Ex: Find the fourier cosine transform of e^{-x^2}.

Sol^n: The fourier cosine transform of a function $f(x) = e^{-x^2}$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cos sx \, dx = I \text{ (say)} \quad \text{------} (1)$$

$$\therefore \frac{dI}{ds} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^\infty (-2xe^{-x^2}) \sin sx \, dx$$

let $-x^2 = z$
$\therefore -2x\,dx = dz$
$\int e^z dz = e^z$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \left[e^{-x^2} \cdot \sin sx \Big|_0^\infty - \int_0^\infty s e^{-x^2} \cos sx \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[(e^{-\infty^2} \cdot \sin s\infty - e^0 \cdot \sin 0) - \int_0^\infty s e^{-x^2} \cos sx \, dx \right]$$

$$= -\frac{1}{\sqrt{2\pi}} \cdot s \int_0^\infty e^{-x^2} \cos sx \, dx$$

$$= -\frac{s}{\sqrt{2\pi}} I \; [\text{from } (1)]$$

$$\Rightarrow \frac{dI}{I} = -\frac{1}{\sqrt{2\pi}} s \, ds$$

$$= -\frac{s}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{2}} I$$

$$\Rightarrow \frac{dI}{I} = -\frac{s}{2}$$

Now, integrating both sides we get

$$\ln I = -\frac{1}{\sqrt{2\pi}} \cdot \frac{s^2}{2} + \ln A$$

$$\therefore I = A \exp\left(-\frac{s^2}{2\sqrt{2\pi}}\right) \quad \text{------} (2)$$

$$\Rightarrow \ln I = -\frac{1}{2} \cdot \frac{s^2}{2} + \ln A$$

$$\therefore I = A \exp\left(-\frac{s^2}{4}\right) \cdots (2)$$

When $s = 0$, then $I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} \cdot \cos 0 \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-z} \frac{dz}{2\sqrt{z}}$$

Let $x^2 = z$
$\therefore 2x\,dx = dz$
$\therefore dx = \frac{dz}{2\sqrt{z}}$

x	0	∞
z	0	∞

$$I = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^\infty e^{-z} z^{-\frac{1}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_0^\infty e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \Gamma\left(\frac{1}{2}\right) \quad \left[\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, n > 0\right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2}}$$

Again, for $s = 0$, (2) gives, $I = A$.

$$\therefore A = \frac{1}{\sqrt{2}}$$

Hence, from (2), $I = \frac{1}{\sqrt{2}} \exp\left(-\frac{s^2}{4}\right)$

$$= \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}. \quad (Ans)$$