System of Linear Equations

What is linear equation?
A linear equation involving two variables x and y has the standard form of ax + by = c, where a, b and c are real number and a and b cannot both equal zero and exponent (power) of variable is one. Some examples of linear equations are:

i) $4x + 7y = 15$ ii) $-x - 2/3\, y = 0$ iii) $3u - 2v = -1/2$

In practice, linear equations occur in more than two variables. A linear equation with n variables has the form of $a_1x_1 + a_2x_2 + a_3x_3 + \dots \dots + a_nx_n = b$; where a_i (i = 1, 2, 3, ..., n) are real numbers and at least one of them is not zero. The main focus here is to solve for x_i (i = 1, 2, 3, ..., n), given the values of a_i and b.

System of Linear Equation:
There is no unique solution for x_n so to find the solution with n variables; we need set of independent linear equations. This set of equations is known as *system of simultaneous linear equations* (or simply, *system of linear equations*).

A system of n linear equations is represented generally

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots \dots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots \dots + a_{2n}x_n = b_2 \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots \dots + a_{nn}x_n = b_n
\end{aligned}
$$

In matrix notion, we can express as Ax = b; where A is an n x n coefficient matrix, b is an n vector and x is a vector of n unknowns.

Existence of Solution
In solving systems of linear equations, we are interested in identifying values of the variables that satisfy all equations in the system simultaneously. Given an arbitrary system of equations, it is difficult to say whether the system has a solution or not. Sometimes there may be solution but it may not be unique. There are four possibilities:

a. System has a unique solution
b. System has no solution
c. System has a solution but not a unique one (i.e. it has infinite solutions)
d. System is ill – conditioned

Unique solution
Consider the system of two linear equations:

$3x + 2y = 18$---(1)
$-x + 2y = 2$ --- (2)

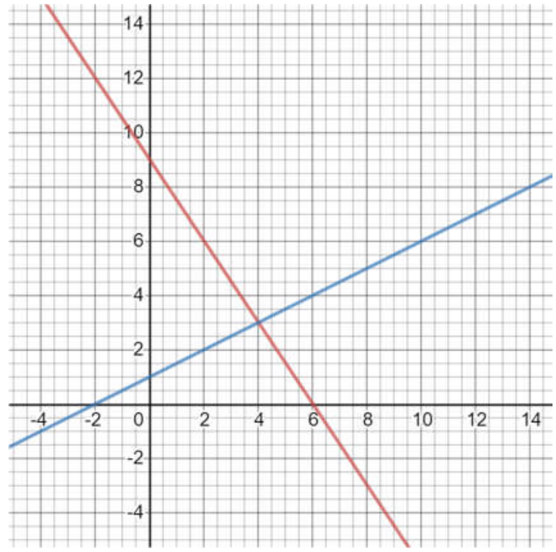

Figure 1 (a)

The red line of figure 1 (a) indicates equation 1 and blue line indicate equation 2. Both the line intersect at $x = 4$ and $y=3$. So the unique solution of x is 4 and y is 3.

No Solution

The equations $-1/2x + y = 1$ and $-1/2x + y = 1/2$ have no solution. These two lines are parallel as shown in Figure 1(b), therefore, they never meet. Such equations are called *inconsistent* equations.

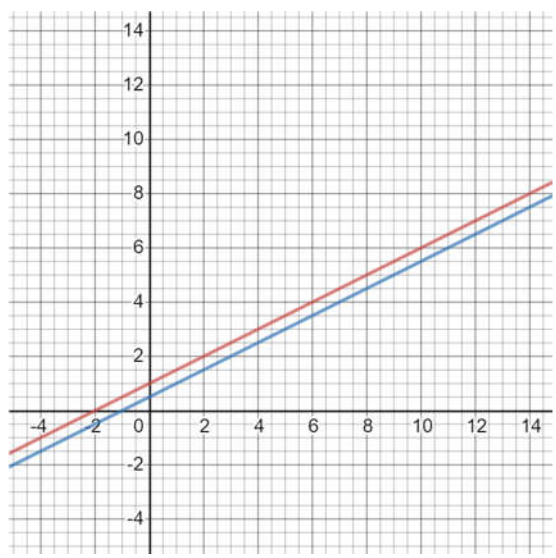

Figure 1(b)

Ill-Conditioned Systems

In ill-conditioned systems slops of equations are so close that it is very difficult to detect the point of intersections. In figure 1(c) the equations $-2.3/5x + y = 1.1$ (red line) and $-1/2x + y = 1$ (blue line) has a solution but it is very difficult to identify the exact point at which the lines intersect. Ill-conditioned systems are very sensitive to roundoff errors and, therefore, may pose problems during computation of the solution.

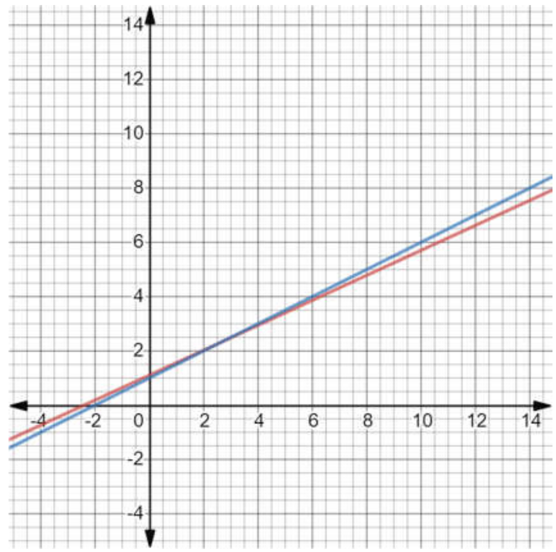

Figure 1(c)

Methods for solving system of linear algebraic equations:
The techniques and methods for solving system of linear algebraic equations belong to two fundamentally different approaches:
a. ***Elimination approach*****:** Elimination approach, also known as direct method, reduces the given system of linear equations to a form from which the solution can be obtained by simple substitution.
b. ***Iterative approach*****:** Iterative approach, as usual, involves assumption of some initial values which are then refined repeatedly till they reach some accepted level of accuracy.

Note:
Let us consider a general form of a system of linear equations of size m x n.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots + a_{2n}x_n = b_2$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$a_{m1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots + a_{mn}x_n = b_m$$

In order to effect a unique solution, the number of equations m should be equal to the number of unknowns n. If m < n, the system is said to be *under-determined* and a unique solution for all unknowns is not possible. On the other hand, if the number of equations is larger than the number of unknowns, then the set is said to be *over-determined*, and a solution may or may not exist.

The system is said to be *homogeneous* when the constants b_i are all zero. Otherwise, the system is called a *non-homogeneous system* (at least one $b_i \neq 0$).

Solution by Elimination

Elimination is a method of solving simultaneous linear equations. It involves elimination of a term containing one of the unknowns in all but one equation. One such step reduces the order of equation by one. Repeated elimination leads finally to one equation with one unknown.

Rule:

a) An equation can be multiplied or divided by a constant.
b) One equation can be added or subtracted from another equation.
c) Equations can be written in any order.

Basic Gauss Elimination Method

This is the elementary elimination method and it reduces the system of equations to an equivalent upper triangular system, which can be solved by back substitution. Although quite general, we shall describe this method by considering a system of three equations for the sake of clarity and simplicity. Let the system be

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

We first form the augmented matrix of the system (1)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix} \qquad \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

To eliminate x_1 from the second equation, we multiply the first equation by $-a21 / a11$ and then add it to the second equation. Similarly, to eliminate x_1 from the third equation, we multiply the first equation by $-a31 / a11$ and then add it to the third equation. $-a21 / a11$ and $-a31 / a11$ are called the *multipliers/pivot* for the *first stage of elimination*. In this stage, we have assumed that $a_{11} \neq 0$. At the end of the first stage of elimination, the augment matrix becomes

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{pmatrix} \qquad \dots\dots\dots\dots\dots\dots\dots\dots (3)$$

where a'_{22}, a'_{23} ... are all changed elements.

Now, to eliminate x_2 from the third equation, we multiply the second equation by $-a'_{32} / a'_{22}$ and then add it to the third equation. Again, we assume that $a_{22} \neq 0$. At the end of the second stage, we have the upper triangular system

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{pmatrix} \qquad \dots\dots\dots\dots\dots\dots\dots\dots (4)$$

from which the values of x_1, x_2 and x_3 can be obtained by the back substitution.

Example: Solve the following system by using Gauss elimination method

$$2x + y + z = 10 \rightarrow R1$$
$$3x + 2y + 3z = 18 \rightarrow R2$$
$$x + 4y + 9z = 16 \rightarrow R3$$

Solution: In the first stage, the multipliers are –3/2 and –1/2. We multiply the first equation successively by –3/2 and –1/2 and add it to the second and third equations respectively to obtain the equations: $1/2\, y + 3/2\, z = 3$ and $7/2\, y + 17/2\, z = 11$

$R2 = R2 + R1*(-3/2)$

$$\begin{array}{lllll} 3x & + & 2y & + 3z & = 18 \\ 2(-3/2)x & + & (-3/2)y & + (-3/2)z & = 10(-3/2) \\ \hline 0 & + & 1/2y & +3/2z & = 3 \end{array}$$

$R2 : 1/2\, y + 3/2\, z = 3$

$R3 = R3 + R1*(-1/2)$

$$\begin{array}{lllll} x & + & 4y & + 9z & = 16 \\ 2(-1/2)x & + & (-1/2)y & + (-1/2)z & = 10(-1/2) \\ \hline 0 & + & 7/2y & +17/2z & = 11 \end{array}$$

$R3 : 7/2\, y + 17/2\, z = 11$

The augmented matrix therefore becomes

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/2 & 11 \end{bmatrix}$$

$$2x + y + z = 10 \rightarrow R1$$
$$1/2y + 3/2z = 3 \rightarrow R'2$$
$$7/2\, y + 17/2\, z = 11 \rightarrow R'3$$

At the second stage, we eliminate y from the third equation by multiplying the second equation by –7 and adding it to the third.

$R3 = R3 + R2*(-7)$

$$\begin{array}{lll} 7/2\, y & + 17/2\, z & = 11 \\ ½ (-7)y & + 3/2*(-7)z & = 3(-7) \\ \hline 0 & - 2z & = -10 \end{array}$$

$R3 : -2\, z = 10$

The resulting system will be upper triangular

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & 10 \end{bmatrix}$$

$$
\begin{aligned}
2x \quad + y \quad + z \quad &= 10 \\
1/2\, y + 3/2\, z &= 3 \\
-2z \quad &= -10
\end{aligned}
$$

Back substitution gives the solution:
$-2z = -10$
$z = 5$
now put value of z in R'2
$1/2\, y + 3/2\,(5) = 3$
$y = -9$
now put value of y and z in R1
$2x - 9 + 5 = 10$
$2x = 14$
$x = 7$

so, $x = 7$, $y = -9$ and $z = 5$

Algorithm: Gauss Elimination (basic) Method

Forward Elimination:

```
//loop over all rows except last
for  k=0 to n−1 do
//loop over all rows bellow the diagonal position
      for i = k+1 to n do
//search for pivot element
//loop over all columns right of the diagonal position
                for j = k+1 to n do
         a_ij = a_ij − a_kj * a_jk / a_kk
         end

     end
end
```

Backward Substitution:

```
//Compute last unknown
xn = bn/ann
//loop over all the row except last row
for i = n−1to 1 do
//loop over all columns to the right of the current row
   for j = i+1 to 1
      xi = 1/aii(b_i − Σ_{j=i+1}^{n} a_ij x_j)
   end
end
```

Gauss Elimination Method with Pivoting

Here, aij, when i = j, is known as a pivot element. Each row is normalized by dividing the coefficient of that row by its pivot element.

i.e. $\mathbf{a_{kj} = a_{kj} / a_{kk}}$ where j = 1, 2, ….. , n

If $a_{kk} = 0$; kth row cannot be normalized. Therefore, the procedure fails as in the above case. And of course, to overcome this problem is to interchange this row with another row below it which does not have a zero element in that position. But, there may be more than one non-zero values in the kth column below the element a_{kk}. So, the question is: which one of them is to be selected? It can be proved that round off error would be reduced if the absolute value of the pivot element is large. Therefore, it is suggested that the row with zero pivot element should be interchanged with the row having the largest (absolute) coefficient in that position. In general, the reordering of equations is done to improve accuracy, even if the pivot element is not zero.
In general: the reordering of equations is done to improve accuracy, even if the pivot element is not zero.
The procedure of reordering involves the following steps:

1. Search and locate the largest absolute value among the coefficients in the first column.
2. Exchange the first row with the row containing that element.
3. Then eliminate the first variable in the other equations as explained earlier.
4. When the second row becomes the pivot row, search for the coefficients in the second column from the second row to the nth row and locate the largest coefficient. Exchange the second row with the row containing the large coefficient.
5. Continue this procedure till (n-1) unknowns are eliminated.

This process is referred to as *partial pivoting*. There is an alternative scheme known as *complete pivoting* in which, at each stage, the largest element at any of the remaining rows is used as the pivot.

Algorithm: Gauss Elimination Method with Partial Pivoting

```
for i = 1 to n do
set pivot = |a_ii|
set r_max = i
//search for maximum coefficient
  for k = i + 1 to n do
     r = |a_ki/a_ii |
     if(r > pivot) then
            pivot = r
            r_max = k
  end
   for k = i + 1 to n do
      swap a_rmax, k and a_ik
    end
end
//Forward Elimination
//Backward Substitution
```

Example: Solve the following system by using Gauss elimination method with partial pivoting

$$2x + 2y + z = 6$$
$$4x + 2y + 3z = 4$$
$$x - y + z = 0$$

Solution:
Original System

2	2	1	6
4	2	3	4
1	-1	1	0

Let,
R1 = 2x + 2y + z = 6
R2 = 4x + 2y + 3z = 4
R3 = x – y + z = 0

Step 1: Interchange first row with row contain highest value of coefficient

4	2	3	4	[Interchange R1 and R2]
2	2	1	6	
1	-1	1	0	

Step 2:
Pivot elements -2/4 and -1/4

Multiply R1 by -2/4:
4*(-2/4) + 2*(-2/4) +3*(-2/4) = 4*(-2/4)
⇨ -2 -1 -3/2 = -2

R2 = R1-R2
-2 -1 -3/2 = -2
2 +2 +1 = 6

---

0 +1 -1/2 = 4

Similarly R3 = R1*(-1/4) – R3

First Derived System

4	2	3	4
0	1	-1/2	4
0	-3/2	1/4	-1

R1 = 2x + 2y + z = 6
R'2 = -3/2 y + ¼ z = 4
R'3 = y -1/2z = 4

Step 2:

4	2	3	4	[Pivot = -2/3]
0	-3/2	1/4	-1	
0	1	-1/2	4	

Now, R’3 = R’2 * -(2/3) - R’3

Second and final Derived System

4	2	3	4
0	-3/2	1/4	-1
0	0	-1/3	10/3

Step 3: Backward Substitution

R1 = 2x + 2y + z = 6
R’2 = -3/2 y + ¼ z = -1
R’’3 = -1/3z = 10/3

From R’’3,
-1/3z = 10/3
z = -10

put the value of z at R’2 we get,
-3/2 y + ¼ (-10) = -1
-3/2 y = -1 + 5/2 = 3/2
y = -1

put value of y and z at R1
we get x = 9

The solution is
z = -10, y = -1 and x = 9

Gauss-Jordan Method

Gauss-Jordan method is another popular method used for solving a system of linear equations. Like Gauss elimination method, Gauss-Jordan method also uses the process of elimination of variables, but there is a major difference between them. In Gauss elimination method, a variable is eliminated from the rows below the pivot equation. But in Gauss-Jordan method, it is eliminated from all other rows (both below and above). This process thus eliminates all the off-diagonal terms producing a diagonal matrix rather than a triangular matrix. Further, all rows are normalized by dividing them by their pivot elements.
Consequently, we can obtain the values of unknowns directly from the b vector, without employing back substitution.

Algorithm 7.3: Gauss-Jordan Elimination Method
1. Normalize the first equation by dividing it by its pivot element.
2. Eliminate x1 term from all the other equations.
3. Now, normalize the second equation by dividing it by its pivot element.
4. Eliminate x2 from all the equations, above and below the normalized pivotal equation.
5. Repeat this process until xn is eliminated from all but the last equation.
6. The resultant b vector is the solution vector.

- The Gauss-Jordan method requires approximately 50 percent more arithmetic operations compared to Gauss method. Therefore, this method is rarely used.
- See following table: for the comparison of computational effort.

	Gauss Method	Gauss-Jordan Method
Multiplication	$1/3\ n^3$	$1/2\ n^3$
Subtraction	$1/3n^3$	$1/2n^3$
Divisions	$1/2n^2$	$1/2n^2$

Algorithm:
For i=1 to n do
  For j-1 to n do
    If $i \neq j$
      $pivot = \frac{a_{ji}}{aii}$
      For k = 1 to n+1 do
        $a_{jk} = a_{jk} - pivot \times a_{ik}$

For i=1 to n+ 1 do
  $x_i = \frac{a_{i,n+1}}{a_{ii}}$

Example: Solve the following system of linear equations using Gauss-Jordan method
R1: 2x + 4y – 6z = –8
R2: x + 3y + z = 10
R3: 2x – 4y – 2z = – 12

Solution:

$$\begin{pmatrix} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & 10 \\ 2 & -4 & -2 & -12 \end{pmatrix}$$

Step 1: Normalize the first equation

R1<-R1*1

$$\begin{pmatrix} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & 10 \\ 2 & -4 & -2 & -12 \end{pmatrix}$$

x + 2y – 3z = –4
x + 3y + z = 10
2x – 4y – 2z = – 12

Step 2: Eliminate x
R2<- R2 – R1
x + 3y + z = 10
– x –2y + 3z = 4

0 + y + 4z = 14

R3<-R3 – 2R1

$$\begin{pmatrix} 1 & 2 & -3 -4 \\ 0 & 1 & 4 \ 14 \\ 0 & -8 & 4 \ -4 \end{pmatrix}$$

x + 2y – 3z = –4
0 + y + 4z = 14
0 – 8y + 4z = – 4

Step 3: Normalize the second equation
R2<-R*1

$$\begin{pmatrix} 1 & 2 & -3 -4 \\ 0 & 1 & 4 \ 14 \\ 0 & -8 & 4 \ -4 \end{pmatrix}$$

x + 2y – 3z = –4
0 + y + 4z = 14
0 – 8y + 4z = – 4

Step 4: Eliminate y
R1<-R1-2R2
R3<-R3+8R2

$$\begin{pmatrix} 1 & 0 & -11 -32 \\ 0 & 1 & 4 \quad 14 \\ 0 & 0 & 36 \ \ 108 \end{pmatrix}$$

~ x + 0 – 11z = –32
0 + y + 4z = 14
0 + 0 + 36z = 108

Step 5: Normalize the third equation
R3<-1/36*R3

$$\begin{pmatrix} 1 & 0 & -11 -32 \\ 0 & 1 & 4 \quad 14 \\ 0 & 0 & 1 \quad 3 \end{pmatrix}$$

~x + 0 – 11z = –32
0 + y + 4z = 14
0 + 0 + z = 3

Step 6: Eliminate z
R1<-R1 + 11R3
R2 <- R2 – 4R3

$$\begin{pmatrix} 1 & 0 & 0-1 \\ 0 & 1 & 0\ 2 \\ 0 & 0 & 1\ 3 \end{pmatrix}$$

~ x + 0 + 0= 1
0 + y + 0 = 2
0 + 0 + z = 3

x=1
y=2
z=3

Gauss-Jordan Matrix Inversion Method
Example: Solve the following system of linear equations using Gauss-Jordan matrix inversion method

2x + y + z = 7	Row 1
x – y + z = 0	Row 2
4x + 2y – 3z = 4	Row 3

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cong \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{bmatrix}$$

Solution:

$$\text{Augmented A} = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 4 & 2 & -3 & 0 & 0 & 1 \end{array}\right)$$

Step 1:
Pivot row-1:
Sub Step 1: Row1 = Row 1 /a [divided by a]
Sub Step 2: Row2 = Row 2 – (d*Row1)
Sub Step 3: Row 3 = Row 3 – (g*Row1)

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & -3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right)$$

Pivot row-2:
Sub Step 1: Row 2 = Row 2 /e
Sub Step 2: Row 1 = Row 1 – (b*Row2)
Sub Step 3: Row 3 = Row 3 – (h*Row2)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & -1/3 & 1/3 & -2/3 & 0 \\ 0 & 0 & -5 & -2 & 0 & 1 \end{array}\right]$$

Pivot row-3:

Sub Step 1: Row 3/i
Sub Step 2: Row 1 – (c*Row3)
Sub Step 3: Row 2 – (f*Row3)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/15 & 1/3 & 2/15 \\ 0 & 1 & 0 & 7/15 & -2/3 & -1/15 \\ 0 & 0 & 1 & 2/5 & 0 & -1/5 \end{array}\right]$$

The last three columns represent the inverse of the matrix.
Therefore, the solution of the system of equations is obtained from $X = A^{-1}B$

References:

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