Ex. Show that the vector field represented by

$\vec{F} = (z^2+2x+3y)\hat{i} + (3x+2y+z)\hat{j} + (y+2zx)\hat{k}$ is irrotational.

Also obtain a scalar Φ function such that grad $\Phi = \vec{F}$.

Sol^n: Curl $\vec{F} = \nabla \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2+2x+3y & 3x+2y+z & y+2zx \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(y+2zx) - \frac{\partial}{\partial z}(3x+2y+z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(y+2zx) - \frac{\partial}{\partial z}(z^2+2x+3y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3x+2y+z) - \frac{\partial}{\partial y}(z^2+2x+3y)\right]$$

$$= \hat{i}\left[(1+0)-(0+0+1)\right] - \hat{j}\left[(0+2z)-(2z+0+0)\right] + \hat{k}\left[(3+0+0)-(0+0+3)\right]$$

$$= \hat{i}(1-1) - \hat{j}(2z-2z) + \hat{k}(3-3)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$$

Since curl $\vec{F} = \vec{0}$, so $\vec{F}$ is irrotational.

Again, given $\vec{F} = \text{grad}\,\Phi = \nabla\Phi$, where Φ is a scalar function.

Now the total differential $d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz$

$$\Rightarrow d\Phi = \left(\hat{i}\frac{\partial \Phi}{\partial x} + \hat{j}\frac{\partial \Phi}{\partial y} + \hat{k}\frac{\partial \Phi}{\partial z}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$\Rightarrow d\Phi = \nabla\Phi \cdot d\vec{r} \quad \left[\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\right]$$

(P.T.O.)

$$\Rightarrow d\Phi = \vec{F}\cdot\vec{dr} \quad [\because \vec{F} = \nabla\Phi]$$

$$\Rightarrow d\Phi = [(z^2+2x+3y)\hat{i} + (3x+2y+z)\hat{j} + (y+2zx)\hat{k}]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$\Rightarrow d\Phi = (z^2+2x+3y)dx + (3x+2y+z)dy + (y+2zx)dz$$

$$\Rightarrow d\Phi = (z^2dx+2x\,dx+3y\,dx) + (3x\,dy+2y\,dy+z\,dy) + (y\,dz+2zx\,dz)$$

$$\Rightarrow d\Phi = (z^2dx+2zx\,dz) + (3y\,dx+3x\,dy) + (z\,dy+y\,dz) + 2x\,dx+2y\,dy$$

$$\Rightarrow d\Phi = [z^2dx + x\cdot(2z\,dz)] + 3(y\,dx+x\,dy) + (y\,dz+z\,dy) + 2x\,dx + 2y\,dy$$

$$\Rightarrow d\Phi = \underbrace{[z^2dx + x\cdot d(z^2)]}_{d(z^2\cdot x)} + 3(x\,dy+y\,dx) + (y\,dz+z\,dy) + 2x\,dx+2y\,dy$$

$$\Rightarrow d\Phi = d(z^2x) + 3d(xy) + d(y\cdot z) + 2x\,dx + 2y\,dy$$

$$[d(u\cdot v) = u\cdot dv + v\cdot du]$$

Now integrating both sides, we get

$$\Rightarrow \int d\Phi = \int d(z^2x) + 3\int d(xy) + \int d(yz) + 2\cdot\int x\,dx + 2\int y\,dy$$

$$\Rightarrow \Phi = z^2x + 3xy + yz + 2\cdot\frac{x^2}{2} + 2\cdot\frac{y^2}{2} + c$$

$$\Rightarrow \Phi = z^2x + 3xy + yz + x^2 + y^2 + c,$$ which is the required scalar function. (Ans.)

Exercise: A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2 y)\hat{j}$. Show that the field is irrotational and find the scalar potential.

[P. 407, Ex. 47]

Hints:

$$d\varphi = \vec{A} \cdot \vec{dr}$$

$$= \left[(x^2 + xy^2)\hat{i} + (y^2 + x^2 y)\hat{j}\right] \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (x^2 + xy^2)\,dx + (y^2 + x^2 y)\,dy + 0$$

$$= x^2 dx + xy^2 dx + y^2 dy + x^2 y\,dy$$

$$= x^2 dx + y^2 dy + \left[xy^2 dx + x^2 y\,dy\right]$$

$$= x^2 dx + y^2 dy + \frac{1}{2}\left[y^2 . 2x\,dx + x^2 . 2y\,dy\right]$$

$$\therefore d\varphi = x^2 dx + y^2 dy + \frac{1}{2}\left[\underbrace{y^2 . d(x^2) + x^2 d(y^2)}_{d(y^2 . x^2)}\right]$$

$$\Rightarrow d\varphi = x^2 dx + y^2 dy + \frac{1}{2} d(y^2 . x^2)$$

$$\Rightarrow \int d\varphi = \int x^2 dx + \int y^2 dy + \frac{1}{2}\int d(x^2 y^2)$$

$$\Rightarrow \varphi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{1}{2}x^2 y^2 + c. \quad \text{(Ans.)}$$

Exercise: Example 48 (Page. 407)