Note : The greatest rate of change of Φ, that is, the maximum directional derivative, takes place in the direction of the vector $\nabla\Phi$ and has the magnitude of the vector $\nabla\Phi$.

Example : (a) In what direction from the point $(2,1,-1)$ is the directional derivative of $\Phi = x^2yz^3$ a maximum ?

(b) What is the magnitude of this maximum ?

Soln: $\nabla\Phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz^3)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2yz^3) + \hat{j}\frac{\partial}{\partial y}(x^2yz^3) + \hat{k}\frac{\partial}{\partial z}(x^2yz^3)$$

$$= 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$$

At the point $(2,1,-1)$, $\nabla\Phi = -4\hat{i} - 4\hat{j} + 12\hat{k}$

(a) the directional derivative is a maximum in the direction $\nabla\Phi = -4\hat{i} - 4\hat{j} + 12\hat{k}$.

(b) The magnitude of this maximum is

$$|\nabla\Phi| = \sqrt{(-4)^2 + (-4)^2 + (12)^2}$$

$$= \sqrt{16 + 16 + 144}$$

$$= \sqrt{176} = 4\sqrt{11}. \quad \text{(Ans.)}$$

Example: The temperature at any point in space is given by $T = xy + yz + zx$. Determine the directional derivative of T in the direction of the vector $3\hat{i} - 4\hat{k}$ at the point $(1,1,1)$.

Sol^n: Given the temperature, $T = xy + yz + zx$

Now, $\nabla T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$= \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(y+x)$

Directional derivative at $(1,1,1) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore$ Directional derivative at $(1,1,1)$ in the direction of $3\hat{i} - 4\hat{k}$

$$= (2\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \frac{3\hat{i} - 4\hat{k}}{\sqrt{9+16}}$$

$$= \frac{2\cdot 3 + 0 - 2\cdot 4}{5} = \frac{6-8}{5} = -\frac{2}{5}. \text{ (Ans.)}$$

Exercise: Find the rate of change of $\Phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point $(1,1,1)$.

Q1. Find the directional derivative of v^2, where $\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$, at the point $(2,0,3)$ in the direction of the outward normal to the sphere $x^2+y^2+z^2=14$ at the point $(3,2,1)$. [Example 29, Page 394]

Sol^n: Here $v^2 = \vec{v}\cdot\vec{v}$

$$= (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k})\cdot(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k})$$

$$= x^2y^4 + z^2y^4 + x^2z^4$$

Now directional derivative of $v^2 = \nabla v^2$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2y^4 + z^2y^4 + x^2z^4) + \hat{j}\frac{\partial}{\partial y}(x^2y^4 + z^2y^4 + x^2z^4) +$$
$$\hat{k}\frac{\partial}{\partial z}(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= \hat{i}(2xy^4 + 0 + 2xz^4) + \hat{j}(4x^2y^3 + 4z^2y^3 + 0) +$$
$$\hat{k}(0 + 2zy^4 + 4x^2z^3)$$

Now directional derivative of v^2 at the point $(2,0,3)$ =

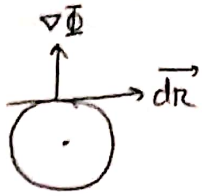
$$\hat{i}(2\cdot2\cdot0^4 + 0 + 2\cdot2\cdot3^4) + \hat{j}(4\cdot2^2\cdot0^3 + 4\cdot3^2\cdot0^3 + 0) +$$
$$\hat{k}(0 + 2\cdot3\cdot0^4 + 4\cdot2^2\cdot3^3)$$

$$= \hat{i}(0 + 0 + 4\times81) + \hat{j}(0+0+0) + \hat{k}(0 + 0 + 4\times4\times27)$$

(P.T.O.)

$= 324\hat{i} + 432\hat{k}$

$= 108(3\hat{i} + 4\hat{k})$



Again, we know normal to the sphere $x^2+y^2+z^2=14$ is

$$= \nabla(x^2+y^2+z^2-14)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-14)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-14) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-14) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-14)$$

$$= \hat{i}(2x+0+0-0) + \hat{j}(0+2y+0-0) + \hat{k}(0+0+2z-0)$$

$$= (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

Now normal to the sphere $x^2+y^2+z^2=14$ at the point $(3,2,1)$ $\cancel{(2,0,3)}$

$$= (2\times3\hat{i} + 2\times2\hat{j} + 2\times1\hat{k})$$

$$= 6\hat{i} + 4\hat{j} + 2\hat{k}$$

Unit normal vector $= \dfrac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{36+16+4}} = \dfrac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{56}}$

$$= \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{4\times14}} = \frac{6\hat{i}+4\hat{j}+2\hat{k}}{2\sqrt{14}} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$

So, directional derivative of r^2 along the normal $=$

$$108(3\hat{i}+4\hat{k}) \cdot \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$

$$= 108 \cdot \frac{(9+0+4)}{\sqrt{14}} = \frac{108\times13}{\sqrt{14}} = \frac{1404}{\sqrt{14}}. \text{ (Ans.)}$$