Echelon Matrix: A matrix $A = [a_{ij}]$ is an echelon matrix if the number of zeros preceding the first non-zero entry of a row increases row by row until only zero rows remain.

A matrix which is in echelon form and the first non-zero element in each non-zero row is the only non-zero element in its column is said to be in reduced echelon form.

Examples of echelon matrices and matrices of reduced echelon form are given below:

(i) $\begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 0 & -13 & 11 \\ 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(echelon matrix)

(ii) $\begin{bmatrix} 2 & 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$

(echelon matrix)

(iii) $\begin{bmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

(reduced echelon form)

Linear dependence and linear independence of vectors:

<u>Definition</u>: Let V be a vector space over the field F. The vectors $v_1, v_2, v_3, \ldots\ldots, v_m \in V$ are said to be linearly dependent over F or simply dependent if there exists a non-trivial linear combination of them equal to the zero vector 0. That is,

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots\cdots\cdots + \alpha_m v_m = 0 \text{ where } \alpha_i \neq 0$$

for at least one i.

On the other hand, the vectors $v_1, v_2, v_3, \ldots\ldots, v_m$ in V are said to be linearly independent over F or simply independent if the only linear combination of them equal to 0 (zero vector) is the trivial one. That is,

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots\cdots\cdots + \alpha_m v_m = 0 \text{ if and only if}$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots\cdots\cdots = \alpha_m = 0.$$

Ex. 1: Prove that the following set of vectors $\{(2,1,2), (0,1,-1), (4,3,3)\}$ is linearly dependent.

Proof: Set a linear combination of the given vectors equal to zero by using unknown scalars x, y, z:

$$x(2,1,2) + y(0,1,-1) + z(4,3,3) = (0,0,0)$$

$$\Rightarrow (2x, x, 2x) + (0, y, -y) + (4z, 3z, 3z) = (0,0,0)$$

$$\Rightarrow (2x + 0 + 4z,\ x + y + 3z,\ 2x - y + 3z) = (0,0,0)$$

Equating corresponding components and forming the linear system, we get

$$\left.\begin{aligned} 2x + 0 + 4z &= 0 \\ x + y + 3z &= 0 \\ 2x - y + 3z &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (1)$$

Reduce the system to echelon form by the elementary transformations.

Interchange first and second equation and get the equivalent system,

$$\sim \left.\begin{aligned} x + y + 3z &= 0 \\ 2x + 0 + 4z &= 0 \\ 2x - y + 3z &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots (2)$$

Now apply $L_2' = L_2 - 2L_1$ and $L_3' = L_3 - 2L_1$ and get the equivalent system

$$\sim \left.\begin{aligned} x + y + 3z &= 0 \\ -2y - 2z &= 0 \\ -3y - 3z &= 0 \end{aligned}\right\} \cdots\cdots\cdots (3)$$

Now apply $L_2' = -\frac{1}{2}L_2$ and $L_3' = -\frac{1}{3}L_3$ and get the equivalent system

$$\sim \left.\begin{aligned} x + y + 3z &= 0 \\ y + z &= 0 \\ y + z &= 0 \end{aligned}\right\} \cdots\cdots\cdots (4)$$

Now apply $L_3' = L_3 - L_2$ and get the equivalent system

$$\sim \left.\begin{aligned} x + y + 3z &= 0 \\ y + z &= 0 \end{aligned}\right\} \cdots\cdots\cdots (5)$$

The system is in echelon form and has only two non-zero equations in three unknowns, hence the system has non-zero solution. Thus the original vectors are linearly dependent. [Proved]

Note:

$$\left.\begin{aligned} x + y + 3z &= 0 \\ y + z &= 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(Echelon form)}$$