Inverse Fourier Transform

1. The inverse fourier transform of a function $F(s)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx} \, ds$$

2. The inverse fourier cosine transform of a function $F(s)$ is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \cdot \cos sx \, ds$$

3. The inverse fourier sine transform of a function $F(s)$ is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \cdot \sin sx \, ds .$$

Ex. 1: Find the inverse Fourier cosine transform of $F(s) = e^{-s}$.

Sol^n: We know the inverse Fourier cosine transform of $F(s)$

is $$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \cdot \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-s} \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-s}}{1+x^2} \left(x \sin sx - 1 \cdot \cos sx \right) \right]_{0}^{\infty}$$

$$\left[\because \int e^{-ax} \cos bx \, dx = \frac{e^{-ax}}{a^2+b^2} \left(b \sin bx - a \cos bx \right) \right]$$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}\left[0 - \left\{\frac{1}{1+x^2}(0-1\cdot 1)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+x^2} \cdot \text{(Ans.)}$$

Ex. 2: Find the inverse fourier sine transform of $F(s) = \frac{e^{-as}}{s}$.

Sol^n: We know the inverse Fourier sine transform of $F(s)$ is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\sin sx \, ds \quad \cdots\cdots\cdots (1)$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx \, ds$$

$$\therefore \frac{df}{dx} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s} \cdot s\cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-as}}{a^2+x^2}(x\sin sx - a\cos sx)\right]_0^\infty$$

$$\left[\because \int e^{-ax}\cos bx \, dx = \frac{e^{-ax}}{a^2+b^2}(b\sin bx - a\cos bx)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \left\{\frac{1}{a^2+x^2}(0 - a\cdot 1)\right\}\right]$$

$$\Rightarrow \frac{df}{dx} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+x^2}$$

$$\Rightarrow df = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+x^2}dx$$

$$\Rightarrow \int df = \sqrt{\frac{2}{\pi}} \cdot a \int \frac{dx}{a^2+x^2}$$

$$\Rightarrow f = \sqrt{\frac{2}{\pi}} \cdot a \cdot \frac{1}{a} \tan^{-1}\frac{x}{a} + c$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{x}{a} + c \quad \cdots\cdots (2)$$

At $x = 0$, from (1) we get $f(0) = 0$, and

at $x = 0$, from (2) we get $f(0) = c$, hence we get $c = 0$.

Thus, the required Fourier sine transform of the given function is $f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{x}{a}$. (Ans.)