CSE2209: Digital Electronics and Pulse Techniques

Course Conducted By:

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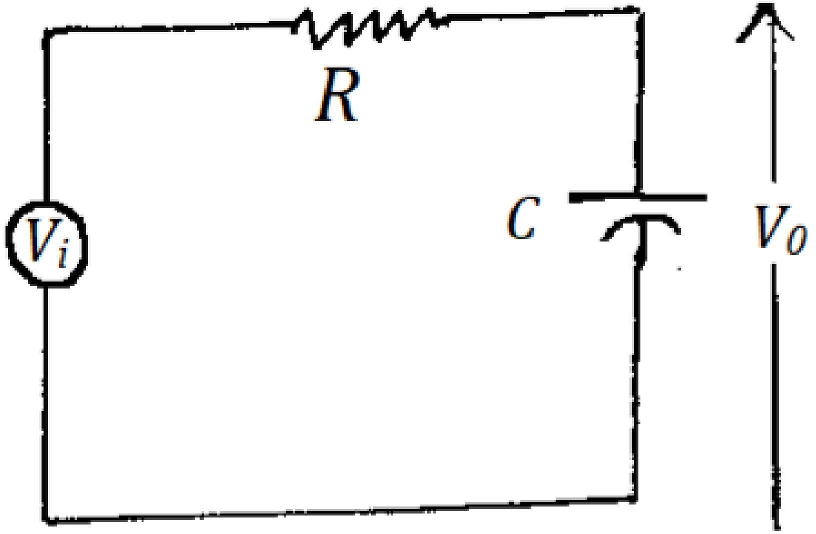
Low Pass RC Circuit

$$V_i(t) = V_R(t) + V_c(t)$$

$$V_i(t) = V_R(t) + V_0(t)$$



$$i_c(t) = C\frac{d}{dt}V_c(t)$$ [As, $V_c = \frac{Q}{C}$ and i = $\frac{Q}{t}$]

$$= C\frac{d}{dt}V_0(t)$$

$$V_i(t) = i(t)R + V_0(t)$$

$$\boxed{\therefore V_i(t) = RC\frac{d}{dt}V_0(t) + V_0\,(t)}$$

$$V_i(t) = RC\frac{d}{dt}V_0(t) + V_0\,(t)$$

1) $\mathcal{L}\left[\frac{d}{dt}f(x)\right] = sf(s) - f(0)$

2) $\mathcal{L}[f(x)] = f(s)$

Now using Laplace transformation

$$V_i(S) = RC\,[S\,V_0(S) - V_0(0)] + V_0\,(S)$$

Here, $V_0(0)$ is the initial capacitor voltage

Now we assume there is no initial capacitor voltage, i.e. $V_0(0) = 0$

$$V_i(S) = SRC\,V_0(S) + V_0\,(S)$$

$$V_i(S) = V_0(S)\,[SRC + 1]$$

$$\boxed{V_0(S) = \frac{V_i(S)}{1 + SRC}}$$

Again, we assume initial capacitor voltage $V_0(0) = V'$, then

$$V_i(S) = RC\,[S\,V_0(S) - V'] + V_0\,(S)$$

$$V_i(S) = SRCV_0(S) - RC\,V' + V_0\,(S)$$

$$V_i(S) = (1 + SRC)\,V_0(S) - RCV'$$

$$V_0(S) = \frac{V_i(S) + RCV'}{1 + SRC} = \frac{V_i(S)}{1+SRC} + \frac{RCV'}{1+SRC}$$

$$\boxed{\therefore V_0\,(S) = \frac{V_i(S)}{1+SRC} + \frac{V'}{S+\frac{1}{RC}}}$$

Step Voltage Input

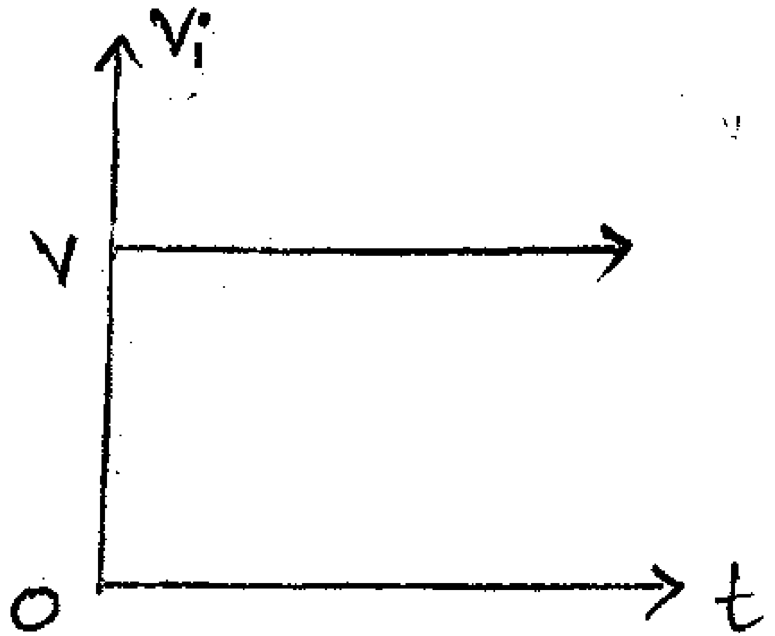
$$V_i(t) = \begin{cases} V, if\ t \geq 0 \\ 0, Otherwise \end{cases}$$

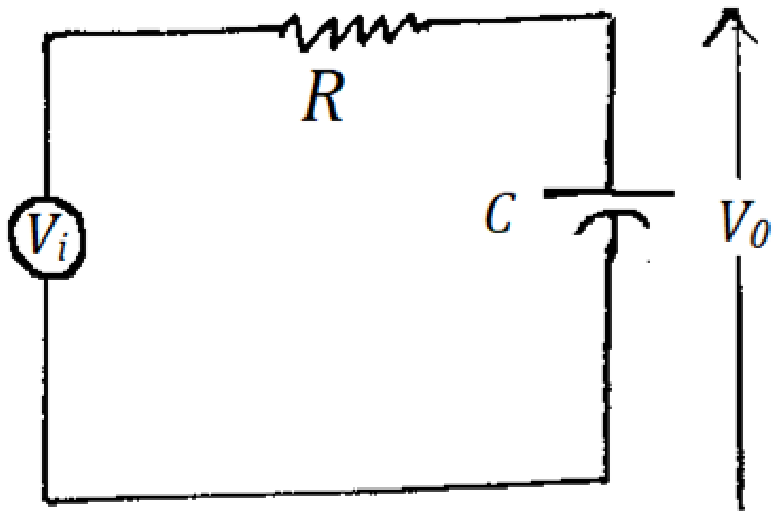
$Now,$

$$V_i(t) = V$$

Now, using Laplace Transformation

$$V_i(S) = \frac{V}{s} \qquad As, \mathcal{L}\{1\} = \frac{1}{S}$$





Now, for low pass RC Circuit, when $V_0(0)$ = 0

$$V_0(S) = \frac{V_i(S)}{1+SRC} \quad = \frac{1}{1+SRC} \cdot \frac{V}{S} \quad = \frac{V}{S(1+SRC)}$$

Using partial fraction,

$$V_0(S) = V\left\{\frac{1}{S} - \frac{RC}{(1+SRC)}\right\} \qquad Hint: \begin{bmatrix} A = 1 \\ B = -\frac{1}{RC} \end{bmatrix}$$

$$= V\left(\frac{1}{S} - \frac{1}{S+\frac{1}{RC}}\right)$$

You must show the partial fraction details in exam

Now, Applying Inverse Laplace transformation

$$V_0(t) = V \cdot \left(1 - \mathrm{e}^{-\frac{t}{Rc}}\right) \qquad \left[\mathcal{L}\left\{\frac{1}{S+a}\right\} = \mathrm{e}^{-at}\right]$$

When $t = 0, V_0(0) =$ V(1-e^0)

=V (1-1)

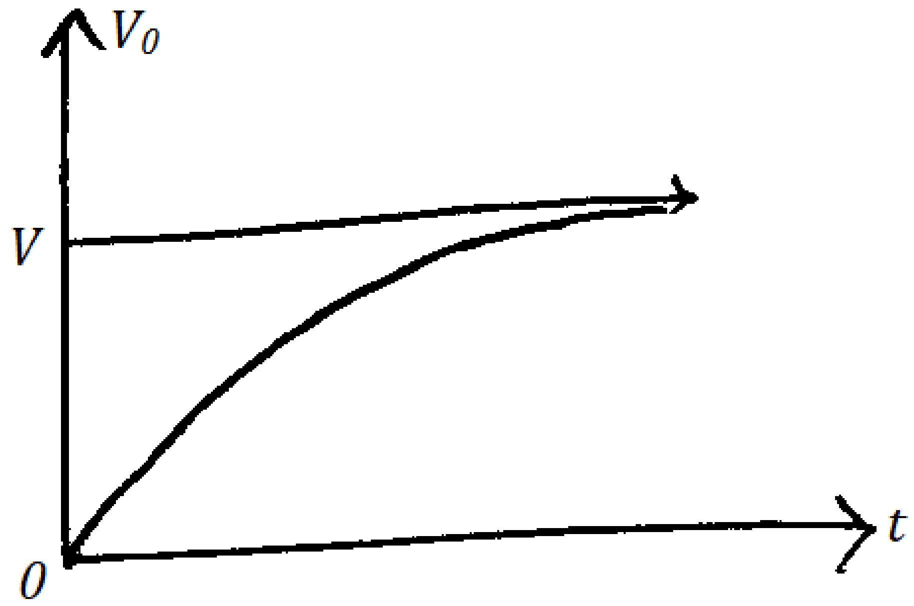
= 0

When $t = \infty, V_0(\infty) =$ V(1-e$^\infty$)

=V (1-0)

= V

Output:



Again, for low pass RC Circuit, when $V_0(0) = V'$

$$V_0(S) = \frac{V_i(S)}{1+SRC} + \frac{V'}{S + \frac{1}{RC}}$$

$$= \frac{V}{S}\,\frac{1}{(1+SRC)} + \frac{V'}{S + \frac{1}{RC}}$$

$$= \text{V}\left\{\frac{1}{S(1+SRC)}\right\} + \frac{V'}{S + \frac{1}{RC}}$$

Using partial fraction,

$$V_0(S) = \mathrm{V}\left(\frac{1}{S} - \frac{1}{S + \frac{1}{RC}}\right) + \frac{V'}{S + \frac{1}{RC}}$$

You must show the partial fraction details in exam

Now, Applying Inverse Laplace transformation

$$V_0(t) = V \cdot \left(1 - \mathrm{e}^{-\frac{t}{Rc}}\right) + V' \cdot \mathrm{e}^{-\frac{t}{Rc}}$$

When $t = 0,$

$$V_0(0) = V \cdot (1 - \mathrm{e}^0) + V' \cdot \mathrm{e}^0$$

$$= \mathrm{V}\,(1\text{-}1) + V' \cdot 1$$

$$= 0 + V'$$

$$\therefore V_0(0) = V'$$

When $t = \infty,$

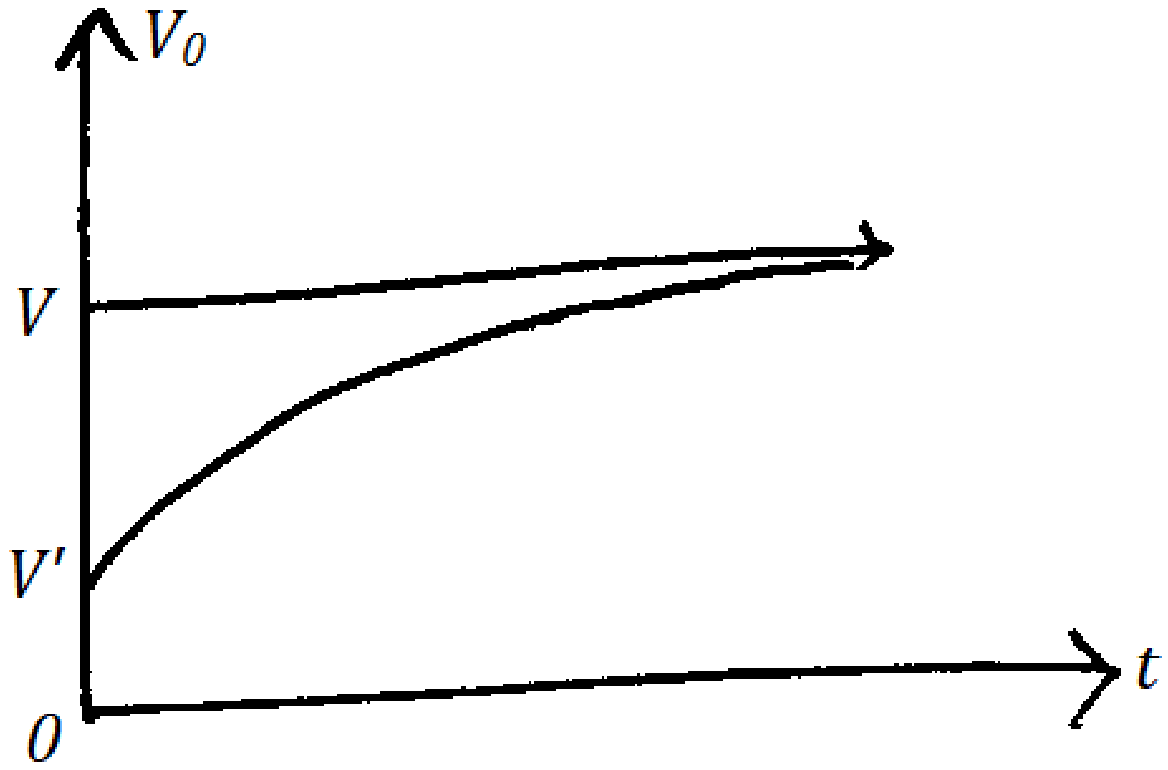
$$V_0(\infty) = V \cdot (1 - \mathrm{e}^\infty) + V' \cdot \mathrm{e}^\infty$$

$$= \mathrm{V}\,(1\text{-}0) + V' \cdot 0$$

$$= \mathrm{V} + 0$$

$$\therefore V_0(\infty) = V$$

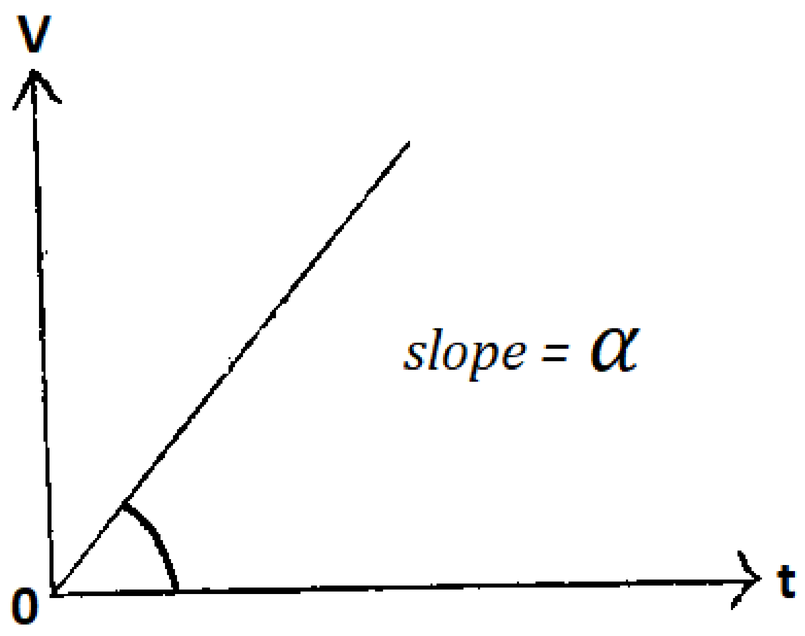
Output:



Ramp Input:

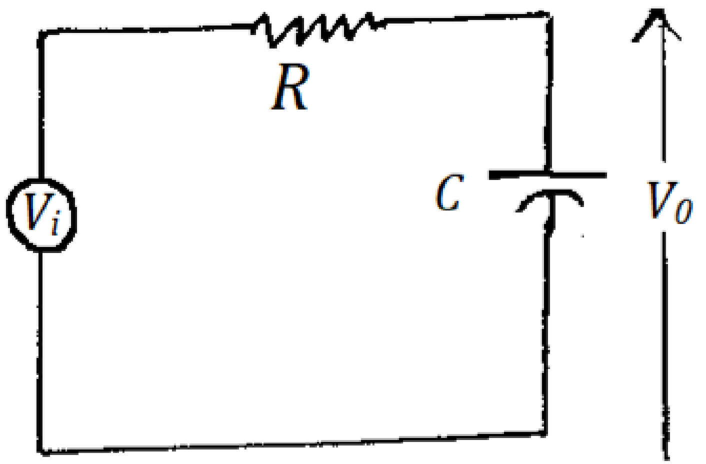
$$V_i(t) = \begin{cases} \alpha t, & if\ t \geq 0 \\ 0, & Otherwise \end{cases}$$

$$V_i(t) = \alpha t$$



Now, using Laplace Transformation

$$\therefore V_i(S) = \frac{\alpha}{S^2} \qquad \left[\mathcal{L}\{t\} = \frac{1}{S^2}\right]$$



We know, for Low Pass RC Circuit

$$V_0(S) = \frac{V_i(S)}{1 + SRC} \quad = \left[\frac{1}{1 + SRC}\right] * \frac{\alpha}{S^2}$$

$$= \alpha \left[\frac{1}{S^2(1 + SRC)}\right]$$

$$= \alpha \left[\frac{\frac{1}{RC}}{\frac{S^2(1 + SRC)}{RC}}\right] \quad = \frac{\alpha}{RC}\left[\frac{1}{S^2\left(S + \frac{1}{RC}\right)}\right]$$

$$V_0(S) = \frac{\alpha}{SRC}\left[\frac{1}{S\left(S+\frac{1}{RC}\right)}\right]$$

Using partial fraction,

$$V_0(S) = \frac{\alpha}{SRC}\left[\frac{RC}{S} - \frac{RC}{S+\frac{1}{RC}}\right] \qquad = \alpha\left[\frac{1}{S^2} - \frac{1}{S\left(S+\frac{1}{RC}\right)}\right]$$

$$V_0(S) = \alpha\left[\frac{1}{S^2} - \frac{1}{S\left(S + \frac{1}{RC}\right)}\right]$$

Again using partial fraction

$$V_0(S) = \alpha\left[\frac{1}{S^2} - \left\{\frac{RC}{S} - \frac{RC}{S + \frac{1}{RC}}\right\}\right]$$

$$= \alpha\left[\frac{1}{S^2} - \frac{RC}{S} + \frac{RC}{S + \frac{1}{RC}}\right]$$

Now, Applying Inverse Laplace transformation

$$V_0(t) = \alpha \cdot \left(\mathrm{t} - \mathrm{RC} + \mathrm{RC}\, \mathrm{e}^{-\frac{t}{Rc}}\right)$$

$$\left[\mathcal{L}\left\{\frac{1}{S+a}\right\} = \mathrm{e}^{-at}\right]$$

$$= \alpha \mathrm{t} - \alpha \mathrm{RC} + \alpha \mathrm{RC}\, \mathrm{e}^{-\frac{t}{Rc}}$$

$$= \alpha \mathrm{t} - \alpha \mathrm{RC}\left(1 - \mathrm{e}^{-\frac{t}{Rc}}\right)$$

Transmission Error

if $\frac{t}{Rc} \gg 1$, then $e^{-\frac{t}{Rc}} = e^{-\frac{\infty}{Rc}} = 0$

$$V_0(t) = \alpha\mathrm{t} \; - \alpha\mathrm{RC}$$

Now transmission error at time t = T

$$e_i(T) = \frac{V_i(T) - V_0(T)}{V_i(T)} \qquad = \frac{\alpha\, T - (\alpha\mathrm{T} - \alpha\mathrm{RC})}{\alpha\, T} = \frac{\alpha\mathrm{RC}}{\alpha\, T}$$

$$\therefore e_i(T) = \frac{RC}{T}$$

Again, if $\frac{t}{Rc} \ll 1$, then

$$V_0(t) = \alpha t - \alpha RC\left(1 - e^{-\frac{t}{Rc}}\right) \qquad \left[e^{-x} = 1 - x + \frac{x^2}{2!} - \cdots \cdots\right]$$

$$= \alpha t - \alpha RC\left\{1 - \left(1 - \frac{t}{Rc} + \frac{t^2}{2!\,(Rc)^2} - \cdots \cdots\right)\right\}$$

$$= \alpha t - \alpha RC\left(\frac{t}{Rc} - \frac{t^2}{2\,R^2c^2}\right)$$ [Neglecting higher terms]

$$= \alpha\left(t - t + \frac{t^2}{2\,RC}\right) \qquad = \frac{\alpha t^2}{2\,RC}$$

Now transmission error at time t = T

$$e_i(T) = \frac{V_i(T) - V_0(T)}{V_i(T)}$$

$$= \frac{\alpha\, T - \frac{\alpha T^2}{2\, RC}}{\alpha\, T}$$

$$= 1 - \frac{\alpha T^2}{\alpha\, T \cdot 2\, RC}$$

$$\therefore e_i(T) = 1 - \frac{T}{2RC}$$

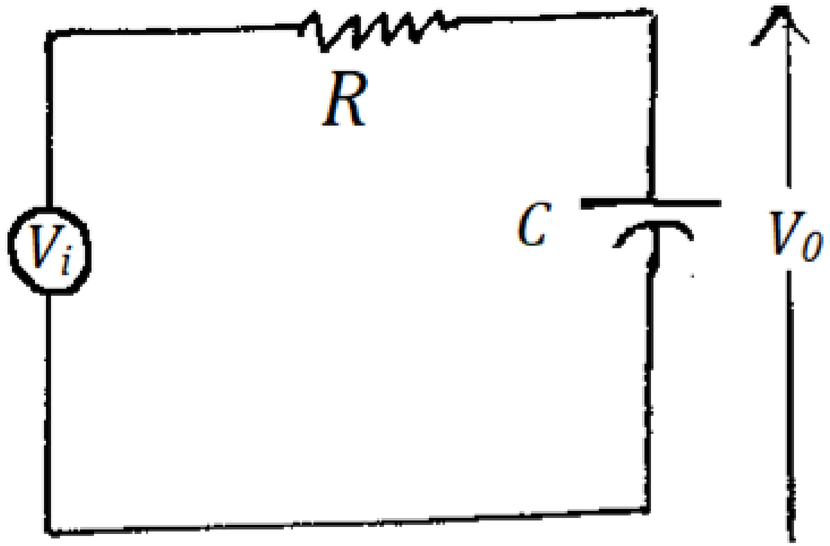
Low Pass RC Circuit as Integrator

If RC is large enough compared to the time required for the input to make an appreciable change then the circuit works like an integrator. Also R>>C

Now,

$$V_i(t) = V_{\dot{c}}(t) + V_R(t)$$

$$V_i(t) \approx V_R(t) \; [\because \text{R>>C}]$$



Again,

$$i_c(t) = C\frac{d}{dt}V_c(t)$$

[As, $V_c = \frac{Q}{C}$ and i = $\frac{Q}{t}$]

$$i\ (t) = \frac{V_R(t)}{R}$$

Now,

$$C\frac{d}{dt}V_c(t) = \frac{V_R(t)}{R}$$

$$\frac{d}{dt}V_c(t) = \frac{1}{RC}V_R(t)$$

$$V_0(t) = \frac{1}{RC} \int V_R(t)\, dt$$

$$V_0(t) = \frac{1}{RC} \int V_i(t)\, dt$$

$$V_0(t) \propto \int V_i(t)\, dt$$

So, the output is proportional to the integration of the input signal.

Now, we need to verify this.

Verification of the Proof that low pass RC circuit works like an integrator.

For Ramp input, $V_i(t) = \alpha t$

Considering R & C are too large, $\frac{t}{Rc} \ll 1$

$$\therefore V_0(t) = \frac{\alpha t^2}{2\,RC} \quad = \frac{\alpha}{RC} \cdot \frac{t^2}{2} \quad = \frac{\alpha}{RC} \int t \cdot dt \quad = \frac{1}{RC} \int \alpha t \cdot dt$$

$$\therefore V_0(t) = \frac{1}{RC} \int V_i(t) \cdot dt \qquad V_0(t) \propto \int V_i(t)\, dt$$

[Verified]

Verification of the Proof that high pass RC circuit works like a differentiator.

For Ramp input, $V_i(t) = \alpha t$

Considering R & C are too small, $\frac{t}{Rc} \gg 1$

$\therefore V_0(t) = \alpha \mathrm{RC} \quad = \alpha \mathrm{RC} \frac{d}{dt}(t) \quad = \mathrm{RC} \frac{d}{dt}(\alpha t) \quad = \mathrm{RC} \frac{d}{dt} V_i(t)$

$\therefore V_0(t) \propto \frac{d}{dt} V_i(t)$

[Verified]

Square Wave Input

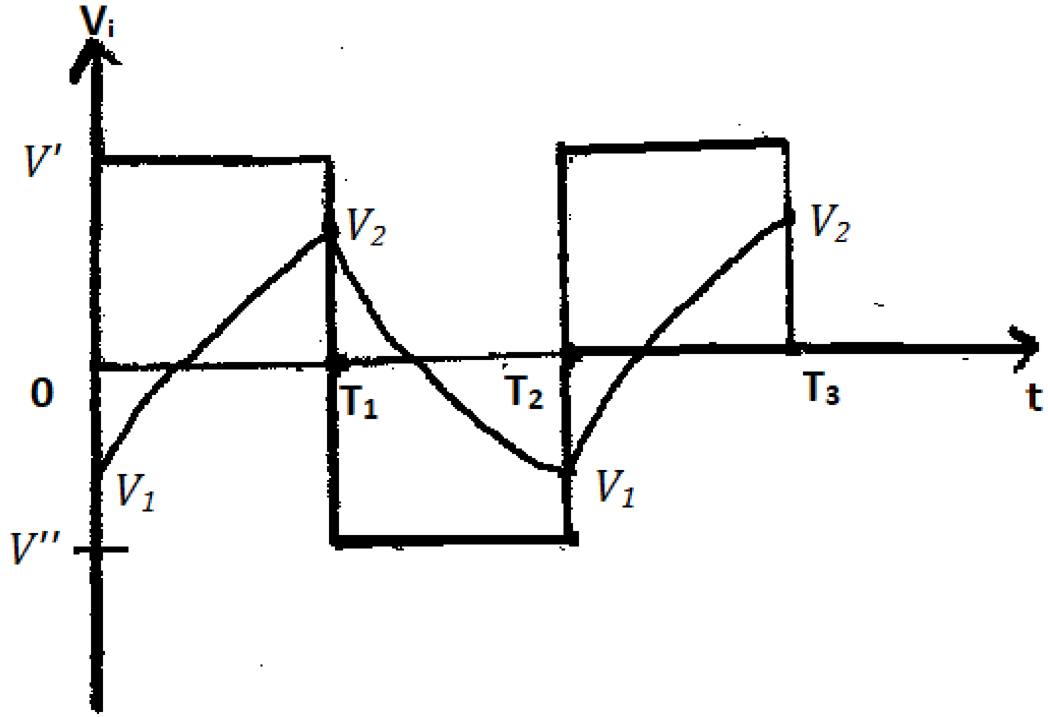
General Facts/Rules:

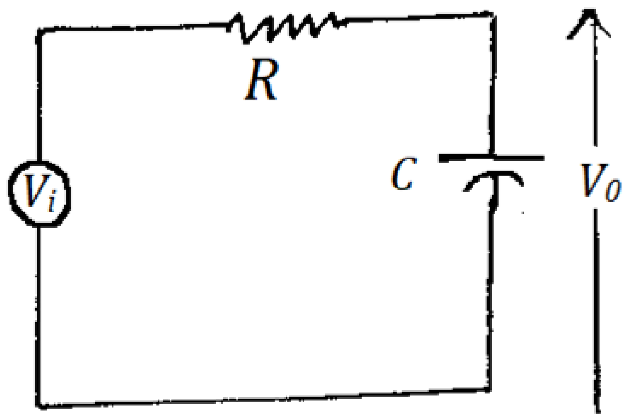
1. At steady state average level of $V_c(t)$ equal to average level of $V_i(t)$.

2. When input remains constant, output $V_c(t)$ exponentially reaches input.

3. $V_c(t)$ will extends on both upper and lower portion of average level.

4. Output will be periodic.



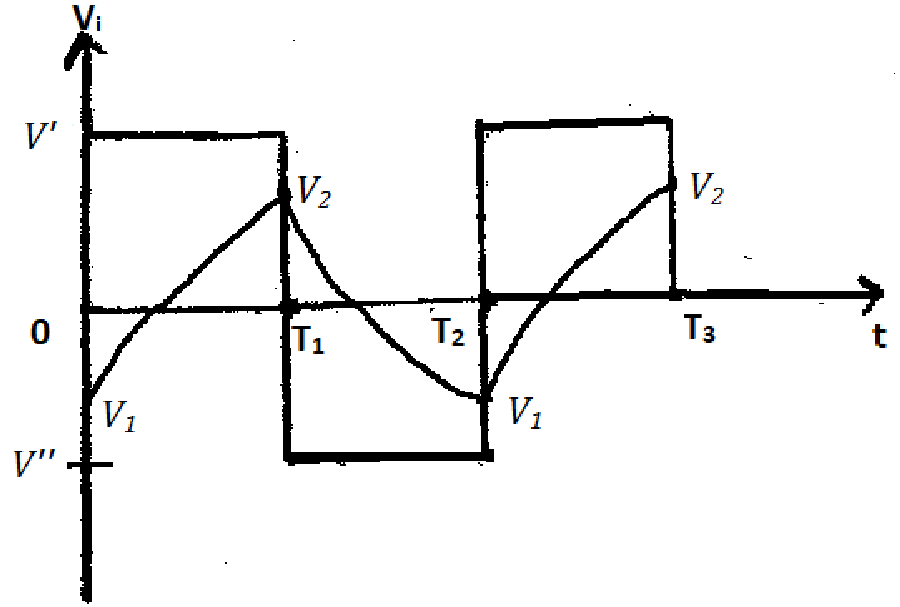


Finding V_1 and V_2

$V_c(t) = V_f + (V_{c_i} - V_f)e^{-\frac{t}{Rc}}$

$V_2 = V' + (V_1 - V')e^{-\frac{T_1}{Rc}}$

$V_1 = V'' + (V_2 - V'')e^{-\frac{T_2}{Rc}}$



Assuming Signal is Symmetric,

$T_1 = T_2 = \frac{T}{2}$ and average level = 0

$$V_1 = V'' + (V_2 - V'')e^{-\frac{T}{2Rc}} \quad \text{——— (i)}$$

$$V_2 = V' + (V_1 - V')e^{-\frac{T}{2Rc}} \quad \text{——— (ii)}$$

Peak to peak value = V

$\therefore V' - V'' = \text{V}$

(i) – (ii),

$$V_1 - V_2 = V'' + (V_2 - V'')e^{-\frac{T}{2Rc}} - \{V' + (V_1 - V')e^{-\frac{T}{2Rc}}\}$$

$$V_1 - V_2 = V'' + (V_2 - V'')e^{-\frac{T}{2Rc}} - V' - (V_1 - V')e^{-\frac{T}{2Rc}}$$

$$V_1 - V_2 = V'' - V' + e^{-\frac{T}{2Rc}}\{(V_2 - V'') - (V_1 - V')\}$$

$$V_1 - V_2 = -V + e^{-\frac{T}{2Rc}}(V_2 - V'' - V_1 + V')$$

$$V_1 - V_2 = -V + e^{-\frac{T}{2Rc}}(V_2 - V_1 + V)$$

$$V_1 - V_2 = -V - e^{-\frac{T}{2Rc}}\{-V + (V_1 - V_2)\}$$

$$V_1 - V_2 = -V + V\,e^{-\frac{T}{2Rc}} - (V_1 - V_2)\,e^{-\frac{T}{2Rc}}$$

$$(V_1 - V_2) + (V_1 - V_2)\,e^{-\frac{T}{2Rc}} = V\,e^{-\frac{T}{2Rc}} - V$$

$$(V_1 - V_2)(\,1 + e^{-\frac{T}{2Rc}}\,) = V(\,e^{-\frac{T}{2Rc}} - 1)$$

$$V_1 - V_2 = \frac{V(\,e^{-\frac{T}{2Rc}} - 1)}{(\,1 + e^{-\frac{T}{2Rc}}\,)}$$

$$V_2 = V_1 - \frac{V(\,e^{-\frac{T}{2Rc}} - 1)}{(\,1 + e^{-\frac{T}{2Rc}}\,)} \qquad \text{———— (iii)}$$

(ii) & (iii),

$$V' + (V_1 - V')e^{-\frac{T}{2Rc}} = V_1 - \frac{V(\,e^{-\frac{T}{2Rc}} - 1)}{(\,1 + e^{-\frac{T}{2Rc}}\,)}$$

$$V' + (V_1 - V')\mathrm{e}^{-\frac{T}{2Rc}} = V_1 + V\,\frac{1 - \mathrm{e}^{-\frac{T}{2Rc}}}{\left(1 + \mathrm{e}^{-\frac{T}{2Rc}}\right)}$$

$$\frac{V}{2} + \left(V_1 - \frac{V}{2}\right)\mathrm{e}^{-\frac{T}{2Rc}} = V_1 + V\,\frac{1 - \mathrm{e}^{-\frac{T}{2Rc}}}{\left(1 + \mathrm{e}^{-\frac{T}{2Rc}}\right)} \qquad \left[\because V' = V'' = \frac{V}{2}\right]$$

$$\frac{V}{2} - \frac{V}{2}\mathrm{e}^{-\frac{T}{2Rc}} + V_1\mathrm{e}^{-\frac{T}{2Rc}} = V_1 + V\,\frac{1 - \mathrm{e}^{-\frac{T}{2Rc}}}{\left(1 + \mathrm{e}^{-\frac{T}{2Rc}}\right)}$$

$$\frac{V}{2}\left(1-\mathrm{e}^{-\frac{T}{2Rc}}\right)+V_1\mathrm{e}^{-\frac{T}{2Rc}}\ -V_1=V\,\frac{1-\mathrm{e}^{-\frac{T}{2Rc}}}{(\,1+\mathrm{e}^{-\frac{T}{2Rc}}\,)}$$

$$\frac{V}{2}\left(1-\mathrm{e}^{-\frac{T}{2Rc}}\right)+V_1(\mathrm{e}^{-\frac{T}{2Rc}}\ -1)=V\,\frac{1-\mathrm{e}^{-\frac{T}{2Rc}}}{(\,1+\mathrm{e}^{-\frac{T}{2Rc}}\,)}$$

$$\frac{V}{2}\left(1-\mathrm{e}^{-\frac{T}{2Rc}}\right)-V_1(1-\mathrm{e}^{-\frac{T}{2Rc}})=V\,\frac{1-\mathrm{e}^{-\frac{T}{2Rc}}}{(\,1+\mathrm{e}^{-\frac{T}{2Rc}}\,)}$$

$$\left(\frac{V}{2} - V_1\right)\left(1 - e^{-\frac{T}{2Rc}}\right) = V \frac{1 - e^{-\frac{T}{2Rc}}}{\left(1 + e^{-\frac{T}{2Rc}}\right)}$$

$$\frac{V}{2} - V_1 = V \frac{1 - e^{-\frac{T}{2Rc}}}{\left(1 - e^{-\frac{T}{2Rc}}\right)\left(1 + e^{-\frac{T}{2Rc}}\right)}$$

$$\frac{V}{2} - V_1 = \frac{V}{\left(1 + e^{-\frac{T}{2Rc}}\right)}$$

$$V_1 = \frac{V}{2} - \frac{V}{(1 + e^{-\frac{T}{2Rc}})}$$

$$V_1 = \frac{V}{2}\left(1 - \frac{2}{1 + e^{-\frac{T}{2Rc}}}\right)$$

$$V_1 = \frac{V}{2}\left(\frac{1 + e^{-\frac{T}{2Rc}} - 2}{1 + e^{-\frac{T}{2Rc}}}\right) = \frac{V}{2}\left(\frac{e^{-\frac{T}{2Rc}} - 1}{1 + e^{-\frac{T}{2Rc}}}\right)$$

$$V_1 = -\frac{V}{2}\left(\frac{1 - \mathrm{e}^{-\frac{T}{2Rc}}}{1 + \mathrm{e}^{-\frac{T}{2Rc}}}\right)$$

Since the signal is symmetric, $V_2 = -V_1$

$$V_2 = \frac{V}{2}\left(\frac{1 - \mathrm{e}^{-\frac{T}{2Rc}}}{1 + \mathrm{e}^{-\frac{T}{2Rc}}}\right)$$

Example:

For low pass RC circuit and symmetric wave input, given $p^2p = 2V$(peak to peak), average level = 0, time constant (RC) = $\frac{T}{2}$ Find output p^2p.

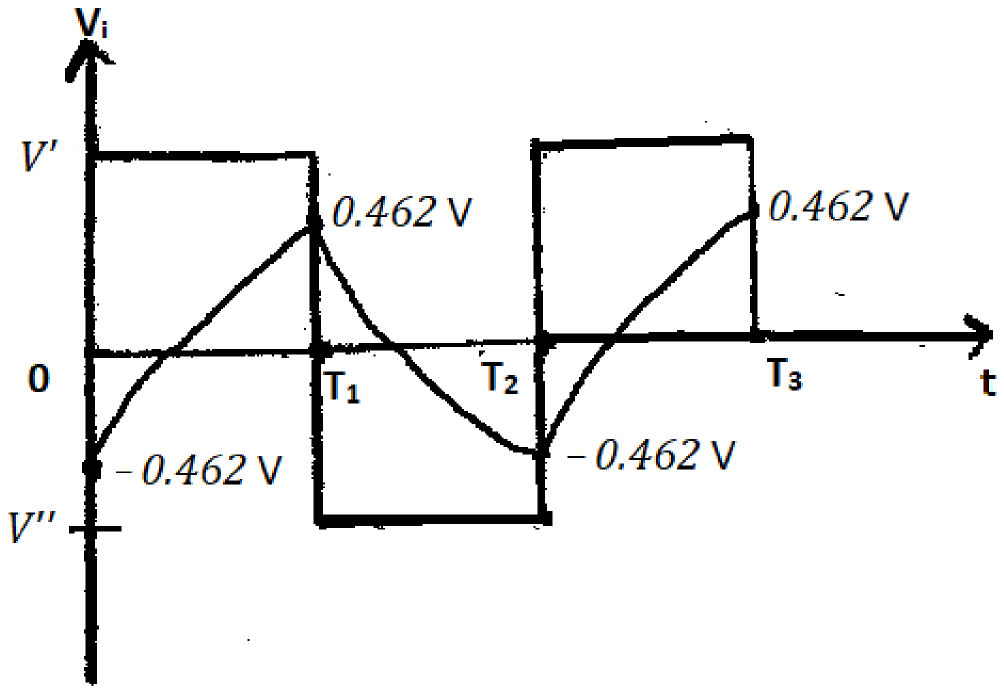
Solution:

$$V_1 = -\frac{V}{2}\left(\frac{1 - e^{-\frac{T}{2Rc}}}{1 + e^{-\frac{T}{2Rc}}}\right)$$

$$V_1 = -\frac{2}{2}\left(\frac{1 - e^{-\frac{T}{2\frac{T}{2}}}}{1 + e^{-\frac{T}{2\frac{T}{2}}}}\right) = -\left(\frac{1 - e^{-1}}{1 + e^{-1}}\right) = -0.462 \text{ V}$$

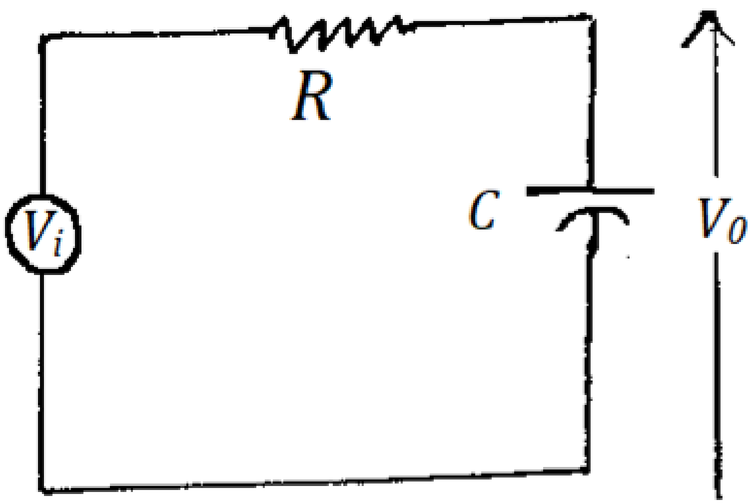
$$\therefore V_2 = 0.462 \text{ V}$$

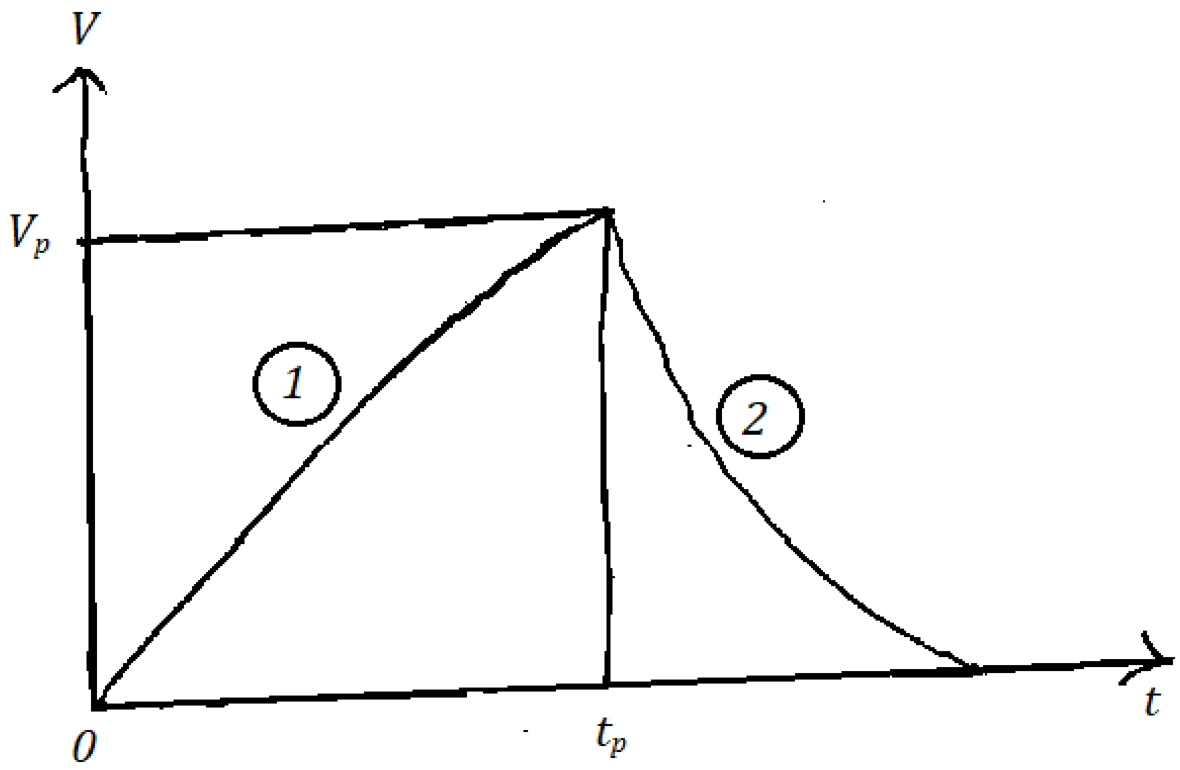
$$\therefore Output\ \ p^2p = 2 * 0.462 \text{ V}$$
$$= 0.924\ V$$



Pulse Input

$$V_i(t) = \begin{cases} V, & 0 \le t \le t_p \\ 0, & Otherwise \end{cases}$$





$$V_0 = V_f + (V_{c_i} - V_f)e^{-\frac{t}{Rc}}$$

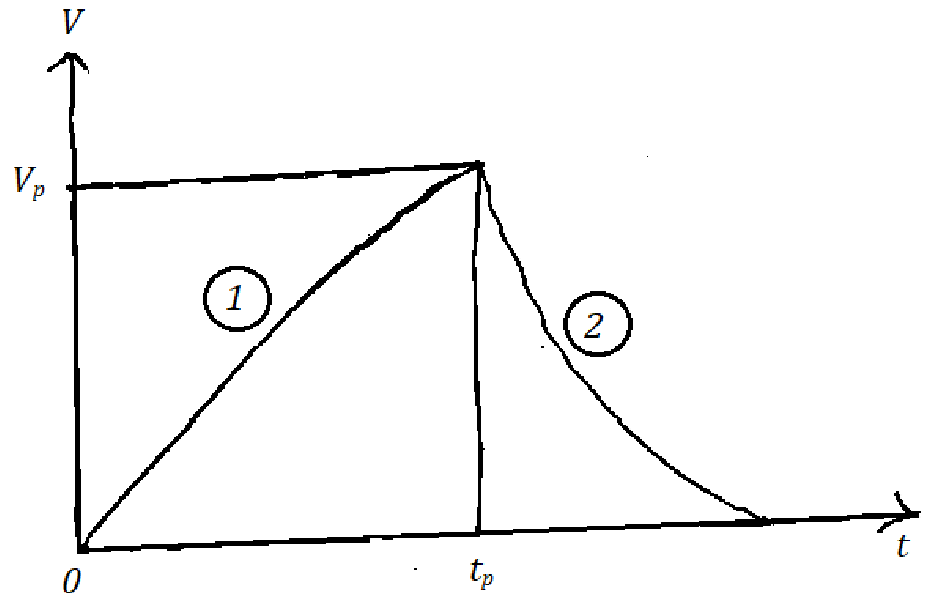
$For\ curve$ 1,

$$V_f = V_p$$

$$V_{c_i} = 0$$

$$V_{0_1}(t) = V_p + (0 - V_p)e^{-\frac{t}{Rc}}$$

$$V_{0_1}(t) = V_p\ (1 - e^{-\frac{t}{Rc}})$$

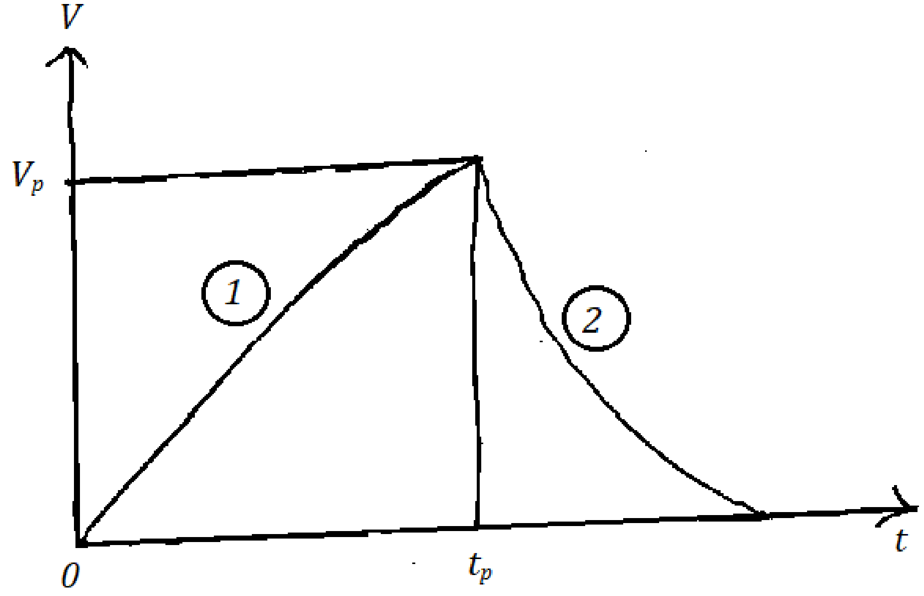


$For\ curve\ 2,$

$V_f = 0$ and, $V_{c_i} = V_p$

$\therefore V_{0_2}(\mathrm{t}) = 0 + (V_p - 0)\mathrm{e}^{-\frac{t}{Rc}}$

$V_{0_2}(\mathrm{t}) = V_p\ \mathrm{e}^{-\frac{t}{Rc}}$



$$\therefore V_0\ (\mathrm{t}) = \begin{cases} V_p\ (1 - \mathrm{e}^{-\frac{t}{Rc}}),\ 0 \le t \le t_p \\ V_p\ \mathrm{e}^{-\frac{t}{Rc}},\ Otherwise \end{cases}$$