Divergence and Stoke's theorem

Example: Find $\iint \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$

and S is the surface of the sphere having centre $(3,-1,2)$ and radius 3.

~~and~~ **Sol^n:** We know the divergence theorem,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv \quad \cdots\cdots\cdots\cdots (1)$$

Here, $\nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}\right]$

$$= \frac{\partial}{\partial x}(2x+3z) - \frac{\partial}{\partial y}(xz+y) + \frac{\partial}{\partial z}(y^2+2z)$$

$$= 2-1+2 = 3$$

Now, from (1), $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V 3\,dv$

$= 3\iiint_V dv = 3V$, where V is the volume

enclosed by the surface S.

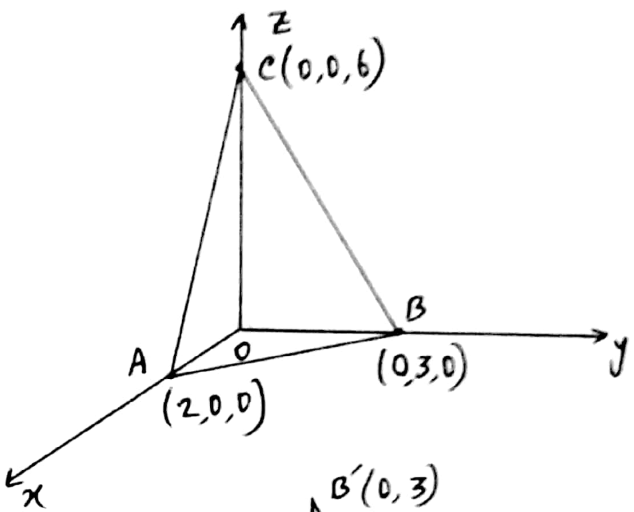
Again, V is the volume of a sphere of radius 3. Therefore,

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3)^3 = 36\pi.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 3V = 3\times 36\pi = 108\pi. \quad \text{(Ans).}$$

Example 2: Use the stoke's theorem to evaluate

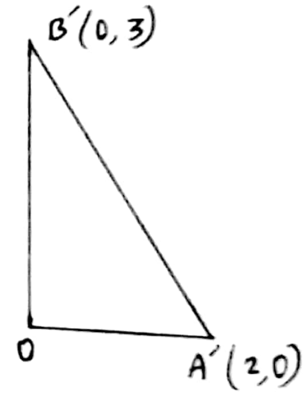
$\int_c [(x+2y)dx + (x-z)dy + (y-z)dz]$ where c is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$ oriented in the anti-clockwise direction.



Sol^n: Given,

$$\int_c [(x+2y)dx + (x-z)dy + (y-z)dz]$$

$$= \int_c [(x+2y)\hat{i} + (x-z)\hat{j} + (y-z)\hat{k}] \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$



$$\therefore \bar{F} = (x+2y)\hat{i} + (x-z)\hat{j} + (y-z)\hat{k}.$$

We know the stoke's theorem,

$$\oint_c \bar{F} \cdot d\bar{r} = \iint_S \text{curl } \bar{F} \cdot \hat{n} \, ds \quad \cdots\cdots\cdots\cdots \quad (1)$$

Now, $\text{curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & x-z & y-z \end{vmatrix}$

$$= \hat{i}(1+1) - \hat{j}(0-0) + \hat{k}(1-2) = 2\hat{i} - \hat{k}.$$

Here, S is the surface of the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$ and $\hat{n}$ is the normal to the plane ABC.

Again, normal vector $= \nabla\Phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{2} + \frac{y}{3} + \frac{z}{6} - 1\right)$$

$$= \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k} = \frac{1}{6}(3\hat{i} + 2\hat{j} + \hat{k})$$

$$\therefore \hat{n} = \frac{\frac{1}{6}(3\hat{i} + 2\hat{j} + \hat{k})}{\sqrt{\frac{1}{36}(9+4+1)}} = \frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})$$

Now, $\text{curl}\,\bar{F}\cdot\hat{n} = (2\hat{i} - \hat{k})\cdot\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})$

$$= \frac{1}{\sqrt{14}}(6-1) = \frac{5}{\sqrt{14}}$$

Now, on applying stoke's theorem,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds$$

$$= \iint_S \frac{5}{\sqrt{14}}\,ds$$

$$= \frac{5}{\sqrt{14}}\iint_R \frac{dx\,dy}{\hat{k}\cdot\hat{n}} \quad \left[\because ds = \frac{dx\cdot dy}{\hat{n}\cdot\hat{k}}\right]$$

$$= \frac{5}{\sqrt{14}}\iint_R \frac{dx\,dy}{\hat{k}\cdot\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k})}$$

$= 5\iint_R dx\cdot dy$, where R is the projection of S on the x-y plane, that is, triangle OAB.

$= 5 \times$ Area of triangle OAB

$= 5 \times \frac{1}{2} \times 2 \times 3 = 15$. (Ans.)

<u>Exercise:</u> Example 87 (Page. 438)