Ordinary Differential Equation

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Overview

- In this Module we will solve ordinary differential equations of the form:
- $\frac{dy}{dx} = f(x, y)$
- According to the initial value problem:

New Value = old value + slope * step size

Or mathematically:

$y_{i+1} = y_i + \emptyset h$, where ∅ is called an incremental function.

- According to the equation, the slope estimate of ∅ is used to extrapolate from an old value y_i to a new value y_{i+1} over a distance h.
- Approach is known as one step method as we use information from only one preceding point

Euler's Method

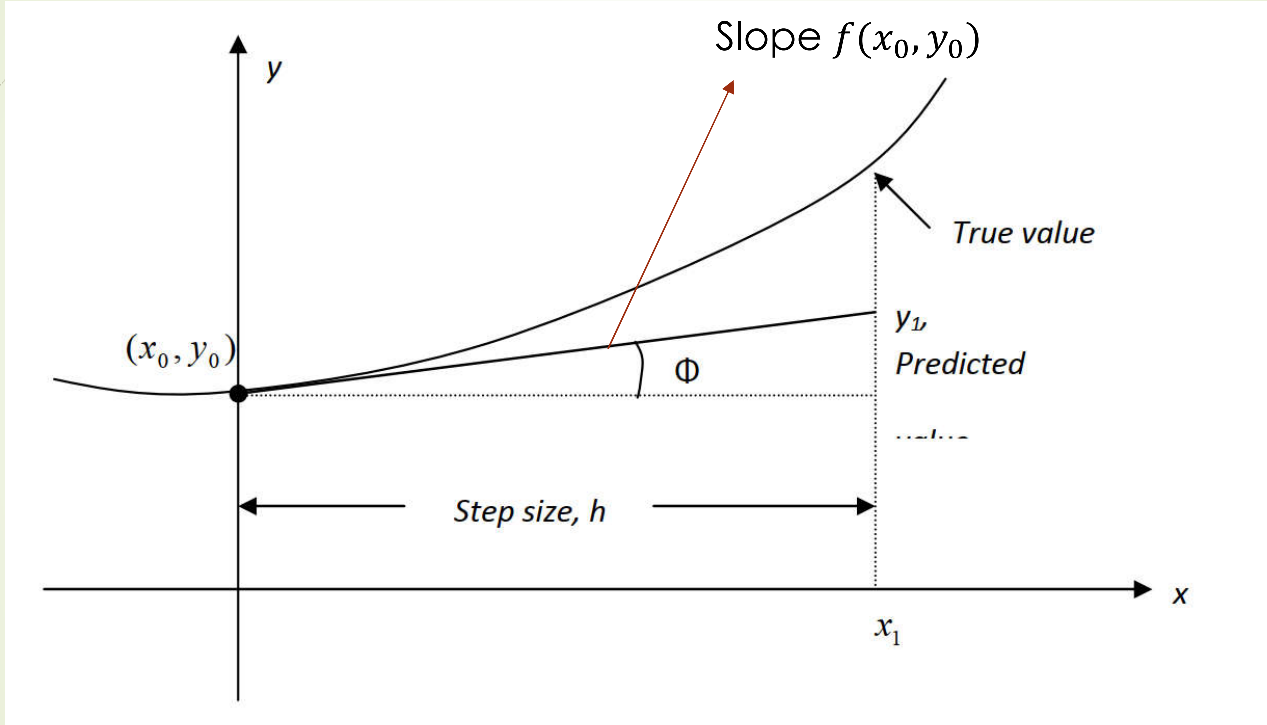
- Euler's Method is one of the simplest one-step methods and it has limited application because of its low accuracy.
- Eular's method is a numerical technique that solve ordinary differential equations in the form of : $\frac{dy}{dx}=f(x, y), y(0) = y0$
- Only first order derivatives can be solved by Euler's Methods.

General form Euler's method:

$y_{i+1} = y_i + f(x_i,y_i)h$, where $f(x_i,y_i)$ is slope at point x_i,y_i.

- A new value of y is predicted using the slope (equal to the first derivative at original value x)to extrapolate linearly over the step size h.

Derivation of Euler's Method



Graphical Representation of Euler's Method

Derivation of Euler's Method

- Initial Condition: At $x = 0$, we are given the value of $y = y_0$.
- Let us call $x = 0$ as x_0 . Now since we know the slope of y with respect to x, that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$.
- Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.
- So the slope at $x = x_0$ as shown in Figure is:

$$slope = \frac{rise}{run} = \frac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0).$$

$$y_1 = y_0 + (x_1 - x_0) f(x_0, y_0)$$

$$(x_1 - x_0) = step\ size\ h$$

$$y_1 = y_0 + hf(x_0, y_0)$$

- One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2,and that would be the predicted value at x_2 , given by $y_2 = y_1 + hf(x_1, y_1)$ where $x = x + h$
- *In general if* $y = y_i$ at x_i, then $y_{i+1} = y_i + hf(x_i, y_i)$

Error Analysis of Euler's Method

- Numerical ODEs involves two types of error:

1. Truncation→ error cause to estimate y locally at each step which also propagate to approximate next value of y.

2. Roundoff

Local Truncation error = $E_t = \frac{f'}{2!}(xi, yi)\ h^2$ proportional to step size raised to the power 2

Or, $E_a = O(h^2)$, E_a is approximate local truncation error.

See lecture note and text book for details analysis.

Example of Euler's Method

Given the Equation:

$\frac{dy}{dx} = 3x^2 + 1$ with $y(1) = 2$ estimate $y(2)$ by Euler's method using:

i. $h = 0.5$

ii. $h = 0.25$

iii. Compute the errors when $h = 0.5$

Example of Euler's Method

Given the Equation:

$\frac{dy}{dx} = 3x^2 + 1 = f(x, y)$ with $y(1) = 2$

i. $h = 0.5$

- $x_0 = 1$
  $y_0(1) = 2$
- $x_1 = 1{+}h = 1 + 0.5 = 1.5$

  $y(1.5) = \; y_0 + hf(x_0, y_0) = 2 + 0.5\left(3{x_0}^2 + 1\right) = 2 + 0.5(3 * (1)^2{+}1) = 4.0$
- $x_2 = 1.5{+}h = 1.5 + 0.5 = 2$

  $y(2) = \; y_1 + hf(x_1, y_1) = 4.0 + 0.5\left(3{x_1}^2 + 1\right) = 2 + 0.5(3 * (1.5)^2{+}1) = 7.875$

$$\therefore y(2) = 7.875$$

Example of Euler's Method

Given the Equation:

$\frac{dy}{dx} = 3x^2 + 1 = f(x, y)$ with $y(1) = 2$

ii. $h = 0.25$

- $x_0 = 1$
  $y_0(1) = 2$
- $x_1 = 1{+}0.25 = 1 + 0.25 = 1.25$
  $y(1.25) = \ y_0 + hf(x_0, y_0) = 2 + 0.25\left(3{x_0}^2 + 1\right) = 2 + 0.25(3 * (1)^2{+}1) = 3.0$
- $x_2 = 1.25{+}h = 1.25 + 0.25 = 1.5$
  $y(1.5) = \ y_1 + hf(x_1, y_1) = 3.0 + 0.25\left(3{x_1}^2 + 1\right) = 3.0 + 0.25(3 * (1.25)^2{+}1) = 5.42188$
- $x_3 = 1.5{+}h = 1.5 + 0.25 = 1.75$
  $y(1.75) = \ y_2 + hf(x_2, y_2) = 5.42188 + 0.25\left(3{x_2}^2 + 1\right) = 5.42188 + 0.25(3 * (1.5)^2{+}1) = 7.35938$
- $x_4 = 1.75{+}h = 1.75 + 0.25 = 2$
  $y(2) = \ y_3 + hf(x_3, y_3) = 7.35938 + 0.25\left(3{x_3}^2 + 1\right) = 7.35938 + 0.25(3 * (1.75)^2{+}1) = 9.90626$

$$\therefore y(2) = 9.90626$$

Example of Euler's Method

Given the Equation:

$\frac{dy}{dx} = 3x^2 + 1 = f(x,y)$ with $y(1) = 2$

iii. Compute error at $h = 0.5$

$y' = 3x^2 + 1$

$y'' = 6x$

$y''' = 6$

- $x_0 = 1$
  $y_0(1) = 2$
- $x_1 = 1{+}h = 1 + 0.5 = 1.5$
  $y(1.5) = \; y_0 + hf(x_0, y_0) = 2 + 0.5\left(3{x_0}^2 + 1\right) = 2 + 0.5(3 * (1)^2{+}1) = 4.0$

  $$E_{t,1} = \frac{y_0''}{2}h^2 + \frac{y_0'''}{6}h^3 = \frac{6(1)}{2}(0.5)^2 + \frac{6}{6}(0.5)^3 = 0.875$$

- $x_2 = 1.5{+}h = 1.5 + 0.5 = 2$
  $y(2) = \; y_1 + hf(x_1, y_1) = 4.0 + 0.5\left(3{x_1}^2 + 1\right) = 2 + 0.5(3 * (1.5)^2{+}1) = 7.875$

  $$E_{t,2} = \frac{y_0''}{2}h^2 + \frac{y_0'''}{6}h^3 = \frac{6(1.5)}{2}(0.5)^2 + \frac{6}{6}(0.5)^3 = 1.25$$

Example of Euler's Method

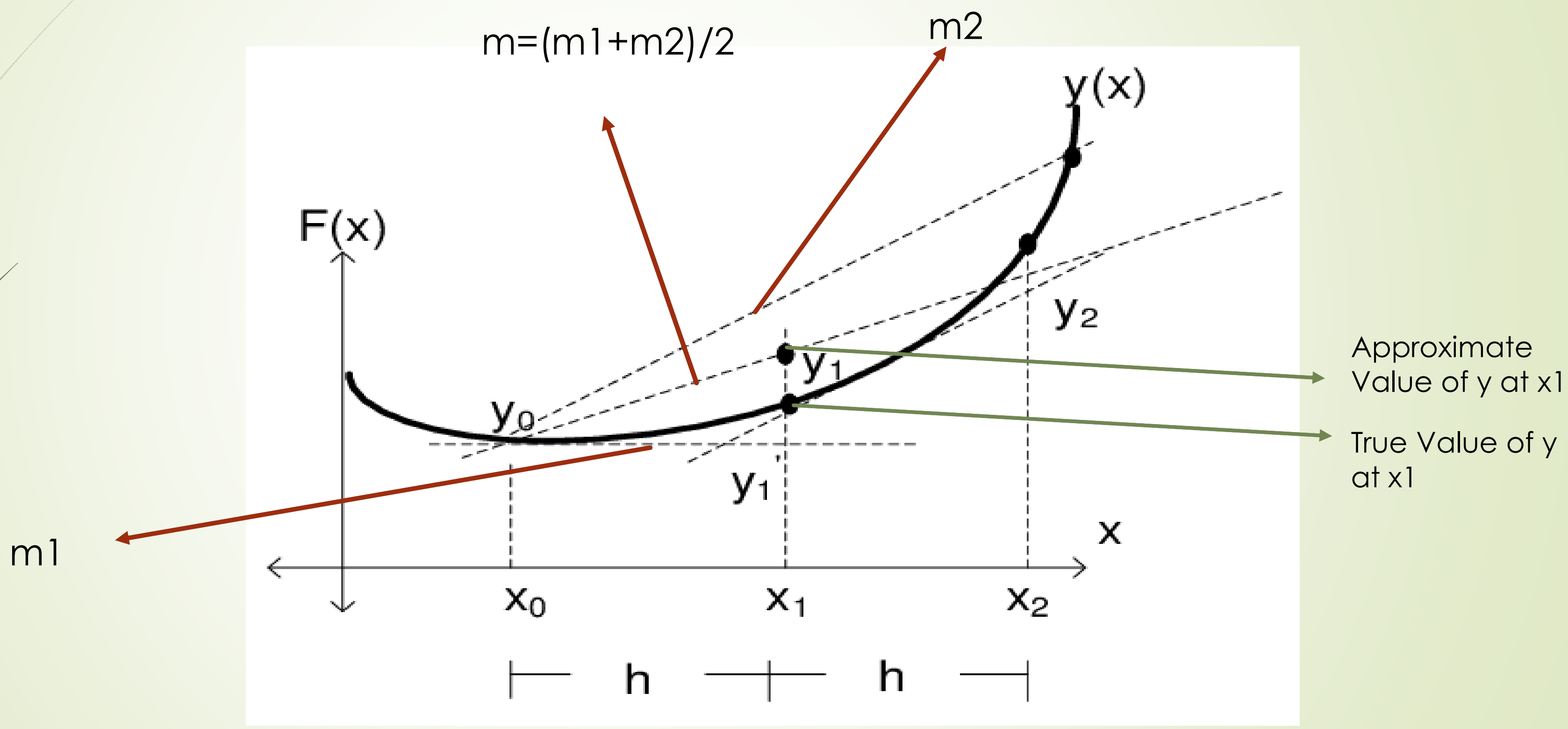
x	Estimated y	True y	E_t	Global Error
2.0	7.875	10.000	1.250	2.125

Huen’s Method

- Limitation of Euler’s Method:
  - Due to its linear characteristics it has large truncation errors.
- Solution:
  Heun’s Method is considered to be an improvement of Euler’s Method.

- Huen’s Methods works as Predictor Corrector Approach. The basic principle of Predictor Corrector Approach is:
  - Predict a solution of given ODE
  - Correct the predictor equation

Huen’s Method



Huen's Method

- Predicted Approach:

Use Euler's Method: $y_1 = y_0 + hm_1$;where $m_1$is the slope at (x_0, y_0)

- As shown in following Figure , y_1 is clearly an underestimate of true value of $y\ (x_1)$.
- Corrector Approach:
  - Draw a line parallel to the tangent at point (x_1, y_1) to extrapolate from y_0 to y_1
  - $i.e. y_1 = y_0 + hm_2$
  - Use slope at point x_0 which is average of $m_1$and m_2 to extrapolate fromy_0 to y_1
  - $i.e. y_1 = y_0 + h\frac{(m_1+m_2)}{2}$

Huen's Formula: $i.e. y_1 = y_0 + h\frac{(m_1+m_2)}{2}$

Huen's Method

- The formula for implementing Heun's method can be constructed easily. Given the equation, y' (x) = f (x, y), we can obtain
- $m_1 = y'(x_i) = f(x_i, y_i)$
- $m_2 = y'(x_{i+1}) = f(x_{i+1}, y_{i+1})$
- $therefore, m = (f(x_i, y_i) + f(x_{i+1}, y_{i+1}))$ / 2
- So, $y_{i+1} = y_i + h/2\,[f(x_i, y_i) + f(x_{i+1}, y_{i+1})]$
- To calculate , y_{i+1}at R.H.S we need to use Euler's method:
  - $y_{i+1} = y_I + hf(x_i, y_i)$
- So Huen's Method become:
  - $y_{i+1} = y_i + \frac{h}{2}(f(x_i, y_i), f(x_i, y_i + hf(x_i, y_i)))$

Error in Huen's Method

- Global Truncation Error:
- $|E_{tg}| = (b - x_0)ch^2$
- Therefore global error is the order of h^2

Example of Huen's Method

Given the Equation:

$\frac{dy}{dx} = x^2 + y^2$ with $y(0) = 2$ estimate $y(1)$ by Huen's method using:'$h = 0.5$

Solution:

Given:

$f(x, y) = \frac{dy}{dx} = x^2 + y^2$

$x_0 = 0;\ y_0 = 2; h = 0.5$

Huen's Formula: $y_1 = y_0 + h\frac{(m_1+m_2)}{2}$

Step 1:

$x_1 = x_0 + h = 0 + 0.5 = 0.5$

$m_1 = f(x_0, y_0) = x_0^2 + y_0^2 = (0)^2 + (2)^2 = 4$

$y_1 = y_0 + hm_1 = 2 + 0.5 * 4 = 4$

$m_2 = f(x_1, y_1) = x_1^2 + y_1^2 = (0.5)^2 + (4)^2 = 16.25$

$\therefore y_1 = y_0 + \frac{h}{2}(m_1 + m_2) = 2 + 0.5 * 10.125 = 7.0625$

Example of Huen's Method

Step 2:

$x_2 = x_1 + h = 0.5 + 0.5 = 1$

$m_1 = f(x_1, y_1) = x_1^2 + y_1^2 = (0.5)^2 + (7.0625)^2 = 50.1289$

$y_2 = y_1 + hm_1 = 7.0625 + 0.5 * 50.1289 = 32.1269$

$m_2 = f(x_2, y_2) = x_2^2 + y_2^2 = (1)^2 + (32.1269)^2 = 1033.1411$

$\therefore y_2 = y_1 + \frac{h}{2}(m_1 + m_2) = 7.0625 + 0.5 * 541.635 = 277.88$

$\therefore y(1) = 277.88 \; at \; h = 0.5$