LDA

Math

① Compute the Linear Discriminant projection for the following two dimensional dataset.

- samples for class ω_1 : $X_1 = (x_1, x_2) = \{(4,2), (2,4), (2,3), (3,6), (4,4)\}$

- Samples for class ω_2 : $X_2 = (x_1, x_2) = \{(9,10), (6,8), (9,5), (8,7), (10,8)\}$

Soln: Step 1: class mean:

- Mean:

$$\mu_1 = \frac{1}{N_1} \sum_{x \in \omega_1} x$$

$$= \frac{1}{5}\left[\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}3\\6\end{pmatrix}+\begin{pmatrix}4\\4\end{pmatrix}\right] = \begin{pmatrix}3\\3.8\end{pmatrix} = \begin{pmatrix}\bar{X_1}\\\bar{X_2}\end{pmatrix}$$

$$\mu_2 = \frac{1}{N_2} \sum_{x \in \omega_2} x$$

$$= \frac{1}{5}\left[\begin{pmatrix}9\\10\end{pmatrix}+\begin{pmatrix}6\\8\end{pmatrix}+\begin{pmatrix}9\\5\end{pmatrix}+\begin{pmatrix}8\\7\end{pmatrix}+\begin{pmatrix}10\\8\end{pmatrix}\right] = \begin{pmatrix}8.4\\7.6\end{pmatrix} = \begin{pmatrix}\bar{X_1}\\\bar{X_2}\end{pmatrix}$$

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Step 2: Covariance Matrix –

$$S_1 = \sum_{x\in\omega_1}(x-\mu_1)(x-\mu_1)^T = \sum_{x\in\omega_1}(x-\mu_1)^2$$

$$= \left[\begin{pmatrix}4\\2\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^2 + \left[\begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^2 + \left[\begin{pmatrix}2\\3\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^2 +$$

$$\left[\begin{pmatrix}3\\6\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^2 + \left[\begin{pmatrix}4\\4\end{pmatrix}-\begin{pmatrix}3\\3.8\end{pmatrix}\right]^2$$

$$= \begin{bmatrix}1\\3.24\end{bmatrix} + \begin{bmatrix}1\\0.04\end{bmatrix} + \begin{bmatrix}1\\0.64\end{bmatrix} + \begin{bmatrix}0\\4.84\end{bmatrix} + \begin{bmatrix}1\\0.04\end{bmatrix}$$

$=$

$$\text{cov} = \begin{pmatrix} Cov(x,x) & Cov(x,y) \\ Cov(y,x) & Cov(y,y) \end{pmatrix}$$

We know,

$$Cov(x,y) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{n-1}$$

$$Cov(x,x) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$$

$$\therefore Cov, S_1 = \begin{pmatrix} 1 & -0.25 \\ -0.25 & 2.2 \end{pmatrix}$$

x	y
4	2
2	4
2	3
3	6
4	4

$\bar{x} = 3$

$\bar{y} = 3.8$

$x_i-\bar{x}$	$y_i-\bar{y}$
1	-1.8
-1	0.2
-1	-0.8
0	2.2
1	0.2

$$Cov, S_2 = \begin{pmatrix} 2.3 & -0.05 \\ -0.05 & 3.3 \end{pmatrix}$$

x	y
9	10
6	8
9	5
8	7
10	8

$\bar{x} = 8.4$
$\bar{y} = 7.6$

$x_i - \bar{x}$	$y_i - \bar{y}$
0.6	2.4
-2.4	0.4
0.6	-2.6
-0.4	-0.6
1.6	0.4

Step-3:

Within class scatter matrix,

$$S_W = S_1 + S_2$$

$$= \begin{pmatrix} 1 & -0.25 \\ -0.25 & 2.2 \end{pmatrix} + \begin{pmatrix} 2.3 & -0.05 \\ -0.05 & 3.3 \end{pmatrix}$$

$$= \begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix}$$

Step-4:

Between class scatter matrix,

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$$

$$= \left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right]\left[\begin{pmatrix} 3 \\ 3.8 \end{pmatrix} - \begin{pmatrix} 8.4 \\ 7.6 \end{pmatrix}\right]^T$$

$$= \begin{pmatrix} -5.4 \\ -3.8 \end{pmatrix} \begin{pmatrix} -5.4 & -3.8 \end{pmatrix}$$

$$= \begin{pmatrix} -5.4 * (-5.4) & (-5.4) * (-3.8) \\ -3.8 * (-5.4) & (-3.8) * (-3.8) \end{pmatrix} = \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix}$$

$$\circledast \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step 5: The LDA projection is then obtained as the sol^n of the generalized eigen value problem,

$$S_W^{-1} S_B W = \lambda W$$

$$\Rightarrow \left| S_W^{-1} S_B - \lambda I \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix}^{-1} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \frac{1}{3.3 * 5.5 - (-0.3)*(-0.3)} \begin{pmatrix} 5.5 & 0.3 \\ 0.3 & 3.3 \end{pmatrix} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \frac{1}{18.06} \begin{pmatrix} 5.5 & 0.3 \\ 0.3 & 3.3 \end{pmatrix} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{pmatrix} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{pmatrix} 9.2199 & 6.488 \\ 4.233 & 2.9788 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 9.2199 - \lambda & 6.488 \\ 4.233 & 2.9788 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (9.2199 - \lambda)(2.9788 - \lambda) - 6.488 * 4.233 = 0$$

$$\Rightarrow \lambda^2 - 9.2199\lambda - 2.9788\lambda + 27.46 - 27.46 = 0$$

$$\Rightarrow \lambda^2 - 12.1987\lambda = 0$$

$\Rightarrow \lambda(\lambda - 12.1987) = 0 \Rightarrow \lambda = 0$ or $\lambda - 12.1987 = 0$

$\therefore \lambda_1 = 0, \quad \lambda_2 = 12.1987$

Hence,

$\cancel{(S_W^{-1} S_B)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda_1 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \text{ and } (S_W^{-1} S_B)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}$

$\therefore \begin{pmatrix} 9.2199 & 6.488 \\ 4.233 & 2.9788 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \cancel{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}$

$\Rightarrow 9.2199\, w_1 + 6.488\, w_2 = 0$

$\Rightarrow w_1 = -\dfrac{6.488\, w_2}{9.2199}$

$= -0.7037\, w_2 \sim \begin{bmatrix} \frac{-0.7037}{A} \\ \frac{1}{A} \end{bmatrix} \sim \begin{bmatrix} -0.5755 \\ 0.8178 \end{bmatrix}$

$A = \sqrt{(-0.7037)^2 + (1)^2} = 1.22278$

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Again, $\begin{pmatrix} 9.2199 & 6.488 \\ 4.233 & 2.9788 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 12.1987\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$\Rightarrow 9.2199\, w_1 + 6.488\, w_2 = 12.1987\, w_1$

$\Rightarrow w_2 = \dfrac{(12.1987 - 9.2199)\, w_1}{6.488}$

$= 0.4591\, w_1$

$\sim \begin{bmatrix} \dfrac{1}{A} \\ \dfrac{0.4591}{A} \end{bmatrix} \sim \begin{bmatrix} 0.9088 \\ 0.41723 \end{bmatrix}$

$A = \sqrt{(0.4591)^2 + 1^2} = 1.10035$

$\therefore\ W_1 = \begin{bmatrix} -0.5755 \\ 0.8178 \end{bmatrix}$ and $W_2 = \begin{bmatrix} 0.9088 \\ 0.41723 \end{bmatrix}$