Fourier Series

Periodic Function :

If $f(t) = f(t+T) = f(t+2T) = \cdots\cdots$, then $f(t)$ is called the periodic function of period T. As for example,

$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \cdots\cdots$

So $\sin x$ is a periodic function with period 2π.

Fourier Series:

A series of sines and cosines of an angle and its multiples of the form :

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots\cdots + a_n \cos nx + \cdots\cdots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots\cdots + b_n \sin nx$$

$= \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx + \sum\limits_{n=1}^{\infty} b_n \sin nx$, is called the Fourier Series, where $a_0, a_1, a_2, \cdots\cdots, a_n, \cdots, b_1, b_2, \cdots, b_n$ are constants.

Here $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\,dx$, or, $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ | $\pi-(-\pi)=2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx,$$

and $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$,

by taking $n = 1, 2, \ldots\ldots$ we can find the values of $a_1, a_2, \ldots,$ $b_1, b_2, \ldots\ldots$

Some useful integrals:

(i) $\int_0^{2\pi} \sin nx \, dx = 0$, (ii) $\int_0^{2\pi} \cos nx \, dx = 0$,

(iii) $\int_0^{2\pi} \sin^2 nx \, dx = \pi$, (iv) $\int_0^{2\pi} \cos^2 nx \, dx = \pi$,

(v) $\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0$, (vi) $\int_0^{2\pi} \cos nx \cdot \cos mx \, dx = 0$,

(vii) $\int_0^{2\pi} \sin nx \cdot \cos mx \, dx = 0$, (viii) $\int_0^{2\pi} \sin nx \cdot \cos nx \, dx = 0$,

(ix) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ where $n \in I$.

(x) $[u \cdot v]_1 = u \cdot v_1 - u' \cdot v_2 + u'' \cdot v_3 - u''' \cdot v_4 + \ldots\ldots\ldots$, where

$[u \cdot v]_1 = \int uv \, dx$, $v_1 = \int v \, dx$, $v_2 = \int v_1 \, dx$ and so on.

$u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on.

Example: Find the Fourier series of $f(x) = x,\ 0 < x < 2\pi$ and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Sol^n: Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x + \cdots\cdots$ (1)

Here, $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$

$$= \frac{1}{\pi} \int_0^{2\pi} x\, dx = \frac{1}{\pi} \left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi} \left[x^2\right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[4\pi^2 - 0\right] = 2\pi.$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \cos nx\, dx$$

$$= \frac{1}{\pi} \left[x \cdot \frac{\sin nx}{n} - 1 \cdot \left(-\frac{\cos nx}{n^2}\right) + 0\right]_0^{2\pi}$$

$$\left[\left[u \cdot v\right]_1 = u \cdot v_1 - u' \cdot v_2 + u'' \cdot v_3 - \cdots\cdots\right]$$

$$= \frac{1}{\pi} \left[x \cdot \frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\left\{2\pi \cdot \frac{\sin 2n\pi}{n} + \frac{\cos 2n\pi}{n^2}\right\} - \left\{0 + \frac{1}{n^2}\right\}\right]$$

$$= \frac{1}{\pi} \left[0 + \frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi} \left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0.$$

Again, $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\cdot\left(-\frac{\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right) + 0\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x\cdot\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-2\pi\cdot\frac{\cos 2n\pi}{n} + \frac{\sin 2n\pi}{n^2}\right\} - \{0+0\}\right]$$

$$= \frac{1}{\pi}\left[-2\pi\cdot\frac{1}{n} + 0\right] = -\frac{2}{n}.$$

Now substituting the values of a_0, a_n and b_n in (1), we get

$$x = \frac{2\pi}{2} - \frac{2}{1}\sin x - \frac{2}{2}\sin 2x - \frac{2}{3}\sin 3x - \cdots\cdots$$

$$= \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\cdots\right]$$ (Ans.)

The graph of $f(x)$ from $x=-4\pi$ to $x=4\pi$ is given below: