Chapter Three

Complex Differentiation and the Cauchy-Riemann Equations

DERIVATIVES

If $f(z)$ is single-valued in some region $\Re$ of the z plane, the *derivative* of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{3.1}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such case we say that $f(z)$ is *differentiable* at z. In the definition (3.1) we sometimes use h instead of Δz. Although differentiability implies continuity, the reverse is not true (see Problem 3.4).

ANALYTIC FUNCTIONS

If the derivative $f'(z)$ exists at all points z of a region $\Re$, then $f(z)$ is said to be *analytic in* $\Re$ and is referred to as an *analytic function in* $\Re$ or a function *analytic in* $\Re$. The terms *regular* and *holomorphic* are sometimes used as synonyms for analytic.

A function $f(z)$ is said to be *analytic at a point* z_0 if there exists a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

CAUCHY-RIEMANN EQUATIONS

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\Re$ is that, in $\Re$, u and v satisfy the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2}$$

If the partial derivatives in (3.2) are continuous in $\Re$, then the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in $\Re$. See Problem 3.5.

The functions $u(x, y)$ and $v(x, y)$ are sometimes called *conjugate functions*. Given u having continuous first partials on a simply connected region $\Re$ (see Section 4.6), we can find v (within an arbitrary additive constant) so that $u + iv = f(z)$ is analytic (see Problems 3.7 and 3.8).

(1) (a) Let $f(z) = \sin z = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x . \frac{e^{i.iy} + e^{-i.iy}}{2} + \cos x . \frac{e^{i.iy} - e^{-i.iy}}{2i}$

$= \sin x \frac{e^{-y} + e^{y}}{2} - i^2 \cos x \frac{e^{-y} - e^{y}}{2i}$

$= \sin x \cosh y + i \cos x \sinh y$

Let $u(x,y) = \sin x \cosh y$, $v(x,y) = \cos x \sinh y$

$\therefore u_x = \cos x \cosh y$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $v_y = \cos x \cosh y$

so, we see that $u_x = v_y$ and $u_y = -v_x$

clearly, partial derivatives are ~~are~~ continuous.

Hence $\sin z$ is analytic.

$\sin z$ ~~co~~ entire functi

Harmonic function : A real valued function H of two real variable x and y are said to be harmonic in a given domain of the xy plane, if through out that domain, it has continuous partial derivatives of the first and second order and satisfies the equation

CH-75 $$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = 0$$ known as Laplace's equation.

Harmonic conjugate : If two given functions u and v are harmonic in a domain D and their first order partial derivate satisfies Cauchy-Riemann equations

$$u_x = v_y \qquad u_y = -v_x$$

throughout D. v is said to be a harmonic conjugate of u.

[1] Show that $u(x, y)$ is harmonic and then find the harmonic conjugate $v(x, y)$ and the corresponding analytic function $f(z) = u + iv$.

(a) $u(x, y) = 2x - x^3 + 3xy^2$ — T 113

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Sol^n: (a) Given, $U = 2x - x^3 + 3xy^2$

$U_x = 2 - 3x^2 + 3y^2$

$U_{xx} = -6x$

$U_y = 6xy$

$U_{yy} = 6x$

clearly partial derivatives of u are continuous

$U_{xx} + U_{yy} = 0$

$\therefore$ U is a harmonic function.

Now from C-R equations, we get $u_x = v_y$ and $u_y = -v_x$

So, $v_y = 2 - 3x^2 + 3y^2$

$\Rightarrow v = \int (2 - 3x^2 + 3y^2)\, \partial y$

$\therefore v = 2y + y^3 - 3x^2y + g(x)$

$\therefore v_x = -6xy + g'(x)$

$\Rightarrow -u_y = -6xy + g'(x)$

$\Rightarrow -6xy = -6xy + g'(x)$

$\Rightarrow g'(x) = 0$

$g(x) = c_1 \to$ (real constant)

Therefore $v(x, y) = 2y - 3x^2y + y^3 + c_1$

This is the required harmonic conjugate of u.

Then

$\therefore f(z) = u + iv = 2x - x^3 + 3xy^2 + i(2y - 3x^2y + y^3 + c_1)$

$= 2(x + iy) - (x^3 - 3xy^2 + i\,3x^2y - iy^3) + ic_1$

$= 2z - \{x^3 + 3x(iy)^2 + 3x^2 \cdot iy + (iy)^3\} + c$

$= 2z - (x + iy)^3 + c$

$= 2z - z^3 + c$

which is the required analytic function.

3.2 Show that $\frac{d}{dz}\bar{z}$ does not exist anywhere, i.e. $f(z) = \bar{z}$ is non-analytic anywhere.

Solution By definition, $\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

if this limit exists independent of the manner in which $\Delta z = \Delta x + i\,\Delta y$ approaches zero.

Then
$$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+iy+\Delta x+i\,\Delta y} - \overline{x+iy}}{\Delta x + i\,\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\,\Delta y - (x - iy)}{\Delta x + i\,\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}$$

If $\Delta y = 0$, the required limit is

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

If $\Delta x = 0$, the required limit is

$$\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e. $f(z) = \bar{z}$ is *non-analytic* anywhere.

3.3 Given $w = f(z) = \frac{(1+z)}{(1-z)}$, find (a) $\frac{dw}{dz}$ and (b) determine where $f(z)$ is non-analytic.

Solution

(a) *Method 1.* Using the definition

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{1+(z+\Delta z)}{1-(z+\Delta z)} - \frac{1+z}{1-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2}{(1-z-\Delta z)(1-z)} = \frac{2}{(1-z)^2}$$

independent of the manner in which $\Delta z \to 0$, provided $z \neq 1$.

Method 2. Using differentiation rules.

By the quotient rule [see Problem 3.10(c)], we have if $z \neq 1$,

$$\frac{d}{dz}\left(\frac{1+z}{1-z}\right) = \frac{(1-z)\frac{d}{dz}(1+z)-(1+z)\frac{d}{dz}(1-z)}{(1-z)^2} = \frac{(1-z)(1)-(1+z)(-1)}{(1-z)^2} = \frac{2}{(1-z)^2}$$

(b) The function $f(z)$ is analytic for all finite values of z except $z = 1$ where the derivative does not exist and the function is non-analytic. The point $z = 1$ is a *singular point* of $f(z)$.

continuous in $\mathcal{R}$. Hence, the above assumptions will not be necessary.

3.7 (a) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.
(b) Find v such that $f(z) = u + iv$ is analytic.

Solution

(a) $$\frac{\partial u}{\partial x} = (e^{-x})(\sin y) + (-e^{-x})(x \sin y - y \cos y) = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y) = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \cos y$$

$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y) = xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y) = -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y$$

Adding (1) and (2) yields $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

(b) From the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y \tag{4}$$

Integrate (3) with respect to y, keeping x constant. Then

$$\begin{aligned} v &= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x) \\ &= ye^{-x}\sin y + xe^{-x}\cos y + F(x) \end{aligned} \tag{5}$$

where $F(x)$ is an arbitrary real function of x.

Substitute (5) into (4) and obtain

$$-ye^{-x}\sin y - xe^{-x}\cos y + e^{-x}\cos y + F'(x) = -ye^{-x}\sin y - xe^{-x}\cos y - ye^{-x}\sin y$$

or $F'(x) = 0$ and $F(x) = c$, a constant. Then from (5),

$$v = e^{-x}(y\sin y + x\cos y) + c$$

For another method, see Problem 3.55.

3.8 Find $f(z)$ in Problem 3.7.

Solution

Method 1. We have $f(z) = f(x + iy) = u(x, y) + i\,v(x, y)$.

Putting $y = 0$, $f(x) = u(x, 0) + i\,v(x, 0)$.

Replacing x by z, $f(z) = u(z, 0) + i\,v(z, 0)$.

Then, from Problem 3.7, $u(z, 0) = 0$, $v(z, 0) = ze^{-z}$ and so $f(z) = u(z, 0) + i\,v(z, 0) = i\,ze^{-z}$, apart from an arbitrary additive constant.

Method 2. Apart from an arbitrary additive constant, we have from the results of Problem 3.7,

$$\begin{aligned} f(z) &= u + iv = e^{-x}(x\sin y - y\cos y) + ie^{-x}(y\sin y + x\cos y) \\ &= e^{-x}\left\{x\left(\frac{e^{iy} - e^{-iy}}{2i}\right) - y\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} + ie^{-x}\left\{y\left(\frac{e^{iy} - e^{-iy}}{2i}\right) + x\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} \\ &= i(x + iy)\,e^{-(x+iy)} = i\,ze^{-z} \end{aligned}$$

Method 3. We have $x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$. Then, substituting into $u(x, y) + i\,v(x, y)$, we find after much tedious labour that $\bar{z}$ disappears and we are left with the result $i\,ze^{-z}$.

In general, method 1 is preferable over methods 2 and 3 when both u and v are known. If only u (or v) is known another procedure is given in Problem 3.116.

3.9 If the potential function is ...

Another procedure is given in Problem 3.116.

3.9 If the potential function is $\log \sqrt{x^2 + y^2}$, find the flux function and the complex potential function.

Solution

If ϕ and ψ be the potential function and flux function respectively, then the complex potential function w is given by

$$w(z) = \varphi(x, y) + i\psi(x, y) \tag{1}$$

where
$$\phi = \log \sqrt{x^2 + y^2} = \frac{1}{2}\log(x^2 + y^2)$$

Now,
$$\frac{\partial \phi}{\partial x} = \frac{x}{(x^2 + y^2)} \tag{2}$$

and $$\frac{\partial \phi}{\partial y} = \frac{y}{x^2 + y^2} \tag{3}$$

From (1) and using C-R equations, we get

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} \tag{4}$$

Substituting values of $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ from (2) and (3) in (4), we get

$$\frac{dw}{dz} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Replacing x by z and y by zero, we obtain

$$\frac{dw}{dz} = \frac{z}{z^2} = \frac{1}{z}$$

Integrating both sides w.r.t. z, we have

$$w = \log z + c, \quad \text{(where } c \text{ is a complex constant)}$$

which is required complex potential function.

Now, $$w = \log(x + iy) + A + iB \quad (\text{Taking } c = A + iB)$$

$\log re^{i\theta}$
$\log r + \log e^{i\theta}$
$\log r + i\theta$

or $$\varphi + i\psi = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x} + A + iB$$

Comparing real and imaginary parts, we have $\psi = \tan^{-1}\left(\frac{y}{x}\right) + B$, the required flux function.

3.10 In a two dimensional fluid flow, the stream functions is

$$\psi = -\frac{y}{x^2 + y^2}, \text{ find the velocity potential } \phi.$$

Solution Since ψ is stream function, it must be harmonic, i.e., it must satisfy

Laplace equation, $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{1}$$

Now, $$\psi = -\frac{y}{x^2 + y^2}, \frac{\partial \psi}{\partial x} = \frac{2xy}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 \psi}{\partial x^2} = 2y\left[\frac{(x^2 + y^2)^2.1 - 2(x^2 + y^2).2x.x}{(x^2 + y^2)^4}\right] = \frac{2y(y^2 - 3x^2)}{(x^2 + y^2)^3} \tag{2}$$

Now $$\frac{\partial \psi}{\partial y} = -\left[\frac{(x^2 + y^2).1 - 2y.y}{(x^2 + y^2)^2}\right] = \frac{-(x^2 - y^2)}{(x^2 + y^2)^2}$$

and $$\frac{\partial^2 \psi}{\partial y^2} = -\left[\frac{(x^2+y^2)^2.(-2y) - 2(x^2+y^2).2y(x^2-y^2)}{(x^2+y^2)^4}\right] = \frac{-2y(y^2-3x^2)}{(x^2+y^2)^3}$$

From (2) and (3), we get $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$. Hence Laplace equation is satisfied.

Now ϕ is the velocity potential, let $w(z) = \phi(x, y) + i\psi(x, y)$

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}$$

Substituting values of $\frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi}{\partial y}$, we get

$$\frac{dw}{dz} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$$

Replacing x by z and y by zero, we get

$$\frac{dw}{dz} = -\frac{z^2}{z^4} = -\frac{1}{z^2}$$

Integrating w.r.t. z we get, $w = \frac{1}{z} + C$, where C is a complex constant.

or $$\phi + i\psi = \frac{1}{x+iy} + C = \frac{x-iy}{x^2+y^2} + C$$

$$= \frac{x-iy}{x^2+y^2} + A + iB \quad (\text{where } C = A + iB)$$

Equating real parts on both sides, we have velocity potential,

$$\phi = \frac{x}{x^2+y^2} + A.$$

$$= i(r^2 e^{2i\theta} - re^{i\theta}) + C + 2i$$

3.16 Find the values of constants a, b, c and d such that the function $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic.

Solution
$$f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2) = u + iv \text{(say)}$$

where
$$u = x^2 + axy + by^2, \quad v = cx^2 + dxy + y^2$$

$\therefore$
$$u_x = 2x + ay, \qquad u_y = ax + 2by$$
$$v_x = 2cx + dy, \qquad v_y = dx + 2y$$

Since $f(z) = u + iv$ is analytic, so Cauchy-Riemann equation must be satisfied.

i.e., $u_x = v_y$ and $u_y = -v_x$

Now,
$$u_x = v_y \Rightarrow 2x + ay = dx + 2y \qquad (1)$$

and
$$u_y = -v_x \Rightarrow ax + 2by = -2cx - dy \qquad (2)$$
$$(1) \Rightarrow 2x - dx + ay - 2y = 0 \Rightarrow (2-d)x + (a-2)y = 0$$
$$(2) \Rightarrow ax + 2cx + 2by + dy = 0 \Rightarrow (a+2c)x + (2b+d)y = 0$$

(1) and (2) will hold good if
$$2 - d = 0, a - 2 = 0$$

and
$$a + 2c = 0, 2b + d = 0$$

i.e.,
$$a = 2, d = 2, c = -1, b = -1$$

3.17 Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is continuous as well as analytic everywhere.

Solution If
$$f(z) = u(x, y) + iv(x, y)$$

then
$$u(x, y) = \sin x \cosh y, \quad \text{and} \quad v(x, y) = \cos x \sinh y$$

Since u and v both are rational functions of x and y, whose denominators are non-zero for all values of x and y, therefore u and v are both continuous everywhere. Hence $f(z) = u + iv$ is also continuous everywhere.

Again,
$$\frac{\partial u}{\partial x} = \cos x \cosh y, \quad \frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y, \quad \frac{\partial v}{\partial y} = \cos x \cosh y.$$

The four partial derivatives are rational functions of x and y with non-zero denominators for all values of x and y, therefore, they are continuous everywhere.

Also, here, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

That is, Cauchy-Riemann conditions are satisfied.

Thus the four partial derivatives being continuous everywhere and Cauchy-Riemann equations being satisfied. The function $f(z)$ is analytic everywhere.

3.18 Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \ (z \neq 0), \ f(0) = 0.$$

Is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist there

Solution Here, $$f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$$

So $$u = \frac{x^3 - y^3}{x^2 + y^2}, \quad v = \frac{x^3 + y^3}{x^2 + y^2} \text{ (where } z \neq 0).$$

Here, we see that both u and v are rational and finite for all values of $z \neq 0$, so u and v are continuous at all those points for which $z \neq 0$. Hence $f(z)$ is continuous where $z \neq 0$.

At the origin $u = 0$, $v = 0$. [since $f(0) = 0$].

Hence u and v are both continuous at the origin; therefore $f(z)$ is continuous at the origin.

Now, at the origin

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0}\left(\frac{x}{x}\right) = 1, \quad \frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0}\left(\frac{-y}{+y}\right) = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0}\left(\frac{x}{x}\right) = 1, \quad \frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0}\left(\frac{y}{y}\right) = +1$$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Therefore, Cauchy-Riemann equations are satisfied at $z = 0$. Again,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0}\left[\frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x + iy}\right]$$

Let $z \to 0$ along $y = x$ then we have

$$f'(0) = \lim_{z \to 0} \frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x + iy} = \lim_{z \to 0} \frac{2i}{2(1+i)} = \frac{1}{2}(1-i)$$ check

Further, let $z \to 0$ along $y = 0$, then we have $f'(0) = \lim_{z \to 0} \frac{x^3(1+i)}{x^3} = 1 + i$. Hence, $f'(0)$ is not unique.

Thus, $f'(z)$ does not exist at the origin.

3.19 Show that the function $f(z) = \sqrt{(|xy|)}$ is not regular at the origin, although Cauchy-Reimann equations are satisfied at the point.

Solution If function be $f(z) = u(x, y) + iv(x, y)$,

then $u(x, y) = \sqrt{(|xy|)}$ and $v(x, y) = 0$. Now, at the origin.

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0, 0)}{x} = \lim_{x\to 0}\left(\frac{0-0}{x}\right) = 0, \quad \frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0, 0)}{y} = \lim_{y\to 0}\left(\frac{0-0}{y}\right) = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0, 0)}{x} = \lim_{x\to 0}\left(\frac{0-0}{x}\right) = 0, \quad \frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0, 0)}{y} = \lim_{y\to 0}\left(\frac{0-0}{y}\right) = 0$$

$\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Hence, C-R equations are satisfied at the origin.

Again, $f'(0) = \lim\limits_{z\to 0} \dfrac{f(z) - f(0)}{z} = \lim\limits_{z\to 0} \dfrac{\sqrt{(|xy|)}}{x+iy} = \lim\limits_{x\to 0} \dfrac{\sqrt{(|mx^2|)}}{x+imx}$,

taking $z \to 0$ along $y = mx$

$= \dfrac{\sqrt{(|m|)}}{1+im}$, which depends on m, that is $f'(0)$ is not unique.

Hence, $f(z)$ is not analytic at the origin although C-R equations are satisfied there.

3.20 Show that the function $u = \cos x \cosh y$ is harmonic and find its harmonic conjugate.

Solution It is given that $u = \cos x \cosh y$

then $\dfrac{\partial u}{\partial x} = -\sin x \cosh y, \quad \dfrac{\partial u}{\partial y} = \cos x \sinh y$

Now, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0 \Rightarrow u$ is a harmonic function.

Let v be its conjugate harmonic function, then we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = -\cos x \sinh y\, dx - \sin x \cosh y\, dy$$

$$= -(\cos x \sinh y\, dx + \sin x \cosh y\, dy).$$

Integrating, we obtain

$v = -\sin x \sinh y + c$, where c is a real constant.

3.21 Prove that $u = y^3 - 3x^2 y$ is a harmonic function. Determine its harmonic conjugate, hence find the corresponding analytic function $f(z)$ in terms of z.

Solution Given $u = y^3 - 3x^2 y$ (1)

$$\Rightarrow \frac{\partial u}{\partial x} = -6xy, \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2, \quad \frac{\partial^2 u}{\partial x^2} = -6y, \quad \frac{\partial^2 u}{\partial y^2} = 6y$$

$\therefore \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial^2 y} = -6y + 6y = 0 \Rightarrow u$ satisfies Laplace's equation, so u is a harmonic function. Further, let v be the harmonic conjugate to u, then we have

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$= -(3y^2 - 3x^2)dx - 6xy\, dy = -(3y^2 dx + 6xy\, dy) + 3x^2 dx.$$

Integrating, $v = -3xy^2 + x^3 + c$.

3.22 If $u(x, y) = x^3 - 3xy^2$, show that there exists a function $v(x, y)$ such that $w = u + iv$ is analytic in a finite region.

Solution
$$u = x^3 - 3xy^2, \Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy, \quad \frac{\partial^2 u}{\partial y^2} = -6x, \text{ Now } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

Thus, the given function u satisfies Laplace's equation and is therefore a harmonic function. Further v is a function of x, y.

$$\therefore dy = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$
$$= 6xy\, dx + (3x^2 - 3y^2)\, dy = (6xy\, dx + 3x^2\, dy) - 3y^2\, dy.$$

Integrating this equation, we get $v = 3x^2y - y^3 + c$, where c is constant.

$$\therefore \quad f(z) = u + iv = x^3 - 3xy^2 + i(3x^2 y - y^3 + c)$$
$$= (x + iy)^3 + ic = z^3 + ic,$$

$\Rightarrow f'(z) = 3z^2$, which exists for all finite values of z.
Hence, $f(z)$ is analytic in any finite region.

3.23 Prove that, if $u = x^2 - y^2$, $v = -y/(x^2 + y^2)$, both u and v satisfy Laplace's equation, but $u + iv$ is not an analytic function of z.

Solution
$$u = x^2 - y^2, \; v = -\frac{y}{x^2 + y^2} \; \therefore \; \frac{\partial u}{\partial x} = 2x, \; \frac{\partial u}{\partial y} = -2y$$

and
$$\frac{\partial v}{\partial y} = \frac{-(x^2 + y^2) + 2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \; \frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2, \; \frac{\partial^2 u}{\partial y^2} = -2, \; \frac{\partial^2 v}{\partial x^2} = \frac{2y(y^2 - 3x^2)}{(x^2 + y^2)^2}, \; \frac{\partial^2 v}{\partial y^2} = \frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3}$$

Now, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$, and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \dfrac{2y(y^2 - 3x^2) - 2y(y^2 - 3x^2)}{(x^2 + y^2)^2} = 0.$

Hence, both u and v satisfy Laplace's equation.

But, we see that $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$ i.e. C-R equations are not satisfied.

Hence, $u + iv$ is not an analytic function of z.

3.14 Construct an analytic function $f(z) = u(x, y) + iv(x, y)$, where $v(x, y) = 6xy - 5x + 3$. Express the result as a function of z.

Solution It is given that $v(x, y) = 6xy - 5x + 3$

$$\frac{\partial v}{\partial x} = 6y - 5, \quad \frac{\partial v}{\partial y} = 6x$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \text{, (using C-R Equation)}$$

$$= 6x\, dx - (6y - 5)\, dy$$

$$u = \int 6x\, dx - \int (6y - 5) dy + C = 3x^2 - 3y^2 + 5y + C$$

$$f(z) = u + iv = 3x^2 - 3y^2 + 5y + i(6xy - 5x + 3) + C$$

$$= 3x^2 - 3y^2 + 5y + 6ixy - 5ix + 3i + C$$

$$= 3(x^2 - y^2 + 2ixy) + (-5ix + 5y) + 3i + C$$

$$= 3(x + iy)^2 - 5i(x + iy) + 3i + C = 3z^2 - 5iz + 3i + C$$