × 30/80 = .150
Must total 1.00
1.000

Bayes' Theorem

- Bayes' Theorem is a method for revising a probability given additional information.
- It is computed using the following formula:

BAYES' THEOREM

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} \quad [5\text{–}7]$$

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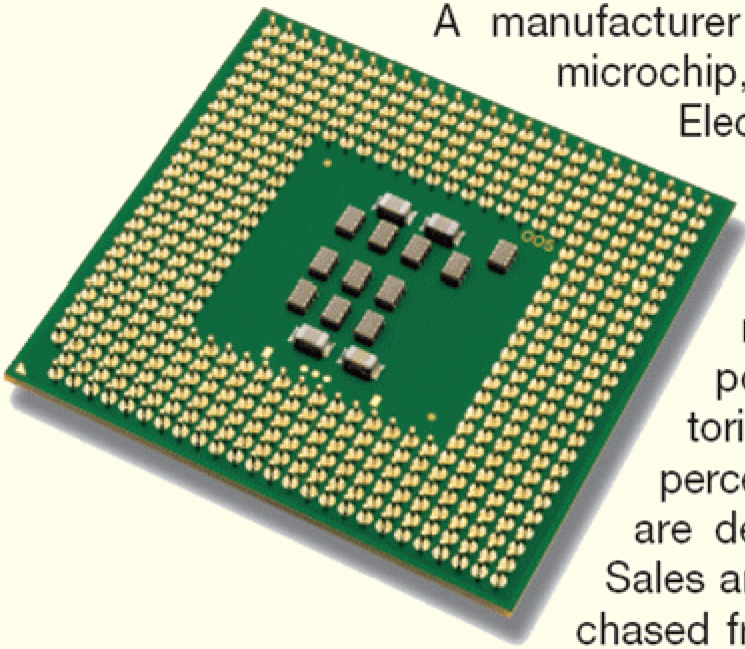
Bayes Theorem - Example



A manufacturer of DVD players purchases a particular microchip, called the LS-24, from three suppliers: Hall Electronics, Schuller Sales, and Crawford Components. Thirty percent of the LS-24 chips are purchased from Hall Electronics, 20 percent from Schuller Sales, and the remaining 50 percent from Crawford Components. The manufacturer has extensive histories on the three suppliers and knows that 3 percent of the LS-24 chips from Hall Electronics are defective, 5 percent of chips from Schuller Sales are defective, and 4 percent of the chips purchased from Crawford Components are defective.

When the LS-24 chips arrive at the manufacturer, they are placed directly in a bin and not inspected or otherwise identified by supplier. A worker selects a chip for installation in a DVD player and finds it defective. What is the probability that it was manufactured by Schuller Sales?

Bayes Theorem – Example *(cont.)*

- There are three mutually exclusive and collectively exhaustive events, that is, three suppliers.

  A_1 The LS-24 was purchased from Hall Electronics.
  A_2 The LS-24 was purchased from Schuller Sales.
  A_3 The LS-24 was purchased from Crawford Components.

- The prior probabilities are:

  $P(A_1) = .30$ The probability the LS-24 was manufactured by Hall Electronics.
  $P(A_2) = .20$ The probability the LS-24 was manufactured by Schuller Sales.
  $P(A_3) = .50$ The probability the LS-24 was manufactured by Crawford Components.

- The additional information can be either:

  B_1 The LS-24 appears defective, or
  B_2 The LS-24 appears not to be defective.

Bayes Theorem – Example *(cont.)*

- The following conditional probabilities are given.

$P(B_1 \mid A_1) = .03$	The probability that an LS-24 chip produced by Hall Electronics is defective.
$P(B_1 \mid A_2) = .05$	The probability that an LS-24 chip produced by Schuller Sales is defective.
$P(B_1 \mid A_3) = .04$	The probability that an LS-24 chip produced by Crawford Components is defective.

- A chip is selected from the bin. Because the chips are not identified by supplier, we are not certain which supplier manufactured the chip. We want to determine the probability that the defective chip was purchased from Schuller Sales. The probability is written $P(A_2 \mid B_1)$.

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Bayes Theorem – Example *(cont.)*

Event, A_i	Prior Probability, $P(A_i)$	Conditional Probability, $P(B_1 \mid A_i)$	Joint Probability, $P(A_i \text{ and } B_1)$	Posterior Probability, $P(A_i \mid B_1)$
Hall	.30	.03	.009	.009/.039 = .2308
Schuller	.20	.05	.010	.010/.039 = .2564
Crawford	.50	.04	.020	.020/.039 = .5128
			$P(B_1)$ = .039	1.0000

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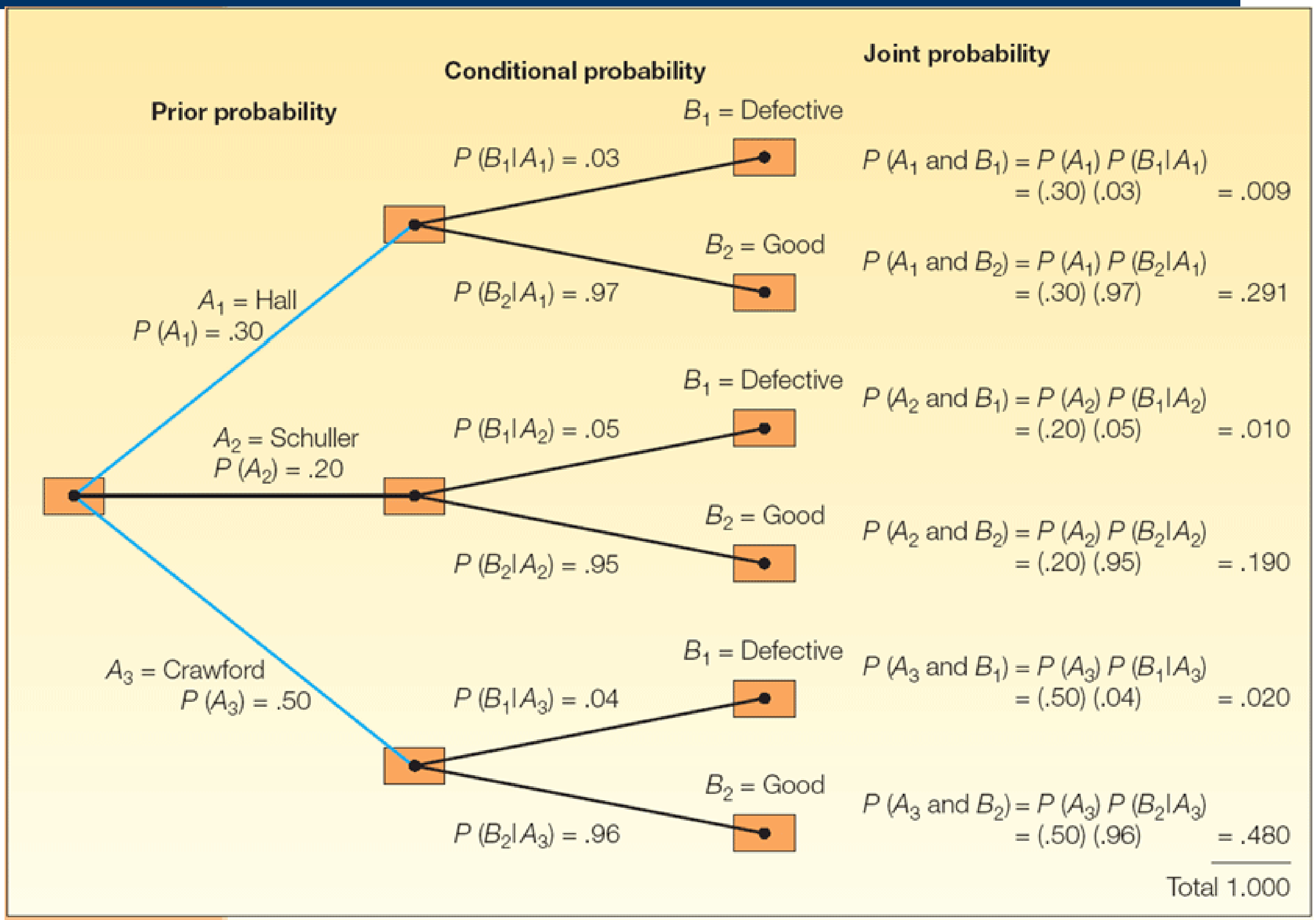

CHART 5–3 Tree Diagram of DVD Manufacturing Problem

Bayes Theorem – Example *(cont.)*

The probability the defective LS-24 chip came from Schuller Sales can be formally found by using Bayes' theorem. We compute $P(A_2|B_1)$, where A_2 refers to Schuller Sales and B_1 to the fact that the selected LS-24 chip was defective.

$$P(A_2|B_1) = \frac{P(A_2)P(B_1|A_2)}{P(A_1)P(B_1|A_1) + P(A_2)P(B_1|A_2) + P(A_3)(B_1|A_3)}$$

$$= \frac{(.20)(.05)}{(.30)(.03) + (.20)(.05) + (.50)(.04)} = \frac{.010}{.039} = .2564$$

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Counting Rules – Multiplication

The multiplication formula indicates that if there are *m* ways of doing one thing and *n* ways of doing another thing, there are *m* x *n* ways of doing both.

Example: Dr. Delong has 10 shirts and 8 ties. How many shirt and tie outfits does he have?

(10)(8) = 80

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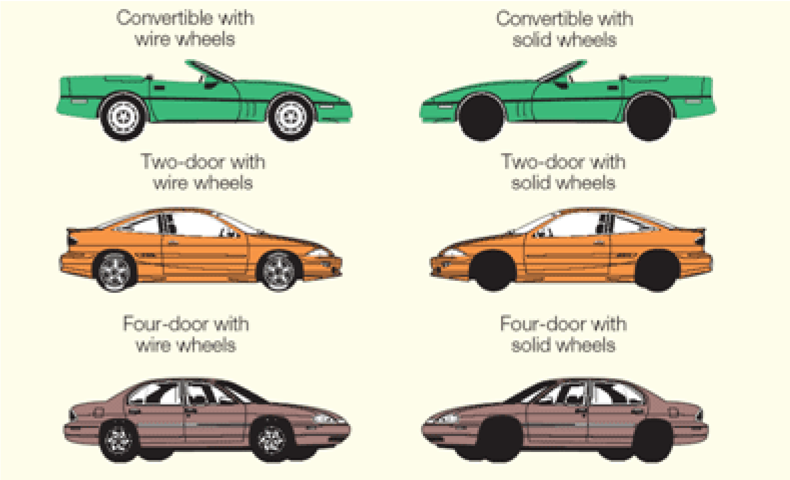
Counting Rules – Multiplication: Example

An automobile dealer wants to advertise that for $29,999 you can buy a convertible, a two-door sedan, or a four-door model with your choice of either wire wheel covers or solid wheel covers. How many different arrangements of models and wheel covers can the dealer offer?



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Counting Rules – Multiplication: Example

MULTIPLICATION FORMULA	Total number of arrangements = $(m)(n)$	**[5–8]**

We can employ the multiplication formula as a check (where m is the number of models and n the wheel cover type). From formula (5–8):

$$\text{Total possible arrangements} = (m)(n) = (3)(2) = 6$$

Counting Rules - Permutation

A permutation is any arrangement of r objects selected from n possible objects. The order of arrangement is important in permutations.

PERMUTATION FORMULA $${}_nP_r = \frac{n!}{(n-r)!} \qquad [5\text{–}9]$$

where:

n is the total number of objects.

r is the number of objects selected.

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Counting - Combination

A combination is the number of ways to choose *r* objects from a group of *n* objects without regard to order.

COMBINATION FORMULA $$_nC_r = \frac{n!}{r!(n-r)!} \qquad [5\text{–}10]$$

where:

n is the total number of objects.

r is the number of objects selected.

Combination - Example

There are 12 players on the Carolina Forest High School basketball team. Coach Thompson must pick five players among the twelve on the team to comprise the starting lineup. How many different groups are possible?

$$_{12}C_5 = \frac{12!}{5!(12-5)!} = 792$$

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Permutation - Example

Suppose that in addition to selecting the group, he must also rank each of the players in that starting lineup according to their ability.

$$_{12}P_5 = \frac{12!}{(12-5)!} = 95,040$$

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End of Chapter 5