Rank of a matrix

Echelon Matrix: A matrix $A = [a_{ij}]$ is an echelon matrix if the number of zeros preceding the first non-zero entry of a row increases row by row until only zero rows remain.

A matrix which is in echelon form and the first non-zero element in each non-zero row is the only non-zero element in its column is said to be in reduced echelon form.

Examples of echelon matrices and matrices of reduced echelon form are given below:

(i)
$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(echelon matrix)

(ii)
$$\begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(echelon matrix)

(iii)
$$\begin{bmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

(reduced echelon form)

Example 1: Find the echelon form and the row reduced echelon form of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}.$$

Soln. Given $A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$

First let us reduce the matrix A to echelon form by the elementary row operations. We multiply 1st row by 2 and 3 and then subtract from 2nd and 3rd rows respectively, then

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix} \quad R_2' = R_2 - 2R_1, \quad R_3' = R_3 - 3R_1$$

Now we multiply 3rd row by 3,

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 15 & -36 & 6 \end{bmatrix} R_3' = 3R_3$$

We multiply 2nd row by 5 and then subtract from the 3rd row,

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix} R_3' = R_3 - 5R_2$$

This matrix is in row echelon form.

Now we subtract 3rd row from the 2nd row. Then

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix} R_2' = R_2 - R_3$$

Now we multiply 3rd row by $-\frac{1}{6}$, and get

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix} R_3' = -\frac{1}{6}R_3$$

Now we multiply 2nd row by $\frac{1}{3}$, and get

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix} R_2' = \frac{1}{3}R_2$$

Now we add 2nd row with the 1st row and get,

$$\sim \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix} \quad R_1' = R_1 + R_2$$

Now we multiply 3rd row by 2 and then subtract from the 1st row and get,

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix} \quad R_1' = R_1 - 2R_3$$

This matrix is in row ~~red~~ reduced echelon form: (Ans)

Rank of a matrix:

Let A be an $m \times n$ matrix and let A_R be the row echelon form of A. Then the rank r of the matrix A is the number of non-zero rows of A_R.

* The rank of a matrix can be determined by the following processes:

Reduce the given matrix A to echelon form using elementary row operations (transformations). Since the non-zero rows of a matrix in echelon form are linearly independent, the number of non-zero rows of the echelon matrix is the rank of the given matrix.

The normal form of a matrix:

By means of elementary transformations any matrix A of rank $r > 0$ can be reduced to one of the forms

$$I_r, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, [I_r, 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

called its normal form.

Here, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ They are of order 3.

$$[I_3 \; 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{3\times 4}$$

Ex 2: Reduce the matrix A to the normal (or canonical form and hence obtain its rank where $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

Sol^n: We will apply both elementary column and row operations to the matrix A for reducing it to the normal form.

Given $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$

We replace c_2 and c_4 by $c_2 - 2c_1$ and $c_4 + c_1$ respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \\ -2 & 7 & 2 & 3 \end{bmatrix}$$

We replace e_2 and e_4 by e_2+2e_3 and e_4-5e_3 respectively.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 11 & 2 & -7 \end{bmatrix}$$

We replace e_1 by ~~e_1 and~~ e_1+e_3 and e_4 by $e_4+\frac{7}{11}e_2$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$$

We replace R_2 by R_2-4R_1, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 2 & 0 \end{bmatrix}$

We replace R_3 by R_3-2R_2, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 11 & 0 & 0 \end{bmatrix}$

We interchange e_2 and e_3, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 11 & 0 \end{bmatrix}$

We replace e_3 by $\frac{1}{11}e_3$, $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$

Where $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence the rank of A is 3. (Ans)