Curl of a vector function

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The curl of a vector point function $\vec{F}$ is defined as below:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} \qquad \left(\text{where } \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

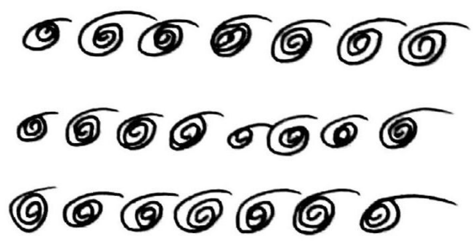
$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

So, curl $\vec{F}$ is a vector quantity.

Note:

1. If curl $\vec{v} \neq \vec{0}$, then $\vec{v}$ is rotational.

Flow $\Rightarrow$



2. If curl $\vec{v} = \vec{0}$, then $\vec{v}$ is irrotational.

Flow $\Rightarrow$

Proof: Show that gradient field describing a motion is irrotational.

Sol^n: Let a field be $f(x, y, z)$.

$\therefore$ Gradient of $f(x,y,z) = \nabla f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f$$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Now curl of Gradient $f(x, y, z) = \nabla \times \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\cdot\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\cdot\frac{\partial f}{\partial y}\right) - \hat{j}\left(\frac{\partial}{\partial x}\cdot\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\cdot\frac{\partial f}{\partial x}\right) + \hat{k}\left(\frac{\partial}{\partial x}\cdot\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\cdot\frac{\partial f}{\partial x}\right)$$

$$= \hat{i}\left(\frac{\partial^2 f}{\partial y\cdot\partial z} - \frac{\partial^2 f}{\partial z\cdot\partial y}\right) - \hat{j}\left(\frac{\partial^2 f}{\partial x\cdot\partial z} - \frac{\partial^2 f}{\partial z\cdot\partial x}\right) + \hat{k}\left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y.\partial x}\right)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k} = \vec{0}$$

Hence, gradient field describing a motion is irrotational.

[Proved]

Ex. Prove that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal and irrotational.

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Proof: For solenoidal, we have to prove $\nabla \cdot \vec{F} = 0$

Now $\nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right]$

$$= \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= (0 - 0 + 0 - 2) + (0 + 2x) + (0 - 2x + 2)$$

$$= -2 + 2x - 2x + 2$$

$$= 0.$$

Since, $\nabla \cdot \vec{F} = 0$, so $\vec{F}$ is solenoidal vector function.

[Proved]

Given $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$.

To prove $\vec{F}$ is irrotational, we have to show that Curl $\vec{F} = \vec{0}$.

Now curl $\vec{F} = \nabla \times \vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right]$$

$$= \hat{i}\left[(3x - 0 + 0) - (3x + 0)\right] - \hat{j}\left[(3y - 2z + 0) - (0 - 2z + 3y - 0)\right] + \hat{k}\left[(3z + 2y) - (2y - 0 + 3z - 0)\right]$$

$$= \hat{i}(3x - 3x) - \hat{j}(3y - 2z + 2z - 3y) + \hat{k}(3z + 2y - 2y - 3z)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

Since, curl $\vec{F} = \vec{0}$, so $\vec{F}$ is irrotational. [Proved]