Polynomials

Let F be a field and λ be an indeterminate, then an expression of the type $f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots\cdots + a_{n-1}\lambda + a_n$ .... (1),

where n is an integer ($n > 0$), $a_0 \neq 0$ and $a_0, a_1, a_2, \ldots\ldots, a_n \in F$ is known as the polynomial of degree n.

Now if A is a square matrix over F, then we define

$$f(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots\cdots + a_{n-1} A + a_n I$$

where I is the identity matrix.

A polynomial is called monic if its leading co-efficient is 1, that is, in the above polynomial when $a_0 = 1$, $f(\lambda)$ will be a monic polynomial.

In particular, we say that A is a root or zero of the polynomial $f(\lambda)$ if $f(A) = 0$.

Ex1: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ and $f(\lambda) = \lambda^2 - 4\lambda + 3$. Find $f(A) = A^2 - 4A + 3I$.

Solⁿ: Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$$\therefore f(A) = A^2 - 4A + 3I$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^2 - 4\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 12 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

Thus A is a root or zero of $f(\lambda)$.