$$|z+w|^2 = (z+w)\overline{(z+w)} \qquad |z|^2 = z\bar{z}$$

$$= (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + z\bar{w} + \cancel{w\bar{z}}\,\bar{z}w + w\bar{w}$$

$$= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2$$

$$= |z|^2 + 2\,\mathrm{Re}\,(z\bar{w}) + |w|^2 \qquad z+\bar{z} = 2\,\mathrm{Re}(z)$$

$$\cancel{\leq |z|^2 + 2|z||\bar{w}|}$$

$$\Rightarrow |z+w|^2 \leq |z|^2 + 2|z\bar{w}| + |w|^2 \qquad \mathrm{Re}(z) \leq |z|$$

$$\Rightarrow |z+w|^2 \leq |z|^2 + 2|z||\bar{w}| + |w|^2 \qquad |zw| = |z||w|$$

$$\Rightarrow |z+w|^2 \leq |z|^2 + 2|z||w| + |w|^2 \qquad |\bar{z}| = |z|$$

$$\Rightarrow |z+w|^2 \leq (|z|+|w|)^2$$

$$\Rightarrow |z+w| \leq |z| + |w| \quad \text{—— } 1$$

$$|z-w|^2 = (z-w)\overline{(z-w)}$$

$$= (z-w)(\bar{z}-\bar{w})$$

$$= (z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w})$$

$$= |z|^2 - z\bar{w} - \overline{z\bar{w}} + |w|^2$$

$$= |z|^2 - (z\bar{w} + \overline{z\bar{w}}) + |w|^2$$

$$= |z|^2 - 2\mathrm{Re}(z\bar{w}) + |w|^2$$

$$|z-w|^2 \geq |z|^2 - 2|z\bar{w}| + |w|^2$$

$$|z-w|^2 \geq |z|^2 - 2|z||\bar{w}| + |w|^2$$

$$\geq |z|^2 - 2|z||w| + |w|^2$$

$$\geq \cancel{(||z|-|w||)^2} \; (|z|-|w|)^2 = ||z|-|w||^2$$

[$a^2 = |a|^2$ $a \in \mathbb{R}$

$$\Rightarrow |z-w| \geq ||z|-|w|| \quad \text{——— (i)}$$

Again

$$||z|-|w|| \geq |z| - |w| \quad \text{——— (ii)}.$$

Now, from (i) and (ii)

$$|z-w| \geq ||z|-|w|| \geq |z|-|w|.$$

let, $z = x+iy$ $\qquad Re(z) = x, \quad Im(z) = y$

$|z| = \sqrt{x^2+y^2}$ $\qquad |Re(z)| = |x|, \quad |Im(z)| = |y|$

we can write,

$$(|x| - |y|)^2 \geq 0$$

$$\Rightarrow |x|^2 - 2|x||y| + |y|^2 \geq 0$$

$$\Rightarrow |x|^2 + |y|^2 \geq 2|x||y|$$

$$\Rightarrow |x|^2 + |y|^2 + |x|^2 + |y|^2 \geq 2|x||y| + |x|^2 + |y|^2$$

$$\Rightarrow 2(|x|^2 + |y|^2) \geq (|x| + |y|)^2$$

$$\Rightarrow \sqrt{2}\,|z| \geq |x| + |y|$$

$$\Rightarrow \sqrt{2}\,|z| \geq |Re(z)| + |Im(z)|$$

$\boxed{|x|^2 = x^2 \text{ [illegible] } x \in \mathbb{R}.}$

If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ then show

i) $$\left|\frac{z_1}{z_2+z_3}\right| \leq \frac{|z_1|}{\big||z_2|-|z_3|\big|} \; ; \quad |z_2| \neq |z_3|$$

ii) $$\left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|} \; ; \quad |z_3| \neq |z_4|$$

<u>Sol^n:</u> We know that,

$$|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$$

Replacing z_2 by $-z_2$, we get

$$|z_1+z_2| \geq \big||z_1| - |z_2|\big| \quad \text{——— (i)}$$

from (i) we can write

$$|z_2+z_3| \geq \big||z_2| - |z_3|\big|$$

$$\Rightarrow \frac{1}{|z_2+z_3|} \leq \frac{1}{\big||z_2|-|z_3|\big|}$$

$$\Rightarrow \frac{|z_1|}{|z_2+z_3|} \leq \frac{|z_1|}{\big||z_2|-|z_3|\big|}$$

$$\Rightarrow \left|\frac{z_1}{z_2+z_3}\right| \leq \frac{|z_1|}{\big||z_2|-|z_3|\big|}.$$

ii) we know that $|z_1+z_2| \le |z_1|+|z_2|$ — (i)

Again,

$$|z_3+z_4| \geq \cancel{|z_3|+z_4} \, \big||z_3|-|z_4|\big|$$

$$\Rightarrow \frac{1}{|z_3+z_4|} \le \frac{1}{\big||z_3|-|z_4|\big|} \quad \text{— (ii)}$$

Combining (i) and (ii) we get,

$$\frac{|z_1+z_2|}{|z_3+z_4|} \le \frac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|}$$

$$\Rightarrow \left|\frac{z_1+z_2}{z_3+z_4}\right| \le \frac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|}$$

1.13 Find an equation for (a) a circle of radius 4 with centre at (–2, 1), (b) an ellipse with major axis of length 10 and foci at (–3, 0) and (3, 0).

Solution

(a) The centre can be represented by the complex number $-2 + i$. If z is any point on the circle (Fig. 1.20), the distance from z to $-2 + i$ is

$$|z - (-2 + i)| = 4$$

Then $|z + 2 - i| = 4$ is the required equation. In rectangular form this is given by

$$|(x + 2) + i(y - 1)| = 4, \text{ i.e. } (x + 2)^2 + (y - 1)^2 = 16$$

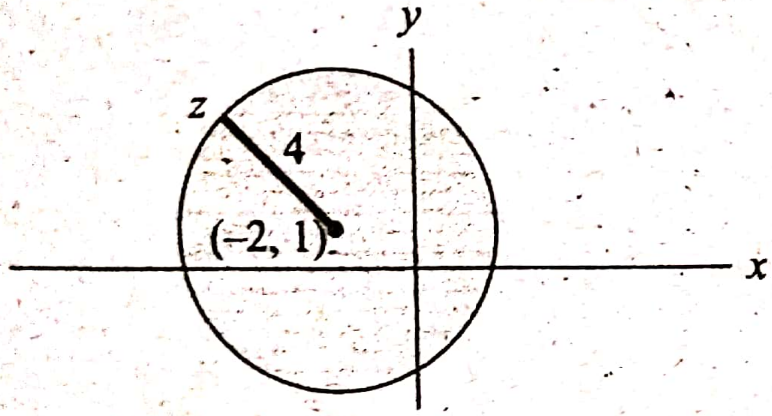


Fig 1.20

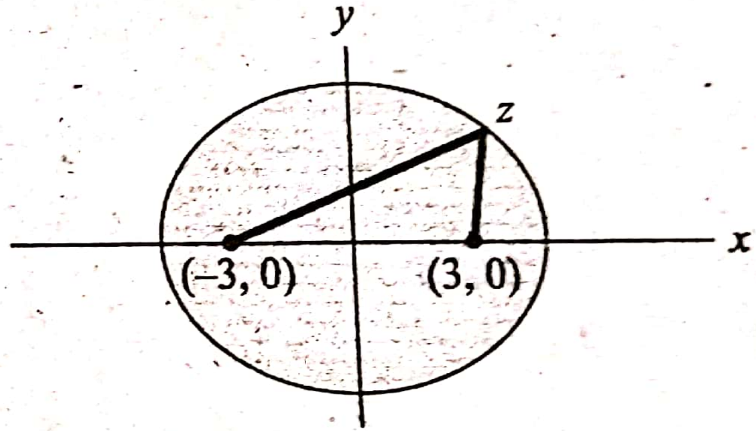


Fig 1.21

(b) The sum of the distances from any point z on the ellipse (Fig. 1.21) to the foci must equal 10. Hence the required equation is

$$|z+3|+|z-3|=10$$

In rectangular form this reduces to $x^2/25 + y^2/16 = 1$ (see Problem 1.74).

Do it

~~Con~~ Describe the following region geometricall

1. $|z-i| = |z+i|$

2. $|z-1| = |z+i|$

3. $|z-2i| = 6$

4. $(z+2)(\bar{z}+2) = 3$

5. $|z+2i| + |z-2i| = 6$

same 6. $|z-4i| + |z+4i| = 4$

7. $\mathrm{Im}(z^2) = 4$

8. $\mathrm{Re}(z^2) = 4$

$\mathrm{Im}(z) \geq 0$

$\mathrm{Im}(1/z) \leq 1/2$

? $\rightarrow$ $0 < \mathrm{Arg}\, z < \pi/2$ $\quad |z| = 2$

$1 < |z+i| \leq 2$

2. Let $z = x+iy$

Given,

$|z-1| = |z+i|$

$\Rightarrow |x+iy-1| = |x+iy+i|$

$\Rightarrow |(x-1)+iy| = |x+i(y+1)|$

$\Rightarrow \sqrt{(x-1)^2+y^2} = \sqrt{x^2+(y+1)^2}$

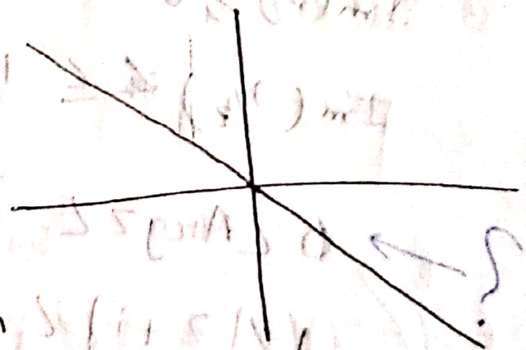
$\Rightarrow (x-1)^2+y^2 = x^2+(y+1)^2$

$\Rightarrow \cancel{x^2}-2x+\cancel{1}+\cancel{y^2} = \cancel{x^2}+\cancel{y^2}+2y+\cancel{1}$

$\Rightarrow -2x = 2y$

$\Rightarrow x = -y.$

$\therefore y+x=0.$



This is the equation of ~~the~~ a straight line passing through the origin and with slope -1.

1. Let $z = x+iy$ $\quad \bar{z} = x-iy$

Given $(z+2)(\bar{z}+2) = 3$

$\Rightarrow z\bar{z} + 2z + 2\bar{z} + 4 = 3$

$\Rightarrow |z|^2 + 2(z+\bar{z}) + 4 = 3$

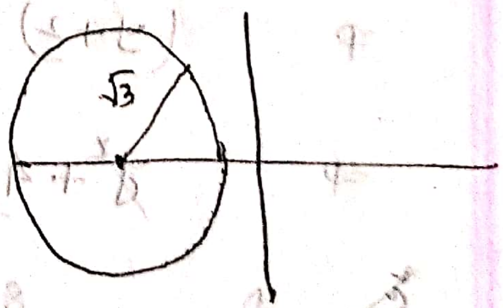
$\Rightarrow |x+iy|^2 + 2(x+\cancel{iy}+x-\cancel{iy}) + 4 = 3$

$\Rightarrow |x+iy|^2 + 4x + 4 = 3$

$\Rightarrow \left(\sqrt{x^2+y^2}\right)^2 + 4x + 4 = 3$

$\Rightarrow x^2 + 4x + 4 + y^2 = 3$

$\Rightarrow (x+2)^2 + y^2 = (\sqrt{3})^2$



the is the equation of a circle with centre at $(-2,0)$ and radius $= \sqrt{3}$.

5. Let $z = x+iy$

Given

$|z+2i| + |z-2i| = 6$

$\Rightarrow |x+iy+2i| + |x+iy-2i| = 6$

$\Rightarrow |x+i(y+2)| + |x+i(y-2)| = 6$

$\Rightarrow \sqrt{x^2+(y^2+2)^2} + \sqrt{x^2+(y-2)^2} = 6$

$\Rightarrow \sqrt{x^2+(y+2)^2} = 6 - \sqrt{x^2+(y-2)^2}$

$\therefore \cancel{x^2}+(y+2)^2 = 36 + \{\cancel{x^2}+(y-2)^2\} - 12\sqrt{x^2+(y-2)^2}$

$\Rightarrow (y+2)^2 = 36 + (y-2)^2 - 12\sqrt{x^2+(y-2)^2}$

$\Rightarrow \cancel{y^2}+4y+\cancel{4} = 36+\cancel{y^2}-4y+\cancel{4}-12\sqrt{x^2+(y-2)^2}$

$\Rightarrow 8y = 36 - 12\sqrt{x^2+(y-2)^2}$

$\Rightarrow 2y = 9 - 3\sqrt{x^2+(y-2)^2}$

$\Rightarrow 3\sqrt{x^2+(y-2)^2} = 9-2y$

$\therefore 9\{x^2+(y-2)^2\} = 81+4y^2-36y$

$\Rightarrow 9(x^2+y^2-4y+4) = 81+4y^2-36y$

$\Rightarrow 9x^2+5y^2 = 81-36 = 45$

$\Rightarrow 9x^2+5y^2 = 45$

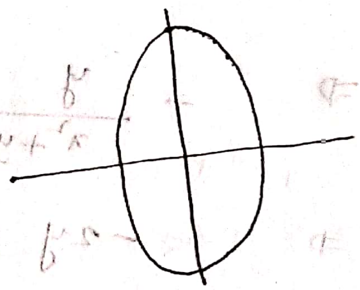
$$\Rightarrow \frac{x^2}{\frac{45}{9}} + \frac{y^2}{\frac{45}{5}} = 1$$

$$\Rightarrow \frac{x^2}{5} + \frac{y^2}{9} = 1$$

$$\Rightarrow \frac{x^2}{(\sqrt{5})^2} + \frac{y^2}{(3)^2} = 1$$

this is an equation of ellipse

centre at $(0,0)$.



$z = x+iy$,

8) $Im(z^2) = 4$;

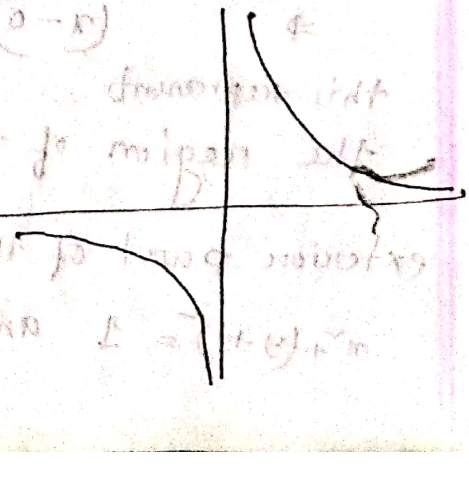
$$\Rightarrow Im\left((x+iy)^2\right) = 4$$

$$\Rightarrow Im\left(x^2+2xyi-y^2\right) = 4$$

$$\Rightarrow Im\left(x^2-y^2+i2xy\right) = 4$$

$$\Rightarrow 2xy = 4$$

$$\Rightarrow xy = 2$$



This is an equation of hyperbola

11. $\operatorname{Im}\left(\frac{1}{z}\right) \leq \frac{1}{2}$

$\Rightarrow \operatorname{Im}\left(\frac{1}{x+iy}\right) \leq \frac{1}{2}$

$\Rightarrow \operatorname{Im}\left(\frac{x-iy}{(x+iy)(x-iy)}\right) \leq \frac{1}{2}$

$\Rightarrow \operatorname{Im}\left(\frac{x-iy}{x^2+y^2}\right) \leq 1/2$

$\Rightarrow \operatorname{Im}\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right) \leq 1/2$

$\Rightarrow -\frac{y}{x^2+y^2} \leq 1/2$

$\Rightarrow -2y \leq x^2+y^2$

$\Rightarrow x^2+y^2 \geq -2y$

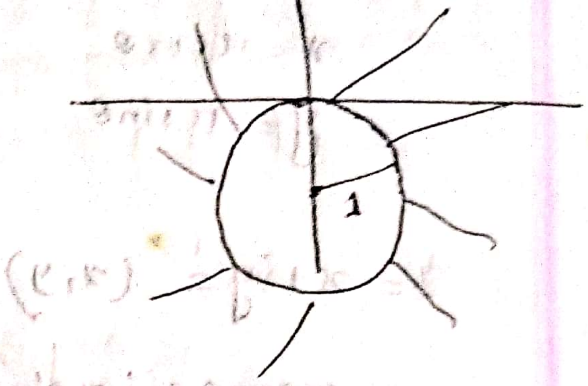
$\Rightarrow x^2+y^2+2y \geq 0$

$\Rightarrow x^2+y^2+2y+1 \geq 1$

$\Rightarrow x^2+(y+1)^2 \geq 1$

$\Rightarrow (x-0)^2+(y+1)^2 \geq 1.$

this represents the region of the whole exterior part of the circle $x^2+(y+1)^2 = 1$ and the circle itself.



$$1 < |z+i| \le 2$$

$\Rightarrow$

$\Rightarrow \quad 1 < x^2+(y+1)^2 \le 2^2$

This represents the annular ([illegible]) region between the concentric circles

$x^2+(y+1)^2=1$ and $x^2+(y+1)^2=2^2$

including the circle $x^2+(y+1)^2=2^2$.