4.25 Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$, where C is the circle $|z+1| = 1$.

Solution If $f(z) = \frac{z+4}{z^2+2z+5}$, then poles of $f(z)$ are given by

$$z^2 + 2z + 5 = 0, \quad \therefore z = -1 + 2i, -1 - 2i$$

when $z = -1 + 2i$, then $|z + 1| = |-1 + 2i + 1| = 2 > 1$

$\therefore$ The pole $z = -1 + 2i$ lies outside the circle $|z + 1| = 1$

Since both the poles lie outside the circle C, hence $f(z)$ is analytic everywhere within C. Also, $f'(z)$ is continuous within and on C.

By applying Cauchy's theorem, we get

$$\int_C f(z)dz = 0 \quad \text{i.e.} \quad \int_C \frac{z+4}{z^2+2z+5}dz = 0.$$

26 Evaluate $\int_C \frac{z^2-z+1}{z-1}dz$, where C is the circle $|z| = \frac{1}{2}$.

Solution Let $f(z) = \frac{z^2-z+1}{z-1}$

Poles of $f(z)$ are given by

$$z - 1 = 0$$

i.e., $\quad z = 1$

Since the pole $z = 1$ lie outside the circle C, hence $f(z)$ is analytic within and on C. Also, $f'(z)$ is continuous at each point within and on C.

So, by applying Cauchy's theorem, we get $\int_C f(z)dz = 0$.

5.5 Evaluate (a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$,

(b) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z| = 3$.

Solution

(a) Since $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

By Cauchy's integral formula with $a = 2$ and $a = 1$ respectively, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz = 2\pi i\{\sin \pi(2)^2 + \cos \pi(2)^2\} = 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz = 2\pi i\{\sin \pi(1)^2 + \cos \pi(1)^2\} = -2\pi i$$?

since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

(b) Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f''(-1) = 8e^{-2}$. Hence (1) becomes

$$8e^{-2} = \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4} dz$$

from which we see that the required integral has the value $8\pi i e^{-2}/3$.

5.6 Evaluate $\int_C \frac{e^{-z}}{z+1}\, dz$, where C is the circle $|z| = \frac{1}{2}$.

Solution Here, $f(z) = e^{-z}$ is an analytic function.

The point $z = -1$ lies outside the circle $|z| = \frac{1}{2}$.

$\therefore$ The function $\frac{e^{-z}}{z+1}$ is analytic within and on C.

By Cauchy's theorem, we have $\int_C \frac{e^{-z}}{z+1}\, dz = 0$.

5.7 Evaluate $\int_C \frac{3z^2+z}{z^2-1}\, dz$, where C is the circle $|z-1| = 1$.

Solution The integrand has singularities, where $z^2 - 1 = 0$ i.e. at $z = 1$ and $z = -1$. The circle $|z-1| = 1$ has center at $z = 1$, $f(z) = 3z^2 + z$, is an analytic function.

Also,

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$\therefore$

$$\int_C \frac{3z^2+z}{z^2-1}\, dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1}\, dz - \frac{1}{2}\int_C \frac{3z^2+z}{z+1}\, dz \qquad (1)$$

By Cauchy's integral formula,

$$\int_C \frac{3z^2+z}{z-1}\, dz = 2\pi i f(1) = 8\pi i \qquad \text{where } f(z) = 3z^2 + z$$

By Cauchy's theorem, $\int_C \frac{3z^2+z}{z+1}\, dz = 0$

$\therefore$ From (1), we have $\int_C \frac{3z^2+z}{z^2-1}\, dz = 4\pi i$.

5.8 Prove C

6.23 Let $f(z) = \ln(1+z)$, where we consider at branch which has the value zero when $z = 0$. (a) Expand $f(z)$ in a Taylor series about $z = 0$. (b) Determine the region of convergence for the series in (a). (c) Expand $\ln\left(\frac{1+z}{1-z}\right)$ in a Taylor series about $z = 0$.

Solution

(a) $f(z) = \ln(1+z)$

$$f'(z) = \frac{1}{1+z} = (1+z)^{-1}, \qquad f(0) = 0$$

$$f''(z) = -(1+z)^{-2} \qquad f'(0) = 1$$

$$f'''(z) = (-1)(-2)(1+z)^{-3} \qquad f''(0) = -1$$

$$f'''(0) = 2!$$

$$\vdots \qquad \vdots$$

$$f^{(n+1)}(z) = (-1)^n n!(1+z)^{-(n+1)} \qquad f^{(n+1)}(0) = (-1)^n n!$$

Then

$$f(z) = \ln(1+z) = f(0) + f'(0)\,z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

Another method. If $|z| < 1$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$$

Then integrating from 0 to z yields

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

(b) The nth term is $u_n = \frac{(-1)^{n-1} z^n}{n}$. Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{u_n+1}{u_n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = |z|$$

and the series converges for $|z| < 1$. The series can be shown to converge for $|z| = 1$ except for $z = -1$.

This result also follows from the fact that the series converges in a circle which extends to the nearest singularity (i.e. $z = -1$) of $f(z)$.

(c) From the result in (a) we have, on replacing z by $-z$,

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

both series convergent for $|z| < 1$. By subtraction, we have

$$\ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

which converges for $|z| < 1$. We can also show that this series converges for $|z| = 1$ except for $z = \pm 1$.

6.24 (a) Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$ (b) Determine the region of convergence of this series.

Solution

(a) $f(z) = \sin z, f'(z) = \cos z, f''(z) = -\sin z, f'''(z) = -\cos z, f^{IV}(z) = \sin z, \ldots$

$f(\pi/4) = \sqrt{2}/2,\ f'(\pi/4) = \sqrt{2}/2,\ f''(\pi/4) = -\sqrt{2}/2,\ f'''(\pi/4) = -\sqrt{2}/2,\ f^{IV}(\pi/4) = \sqrt{2}/2, \ldots$

Then, since $a = \pi/4$,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + \cdots$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(z-\pi/4) - \frac{\sqrt{2}}{2\cdot 2!}(z-\pi/4)^2 - \frac{\sqrt{2}}{2\cdot 3!}(z-\pi/4)^3 + \cdots$$

$$= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\}.$$

Another method. Let $u = z - \pi/4$ or $z = u + \pi/4$. Then we have,

$$\sin z = \sin(u + \pi/4) = \sin u \cos(\pi/4) + \cos u \sin(\pi/4)$$

$$= \frac{\sqrt{2}}{2}(\sin u + \cos u)$$

$$= \frac{\sqrt{2}}{2}\left\{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right) + \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots\right)\right\}$$

$$= \frac{\sqrt{2}}{2}\left\{1 + u - \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots\right\}$$

$$= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\}$$

(b) Since the singularity of $\sin z$ nearest to $\pi/4$ is at infinity, the series converges for all finite values of z, i.e. $|z| < \infty$. This can also be established by the ratio test.

6.27 Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for: (a) $1 < |z| < 3$, (b) $|z| > 3$, (c) $0 < |z+1| < 2$, (d) $|z| < 1$.

Solution

(a) Resolving into partial fractions,

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

If $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots.$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$, i.e. $1 < |z| < 3$, is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(b) If $|z| > 1$, we have as in part (a),

$$\frac{1}{2(z+1)} = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| > 3$,

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e. $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

Lecture থেকে Laurent series এর [illegible]

(c) Let $z + 1 = u$. Then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

valid for $|u| < 2$, $u \neq 0$ or $0 < |z+1| < 2$.

(d) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (a),

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion, valid for both $|z|<1$ and $|z|<3$, i.e. $|z|<1$, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

This is a *Taylor series.*