Determining All Possible Roots:

All the methods discussed so far estimate only one root. If we want to locate all the roots in the given interval then one option is to *plot a graph of the function* and then identify various independent intervals that bracket the roots. These intervals can be used to locate the various roots. Another approach is to use an *incremental search* technique covering the entire interval containing the roots. This means that search for a root continues even after the first root is found. The procedure consists of starting at one end of the interval, say at point a, and then searching for a root at every incremental interval till the other end, say point b, is searched. The end point of each incremental interval can serve initial points for one of the bracketing techniques discussed. The following algorithm describes the steps for implementing an incremental search technique using the bisection method for locating all roots.

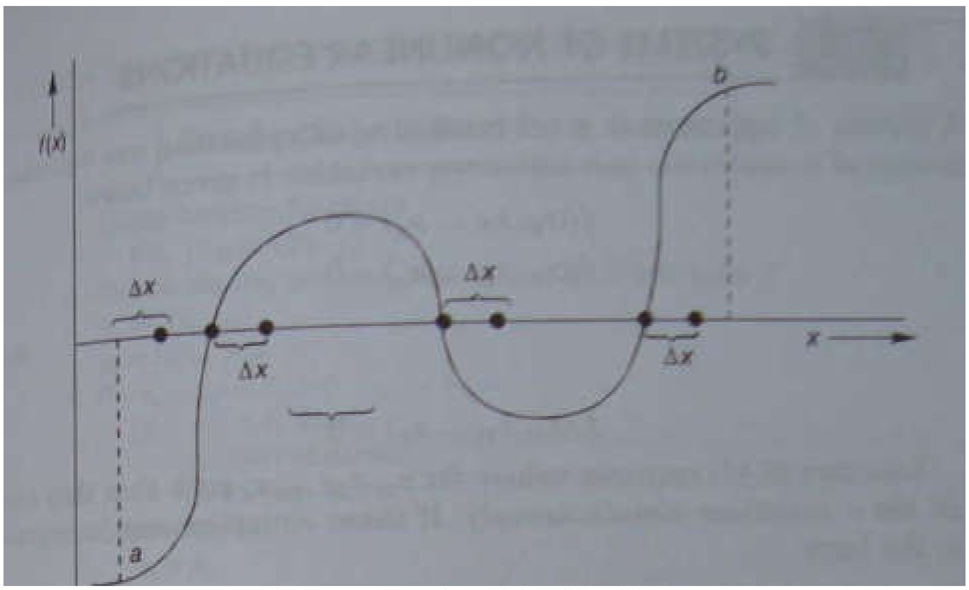


Figure 1: Incremental Search for All Possible Roots

Algorithm

1. Choose lower limit a and upper limit b of the interval covering all the roots.
2. Decide the size of the increment interval Δx
3. set $x_1 = a$ and $x_2 = x_1 + \Delta x$
4. Compute $f_1 = f(x_1)$ and $f_2 = f(x_2)$
5. If $(f_1 * f_2)>0$, then the interval does not bracket any root and go to step 9
6. Compute $x_0 = (x_1 + x_2)/2$ and $f_0 = f(x_0)$
7. If $(f_1 * f_2)<0$, then set $x_2 = x_0$
   Else set $x_1 = x_0$ and $f_1 = f_0$
8. If $| (x_2 - x_1)/ x_2| < E$, then
   root = $(x_1+x_2) / 2$
   write the value of root
   go to step 9
   Else
   go to step 6
9. If $x_2 < b$, then set $a = x_2$ and go to step 3
10. Stop.

- A major problem is to decide the increment size. A small size may mean more iterations and more execution time. If the size is large, then there is a possibility of missing the closely spaced roots.

Roots of Polynomials
Polynomial could have multiple real or complex roots.
The properties of nth degree of polynomials:

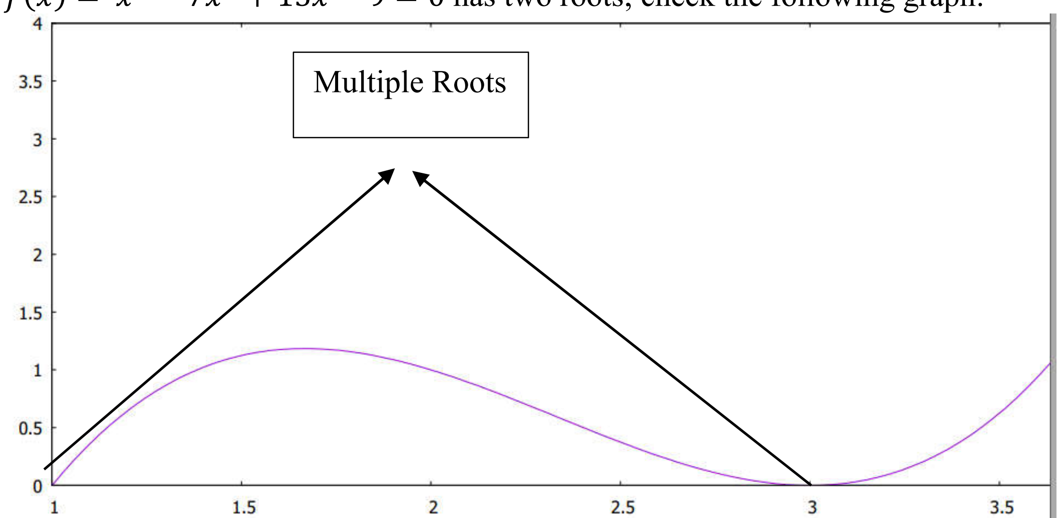
1. There are n roots (real or complex)
2. A root may be repeated (multiple roots)
3. Complex roots occur in conjugate pairs.
4. If n is odd and all the coefficients are real, then there is at least one real root.
5. The polynomial can be expressed as:
   $\mathbf{p(x) = (x - x_r)\ q(x)}$,

where x_r is a root of the polynomial P(x), and q(x) is the quotient polynomial of degree n – 1.

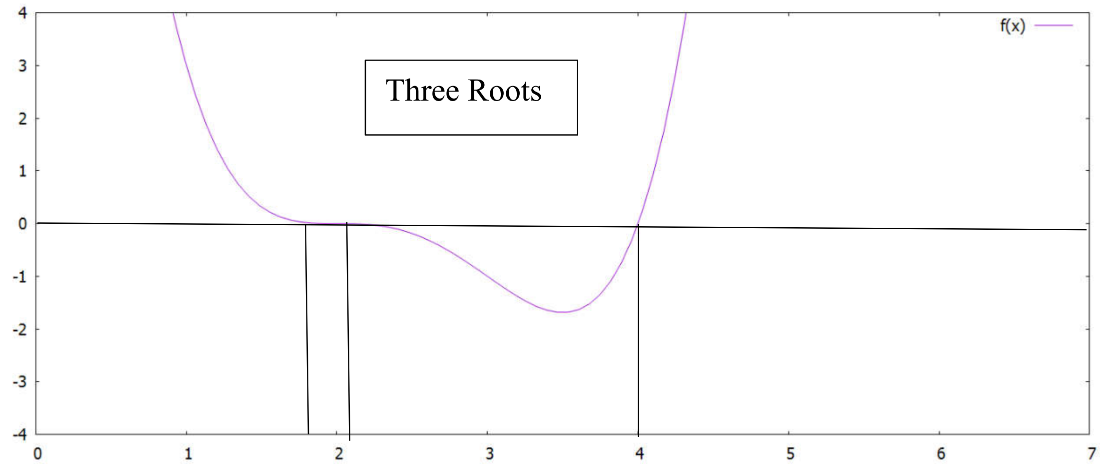
Descartes'Rule:
The number of real roots can be obtained using Descarte's rule of sign. This rule states that

1. The number of positive real roots is equal (or less than by an even integer) to the number of sign changes in the co-efficient of the equation.
2. The number of negative real roots is equal (or less than by an even integer) to the number of sign changes in the co-efficient if x is replaced by –x.

Multiple Roots:
A polynomial function contains a multiple root at a point when the function is tangential to the x-axis at the point.
$f(x) = x^3 - 7x^2 + 15x - 9 = 0$ has two roots; check the following graph:



$\boldsymbol{f(x)} = x^4 - 10x^3 + 36x^2 - 56x = 32 = 0$



Note:

1. Curve crosses the x-axis for odd multiple roots and turns back for even multiple roots.
2. Bracketing methods will have problem for relocating he even multiple roots.
3. Both f(x) and f'(x) will be zero at the points of multiple roots.

Deflation and Synthetic Division:

The polynomial can be expressed as:

p(x) = (x – x_r) q(x),

where x_r is a root of the polynomial P(x), and q(x) is the quotient polynomial of degree n – 1.

- We can find the q(x) with degree n – 1 by dividing p(x) by (x – x_r) using a process known as synthetic division.
- The name synthetic is used because the quotient polynomial q(x) is obtained without actually performing the division.
- The activity of reducing the degree of a polynomial is referred to as deflation.

Example of Synthetic /Polynomial Division:

Divide: $\frac{2x^3 \;\; - 5x^2 + 3x + 7}{x-2}$

To set up the problem, first, set the denominator equal to zero to find the number to put in the division box. Next, make sure the numerator is written in descending order and if any terms are missing you must use a zero to fill in the missing term, finally list only the coefficient in the division problem.

Step 1: Carry down the 2 that indicates the leading coefficient	$\begin{array}{c\|cccc} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & & & \\ \hline & 2 & & & \end{array}$

Step 2: Multiply by the number on the left, and carry the result into the next column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & & \\ \hline & 2 & & & \end{array}$
Step 3: Add down the column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & & \\ \hline & 2 & -1 & & \end{array}$
Step 4: Multiply by the number on the left, and carry the result into the next column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & -2 & \\ \hline & 2 & -1 & & \end{array}$
Step 5: Add down the column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & -2 & \\ \hline & 2 & -1 & 1 & \end{array}$
Step 6: Multiply by the number on the left, and carry the result into the next column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & -2 & 2 \\ \hline & 2 & -1 & 1 & \end{array}$
Step 7: Add down the column	$\begin{array}{r\|rrrr} 2 & 2 & -5 & 3 & 7 \\ & \downarrow & 4 & -2 & 2 \\ \hline & 2 & -1 & 1 & 9 \end{array}$
Step 8: Write the final answer	$2x^2 - x + 1 + \frac{9}{x-2}$

- The quotient polynomial q(x) can be used to determine the other roots of p(x), because the remaining roots of p(x) are the roots of q(x).
- When the root is found for q(x), further deflation is performed and the process can be continued until the degree is reduced to one.

Process of Synthetic division:
Let $\boldsymbol{p(x) = \sum_{i=0}^{n} a_i x^i}$
And

$$q(x) = \sum_{i=0}^{n-1} b_i x^i$$

If $p(x) = (x - x_r)q(x)$, then

$$a_n x^n + a_{n-1}x^{n-1} + \cdots.. + a_1 x + a_0 = (x - x_r)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots ... + b_1 x + b_0)$$

By comparing the coefficients of like powers of x on both sides of the above equation, we get the following relations between them:

$$\boldsymbol{a_n = b_{n-1}}$$
$$a_{n-1} = b_{n-2} - x_r b_{n-1}$$
$$\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots$$
$$a_1 = b_0 - x_r b_1$$
$$a_0 = -x_r b_0$$

That is,

$$a_i = b_{i-1} - x_r b_i \qquad i = n, n-1, \dots\dots 0$$

Where, $b_n = b_{n-1} = 0$

Then, $\boxed{\begin{array}{ll} b_{i-1} = a_i + x_r b_i & i = n \dots\dots 1 \\ & b_n = 0 \end{array}}$

- We can determine the coefficients of *q(x)* (i.e $b_{n-1} + b_{n-2} + \cdots \dots + b_1 + b_0$) from the coefficients of *p(x)*($\boldsymbol{i.e}$ $a_n + a_{n-1} + \cdots .. + a_1 + a_0$) recursively.

Example:

The polynomial equation $p(x) = x^3 - 7x^2 + 15x - 9 = 0$ has a root at $x = 3$. Find the quotient polynomial $q(x)$ such that $p(x) = (x-3)q(x)$

Solution: form $p(x)$, we have

$$a_3 = 1, a_2 = -7, a_1 = 15, a_0 = -9$$
$$b_3 = 0$$
$$b_2 = a_3 + b_3 * 3 = 1 + 0 = 1$$
$$b_1 = a_2 + b_2 * 3 = -7 + 3 = -4$$
$$b_0 = a_1 + b_1 * 3 = 15 - 12 = 3$$

Thus, $q(x) = x^2 - 4x + 3$

Purification of Roots:

Purification, as the name indicates, is the process of refining the roots that do not satisfy the required accuracy conditions. These roots may be used again for testing the original problem and improving their approximation.

The Newton-Raphson method is a popular one used for purification of roots. The values of the roots obtained through other methods (for example bisection method) are used as initial values to the Newton method.

Multiple Roots by Newton's Methods

A polynomial functions contains a multiple root at a point when the function is tangential to the x-axis at that point. For example, the equations: $x^3 - 7x^2 + 15x - 9 = 0$ has a double root at x = 3 (see figure 2).

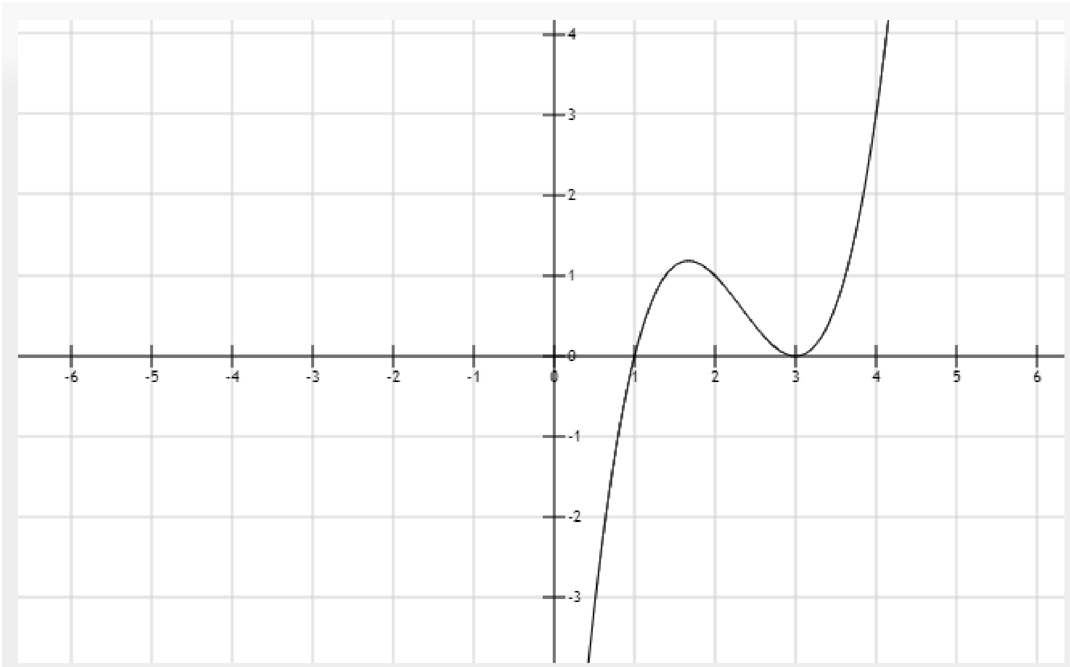


Figure 2

- We can locate all real roots of a polynomial by repeatedly applying Newton-Raphson method and polynomial deflation to obtain polynomials of lower and lower degree.
- The deflation process is performed $(n-1)$ times where n is the degree of the given polynomial. After $(n-1)$ deflations, the quotient is the linear polynomial of type: $a_1x + a_0 = 0$, and therefore the final root is given by, $x_r = -\frac{a_0}{a_1}$

Algorithm: Multiple Roots by Newton-Raphson Method

1. Obtain degree and co-efficient of polynomial (n and a_i).
2. Decide an initial estimate for the first root (x_0) and error criterion, E.
   Do while n >1
3. Find the root using Newton-Raphson algorithm
   $x_r = x_0 - f(x_0) / f'(x_0)$
4. Root (n) = x_r
5. Deflate the polynomial using synthetic division algorithm and make the factor polynomial a the new polynomial of order n-1.
6. Set $x_0 = x_r$ [Initial value of the new root]
   End of Do
7. Root (1) = $- a_0 / a_1$
8. Stop.