Determinates and Crammer Rule

Determinants

Every square matrix A is associated with a number called its *determinant*, which is denoted by | D |. The determinate can be illustrated for a set of three equations: [A]{X} = {B}

Where [A] is the coefficient matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The value of 2nd order determinant: $| D | = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$.

Similarly, The value of 3rd order determinant:

$$| D | = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} + a_{11}a_{32}a_{23} + a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}$$

For large matrices, the determinant is much more difficult to define and compute manually. In general, for n x n matrix, the determinant will contain a sum of n! signed product terms, each having n elements. [See page-508, Balagurusamy]

- For a *triangular matrix* (in which all the elements below (or above) the diagonal are zero), the determinant is the product of the diagonal elements.
- If | A | = 0, then A said to be a *singular matrix*; otherwise, it is said to be *nonsingular*.

Minors and Cofactors

The *minor* of a particular element is the determinant, after the row and the column which contain the element have been deleted.

Example: Consider the following 3* 3 matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The Minor entry M_{11} is the determinate of the 2 * 2 matrix after deleting 1st row and column. So,

$M_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = 5 * 9 - 6 * 8 = 45 - 48 = -3$

$M_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = 1 * 8 - 2 * 7 = 8 - 14 = -6$

Similarly,

$M_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = 1 * 5 - 2 * 4 = 5 - 8 = -3$

The *cofactor* of an element is its minor with a sign attached. Once you find a minor $M_{i,j}$, you take the subscript on the name of the minor (the "*i, j*" part) and add the two numbers *i* and *j*. Whatever result you get from this addition, make this value the power on –1, so you get "+1" or "–1", depending on whether $i + j$ is even or odd. If i+j is even the sign is + and if odd then the sign is –
The cofactor d_{ij} of an element a_{ij} is given by $d_{ij} = (-1)^{i+j} M_{ij}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For example $d_{12} = -M_{12}$. Of course if you forget, you can always use the formula $d_{ij} = (-1)^{i+j} M_{ij}$, for example $d_{12}=(-1)^{1+2}M_{ij} = (-1)^3 M_{ij} = -M_{ij}$
Here,

$M_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} = 4 * 9 - 6 * 7 = 36 - 42 = -3$

So, $d_{12} = -M_{12} = 3$

Adjoint Matrix

If d_{ij} is the cofactor of the element a_{ij} of the square matrix A, then, by definition, the *adjoint matrix* of A is given by adj (A) = D^T

Example:

Consider the following 3* 3 matrix

$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Calculate the cofactors:

$d_{11} = +\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} = 5 * 9 - 6 * 8 = 45 - 48 = -3$

$d_{12} = -\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} = 4 * 9 - 6 * 7 = 36 - 42 = 6$

$d_{13} = +\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} = 4 * 8 - 5 * 7 = 32 - 35 = -3$

$d_{21} = -\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} = 2 * 9 - 3 * 8 = 18 - 24 = -3$

$d_{22} = +\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} = 1 * 9 - 3 * 7 = 9 - 21 = -12$

$d_{23} = -\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = 1 * 8 - 2 * 7 = 8 - 14 = 6$

$d_{31} = +\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} = 2 * 6 - 3 * 5 = 12 - 15 = -3$

$d_{32} = -\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = 1 * 6 - 3 * 4 = 6 - 12 = 6$

$d_{33} = +\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = 1 * 5 - 2 * 4 = 5 - 8 = -3$

The cofactors of the matrix = $d_{ij} = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$

Adjoint matrix of A is transpose matrix of cofactor matrix of A that is $[\boldsymbol{d}_{ij}]^T$

$$\text{Adj}A = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}^T = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Inverse Matrix

If B and C are two n * n square matrices such that BC = CB = I (identity matrix) then, B is called the *inverse* of C and C is the inverse of B. The common notation for inverses is B^{-1} and C^{-1}. That is, $B^{-1}B = I$ and $C^{-1}C = I$.

$B^{-1} = \frac{1}{\det B}(Adj\ B) = \frac{1}{\det B}(cofactor\ matrix\ of\ B)^T$

Example:

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

Cofactor matrix for B = $\begin{bmatrix} \mathbf{24} & \mathbf{5} & \mathbf{-4} \\ \mathbf{-12} & \mathbf{3} & \mathbf{2} \\ \mathbf{-2} & \mathbf{-5} & \mathbf{4} \end{bmatrix}$

Adj B = $\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$

detB = 22

$$B^{-1} = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

Cramer's Rule

A linear system of n equations in n unknowns can be represented in matrix form as AX = B, where A, X and B are matrices and are given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & \dots & \dots & a_{nn} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$$

A is called *coefficient matrix* and B is known as *constant vector*. X is the required solution and, therefore, it is called the *solution vector*.

According to Cramer's rule, the solution can be given in terms of determinants, i.e.

$X_i = \frac{\det A_i}{\det A}$
where A_i is the matrix obtained from A by replacing the *i*th column with the vector B.

- Obviously, Cramer's rule only works for det A ≠ 0. One can show that the system has a unique solution if and only if det A ≠ 0. Otherwise, the system has either no solution or infinitely many solutions.
- One would not use Cramer's rule to solve a large system of linear equations, simply because calculating a single determinant is as time consuming as solving the complete system by a more efficient algorithm.

Example: Solve the following system of equations using Cramer's rule:

$$2x_1 + 3x_2 = 12$$
$$4x_1 - x_2 = 10$$

Solution:

$$x_1 = \frac{\begin{vmatrix} 12 & 3 \\ 10 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}} = \frac{-12 - 30}{-2 - 12} = \frac{42}{14} = 3$$

$$x_2 = \frac{\begin{vmatrix} 2 & 12 \\ 4 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}} = \frac{20 - 48}{-2 - 12} = \frac{28}{14} = 2$$

References:

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3. Rao, G. Shankar. Numerical Analysis. New Age International Publisher, 2002. 3rd edition