Total differential, $d\varphi$:

Total differential of a scalar potential function, φ which is

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$

Note:
$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz$$
$$= \left(\hat{i}\frac{\partial\varphi}{\partial x} + \hat{j}\frac{\partial\varphi}{\partial y} + \hat{k}\frac{\partial\varphi}{\partial z}\right)\cdot\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$
$$= \nabla\varphi\cdot\vec{dr} \quad \left(\text{where } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k};\ \vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

That is, $d\varphi = \nabla\varphi\cdot\vec{dr}$ ---------- (1)

Note: If $\vec{F}$ is irrotational (that is, curl $\vec{F} = \nabla\times\vec{F} = \vec{0}$), then

$\vec{F} = \nabla\varphi$, where φ is a scalar potential function. Then equation (1) can be written as:

$$d\varphi = \vec{F}\cdot\vec{dr} \text{ ------------- } (2)$$

To find φ from equation (2), we have to integrate both sides:

$$\int d\varphi = \int \vec{F}\cdot\vec{dr}$$

That is, $\varphi = \int \vec{F}\cdot\vec{dr} + c$, where c is the integrating constant.

Thus, we can get the scalar potential function, φ from the total differential, $d\varphi$.

Rule: $d(u \cdot v) = u\,dv + v\,du$

Ex. Find the scalar potential function, φ for $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$.

Sol^n: Given, $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

Now curl $\vec{F} = \nabla \times \vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(z+x)\right] - \hat{j}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$= \hat{i}\left[(0+1) - (1+0)\right] - \hat{j}\left[(1+0) - (0+1)\right] + \hat{k}\left[(0+1) - (1+0)\right]$$

$$= \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$$

Since curl $\vec{F} = \vec{0}$ so $\vec{F}$ is irrotational. Hence $\vec{F} = \nabla\varphi$, where φ is the scalar potential function.

[P.T.O.]

Again, total differential $d\varphi = \frac{\partial \varphi}{\partial x} \cdot dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz$

$$\Rightarrow d\varphi = \left(\hat{i}\frac{\partial \varphi}{\partial x} + \hat{j}\frac{\partial \varphi}{\partial y} + \hat{k}\frac{\partial \varphi}{\partial z}\right) \cdot \left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$\Rightarrow d\varphi = \nabla\varphi \cdot d\vec{r} \quad [\because \vec{r} = \hat{i}x + \hat{j}y + \hat{k}z]$$

$$\Rightarrow d\varphi = \vec{F} \cdot d\vec{r}$$

$$\Rightarrow d\varphi = [(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}] \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\Rightarrow d\varphi = (y+z)dx + (z+x)dy + (x+y)dz$$

$$\Rightarrow d\varphi = y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz$$

$$\Rightarrow d\varphi = (y\,dx + x\,dy) + (z\,dy + y\,dz) + (z\,dx + x\,dz)$$

$$\Rightarrow d\varphi = (x\,dy + y\,dx) + (y\,dz + z\,dy) + (z\,dx + x\,dz)$$

$$\Rightarrow d\varphi = d(xy) + d(yz) + d(zx) \quad [\because d(u \cdot v) = u\,dv + v\,du]$$

$$\Rightarrow \int d\varphi = \int d(xy) + \int d(yz) + \int d(zx)$$

$$\Rightarrow \varphi = xy + yz + zx + c, \qquad [\because \int dx = \int d(x) = x + c]$$

which is the required scalar potential function. (Ans)

<u>Exercise</u>: Find the scalar potential function f for $\vec{A} = y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}$.

<u>Hints</u>: Curl $\vec{A} = \nabla \times \vec{A} = \vec{0}$ (verify it)

Hence $\vec{A}$ is irrotational and so $\vec{A} = \nabla f$

Now total differential, $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

$\Rightarrow df = \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \cdot \left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$

$\Rightarrow df = \nabla f \cdot \vec{dr}$

$\Rightarrow df = \vec{A} \cdot \vec{dr}$ $[\because \vec{A} = \nabla f]$

$\Rightarrow df = \left(y^2\hat{i} + 2xy\hat{j} - z^2\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$

$\Rightarrow df = y^2 dx + 2xy\,dy - z^2 dz$

$\Rightarrow df = \left(y^2 dx + x \cdot 2y\,dy\right) - z^2 dz$

$\Rightarrow df = \left[y^2 dx + x \cdot d(y^2)\right] - z^2 dz$ $[\because d(y^2) = 2y\,dy]$

$\Rightarrow df = d(y^2 \cdot x) - z^2 dz$ $[\because d(u \cdot v) = u\,dv + v\,du]$

$\Rightarrow \int df = \int d(xy^2) - \int z^2 dz$

$\Rightarrow f = xy^2 - \frac{z^3}{3} + c$, which is the required scalar potential function.

(Ans)

Note:

$d(y^2 \cdot x)$

$= y^2 dx + x\,d(y^2)$

$= y^2 dx + x \cdot 2y\,dy$

$= y^2 dx + 2xy\,dy$