Fourier Sine Transform

The fourier sine transform of a function $f(x)$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

Ex.1: Find the Fourier sine transform of the following function:

$$f(x) = \begin{cases} \sin x & \text{when } 0 < x < a \\ 0 & \text{when } x > a. \end{cases}$$

Soln: We know the fourier sine transform of a function $f(x)$ is

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a f(x) \sin sx \, dx + \sqrt{\frac{2}{\pi}} \int_a^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^a \sin x \cdot \sin sx \, dx + \int_a^\infty 0 \cdot \sin sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \sin x \cdot \sin sx \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^a 2 \sin x \cdot \sin sx \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^a \left[\cos(x - sx) - \cos(x + sx) \right] dx$$

$$\left[\because 2 \sin a \cdot \sin b = \cos(a-b) - \cos(a+b) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^a \left[\cos(1-s)x - \cos(1+s)x\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1-s)x}{(1-s)} - \frac{\sin(1+s)x}{(1+s)}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{\frac{\sin(1-s)a}{(1-s)} - \frac{\sin(1+s)a}{(1+s)}\right\} - 0\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(1-s)a}{(1-s)} - \frac{\sin(1+s)a}{(1+s)}\right]. \quad \text{(Ans.)}$$

Ex.2: Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

Sol^n: We know the Fourier sine transform of a function

$f(x) = \frac{e^{-ax}}{x}$ is $F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin sx \, dx = I$ (say)

Then $I = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cdot \sin sx \, dx$ .... (1)

Now differentiating (1) with respect to 's' we get,

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cdot \cos sx \cdot x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(s \sin sx - a \cos sx)\right]_0^\infty \quad \left[\because \int e^{-ax} \cos bx \, dx = \frac{e^{-ax}}{a^2+b^2}(b \sin bx - a \cos bx)\right]$$

$$\frac{dI}{ds} = \sqrt{\frac{2}{\pi}}\left[0 - \left\{\frac{1}{a^2+s^2}\left(0 - a.1\right)\right\}\right]$$

$$\Rightarrow \frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$$

$$\Rightarrow dI = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}ds$$

$$\Rightarrow \int dI = \sqrt{\frac{2}{\pi}} \cdot a\int\frac{ds}{a^2+s^2}$$

$$\Rightarrow I = \sqrt{\frac{2}{\pi}} \cdot a \cdot \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + A \quad \left[\because \int\frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$\Rightarrow I = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right) + A$$

For $s=0$, this gives $I = A$ .... (2)

Again, from (1), for $s=0$, we get $I=0$ ..... (3)

Hence, from (2) and (3) we get, $A=0$.

So, the required fourier sine transform of the given function is $I = \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{s}{a}$. (Ans).