Divergence of a vector function:

The divergence of a vector point function $\vec{F}$ is denoted by $\text{div}\vec{F}$ and is defined as below:

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$, then

$$\text{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

It is evident that $\text{div}\vec{F}$ is a scalar function.

<u>Note:</u> If $\text{div}\vec{F} = 0$, then $\vec{F}$ is called a solenoidal vector function.

The equation $\text{div}\vec{F} = 0$

$$\Rightarrow \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$ is also called the equation of continuity or conservation of mass.

Q1. Examine whether the vector field represented by

$\vec{F} = (z^2+2x+3y)\hat{i} + (3x+2y+z)\hat{j} + (y+2zx)\hat{k}$ is solenoidal or not.

Sol^n: $\text{div}\,\vec{F} = \nabla\cdot\vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(z^2+2x+3y)\hat{i} + (3x+2y+z)\hat{j} + (y+2zx)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(z^2+2x+3y) + \frac{\partial}{\partial y}(3x+2y+z) + \frac{\partial}{\partial z}(y+2zx)$$

$$= (0+2+0) + (0+2+0) + (0+2x)$$

$$= 2+2+2x$$

$$= 4+2x \neq 0$$

Since $\text{div}\,\vec{F} \neq 0$, so $\vec{F}$ is not solenoidal.

Q2. If $u = x^2+y^2+z^2$, and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $div(u\vec{r})$ in terms of u.

Sol^n: $div(u\vec{r}) = \nabla(u\vec{r})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k})\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[x(x^2+y^2+z^2)\hat{i} + y(x^2+y^2+z^2)\hat{j} + z(x^2+y^2+z^2)\hat{k}\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[(x^3+xy^2+xz^2)\hat{i} + (x^2y+y^3+z^2y)\hat{j} + (x^2z+y^2z+z^3)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x^3+xy^2+xz^2) + \frac{\partial}{\partial y}(x^2y+y^3+z^2y) + \frac{\partial}{\partial z}(x^2z+y^2z+z^3)$$

$$= (3x^2+y^2+z^2) + (x^2+3y^2+z^2) + (x^2+y^2+3z^2)$$

$$= (3x^2+x^2+x^2) + (y^2+3y^2+y^2) + (z^2+z^2+3z^2)$$

$$= 5x^2+5y^2+5z^2$$

$$= 5(x^2+y^2+z^2)$$

$$= 5u \quad [\text{since } u = x^2+y^2+z^2]$$

(Ans)