Inverse Matrix:

Ex.1 Find the inverse of the following matrix by using row canonical form: $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$.

Solⁿ: $[A\,I_3] = \left[\begin{array}{ccc:ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$ Interchange first and second rows.

$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$ We multiply first row by 3 and 2 and then subtract from the second and third rows respectively.

$R_2' = R_2 - 3R_1$ and $R_3' = R_3 - 2R_1$

$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$ Subtract third row from the second row.

$R_2' = R_2 - R_3$

$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & -1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$ Multiply the second row by (-1).

$R_2' = -R_2$

$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right]$ We multiply second row by 5 and then subtract from the third row.

$R_3' = R_3 - 5R_2$

$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & -10 & 5 & -7 & -4 \end{array}\right]$ We multiply third row by $\left(-\frac{1}{10}\right)$

$R_3' = -\frac{1}{10}R_3$

$1 - \frac{}{10} = - \frac{}{10}$

$R_1' = R_1 - 3R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right]$$

We multiply third row by 3 and then subtract from the first row.

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{array}\right] = \left[I_3 \; A^{-1}\right]$$

Hence A is invertible and $A^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$.

Exercise 1: Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ -6 & 0 & 1 & -2 \\ 8 & 1 & -2 & 1 \end{bmatrix}$

by using only row transformations to reduce A to I_4.

Exercise 2: Prove that the matrix

$$A = \frac{1}{6}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & -5 & 1 & 1 \\ 3 & 1 & 1 & -5 \\ 3 & 1 & -5 & 1 \end{bmatrix}$$

is orthogonal.

[Orthogonal matrix: A real ~~matrix~~ square matrix A is said to be orthogonal if $AA^T = A^TA = I$].

Ex.3 Find the inverse of the matrix $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ by using row canonical form.

Sol^n: $[A\ I_2] = \left[\begin{array}{cc:cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$

Interchange first and second rows,

$\sim \left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right]$

Apply $R_2' = R_2 - 2R_1$, $\sim \left[\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$

Now apply, $R_1' = R_1 + 3R_2$, $\sim \left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & -1 & 1 & -2 \end{array}\right]$

Now, apply $R_2' = -R_2$, $\sim \left[\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right] = [I_2\ A^{-1}]$

Hence A is invertible and $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.