Elimination Methods

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Naïve Gauss Elimination

Very close to Elimination of Unknowns

1. Forward Elimination
2. Backward Substitution

Naïve because we don't consider division by zero to be a possibility

Consider the following system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

It can be written as: [A][X] = [b]; A is the coefficient Matrix, X is vector of unknown and b is the vector of constant.

Naïve Gauss Elimination

1. Forward Elimination of Unknowns
   - Reduce the coefficient matrix [A] to an upper triangular system
   - Eliminate x_1 from the 2nd to nth Equations.
   - Eliminate x_2 from the 3rd to nth Equations.
   - Continue process until the nth equation has only 1 Non-Zero coefficient

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

Naïve Gauss Elimination

1. Forward Elimination of Unknowns

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ ....(1) → **(Pivot row)**

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ ....(2) → **(Elimination row)**

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ ....(3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Pivot

To eliminate x_1 from the second equation, we multiply the first equation by $-a_{21} / a_{11}$ and then add it to the second equation

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{11} (-a_{21} / a_{11})x_1 + a_{12} (-a_{21} / a_{11}) x_2 + a_{13} (-a_{21} / a_{11}) x_3 = b_1 (-a_{21} / a_{11})$

$0 + (a_{22} - a_{12} (a_{21} / a_{11}))x_2 + (a_{23} - a_{13} (a_{21} / a_{11})) x_3 = b_2 - b_1 (a_{21} / a_{11})$

Let,

$(a_{22} - a_{12} (a_{21} / a_{11})) = a'_{22}$

$(a_{23} - a_{13} (a_{21} / a_{11})) = a'_{23}$

$b_2 - b_1 (a_{21} / a_{11}) = b'_2$

$a'_{22} x_2 + a'_{23} x_3 = b'_2$ ....(2')

Replace equation 2 by equation 2'

Naïve Gauss Elimination

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ ....(1) **(Pivot row)**

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ ....(2)

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ ....(3) **(Elimination row)**

Pivot Element

To eliminate x_1 from the second equation, we multiply the first equation by $-a_{31} / a_{11}$ and then add it to the third equation

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$a_{11}(-a_{31}/a_{11})x_1 + a_{12}(-a_{31}/a_{11})x_2 + a_{13}(-a_{31}/a_{11})x_3 = b_1(-a_{31}/a_{11})$

$0 + (a_{32} - a_{12}(a_{31}/a_{11}))x_2 + (a_{33} - a_{13}(a_{31}/a_{11}))x_3 = b_3 - b_1(a_{31}/a_{11})$

Let,

$(a_{32} - a_{12}(a_{31}/a_{11})) = a'_{32}$ $a'_{32}x_2 + a'_{33}x_3 = b'_3$ ....(3')

$(a_{23} - a_{13}(a_{21}/a_{11})) = a'_{33}$

$b_2 - b_1(a_{21}/a_{11}) = b'_3$ Replace equation 3 by equation 3'

Naïve Gauss Elimination

1. Forward Elimination of Unknowns

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$..............................(1)

$a'_{22}\, x_2 + a'_{23}\, x_3 = b'_2$ ....(2') (Pivot row)

$a'_{32}\, x_2 + a'_{33}\, x_3 = b'_3$ ....(3') (Elimination row)

Now, to eliminate x_2 from the 3' equation, we multiply the 2'equation by $-a'_{32} / a'_{22}$ and then add it to the 3' equation.

$a'_{32}\, x_2 + a'_{33}\, x_3 = b'_3$

$a'_{22}\,(-\,a'_{32} / a'_{22})x_2 + a'_{23}\,(-\,a'_{32} / a'_{22})\, x_3 = (-\,a'_{32} / a'_{22})\, b'_2$

$0 + (a'_{33} - a'_{23}\,(a'_{32} / a'_{22}))\, x_3 = b'_3 - (a'_{32} / a'_{22})\, b'_2$

Let

$a'_{33} - a'_{23}\,(a'_{32} / a'_{22})) = a''_{33}$ $\qquad$ $a''_{33}\, x_3 = b''_3$ ....(3'')

$b'_3 - (a'_{32} / a'_{22})\, b'_2 = b''_3$

Naïve Gauss Elimination

1. Forward Elimination

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$……………………………(1)

$a'_{22}\, x_2 + x_3 = b'_2$ ……………………….(2')

$a''_{33}\, x_3 = b''_3$ …………………….(3'')

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'22 & a'23 \\ 0 & 0 & a''33 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'2 \\ b''3 \end{bmatrix}$$

Naïve Gauss Elimination

2. Backward Substitution

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$............................(1)

$a'_{22}\, x_2 + a'_{23}\, x_3 = b'_2$ ....(2')

$a''_{33}\, x_3 = b''_3$ ....(3'')

$x_3 = \frac{b_3''}{a_3''}$

From 2'

$x_2 = \frac{b_2' - a'23\ x_3}{a_{22}'}$

From 1

$x_1 = \frac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$

Naïve Gauss Elimination

General Form

$$x_n = \frac{b_n^{n-1}}{a_{nn}^{n-1}}$$

$$x_i = \frac{b_i^{i-1} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{i-1}}$$

Naïve Gauss Elimination

Example: Solve the following system using Naïve Gauss Elimination Method

$x - 3y + z = 4$……………….(1)

$2x \ - 8y + 8z = -2$……….(2)

$-6x + 3y - 15z = 9$……….(3)

Naïve Gauss Elimination

$Row\ 1$: $x - 3y + z = 4$……………….(1) (Pivot Row)

$Row\ 2$: $2x\ - 8y + 8z = -2$……….(2) (Elimination Row)

$Row\ 3$: $-6x + 3y - 15z = 9$……….(3)

Forward Elimination:

Step 1: Eliminate x from Row 2

Row 2 = Row 2+ (-2)* Row 1

$$2x\ - 8y + 8z = -2$$
$$-2x + 6y - 2z = -8$$

$$-2y + 6z = -10$$
$$\Rightarrow -y + 3z = -5$$ …………..(2’)

Naïve Gauss Elimination

$Row\ 1$: $x - 3y + z = 4$……………….(1) (Pivot Row)

$Row\ 2$: $2x\ - 8y + 8z = -2$……….(2)

$Row\ 3$: $-6x + 3y - 15z = 9$……….(3) (Elimination Row)

Step 2: Eliminate x from Row 3

Row 3 = Row 3+ (6) * Row 1

$$-6x + 3y - 15z = 9$$

$$6x - 18y + 6z = 24$$

$$-15y\ - 9z = 33$$

$$\Rightarrow\ -5y\ - 3z = 11$$ ………….(3’)

Naïve Gauss Elimination

$Row\ 1$: $x - 3y + z = 4$……………….(1)

$Row\ 2'$: $-y + 3z = -5$…………. .(2') (Pivot Row)

Row $3'$: $-5y - 3z = 11$…………..(3') (Elimination Row)

Step 3: Eliminate y from 3'

Row $3'$= Row $3'$ + ($-$ 5) $Row\ 2'$

$$-5y - 3z = 11$$
$$5y - 15z = 25$$

$$-18z = 36 \text{……………………………..(3'')}$$

Naïve Gauss Elimination

$Row\ 1$: $x - 3y + z = 4$..................(1)
$Row\ 2'$: $-y + 3z = -5$............. .(2')
Row $3''$: $-18z = 36$.............(3'')

Back Substitution:

From (3'')
$-18z = 36$
$z = -2$

From (2')
$-y + 3(-2) = -5$
$y = -1$

From (1)
$x - 3(-1) - 2 = 4$
$x = 1$

Solution: $x = 1; y = -1; z = -2$

Gauss Elimination With Pivoting

Problem of Naïve Gauss Elimination:

→during both the elimination and back substitution a division by zero may occur

Consider the following system:

$$
\begin{aligned}
2x_2 + 3x_3 &= 8 \\
4x_1 + 6x_2 + 7x_3 &= -3 \\
2x_1 - 3x_2 + 6x_3 &= 5
\end{aligned}
$$

The normalization of the first row would involve division by zero

If magnitude of pivot element is small compared to other elements, then round of errors may occur.

Solution:

The row with zero pivot element should be interchanged with the row having the largest (absolute) coefficient in that position

Gauss Elimination With Partial Pivoting

Steps of Gauss Elimination with Pivoting:

1. Search and locate the largest absolute value among the coefficients in the first column.
2. Exchange the first row with the row containing that element.
3. Then eliminate the first variable in the other equations as explained earlier.
4. Continue this procedure till (n-1) unknowns are eliminated.

This process is referred to as *partial pivoting*. There is an alternative scheme known as *complete pivoting* in which, at each stage, the largest element at any of the remaining rows is used as the pivot.

Gauss Elimination With Partial Pivoting

Solve the following system using Gauss Elimination With Partial Pivoting.

$$
\begin{aligned}
2x_1 + x_2 + x_3 &= 5 \quad &(1)\\
4x_1 - 6x_2 \quad &= -2 \quad &(2)\\
-2x_1 + 7x_2 + 2x_3 &= 9 \quad &(3)
\end{aligned}
$$

$$
\begin{vmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{vmatrix}
\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}
=
\begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}
$$

Gauss Elimination With Partial Pivoting

Step 1: Find the row with largest coefficient at 1st column

Row 1: $2x_1 + x_2 + x_3 = 5$……………………(1)

Row 2: $4x_1 - 6x_2 \quad = -2$…………………(2)

Row 3: $-2x_1 + 7x_2 + 2x_3 = 9$……………(3)

Row 2 has largest coefficient so, interchange row 1 and row 2 and system become:

Row 1: $4x_1 - 6x_2 \quad = -2$ ………………(1)

Row 2: $2x_1 + x_2 + x_3 = 5$ …………………(2)

Row 3: $-2x_1 + 7x_2 + 2x_3 = 9$……………(3)

Gauss Elimination With Partial Pivoting

$-\frac{a_{21}}{a_{11}}$

Row 1: $4x_1 - 6x_2 \quad = -2$ ………………(1) (Pivot Row)

Row 2: $2x_1 + x_2 + x_3 = 5$ …………………(2) (Elimination row)

Row 3: $-2x_1 + 7x_2 + 2x_3 = 9$……………(3)

pivot $= -\frac{2}{4} = -\frac{1}{2}$

$4 \times (-\frac{1}{2}) + 6(\frac{1}{2}) =$

Forward Elimination:

Step 2:

Row 2 = Row 2+ $(-1/2)$* Row 1

$$\cancel{2x_1} + x_2 + x_3 = 5$$

$$\cancel{-2x_1} + 3x_2 \quad = 1$$

$4x_2 + x_3 = 6$ ………….(2’)

Gauss Elimination With Partial Pivoting

Row 1: $4x_1 - 6x_2 = -2$ ....(1) (Pivot Row)

Row 2: $2x_1 + x_2 + x_3 = 5$ ....(2)

$Row\ 3$: $-2x_1 + 7x_2 + 2x_3 = 9$...............(3) (Elimination row)

Step 3:

Row 3 = Row 3+ (1/2)* Row 1

Pivot = $\frac{1}{2}$

$$-2x_1 + 7x_2 + 2x_3 = 9$$

$$2x_1 - 3x_2 = -1$$

$$\left[\begin{array}{ccc|c} 4 & -6 & 0 & -2 \\ 0 & 4 & 1 & 6 \\ 0 & 4 & 2 & 8 \end{array}\right]$$

$$4x_2 + 2x_3 = 8 \quad ....(3')$$

Gauss Elimination With Partial Pivoting

Row 1: $4x_1 - 6x_2 \quad = -2$ ....(1)

Row 2': $4x_2 + x_3 = 6$ ....(2')(Pivot Row)

$Row\ 3'$: $4x_2 + 2x_3 = 8$................(3') (Elimination row)

Step 4:

Row 3' = Row 3'+ (− 1)* Row 2'

$$\cancel{4x_2} + 2x_3 = 8$$

$$-\cancel{4x_2} - x_3 = -6$$

$$x_3 = 2 \quad ....(3'')$$

Pivot = −1

Row 2' * (−1)

$= -4x_2 - x_3 = -6$

$$\begin{bmatrix} 4 & -6 & 0 & -2 \\ 0 & 4 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Gauss Elimination With Partial Pivoting

Upper Triangular Matrix

Row 1: $4x_1 - 6x_2 \quad = -2$ ……………….(1)

Row 2': $4x_2 + x_3 = 6$ ……………….(2') =>

Row 3'' : $x_3 = 2$……………….(3'')

$$\begin{bmatrix} 4 & -6 & 0 & \vdots & -2 \\ 0 & 4 & 1 & \vdots & 6 \\ 0 & 0 & 1 & & 2 \end{bmatrix}$$

Back Substitution:

From $Row\ 3''$

$x_3 = 2$

From $Row\ 2'$

$4x_2 + 2 = 6$

=> $x_2 = 1$

From $Row\ 1$

$4x_1 - 6*1 \quad = -2$

=>$x_1 = 1$

Solution: $x_1 = 1;\ x_2 = 1;\ x_3 = \cancel{-}2$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Complexity Analysis of Gauss Elimination Method

- Total Number of Flops (Floating point operations):
  - Total Number of multiplication/division (n-1)(n+1)
  - Total Number of addition/ subtraction (n-1)(n)
- Total Complexity:

$$\frac{2n^3}{2} + o(n^2) + n^2 + \mathrm{o(n)}$$

Forward Elimination Backward Substitution

$$=> \frac{2n^3}{2} + o(n^2)$$

Algorithm: Gauss Elimination With Partial Pivoting

```
//Search for Pivot Element
for i = 1 to n do
set pivot =  |a_ii|
set r_max =  i
//search for maximum coefficient
  for k = i + 1 to n do
    r = |a_ki/a_ii |
    if(r > pivot) then
           pivot = r
           r_max = k
  end
   for k = i + 1 to n do
      swap a_rmax, k and a_ik
   end
end
```

```
//Forward Elimination:
//loop over all rows except last
for  k=0 to n—1 do
//loop over all rows bellow the diagonal position
      for i = k+1 to n do
//search for pivot element
//loop over all columns right of the diagonal position
              for j = k+1 to n do
                a_ij = a_ij − a_kj *a_jk/a_kk
                end

      end
end
```

```
//Backward Substitution:
//Compute last unknown
xn = bn/ann
//loop over all the row except last row
for i = n—1to 1 do
//loop over all columns to the right of the current row
   for j = i+1 to 1
       xi = 1/aii(b_i − Σ_{j=i+1}^{n} a_ij x_j)
   end
end
```

Gauss Jordan Method

- Another popular approach for solving system of linear equations.
- In Gauss-Jordan method, eliminates all the off-diagonal unknowns producing a diagonal matrix.
- All rows are normalized by dividing them by their pivot elements.
- Obtain the values of unknowns directly from the b vector, without employing back substitution.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a'_{22} & 0 \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

Gauss Jordan Method

- Normalize the first equation by dividing it by its pivot element.
- Eliminate x_1 term from all the other equations.
- Now, normalize the second equation by dividing it by its pivot element.
- Eliminate x_2 from all the equations, above and below the normalized pivotal equation.
- Repeat this process until x_n is eliminated from all but the last equation.
- The resultant b vector is the solution vector.

Gauss Jordan Method

- Solve the following system of linear equations using Gauss-Jordan method

$$2x_1 + 4x_2 + 6x_3 = 18$$
$$4x_1 + 5x_2 + 6x_3 = 24$$
$$3x_1 + x_2 - 2x_3 = 4$$

Gauss Jordan Method

Step	Equation Form	Augmented Matrix Form	Next Step
	R1: $2x_1 + 4x_2 + 6x_3 = 18$ R2: $4x_1 + 5x_2 + 6x_3 = 24$ R3: $3x_1 + x_2 - 2x_3 = 4$	$\begin{bmatrix} 2 & 4 & 6 & \vdots & 18 \\ 4 & 5 & 6 & \vdots & 24 \\ 3 & 1 & -2 & & 4 \end{bmatrix}$	Normalize R1 R1 → 1/2 R1
1	R1: $x_1 + 2x_2 + 3x_3 = 9$ R2: $4x_1 + 5x_2 + 6x_3 = 24$ R3: $3x_1 + x_2 - 2x_3 = 4$	$\begin{bmatrix} 1 & 2 & 3 & \vdots & 9 \\ 4 & 5 & 6 & \vdots & 24 \\ 3 & 1 & -2 & & 4 \end{bmatrix}$	Eliminate x_1 from R3 and R2 R2 → R2 – 4R1 R3 → R3 – 3R1
2	R1: $x_1 + 2x_2 + 3x_3 = 9$ R2: $-3x_2 - 6x_3 = -12$ R3: $-5x_2 - 11x_3 = -23$	$\begin{bmatrix} 1 & 2 & 3 & \vdots & 9 \\ 0 & -3 & -6 & \vdots & -12 \\ 0 & -5 & -11 & & -23 \end{bmatrix}$	Normalize R2 R2 → −1/3 R2

Gauss Jordan Method

Step	Equation Form	Augmented Matrix Form	Next Step
3	R1: $x_1 + 2x_2 + 3x_3 = 9$ R2: $x_2 + 2x_3 = 4$ R3: $-5x_2 - 11x_3 = -23$	$\begin{bmatrix} 1 & 2 & 3 & \vdots & 9 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & -5 & -11 & & -23 \end{bmatrix}$	Eliminate x_2 from R1 and R3 R1 → R1 − (2) R2 R3 → R3 − (−5) R2
4	R1: $x_1 - x_3 = 1$ R2: : $x_2 + 2x_3 = 4$ R3: $-x_3 = -3$	$\begin{bmatrix} 1 & 0 & -1 & & 1 \\ 0 & 1 & 2 & \vdots & 4 \\ 0 & 0 & -1 & & -3 \end{bmatrix}$	Eliminate x3 from R1 and R2 R2 → R2 − 2 R3 R1 → R1 − (−1)R3
6	R1: $x_1 = 4$ R2: : $x_2 = -2$ R3: $-x_3 = -3$	$\begin{bmatrix} 1 & 0 & 0 & & 4 \\ 0 & 1 & 0 & \vdots & -2 \\ 0 & 0 & -1 & & -3 \end{bmatrix}$	

Gauss Jordan Method

$$\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} = \begin{bmatrix}4 \\ -2 \\ -3\end{bmatrix}$$

Solution for the system:

x_1 = 4

$x2$ = -2

x_3 = 3

Gauss Jordan Method

- The Gauss-Jordan method requires approximately 50 percent more arithmetic operations compared to Gauss method. Therefore, this method is rarely used.
- See following table: for the comparison of computational effort.

	Gauss Method	Gauss-Jordan Method
Multiplication	$1/3\ n^3$	$1/2\ n^3$
Subtraction	$1/3n^3$	$1/2n^3$
Divisions	$1/2n^2$	$1/2n^2$