Find standard form and principal argument of

a) $(\sqrt{3}+i)^7$ (b) $(\sqrt{3}-i)^6$

a) $(\sqrt{3}+i)^7$

$\cos 210$
$\cos 180+30$
$-\cos 30$
$= -\sqrt{3}/2$

$r = 2.$

$\operatorname{Arg}(\sqrt{3}+i) = \frac{\pi}{6}$

$$\therefore (\sqrt{3}+i)^7 = \left(2e^{i.\frac{\pi}{6}}\right)^7 = 2^7 . e^{i.\frac{7\pi}{6}}$$

$$= 2^7\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

$$= 2^7\left(-\frac{\sqrt{3}}{2} - i.\frac{1}{2}\right)$$

$$= 2^6(-\sqrt{3} - i)$$

$$\operatorname{Arg}(\sqrt{3}+i)^7 = -\frac{5\pi}{6},$$

$$\therefore \arg(\sqrt{3}+i)^7 = -\frac{5\pi}{6} + 2n\pi.$$

$n = 0, \pm 1, \pm 2, \ldots$

$6)$
$\frac{5\pi}{6}$

De Moivre's Th^m:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$n = 0, \pm1, \pm2, \ldots$

আর, n, যেকোন rational number ও হবে

$\therefore \cancel{n \in} \; . \; n \in Q.$

So,

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$z^n = r^n e^{in\theta}$$

In n th root of $z_0 = r_0 e^{i\theta_0}$ is $z = re^{i\theta}$ then

$$\sqrt[n]{z_0} = z$$

$$\Rightarrow z^n = z_0$$

$$\Rightarrow r^n e^{in\theta} = r_0 e^{i\theta_0}$$

$\therefore \quad r^n = r_0$ and $\theta n = \theta_0 + 2k\pi \quad k = 0, \pm 1, \pm 2 \ldots$

$$\therefore \quad r = \sqrt[n]{r_0} \qquad \theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n}.$$

$$\therefore\ z = \sqrt[n]{z_0}$$

$$= \sqrt[n]{z_0}\ \exp\left\{ i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right\} \quad k = 0, \pm 1, \pm 2, \ldots$$

If n th root of $z_0 = r_0 e^{i\theta_0}$ is z then,

$$z = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right], \quad k = 0, \pm 1, \pm 2, \ldots$$

we can see from the exponential form of the roots that they all lie on the circle $|z| = \sqrt[n]{r_0}$, about the origin and are equally spaced every $\frac{2\pi}{n}$ radians, starting with $\frac{\theta_0}{n}$.

Evidently, all of the distinct roots are obtained when $k = 0, 1, 2, \ldots, n-1$ and no further roots arise with other values of k.

we let c_k $(k = 0, 1, 2, \ldots, n-1)$ denote these distinct roots and write

$$c_k = \sqrt[n]{r_0} \exp\left[i \left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right] ; \quad k = 0, 1, 2, \ldots, n-1.$$

Hints: Write z_0 in the general form $z_0 = r_0 e^{i(\theta_0 + 2k\pi)}$

Here $n \in \mathbb{N}$

Then,

$$z_0^{1/n} = \left[r_0 e^{i(\theta_0 + 2k\pi)} \right]^{1/n}$$

$$= \sqrt[n]{r_0} \exp\left[i \left(\frac{\theta_0 + 2k\pi}{n} \right) \right]$$

$$= \sqrt[n]{r_0} \exp\left[i \left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right] ; \quad k = 0, 1, 2, \ldots, n-1$$

1) Find all values of $(8i)^{1/3}$

$\tan^{-1}\frac{8}{0}$, ~~$\tan^{-1}\tan$~~, $\tan \infty$ then

Let, $z = 8i$, $r = 8$

$\text{Arg } z = \frac{\pi}{2}$

$\therefore z = 8i = 8\, e^{i\left(\frac{\pi}{2} + 2k\pi\right)}$, $k = 0, \pm 1, \pm 2, \dots$

$z^{1/3} = (8i)^{1/3} = \left[8 \exp\left\{i\left(\frac{\pi}{2} + 2k\pi\right)\right\}\right]^{1/3}$

$= \underline{\underline{2}} \exp\left\{i\left(\frac{\pi/2 + 2k\pi}{3}\right)\right\}$

radius

$= 2 \exp\left\{i\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right)\right\}$, $k = 0, \pm 1, \pm 2, \dots$

[k এর value (-) ve নিলেও একই, যেগুলো, distinct answer আসবে।]

Let the cube root of $8i$ are c_0, c_1, c_2. Then

[k=0, হলে, যে root তাকে principle root বলে]

$c_0 = 2 \exp\left\{i\left(\frac{\pi}{6}\right)\right\} = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$

$= 2\left(\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right)$

$= \sqrt{3} + i$

$c_1 = 2 \exp\left\{i\left(\frac{\pi}{6} + \frac{2}{3}\pi\right)\right\} = 2\exp\left\{i\left(\frac{5\pi}{6}\right)\right\}$

$= 2\left(\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right)$

$= 2\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right)$

$= -\sqrt{3} + i$

$$c_2 = 2\exp\left\{ i\left(\frac{\pi}{6} + \frac{4}{3}\pi\right)\right\}$$

$$= 2\exp\left\{ i\left(\frac{9\pi}{6}\right)\right\}$$

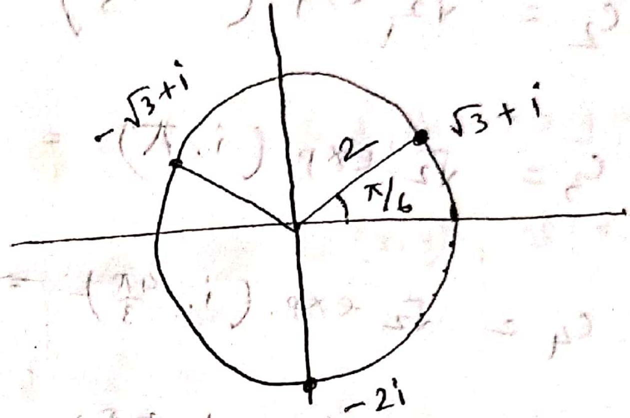
$$= 2\exp\left\{ i\left(\frac{3\pi}{2}\right)\right\}$$

$$= 2.\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$= 2.\left(+ - i\right)$$

$$= -2i.$$

$2\pi/3 = 120^\circ$



Q. Find all values of $(8)^{1/6}$

Let $z = 8$ $\qquad r = 8$

$\qquad \operatorname{Arg} z = 0$

$$\therefore z = 8 = 8.e^{i(0+2k\pi)}, \quad k = 0, \pm1, \pm2, \dots$$

$$z^{1/6} = (8)^{1/6} = \left\{8\, e^{i(0+2k\pi)}\right\}^{1/6}$$

$$= 8^{1/6}\exp\left\{ i\left(\frac{2k\pi}{6}\right)\right\}$$

$$= 8^{1/6}\exp\left\{ i\,\frac{k\pi}{3}\right\} \qquad k = 0, \pm1, \pm2, \dots$$

$$= \sqrt{2}\ \exp\left\{ i \frac{k\pi}{3} \right\}$$

Let the roots are $c_0, c_1, c_2, c_3, c_4, c_5$

$$c_0 = \sqrt{2}\ \exp(i.0) = \sqrt{2}$$

$$c_1 = \sqrt{2}\ \exp\left\{ i \frac{\pi}{3} \right\} = \sqrt{2}\left(\cos\frac{\pi}{3} + i.\sin\frac{\pi}{3} \right)$$

$$= \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)$$

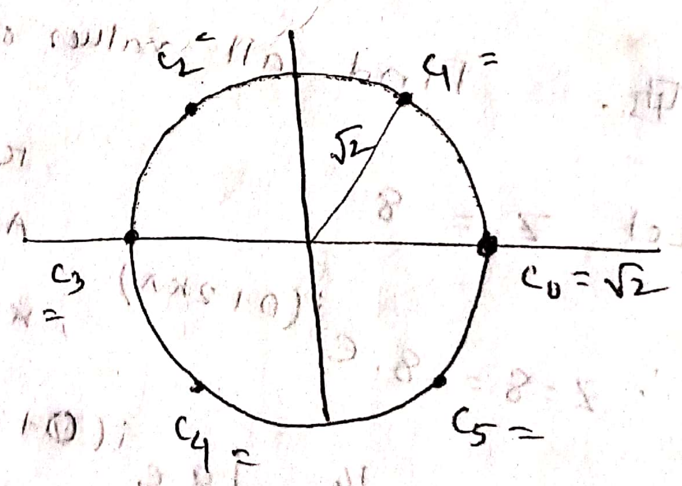
$$c_2 = \sqrt{2}\ \exp\left(i.\ \frac{2\pi}{3} \right) =$$

$$c_3 = \sqrt{2}\ \exp\left(i.\ \pi \right) =$$

$$c_4 = \sqrt{2}\ \exp\left(i.\ \frac{4\pi}{3} \right) =$$

$$c_5 = \sqrt{2}\ \exp\left(i.\ \frac{5\pi}{3} \right) =$$

angle difference $= \frac{\pi}{3}$



~~Do it~~

Find all values of

a) $(\sqrt{3}+i)^{1/2}$ (b) $8^{1/3}$ (c) $(-24)^{1/4}$ (d) $(-8-8\sqrt{3}i)^{1/4}$

ROOTS OF COMPLEX NUMBERS

1.28 (a) Find all values of z for which $z^5 = -32$, and (b) locate these values in the complex plane.

Solution

(a) In polar form, $-32 = 32\{\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)\}$, $k = 0, \pm 1, \pm 2, \ldots$.
Let $z = r(\cos\theta + i \sin\theta)$. Then by De Moivre's theorem,

$$z^5 = r^5(\cos 5\theta + i \sin 5\theta) = 32\{\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)\}$$

and so $\quad r_5 = 32$, $5\theta = \pi + 2k\pi$, from which $r = 2$, $\theta = (\pi + 2k\pi)/5$. Hence

$$z = 2\left\{\cos\left(\frac{\pi + 2k\pi}{5}\right)\right\} + i \sin\left(\frac{\pi + 2k\pi}{5}\right)$$

If $k = 0$, $z = z_1 = 2(\cos \pi/5 + i \sin \pi/5)$.
If $k = 1$, $z = z_2 = 2(\cos 3\pi/5 + i \sin 3\pi/5)$.

If $k = 2$, $z = z_3 = 2(\cos 5\pi/5 + i \sin 5\pi/5) = -2$.
If $k = 3$, $z = z_4 = 2(\cos 7\pi/5 + i \sin 7\pi/5)$.
If $k = 4$, $z = z_5 = 2(\cos 9\pi/5 + i \sin 9\pi/5)$.

By considering $k = 5, 6, \ldots$ as well as negative values, $-1, -2, \ldots$, repetitions of the above five values of z are obtained, Hence these are the only *solutions* or *roots* of the given equation. These five roots are called the *fifth roots of* -32 and are collectively denoted by $(-32)^{1/5}$. In general, $a^{1/n}$ represents the nth roots of a and there are n such roots.

(b) The values of z are indicated in Fig. 1.31. Note that they are equally spaced along the circumference of a circle with centre at the origin and radius 2. Another way of saying this is that the roots are represented by the vertices of a regular polygon.

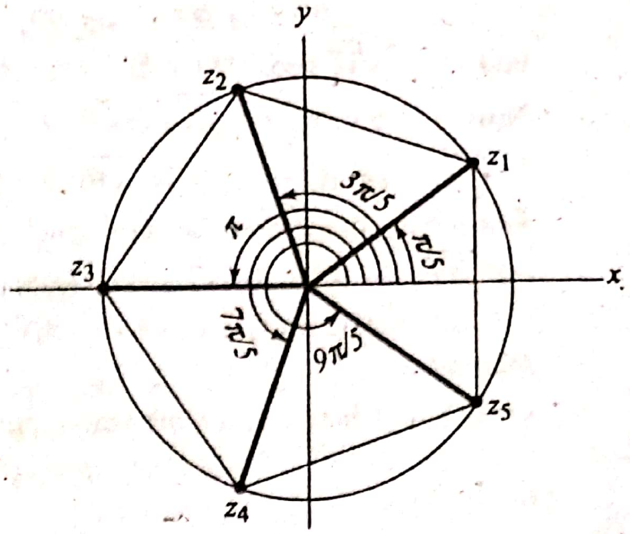


Fig. 1.31

1.29 Find each of the indicated roots and locate them graphically.

(a) $(-1 + i)^{1/3}$ (b) $(-2\sqrt{3} - 2i)^{1/4}$

Solution

(a) $(-1 + i)^{1/3}$

$-1 + i = \sqrt{2}\,\{\cos(3\pi/4 + 2k\pi) + i \sin(3\pi/4 + 2k\pi)\}$

$$(-1+i)^{1/3} = 2^{1/6}\left\{\cos\left(\frac{3\pi/4 + 2k\pi}{3}\right) + i\sin\left(\frac{3\pi/4 + 2k\pi}{3}\right)\right\}$$

If $k = 0$, $z_1 = 2^{1/6}(\cos \pi/4 + i \sin \pi/4)$.
If $k = 1$, $z_2 = 2^{1/6}(\cos 11\pi/12 + i \sin 11\pi/12)$.
If $k = 2$, $z_3 = 2^{1/6}(\cos 19\pi/12 + i \sin 19\pi/12)$.
These are represented graphically in Fig. 1.32.

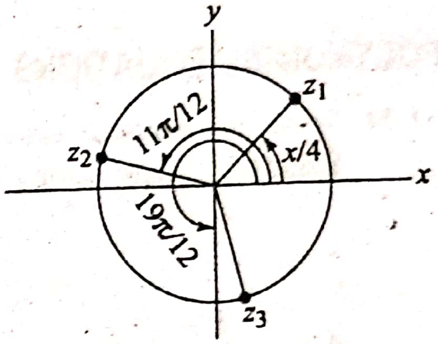


Fig. 1.32

(b) $(-2\sqrt{3} - 2i)^{1/4}$

$-2\sqrt{3} - 2i = 4\{\cos(7\pi/6 + 2k\pi) + i \sin(7\pi/6 + 2k\pi)\}$

$$(-2\sqrt{3} - 2i)^{1/4} = 4^{1/4}\left\{\cos\left(\frac{7\pi/6 + 2k\pi}{4}\right) + i\sin\left(\frac{7\pi/6 + 2k\pi}{4}\right)\right\}$$

If $k = 0$, $z_1 = \sqrt{2}\,(\cos 7\pi/24 + i \sin 7\pi/24)$.
If $k = 1$, $z_2 = \sqrt{2}\,(\cos 19\pi/24 + i \sin 19\pi/24)$.
If $k = 2$, $z_3 = \sqrt{2}\,(\cos 31\pi/24 + i \sin 31\pi/24)$.
If $k = 3$, $z_4 = \sqrt{2}\,(\cos 43\pi/24 + i \sin 43\pi/24)$.
These are represented graphically in Fig. 1.33.

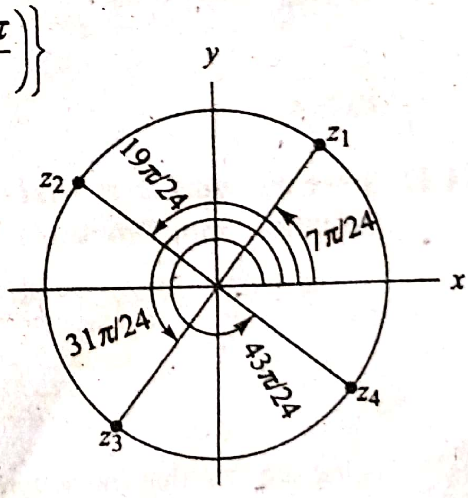


Fig. 1.33

1.30 Find the square roots of $-15 - 8i$.

Solution

Method 1.

$-15 - 8i = 17\{\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)\}$
where $\cos\theta = -15/17$, $\sin\theta = -8/17$.
Then the square roots of $-15 - 8i$ are

$$\sqrt{17}\,(\cos \theta/2 + i \sin \theta/2) \quad (1)$$

and $$\sqrt{17}\,\{\cos (\theta/2 + \pi) + i \sin (\theta/2 + \pi)\} = -\sqrt{17}\,(\cos \theta/2 + i \sin \theta/2) \quad (2)$$

Now $$\cos \theta/2 = \pm\sqrt{(1+\cos \theta)/2} = \pm\sqrt{(1-15/17)/2} = \pm 1/\sqrt{17}$$

$$\sin \theta/2 = \pm\sqrt{(1-\cos \theta)/2} = \pm\sqrt{(1-15/17)/2} = \pm 4/\sqrt{17}$$

Since θ is an angle in the third quadrant, $\theta/2$ is an angle in the second quadrant. Hence $\cos \theta/2 = -1/\sqrt{17}$, $\sin \theta/2 = 4/\sqrt{17}$ and so from (1) and (2) the required square roots are $-1 + 4i$ and $1 - 4i$. As a check, note that $(-1 + 4i)^2 = (1 - 4i)^2 = -15 - 8i$.

***Method* 2.**

Let $p + iq$, where p and q are real, represent the required square roots. Then

$$(p + iq)^2 = p^2 - q^2 + 2pqi = -15 - 8i$$

$$p^2 - q^2 = -15 \quad (1)$$

or $$pq = -4 \quad (2)$$

Substituting $q = -4/p$ from (4) into (3), it becomes $p^2 - 16/p^2 = -15$ or $p^4 + 15p^2 - 16 = 0$, i.e., $(p^2 + 16)(p^2 - 1) = 0$ or $p^2 = -16$, $p^2 = 1$. Since p is real, $p = \pm 1$. From (4) if $p = 1$, $q = -4$; if $p = -1$, $q = 4$. Thus the roots are $-1 + 4i$ and $1 - 4i$.

Functions of Complex Variables

Let $S \subset \mathbb{C}$, $S \neq \phi$. A function $f : S \to \mathbb{C}$ is a rule which assigns to each $z \in S$ a complex number $f(z)$. The number $w = f(z)$ is called the value of f at z.

Write $z = x + iy$. Then it is equivalent to saying there are two real valued function $u(x,y)$, $v(x,y)$ of the real variables x, y such that

$$f(z) = u(x,y) + i\,v(x,y).$$

In polar co-ordinates if we write $z = re^{i\theta}$, then

$$f(z) = u(r,\theta) + i\,v(r,\theta)$$

Ex: Write the function $f(z) = z^2$ in the form

$$f(z) = u(x,y) + i\,v(x,y).$$

Let, $z = x + iy$

$\therefore f(z) = f(x+iy) = (x+iy)^2 = x^2 + 2xyi - y^2$

$= x^2 - y^2 + i\, 2xy$

$\therefore \quad u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy.$

Therefore

~~$\therefore f(z) = (x^2 - y^2) + i$~~

* $\boxed{\text{Write the function } f(z) = z^2 \text{ in the form } f(z) = u(r,\theta) + iv(r,\theta)}$

Let $z = re^{i\theta}$

$\therefore f(z) = f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$= r^2(\cos 2\theta + i\sin 2\theta)$

$\therefore \quad u(r,\theta) = r^2\cos 2\theta, \quad v(r,\theta) = r^2\sin 2\theta.$

Neighbourhood of a point:

(Real এর মত Main term same)

The neighbourhood of a point $z_0 \in \mathbb{C}$, is the set of all points $z \in \mathbb{C}$ such that $|z - z_0| < \varepsilon$ where ε is a positive small number.

যেটি circle এর Inner ~~part~~ side represent করে।

$$N_\varepsilon(z_0) = \{ z : |z - z_0| < \varepsilon \}$$

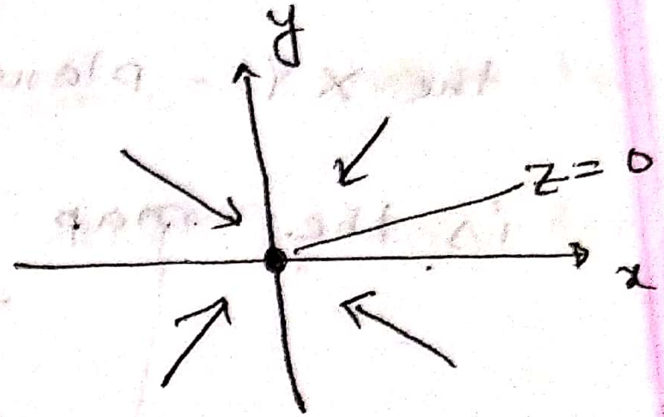
Limit of a complex function: Let a function f be defined in some neighbourhood of z_0 except possibly for the point z_0, then the limit of $f(z)$ as z approaches z_0 is a number w_0, or that

$$\lim_{z \to z_0} f(z) = w_0 \quad \text{---(i)}$$

means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but different (distinct) from it.

S.S-53

If $f(z) = \frac{z}{\bar{z}}$, then prove that the limit $\lim_{z \to 0} f(z)$ does not exist.



When $z = (x, 0)$ is a non zero point on the real axis

$$f(z) = \frac{x + 0.i}{x - 0.i} = 1.$$

And when $z = (0, y)$ is a non zero point on the imaginary axis, then

$$f(z) = \frac{0 + iy}{0 - iy} = -1$$

Thus by letting z approaches the origin along the real axis, we find the limit is 1. When along imaginary axis the limit is -1.

Since a limit should be unique, so the limit doesn't exists.

Continuity: A function f is continuous at a point z_0 if all three of the following conditions are satisfied.

a) $\lim_{z \to z_0} f(z)$ exists

(b) $f(z_0)$ is defined

(c) $\lim_{z \to z_0} f(z) = f(z_0)$

Derivatives : Let $w = f(z)$ be a function whose domain of definition contains a neighbourhood of a point z_0. The derivative of ~~two~~ $f(z)$ at z_0, written as $f'(z_0)$ and defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided the limit exists. The function f is said to be differentiable at z_0 when its derivatives at z_0 exists.

Let $\Delta z = z - z_0$ then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

■ Using definition find $\frac{dw}{dz}$ at any point z where $w = f(z) = z^2$.
(মূল নিয়মে অন্তরীকরণ)

By definition

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + 2z.\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z) = 2z.$$

using definition find $\frac{dw}{dz}$ where $\cancel{z=}$ $w = f(z) = z^2$ at $z = i$

আগের মত করতে হবে।

Then, $\therefore \left.\frac{dw}{dz}\right|_{z=i} = 2i$

■ using definition, find derivative of the following functions at the indicated points.

(a) $f(z) = \frac{2z - i}{z + 2i}$ at $z = -i$ ~~12-11~~ Ans. $-5i$

(b) $f(z) = 3z^{-2}$ at $z = 1 + i$ → Ans $\frac{6}{(1+i)^3}$ $= \frac{3}{1-i}$

See example 2
ch- 63 page,
alr, 65 page

$|z|^2 = z\bar{z}$

Test the differentiability of $f(z) = |z|^2$

Here

$$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \frac{(z+\Delta z).(\bar{z}+\overline{\Delta z}) - z.\bar{z}}{\Delta z}$$

$$= \frac{z\bar{z} + \Delta z\overline{\Delta z} + z\overline{\Delta z} + \bar{z}\Delta z - z\bar{z}}{\Delta z}$$

$$= \bar{z} + \overline{\Delta z} + \frac{\overline{\Delta z}}{\Delta z}.z$$

When Δz approaches the origin horizontally through the real axis, then points $(\Delta x, 0)$ on the

$$\overline{\Delta z} = \overline{\Delta x + 0.i} = \Delta x - 0.i = \Delta x + 0.i = \Delta z$$

Then, $\frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} + z$

and $\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = z + \bar{z}$

When Δz approaches the origin vertically through the imaginary axis, then points $(0, \Delta y)$ on the

$$\overline{\Delta z} = \overline{0 + i.\Delta y} = 0 - i.\Delta y = -(0 + i.\Delta y) = -\Delta z$$

stion: Show that $f(z) = |z|^2$ is continuous everywhere — Proof — Kuddus 89 page

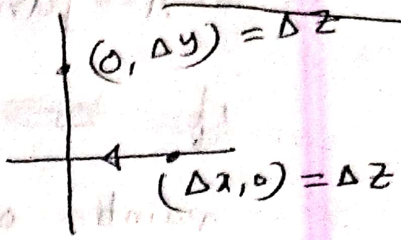
but not differentiable expect at the origin → (Sir Lecture).

OR, $f(z) = |z|^2$ is differentiable at $z=0$ but not analytic there

[Proof Titas 89 page

Then,

$$\frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} - z$$



and $$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \bar{z} - z.$$

Since limits are unique, so it must satisfy

$$z + \bar{z} = \bar{z} - z$$

$$\Rightarrow 2z = 0$$

$\therefore$ $z = 0$, if $\frac{dw}{dz}$ to exists.

একমাত্র $z=0$ তে $f(z) = |z|^2$ differentiable হয়।

at $z = 0$, $\frac{\Delta w}{\Delta z} = \overline{\Delta z}$

and $$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = 0 = \frac{dw}{dz}.$$

so, $f(z)$ is only differentiable at $z=0$ and value is also 0.

If the functions f, g are analytic ... the sum $f(z) + g(z)$ ...