14

Integral Transforms

14.1 INTRODUCTION

Integral transforms are used in the solution of partial differential equations. The choice of a particular transform to be used for the solution of a differential equations depends upon the nature of the boundary conditions of the equation and the facility with which the transform $F(s)$ can be converted to give $f(x)$.

14.2 INTEGRAL TRANSFORMS

The integral transform $F(s)$ of a function $f(x)$ with the Kernel $k(s, k)$ is defined as

$$I[f(x)] = F(s) = \int_a^b f(x)\, k(s,x)\, dx.$$

For example

1. Laplace transform with the kernel $k(s, x) = e^{-sx}$

$$L[f(x)] = F(s) = \int_0^\infty f(x).e^{-sx} dx$$

2. Fourier Complex transform with the kernel $k(s, k) = e^{-isx}$

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx} dx \qquad \text{(Inversion formula)}$$

3. Fourier Sine transform with the kernel $k(s, x) = \sin sx$

$$F_s[f(x)] = F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx.dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx\, ds \qquad \text{(Inversion formula)}$$

4. Fourier Cosine transform with the kernel $k(s, x) = \cos sx$

$$F_c[f(x)] = F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx.dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx\, ds \qquad \text{(Inversion formula)}$$

5. Hankel Transform with the kernel $(k, s) = x\,J_n(sx)$

$$H[f(x)] = F(s) = \int_0^{\infty} f(x).x\,J_n(sx)\,dx$$

$$f(x) = \int_0^{\infty} s\;F(s)\;J_n(sx)\,dx \qquad \text{(Inversion formula)}$$

6. Hilbert Transform with the kernel $k(s, x) = \dfrac{1}{s-x}$

$$F(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{s-x}\,dx$$

$$f(x) = \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{F(s)}{s-x}\,ds \qquad \text{(Inversion formula)}$$

7. Mellin transform with the kernel $k(s, x) = x^{s-1}$

$$M[f(x)] = F(s) = \int_0^{\infty} f(x).x^{s-1}dx.$$

The students have already done "Laplace transform" and also learnt to solve the ordinary differential equations by using Laplace transforms.

Integral transforms are used in solving the partial differential equation with boundary conditions.

List of Formulae of Fourier Integrals

1. Fourier Integral for $f(x)$ is $\quad f(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,\cos u(t-x)\,du\,dt$

2. Fourier Sine Integral for $f(x)$ is $\quad f(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\int_0^{\infty} f(t)\,\sin ut\,\sin ux\;du\,dt$

3. Fourier Cosine Integral for $f(x)$ is $\quad f(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty}\int_0^{\infty} f(t)\,\cos ut\,\cos ux\;du\,dt$

14.3 FOURIER INTEGRAL THEOREM

It states that $f(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt\,du$

Proof. We know that Fourier series of a function $f(x)$ in $(-c, c)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c} \qquad \text{... (1)}$$

where a_0, a_n and b_n are given by

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(t)dt,\quad a_n = \frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\,dt$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\,dt$$

Substituting the values of a_0, a_n and b_n in (1) we get

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\cos\frac{n\pi t}{c}\cos\frac{n\pi x}{c}\,dt \;+\; \sum_{n=1}^{\infty}\frac{1}{c}\int_{-c}^{c} f(t)\sin\frac{n\pi t}{c}\sin\frac{n\pi x}{c}dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty} \frac{1}{c}\int_{-c}^{c} f(t)\left[\cos\frac{n\pi t}{c}\cos\frac{n\pi x}{c} + \sin\frac{n\pi t}{c}\sin\frac{n\pi x}{c}\right]dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\,dt + \sum_{n=1}^{\infty}\frac{1}{c} f(t)\cos\frac{n\pi}{c}(t-x)\,dt$$

$$= \frac{1}{2c}\int_{-c}^{c} f(t)\left\{1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\}dt \qquad \text{... (2)}$$

Since cosine functions are even functions *i.e.*, $\cos\,(-\theta) = \cos\,\theta$ the expression

$$1 + 2\sum_{n=1}^{\infty}\cos\frac{n\pi}{c}(t-x) = \sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{c}(t-x)$$

Therefore, (2) becomes

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\left\{\sum_{-\infty}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\}dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left\{\frac{\pi}{c}\sum_{-\infty}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\}dt \qquad \text{... (3)}$$

Let us now assume that c increases indefinitely, so that we may write $\frac{n\pi}{c} = u$ and $\frac{\pi}{c} = du$. This assumption gives

$$\lim_{c\to\infty}\left\{\frac{\pi}{c}\sum_{-\infty}^{\infty}\cos\frac{n\pi}{c}(t-x)\right\} = \int_{-\infty}^{\infty}\cos u(t-x)\,du$$

$$= 2\int_{0}^{\infty}\cos u(t-x)\,du \qquad \text{... (4)}$$

Substituting in (3) from (4), we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\left\{2\int_{0}^{\infty}\cos u(t-x)\,du\right\}dt \qquad \text{...(5)}$$

Thus $$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\,\cos u(t-x)\,du\,dt$$ **Proved.**

Note. We have assumed the following conditions on $f(x)$.

(*i*) $f(x)$ is defined as single-valued except at finite points in $(-c, c)$.

(*ii*) $f(x)$ is periodic outside $(-c, c)$ with period $2c$.

(*iii*) $f(x)$ and $f'(x)$ are sectionally continuous in $(-c, c)$.

(*iv*) $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges, *i.e.,* $f(x)$ is absolutely integrable in $(-\infty, \infty)$.

14.4 FOURIER SINE AND COSINE INTEGRALS

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\sin ux\,du\int_{0}^{\infty} f(t)\,\sin ut\,dt \qquad \textbf{(Fourier Sine Integral)}$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\cos ux\,du\int_{0}^{\infty} f(t)\,\cos ut\,dt \qquad \textbf{(Fourier Cosine Integral)}$$

Proof. We know that

$$\cos u(t-x)=\cos(ut-ux)$$

or
$$\cos u(t-x)=\cos ut\cos ux+\sin ut\sin ux$$

Then equation (5) of article 14.3, can be written as

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\,(\cos ut\cos ux+\sin ut\sin ux)\,du\,dt$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\,\cos ut\,\cos dx\,du\,dt+\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\,\sin ut\,\sin ux\,du\,dt \quad \ldots$$

(6)

Case 1. When $f(t)$ is odd.

$\therefore$ $f(t)\cos ut$ is odd hence $\int_0^\infty\int_{-\infty}^\infty f(t)\,\cos ut\,\cos ux\,du\,dt=0$

$$\left[\begin{array}{ll}\text{For odd function} & \int_{-a}^{a} f(x)dx=0\\ \text{For even function} & \int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx\end{array}\right]$$

From (6) we have

$$\therefore \qquad f(x)=\frac{2}{\pi}\int_0^\infty \sin ux\,du\int_0^\infty f(t)\,\sin ut\,dt \qquad \ldots(7)$$

The relation (7) is called **Fourier sine integral**.
Case 2. When $f(t)$ is even.

$\therefore$ $f(t)\sin ut$ is odd. $\int_0^\infty\int_{-\infty}^\infty f(t)\sin ut\,\sin ux\,du\,dt=0$

$\therefore$ $f(t)\cos ut$ is even.
From (6) we have

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos ux\,du\int_0^\infty f(t)\cos ut\,dt \qquad \ldots(8)$$

The relation (8) is known as Fourier cosine integral.

14.5 FOURIER'S COMPLEX INTEGRAL

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{-iux}du\int_{-\infty}^\infty f(t)\,e^{-iut}\,dt$$

Proof. We know that $\int_{-a}^{a} f(x)dx=0$ if $f(x)$ is odd function.

$$\therefore \qquad \int_{-\infty}^\infty \sin u(t-x)\,du=0 \qquad \text{[since } \sin u\,(t-x) \text{ is odd.]}$$

Obviously we have

$$\frac{1}{2\pi}\int_{-\infty}^\infty f(t)\,dt\int_{-\infty}^\infty \sin u(t-x)\,du=0$$

or
$$\frac{i}{2\pi}\int_{-\infty}^\infty f(t)\,dt\int_{-\infty}^\infty \sin u(t-x)\,du=0 \quad \text{(Multiplying by } i\text{)} \qquad \ldots(9)$$

On adding (5) and (9) we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\ \cos u(t-x)\,du\,dt+\frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty} \sin u(t-x)\,du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty}[\cos u(t-x)+i\sin u(t-x)]\,du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty} e^{iu(t-x)}\,du$$

or
$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}du\int_{-\infty}^{\infty} f(t)e^{iut}\,dt \qquad \ldots (10)$$

Relation (10) is called Fourier's Complex Integral.

Example 1. *Express the function*

$$f(x)=\begin{cases}1 & \text{when } |x|\le 1\\ 0 & \text{when} |x|>1\end{cases}$$

as a Fourier integral. Hence evaluate

$$\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda$$ (*Mysore 1975S*)

Solution. The Fourier Integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\ \cos\lambda(t-x)\,dt\,d\lambda$$

$$=\frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1}\cos\lambda(t-x)\,dt\,d\lambda \qquad (\text{since } f(t)=1)$$

$$=\frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{-1}^{1}d\lambda$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{\sin\lambda(1-x)+\sin\lambda(1+x)}{\lambda}d\lambda \qquad \text{By } \sin C+\sin D \text{ formula}$$

Thus
$$f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda$$ **Ans.**

or
$$\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{2}f(x)$$

or
$$\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\begin{cases}\frac{\pi}{2} & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$$

For $|x|=1$, which is a point of discontinuity of $f(x)$, value of integral $=\dfrac{\pi/2+0}{2}=\dfrac{\pi}{4}$ **Ans.**

Example 2. *Find the Fourier sine integral for*

$$f(x)=e^{-\beta x} \qquad (\beta>0)$$

hence show that $\dfrac{\pi}{2}e^{-\beta x}=\displaystyle\int_0^{\infty}\frac{\lambda\sin\lambda x}{\beta^2+\lambda^2}d\lambda$ (*Gulbarga 1996*)

Solution. The Fourier sine transform of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda\, x\, d\lambda \int_0^\infty f(t)\sin\lambda\, t\, dt \qquad \text{... (1)}$$

Putting the value of $f(x)$ in (1) we get

$$e^{-\beta x} = \frac{2}{\pi}\int_0^\infty \sin\lambda\, x\, d\lambda \int_0^\infty e^{-\beta t}\sin\lambda t\, dt$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda\, x\, d\lambda\left[\frac{e^{-\beta t}}{(\beta^2+\lambda^2)}\left(-\beta\sin\lambda t - \lambda\cos\lambda t\right)_0^\infty\right]$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x\, d\lambda\left[0 + \frac{\lambda}{\beta^2+\lambda^2}\right]$$

$$e^{-\beta x} = \frac{2}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{\beta^2+\lambda^2}d\lambda \qquad \text{or} \qquad \frac{\pi}{2}e^{-\beta x} = \int_0^\infty \frac{\lambda\sin\lambda x}{\beta^2+\lambda^2}d\lambda. \qquad \textbf{Proved.}$$

Example 3. *Using Fourier cosine integral representation of an appropriate function, show that*

$$\int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw = \frac{\pi e^{-kx}}{2k}, \qquad x > 0, k > 0.$$

Solution. We know that Fourier integral is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos u\, x\, du \int_0^\infty f(t)\cos u\, t\, dt$$

Putting the value of $f(t)$ and replacing u by w we get

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \cos wx\, dw \int_0^\infty e^{-kt}\cos w\, t\, dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos w\, x\, dw\left[\frac{e^{-kt}}{k^2+w^2}\{-k\cos wt + w\sin wt\}_0^\infty\right]$$

$$= \frac{2}{\pi}\int_0^\infty \cos w\, x\, dw\left[0 + \frac{k}{k^2+w^2}\right] = \frac{2k}{\pi}\int_0^\infty \frac{\cos w\, x\, dw}{k^2+w^2}$$

or $$\int_0^\infty \frac{\cos wx}{k^2+w^2} = \frac{\pi e^{-kx}}{2k} \qquad \textbf{Proved.}$$

14.6 FOURIER TRANSFORMS

We have done in Article 14.5 that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-iux}du \int_{-\infty}^\infty f(t)e^{iut}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-isx}ds\int_{-\infty}^\infty f(t)e^{ist}dt \qquad (u = s) \quad \text{... (1)}$$

Putting $\int_{-\infty}^\infty f(t)e^{ist}dt = F(s)$ in (1) we get ... (2)

or $$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-isx}.F(s)\, ds \qquad \text{... (3)}$$

In (2) $F(s)$ is called the **Fourier transform** of $f(x)$.

In (3) $f(x)$ is called the **inverse Fourier transform** of $F(s)$.

For reasons of symmetry, we multiply both $F(x)$ and $F(s)$ by $\sqrt{\frac{1}{2\pi}}$ instead of having the factor $\frac{1}{2\pi}$ in only one function. Thus, we obtain the definition of Fourier transforms as

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

14.7 FOURIER SINE AND COSINE TRANSFORMS

From equation (7) of Article 14.4 we know that

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin sx\, ds \int_0^{\infty} f(t) \sin st\, dt \qquad (s = u)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sin sx\, ds\, F(s) \qquad \text{... (1)}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin st\, dt \qquad \text{... (2)}$$

In equation (2), $F(s)$ is called **Fourier sine transform** of $f(x)$.
In equation (1), $f(x)$ is called the **Inverse Fourier sine transform** of $F(s)$
From equation (8) of Article 14.4, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos sx\, F(s)\, dx \qquad \text{... (3)}$$

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos st\, dt \qquad \text{... (4)}$$

In equation (4), $F(s)$ is called **Fourier cosine transform** of $F(x)$.
In equation (3), $f(x)$ is called the **inverse Fourier cosine transform of** $F(s)$.

Example 4. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 & for \quad |x| < a \\ 0 & for \quad |x| > a \end{cases}$$

Solution. The Fourier transform of a function $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) . e^{isx} dx$$

Substituting the value of $f(x)$, we get

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 . e^{isx} dx = \left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{1}{(is)} \left[e^{ias} - e^{-ias}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{s} . \frac{e^{ias} - e^{-ias}}{2i} = \frac{1}{\sqrt{2\pi}} \frac{2 \sin sa}{s} = \sqrt{\frac{2}{\pi}} \frac{\sin sa}{s} \qquad \textbf{Ans.}$$

Example 5. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 - x^2 & if \quad |x| \le 1. \\ 0 & if \quad |x| > 1. \end{cases}$$

Solution. $f(x)=\begin{cases}1-x^2 & -1<x<1\\ 0 & |x|>1\end{cases}$

The Fourier transform of a function $f(x)$ is given by

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{isx}.dx \qquad \dots (1)$$

Substituting the values of $f(x)$ in (1), we get

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)\,e^{isx}dx$$

Integrating by parts, we get $\left[\int [uv]_1 = uv_1 - u'v_2 + u''u_3\dots\right]$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{isx}}{is}-(-2x)\frac{e^{isx}}{(is)^2}+(-2)\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$=\frac{1}{\sqrt{2\pi}}\left[-2\frac{e^{is}}{s^2}+2\frac{e^{is}}{is^3}-2\frac{e^{is}}{s^2}-\frac{e^{-is}}{is^3}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[-\frac{2}{s^2}(e^{is}+e^{-is})+\frac{2}{is^3}(e^{is}-e^{-is})\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[-\frac{2}{s^2}(2\cos s)+\frac{2}{is^3}(2i\sin s)\right]$$

$$=\frac{1}{\sqrt{2\pi}}\frac{4}{s^3}[-s\cos s+\sin s]$$ **Ans.**

Example 6. *Find the Fourier sine and cosine transforms of* $f(x)=e^{-ax}$.

Solution. The Fourier sine transform of $f(x)$ is given by

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$$

Putting the value of $f(x)$ we get

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{e^{-ax}}{a^2+s^2}[-a\sin sx-s\cos sx]_0^{\infty}$$

$$=-\sqrt{\frac{2}{\pi}}\,\frac{e^{-ax}}{a^2+s^2}[a\sin sx+s\cos sx]_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\left[-0+\frac{1}{a^2+s^2}\times s\right]=\sqrt{\frac{2}{\pi}}\,\frac{s}{a^2+s^2}$$ **Ans.**

The Fourier cosine transform is

$$F(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}\{-a\cos sx + s\sin sx\}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\left[0+\frac{a}{a^2+s^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2} \qquad \textbf{Ans.}$$

Example 7. *Find Fourier sine transform of* $\frac{1}{x}$.

Solution. $F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin sx}{x}dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin\theta}{\frac{\theta}{s}}\frac{d\theta}{s}$$ Putting s $x = \theta$ so that $s\,dx = d\theta$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin\theta}{\theta}d\theta = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}} \qquad \textbf{Ans.}$$

Example 8. *Find the Fourier cosine transform of*
$f(x) = e^{-2x} + 4\,e^{-3x}$

Solution. The Fourier cosine transform of $f(x)$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$$

Putting the value of $f(x)$ we get

$$F(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}(e^{-2x}+4e^{-3x})\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-2x}\cos sx\,dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty}4\,e^{-3x}\cos sx\,dx$$

$$\left(\because \int e^{-ax}\cdot\cos bx\,dx = \frac{e^{-ax}}{a^2+b^2}[b\sin bx - a\cos bx]\right)$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2}{s^2+4}+4\cdot\frac{3}{s^2+9}\right] = 2\sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+4}+\frac{6}{s^2+9}\right] \qquad \textbf{Ans.}$$

Example 9. *Find the Fourier sine transform of*

$$f(x) = \frac{e^{-ax}}{x}.$$

Solution. The sine transform of the function $f(x)$ is given by

$$F(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \sin sx\ dx$$

Substituting the value of $f(x)$, we get

$$F(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{x}\sin sx\,dx$$

Differentiating both sides w.r.t. 's' we get

$$\frac{d}{dx}[F(s)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{x}(x\cos sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos ax \, dx = \sqrt{\frac{2}{\pi}} \frac{a}{s^2 + a^2}$$

Integrating w.r.t. 's' we get

$$F(s) = \sqrt{\frac{2}{\pi}} \int \frac{a}{s^2 + a^2} ds = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{s}{a} + c$$

For $s = 0$, $F(s) = 0$

Putting these values in the above equation we get

$$0 = 0 + c \quad \text{or} \quad c = 0 \qquad \therefore \quad F(s) = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{s}{a}$$ **Ans.**

Example 10. *Find the Fourier sine and cosine transforms of*

$$f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$$

Solution. Fourier sine Transform

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a 1 \sin sx \, dx$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sx}{s} \right]_0^a = \sqrt{\frac{2}{\pi}} \left[-\frac{\cos as}{s} + \frac{1}{s} \right]$$

Fourier cosine transform

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a 1 \cos sx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$$ **Ans.**

Example 11. *Find the Fourier cosine transform of*

$$f(x) = \begin{cases} x & \text{for} \quad 0 < x < \frac{1}{2} \\ 1 - x & \text{for} \quad \frac{1}{2} < x < 1 \\ 0 & \text{for} \quad x > 1. \end{cases}$$

Write the inverse transform.

Solution. Fourier cosine transform

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1/2} x \cos sx \, dx + \sqrt{\frac{2}{\pi}} \int_{1/2}^1 (1 - x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[x \frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2} \right) \right]_0^{1/2} + \sqrt{\frac{2}{\pi}} \left[(1 - x) \frac{\sin sx}{s} - (-1) \frac{(-\cos sx)}{s^2} \right]_{1/2}^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{2} \frac{\sin s/2}{s} + \frac{\cos s/2}{s^2} - \frac{1}{s^2} \right] + \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s^2} - \frac{1}{2} \frac{\sin s/2}{s} + \frac{\cos s/2}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s^2} + \frac{2 \cos s/2}{s^2} - \frac{1}{s^2} \right]$$ **Ans.**

Example 12. *Find the Fourier transform of the function*

$$f(x)=\begin{cases} 1+\dfrac{x}{a}, & -a<x<0 \\ 1-\dfrac{x}{a} & 0<x<a \\ 0 & otherwise \end{cases}$$

(*U.P., III Semester, Summer 2002*)

Solution. Fourier tranform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{0}\left(1+\frac{x}{a}\right)e^{isx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{a}\left(1-\frac{x}{a}\right)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(1+\frac{x}{a}\right)\times\frac{e^{isx}}{is}-\left(\frac{1}{a}\right)\frac{e^{isx}}{-s^2}\right]_{-a}^{0} + \frac{1}{\sqrt{2\pi}}\left[\left(1-\frac{x}{a}\right)\frac{e^{isx}}{is}-\left(-\frac{1}{a}\right)\frac{e^{isx}}{-s^2}\right]_{0}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{is}+\frac{1}{a}\frac{1}{s^2}+\frac{1}{a}\frac{e^{-isa}}{-s^2}\right]+\frac{1}{\sqrt{2\pi}}\left[\frac{1}{a}\cdot\frac{e^{isa}}{-s^2}-\frac{1}{is}+\frac{1}{a}\frac{1}{s^2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2}+\frac{1}{-as^2}\left(e^{isa}+e^{-isa}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2}-\frac{2}{as^2}\cos sa\right] = \frac{1}{\sqrt{2\pi}}\,\frac{2}{as^2}\,[1-\cos as]$$

$$= \frac{2}{\sqrt{2\pi}\,as^2}\,2\sin^2\frac{as}{2} = \frac{2\sqrt{2}\,\sin^2\dfrac{as}{2}}{\sqrt{\pi}\,as^2}$$ **Ans.**

Example 13. *Find Fourier sine and cosine transform of (a)* x^{n-1}. *(b)* $\dfrac{1}{\sqrt{x}}$.

Solution. (*a*) $\quad F_s(x^{n-1}) = \sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} \sin sx\cdot x^{n-1}\,dx$

$$F_c(x^{n-1}) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\cos sx\cdot x^{n-1}dx$$

$$F_c(x^{n-1}) + F_s(x^{n-1}) = \frac{2}{\sqrt{\pi}}\int_0^{\infty}(\cos sx + i\sin sx)\,x^{n-1}dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{isx}x^{n-1}dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-t}\left(-\frac{t}{is}\right)^{n-1}\left(-\frac{dt}{is}\right)$$

$$\left[\begin{aligned} \text{Putting}\quad isx &= -t, \\ x &= -\frac{t}{is}, \\ dx &= -\frac{dt}{is} \end{aligned}\right.$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{(is)^n} (-1)^n \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{(i)^{2n}}{(i)^n s^n} \Gamma n = \sqrt{\frac{2}{\pi}} \frac{(i)^n}{s^n} \Gamma n$$

$$= \sqrt{\frac{2}{\pi}} \frac{\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n \Gamma n}{s^n} = \sqrt{\frac{2}{\pi}} \frac{\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)\Gamma n}{s^n}$$

Equating real and imaginary parts we get

$$F_c(x^{n-1}) = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos\frac{n\pi}{2}$$

$$F_s(x^{n-1}) = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin\frac{n\pi}{2}$$ **Ans.**

(*b*) $n = \frac{1}{2}$

$$F_c\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}} \cos\frac{\pi}{4} = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$

$$F_s\left(\frac{1}{\sqrt{x}}\right) = \sqrt{\frac{2}{\pi}} \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}} \sin\frac{\pi}{4} = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{\sqrt{s}} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{s}}$$ **Ans.**

Example 14. *Find the complex Fourier transform of dirac delta function* $\delta(t-a)$.

$$F\{\delta(t-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \delta(t-a)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \operatorname*{Lt}_{h\to 0} \int_a^{a+h} \frac{1}{h} e^{ist} dt$$

$$= \frac{1}{\sqrt{2\pi}} \operatorname*{Lt}_{h\to 0} \frac{1}{h} \left(\frac{e^{ist}}{is}\right)_a^{a+h}$$

$$= \frac{1}{\sqrt{2\pi}} \operatorname*{Lt}_{h\to 0} e^{isa} \left(\frac{e^{ish}-1}{ish}\right)$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \qquad \text{since } \operatorname*{Lt}_{\theta\to 0} \frac{e^\theta - 1}{\theta} = 1$$ **Ans.**

Note. Dirac delta function $\delta(t-a)$ is defined as

$$\delta(t-a) = \operatorname*{Lt}_{h\to 0} I(h\ t-a) \text{ where}$$

$$I(h, t-a) = \frac{1}{h} \qquad \text{for } a < t < a+h$$

$$= 0 \qquad \text{for } t < a \text{ and } t > a+h$$ **Ans.**

Example 15. *Show that*

(*a*) $F_s\,[x\, f(x)] = -\dfrac{d}{ds} F_c(s)$ (*b*) $F_c\,[x\, f(x)] = \dfrac{d}{ds} F_s(s)$

and hence find Fourier cosine and sine transform of $xe.^{-ax}$

Solution. (*a*) $F_c\,(s) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x) \cos sx\, dx$

$$\frac{d}{ds} F_c(s) = -\sqrt{\frac{2}{\pi}} \int_0^\infty x\, f(x) \sin sx\, dx$$

$$= -\sqrt{\frac{2}{\pi}}\; F_s\{x\; f(x)\}$$

(*b*) $F_s\,(s) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x) \sin sx\, dx$

$$\frac{d}{ds} F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty x\, f(x)\, \cos sx\; dx$$

$$= \sqrt{\frac{2}{\pi}}\; F_c\,\{x\, f(x)\}$$

(*c*) $F_c\,(xe^{-ax}) = \dfrac{d}{ds}\, F_s(e^{-ax})$

$$= \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\; \frac{s}{a^2+s^2}\right] \qquad \text{(Using example 6)}$$

$$= \sqrt{\frac{2}{\pi}}\; \frac{(a^2+s^2) - s(2s)}{(a^2+s^2)^2} = \sqrt{\frac{2}{\pi}}\, \frac{a^2-s^2}{(a^2+s^2)^2}$$

(*d*) $F_s\,(xe^{-ax}) = -\dfrac{d}{dx} F_c(e^{-ax}) = -\dfrac{d}{ds}\left[\sqrt{\dfrac{2}{\pi}}\, \dfrac{a}{a^2+s^2}\right]$ (Using example 6)

$$= \sqrt{\frac{2}{\pi}}\; \frac{2as}{(a^2+s^2)^2}$$ **Ans.**

Example 16. *Find Fourier cosine transform of* $e^{-a^2x^2}$ *and hence evaluate Fourier sine transform of* $xe^{-a^2x^2}$.

Solution. $F_c(e^{-a^2x^2}) = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty e^{-a^2x^2} \cdot \cos sx\, ds$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cdot e^{isx}\, ds$$

$$= \text{Real part of } \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2+isx} ds$$

$$= \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \quad \text{(Refer example)}$$

We know that

$$F_s\,(x\,f(x) \;=\; -\frac{d}{ds} F_c f(x)$$

$$F_s(xe^{-a^2x^2}) \;=\; -\frac{d}{ds} F_c\,(e^{-a^2x^2}) \qquad\qquad f(x) = e^{-a^2x^2}$$

$$= -\frac{d}{ds}\frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \;=\; \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \frac{s}{2a^2}$$

$$= \frac{1}{2a^3\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

Ans.

EXERCISE 14.1

1. Express $f(x) = \begin{bmatrix} 1 \text{ for } 0 \le x \le \pi \\ 0 \text{ for } x > \pi \end{bmatrix}$ as a Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1-\cos\pi\lambda}{\lambda} \sin\lambda\, x\, d\lambda$$

Ans. $\frac{\pi}{4}$

2. Find the Fourier's cosine integral of the function e^{-ax}. Hence show that

$$\int_0^\infty \frac{\cos\lambda x}{\lambda^2+1^2} d\lambda = \frac{\pi}{2} e^{-x}, \quad x \ge 0$$

Ans. $\frac{2a}{\pi}\int_0^\infty \frac{\cos\lambda x}{\lambda^2+a^2} d\lambda.$

3. Show that the Fourier transform of

$$\begin{aligned} f(x) &= 0 \text{ for } x < \alpha \\ &= 1 \text{ for } \alpha < x < \beta \\ &= 0 \text{ for } x > \beta \end{aligned}$$

is $\frac{1}{\sqrt{2\pi}}\left(\frac{e^{i\beta s} - e^{i\alpha s}}{is}\right)$

4. Find the Fourier transform of $f(x)$ if

$$f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Ans. $\frac{1}{\sqrt{2\pi}}\frac{2i}{s^2}(as\cos as - \sin as)$

5. Show that the Fourier transform of

$$f(x) = \begin{cases} a - |x| & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$$

is $\sqrt{\frac{2}{\pi}}\,\frac{1-\cos as}{s^2}$. Hence show that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

6. Show that the Fourier transform of

$$f(x)=\begin{cases}\dfrac{\sqrt{2\pi}}{2a} & \text{for } |x|\le a\\ 0 & \text{for } |x|>a\end{cases}$$

is $\dfrac{\sin sa}{sa}$

7. Show that the transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{s^2}{2}}$ by finding the Fourier transform of $e^{-a^2x^2}$, $a > 0$.

8. Show that the Fourier transform of $e^{\frac{-x^2}{2}}$ is self-reciprocal.

9. Find Fourier transform of $e^{-a|x|}$ is $a > 0$. **Ans.** $\dfrac{2a}{a^2+s^2}$

10. Find Fourier transform of $\dfrac{1}{\sqrt{|x|}}$.

11. Find the Fourier tranform of $f(x) = e^{i\,kx}$, $a < x < b$
$= 0$, $x < a$ and $x > b$

Ans. $\dfrac{i}{\sqrt{2\pi}\,(k+s)}\left[e^{i(k+s)a}-e^{i(k+s)b}\right]$

12. Find the Fourier sine transform of $e^{-|x|}$. Hence evaluate

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx.$$

Ans. $\dfrac{s}{1+s^2}$, $\dfrac{\pi}{2}e^{-m}$

13. Show that the Fourier sine transform of

$$f(x)=\begin{cases}x & \text{for } 0 < x < 1\\ 2-x & \text{for } 1 < x < 2\\ 0 & \text{for } x > 2\end{cases}$$

is $2\sin s\,(1-\cos s)/s^2$.

14. Show that the Fourier sine transform of $\dfrac{x}{1+x^2}$ is $\sqrt{\dfrac{\pi}{2}}\,as\,e^{-as}$

15. Show that the Fourier cosine transform of $\dfrac{1}{1+x^2}$ is $\sqrt{\dfrac{\pi}{2}}\,e^{-s}$

16. Find the sine and cosine transforms of e^{-ax} $(a > 0)$ **Ans.** $\dfrac{s}{a^2+s^2}$, $\dfrac{a}{a^2+s^2}$

17. Find the Fourier sine and cosine transform of $\cosh x - \sinh x$

18. Find the Fourier sine and cosine transform of $ae^{-\alpha x}+be^{-\beta x}$, $\alpha, \beta > 0$.

Ans. $\dfrac{as}{s^2+\alpha^2}+\dfrac{bs}{s^2+\beta^2}$, $\dfrac{a\alpha}{s^2+\alpha^2}+\dfrac{b\beta}{s^2+\beta^2}$

14.8 PROPERTIES OF FOURIER TRANSFORMS

(1) **LINEAR PROPERTY.** I f $F_1(s)$ and $F_2(s)$ are Fourier transforms of $f_1(x)$ and $f_2(x)$ respectively, then

$$F[af_1(x)+b\,f_2(x)] = a\,F_1(s)+b\,F_2(s)$$

where a and b are constants.

We know that $$F_1(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \cdot f_1(x)\, dx$$

and $$F_2(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f_2(x)\, dx$$

$$F[af_1(x) + b f_2(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \left[af_1(x) + b f_2(x)\right] dx$$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f_1(x)dx + b\int_{-\infty}^{\infty} e^{isx} f_2(x)dx$$

$$= aF_1(s) + bF_2(s)$$ **Proved.**

(2) **CHANGE OF SCALE PROPERTY**

If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Proof. We know that $F(s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

$$F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(ax)\, dx$$ Put $ax = t \Rightarrow dx = \dfrac{dt}{a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\frac{t}{a}} f(t) \frac{dt}{a}$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\frac{s}{a}\right)t} f(t)\, dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$ **Proved.**

(3) **SHIFTING PROPERTY**

If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F\{f(x-a)\} = e^{isa} F(s)$$

Proof. $$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx$$ [Put $x - a = t$, so that $dx = dt$]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(t+a)} f(t) dt = \frac{e^{isa}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t) dt$$

$$= e^{isa} F(s)$$ **Proved.**

(4) $\boldsymbol{F\left\{e^{iax} f(x)\right\} = F(s+a)}$

Proof. $$F\{e^{iax} f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx$$
$$= F(s+a)$$ **Proved.**

(5) **MODULATION THEOREM**

If $F(s)$ is the complex Fourier transform of $f(x)$, then

$$F\{f(x)\cos ax\} = \frac{1}{2}[F(s+a) + F(s-a)]$$

Proof. We know that $$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \frac{e^{iax} + e^{-iax}}{2} dx$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) e^{iax} dx + \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} e^{-iax} dx$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx + \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s-a)x} f(x) dx$$

$$= \frac{1}{2} F(s+a) + \frac{1}{2} F(s-a)$$

$$= \frac{1}{2}[F(s+a) + F(s-a)]$$ **Proved.**

(6) If $\boldsymbol{F\{f(x)\} = F(s)}$, **then** $\boldsymbol{F\{x^n f(x)\} = (-i)^n \dfrac{d^n}{ds^n} F(s)}$.

Proof. We know that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) dx \quad \text{... (1)}$$

Differentiating (1) w.r.t. s both sides, n times, we get

$$\frac{d^n F(s)}{ds^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n e^{isx} f(x) dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{isx} \cdot f(x) \cdot dx$$

$$= (i)^n F(x^n f(x))$$

$$F(x^n f(x)) = (-i)^n \frac{d^n \{F(s)\}}{ds^n}$$

(7) $\boldsymbol{F\{f'(x)\} = is F(s)}$ **if** $\boldsymbol{f(x) \to 0}$ **as** $\boldsymbol{x \to \pm\ \infty}$

Proof. $$F\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f'(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} d\{f(x)\} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left\{ e^{isx} f(x) \right\}_{-\infty}^{\infty} e^{isx} f(x) dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - is \int_{-\infty}^{\infty} e^{is} f(x) dx \right]$$

$$= -is F(s).$$ **Proved.**

(8) $\boldsymbol{F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(s)}{(-is)}}$

Proof. Let $f_1(x) = \int_a^x f(x)\,dx \quad \Rightarrow \quad f_1'(x) = f(x)$

$$F\{f'(x)\} = (-is) F_1(s) = (-is) F\{f_1(x)\}$$

$$= -is F\left\{\int_a^x f(x)\,dx\right\}$$

$$F\left\{\int_a^x f(x)\,dx\right\} = \frac{1}{(-is)} F\{f_1'(x)\}$$

$$= \frac{1}{(-is)} F\{f(x)\} = \frac{F(s)}{(-is)}$$ **Proved.**

Note. $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively.

Properties.

1. $F_s\{af(x) + bg(x)\} = aF_s\{f(x)\} + b\,F_s\{g(x)\}$
2. $F_c\{af(x) + bg(x)\} = aF_c\{f\{x\}\} + b\,F_c\{g(x)\}$
3. $F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right)$
4. $F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{s}{a}\right)$
5. $F_s[f(x)\sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$
6. $F_c\{f(x)\sin ax\} = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$
7. $F_s\{f(x)\cos ax\} = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

Proof of (5) : $F_s\{f(x)\sin ax\} = \int_0^{\infty} f(x) \sin ax \,.\, \sin sx\,dx$

$$= \frac{1}{2} \int_0^{\infty} f(x)\{\cos(s-a)x - \cos(s+a)x\}\,dx$$

$$= \frac{1}{2}\left[\int_0^{\infty} f(x)\cos(s-a)x\,dx - \int_{0.}^{\infty} f(x)\cos(s+a)x\,dx\right]$$

$$= \frac{1}{2}[F_c(s-a) - F_c(s+a)]$$ **Proved**

14.9 CONVOLUTION

The Convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x)\, g(x-u)\, du$$

Convolution Theorem on Fourier Transform

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms, *i.e.*,

$$F[f(x) * g(x)] = F[f(x)] \cdot F[g(x)]$$

Proof. We know that

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \cdot g(x-u)\, du \qquad \text{... (1)}$$

Taking Fourier transform of both sides of (1), we have

$$F[f(x) * g(x)] = F\left[\int_{-\infty}^{\infty} f(u) \cdot g(x-u)\, du\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \cdot g(x-u)\, du\right] e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, du\, \frac{1}{\sqrt{2\pi}} \left\{\int_{-\infty}^{\infty} g(x-u) e^{isx} dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{f(u) \cdot du \cdot Fg(x-u)\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) du \cdot e^{ius} G(s) \qquad \text{(using shifting property)}$$

$$= G(s) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{ius} du$$

$$= G(s) \cdot F(s)$$

$$= F(s) \cdot G(s)$$ **Proved.**

By inversion

$$F^{-1}\{F(s) \cdot G(s)\} = f * g = F^{-1}\{F(s)\} * F^{-1}\{G(s)\}$$

14.10 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the fourier transforfm of $f(x)$ and $g(x)$ be $F(s)$ and $G(s)$ respectively, then

(*i*) $$\int_{-\infty}^{\infty} F(s)\bar{G}(s)\, ds = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\, dx$$

where $\overline{G}(s)$ is the complex conjugate of $G(s)$ and $\bar{g}(x)$ is the complex conjugate of $g(x)$

(*ii*) $$\int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Proof. (*i*) $$\int_{-\infty}^{\infty} [f(x)\bar{g}(x) dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{G}(s) e^{isx} ds\right] ds$$

Since $$\bar{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{G}(s) e^{isx} ds$$

$$\int_{-\infty}^{\infty} f(x)\bar{g}(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{G}(s) ds \cdot \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\left[\text{since } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = F(s)\right] \text{Fourier Transform}$$

$$= \int_{-\infty}^{\infty} \overline{G}(s) F(s)\, ds \qquad \dots (1)$$

Putting $g(x) = f(x)$ in (1) we get

$$\int_{-\infty}^{\infty} F(s) \cdot \overline{F}(s) = \int_{-\infty}^{\infty} f(x) \cdot \bar{f}(x)\, dx$$

or $$\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$ **Proved.**

14.11 PARSEVAL'S IDENTITY FOR COSINE TRANSFORM

(*i*) $\frac{2}{\pi} \int_0^{\infty} F_C(s) \cdot G_C(s)\, ds = \int_0^{\infty} f(x) \cdot g(x)\, dx$ (*ii*) $\frac{2}{\pi} \int_0^{\infty} |F_c(s)|^2\, dx = \int_0^{\infty} |f(x)|^2\, dx$

14.12 PARSEVAL'S IDENTITY FOR SINE TRANSFORM

(*i*) $\frac{2}{\pi} \int_0^{\infty} F_s(s) \cdot G_s(s)\, ds = \int_0^{\infty} f(x) \cdot g(x)\, dx$ (*ii*) $\frac{2}{\pi} \int_0^{\infty} |F_s(s)|^2\, ds = \int_0^{\infty} |f(x)|^2\, dx$

Example. 17. *Using Parseval's identity, show that*

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

Solution. Let $f(x) = e^{-x}$ so that $F_c(s) = \frac{1}{1+s^2}$

By Parseval's identity for cosine transformation

$$\frac{2}{\pi} \int_0^{\infty} [F_c(s)]^2\, ds = \int_0^{\infty} |f(x)|^2\, dx$$

$$\frac{2}{\pi} \int_0^{\infty} \left| \frac{1}{(1+s^2)^2} \right|^2 ds = \int_0^{\infty} |e^{-x}|^2\, dx = \int_0^{\infty} |e^{-2x}|\, dx = \left[\frac{e^{-2x}}{-2}\right]_0^{\infty}$$

$$\int_0^{\infty} \left| \frac{1}{(1+s^2)} \right|^2 ds = \frac{1}{2} \frac{\pi}{2} = \frac{\pi}{4}$$ **Ans.**

Example 18. *Using Parseval's identity, show that*

$$\int_0^{\infty} \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4}$$

Solution. Let $f(x) = \frac{x}{x^2+1}$ so that $F_s(s) = \frac{\pi}{2} e^{(-s)}$

By Parseval's identity for sine transformation

$$\frac{2}{\pi} \int_0^{\infty} |F_s(s)|^2\, ds = \int_0^{\infty} |f(x)|^2\, dx$$

$$\int_0^\infty \left|\frac{x}{x^2+1}\right|^2 dx = \frac{2}{\pi}\int_0^\infty \left|\frac{\pi}{2}e^{-x}\right|^2 ds$$

$$= \left(\frac{2}{\pi}\right)\left(\frac{\pi^2}{4}\right)\int_0^\infty \left|e^{-2s}\right| ds = \frac{\pi}{2}\left[\frac{e^{-2s}}{-2}\right]_0^\infty$$

$$= \frac{\pi}{2}\left[0+\frac{1}{2}\right] = \frac{\pi}{4}$$ **Proved**

Example 19. *Using Parseval's identity, prove that*

$$\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

Solution. Let $f(x) = e^{-ax}$, $g(x) = e^{-bx}$

Then $F_c(s) = \frac{a}{a^2+s^2}$, $G_c(s) = \frac{b}{b^2+s^2}$

By Parseval's identity for Fourier cosine transformation

$$\frac{2}{\pi}\int_0^\infty F_c(s)G_c(s)\,ds = \int_0^\infty f(x)\cdot g(x)\,dx \qquad \dots (1)$$

On substitution in (1), we get

$$\frac{2}{\pi}\int_0^\infty \left(\frac{a}{a^2+s^2}\right)\left(\frac{b}{b^2+s^2}\right) ds = \int_0^\infty e^{-ax}\cdot e^{-bx} dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{ab}{(a^2+s^2)}\frac{ds}{(b^2+s^2)} = \int_0^\infty e^{-(a+b)x} dx$$

$$= \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \left[0+\frac{1}{a+b}\right]$$

$$\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)} = \frac{\pi}{2ab}\frac{1}{a+b}$$

$$\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$ **Proved**

Example 20. *Using Parseval's identity, prove* $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$.

Solution. By example we know that

if $f(x) = \begin{cases} 1 & \text{for } |x| < 0 \\ 0 & \text{for } |x| > a > 0 \end{cases}$

then $F(s) = \sqrt{\frac{2}{\pi}}\frac{\sin as}{s}$

Using Parseval's identity

$$\int_{-\infty}^\infty |f(t)|^2\,dt = \int_{-\infty}^\infty |F(s)|^2\,ds$$

$$\therefore \qquad \int_{-a}^{a} (1)^2\, dt = \int_{-\infty}^{\infty} \frac{2}{\pi}\left(\frac{\sin a s}{s}\right)^2 ds$$

$$2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

Putting $as = t$, we get $$2a = \frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin t}{\frac{t}{a}}\right)^2 \frac{dt}{a} \qquad \left(a\, ds = dt,\ ds = \frac{dt}{a}\right)$$

$$a\pi = a\int_{-\infty}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \pi$$ **Proved.**

Example 21. *Find the Fourier transform of*

$f(x) = 1 - |x|$ if $|x| < 1$

$= 0$ for $|x| > 1$

and hence find the value of $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt$.

Solution. $$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)\cos sx\, dx + \frac{i}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)\sin sx\, dx$$

(Even function) (odd function)

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{1} (1-x)\cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{(1-x)\frac{\sin sx}{s}\right\}_0^1 + \int_0^1 \frac{\sin sx}{s}\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 + \left\{\frac{-\cos sx}{s^2}\right\}_0^1\right] = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)$$

Using Parseval's identity, we get

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(t)|^2\, dt$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = \int_{-1}^{+1} (1-|x|)^2 dx$$

$$\frac{4}{\pi}\int_0^{\infty}\frac{\left(1-1+2\sin^2\frac{s}{2}\right)^2}{s^4}\,ds=\int_{-1}^{+1}(1+x^2-2x)\,dx \quad \text{(Odd function)}$$

$$\frac{16}{\pi}\int_0^{\infty}\frac{\sin^4\frac{s}{2}}{s^4}\,ds=2\int_0^1(1+x^2)\,dx=2\left(x+\frac{x^3}{3}\right)_0^1=\frac{2}{3}$$

Putting $\frac{s}{2}=x$, we get

$$\int_0^{\infty}\frac{\sin^4 x}{x^4}\,dx=\frac{\pi}{3} \qquad \textbf{Ans.}$$

Example 22. *Solve for f(x) from the integral equation*

$$\int_0^{\infty} f(x)\cos sx\,dx=e^{-s}$$

Solution. $$\int_0^{\infty} f(x)\cos x\,dx=e^{-s} \qquad \text{... (1)}$$

Multiplying (1) by $\sqrt{\frac{2}{\pi}}$, we get

$$\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx=\sqrt{\frac{2}{\pi}}\,e^{-s}$$

$$F_c\{f(x)\}=\sqrt{\frac{2}{\pi}}\,e^{-s}$$

$$f(x)=F_c^{-1}\left[\sqrt{\frac{2}{\pi}}e^{-s}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty}e^{-s}\cos sx\,ds\right]$$

$$=\frac{2}{\pi}\int_0^{\infty}e^{-s}\cos sx\,ds$$

$$=\frac{2}{\pi}\left[\frac{e^{-s}}{1+x^2}\{\cos sx+s\sin sx\}\right]_0^{\infty}$$

$$=\frac{2}{\pi}\frac{1}{1+x^2} \qquad \textbf{Ans.}$$

Example. 23. *Solve for f (x) from the integral equation*

$$\int_0^{\infty} f(x)\sin sx\,dx=\begin{cases}1 \text{ for } 0\le s<1\\ 2 \text{ for } t\le s<2\\ 0 \text{ for } \quad s\ge 2\end{cases}$$

Solution. Multiplying by $\sqrt{\frac{2}{\pi}}$ both sides of the given equation, we get

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{for } 0 \le s < 1 \\ 2\sqrt{\frac{2}{\pi}} & \text{for } 1 \le s < 2 \\ 0 & \text{for } s \ge 2 \end{cases}$$

$$F_s f(x) = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{for } 0 \le s < 1 \\ 2\sqrt{\frac{2}{\pi}} & \text{for } 1 \le s < 2 \\ 0 & \text{for } s \ge 2 \end{cases}$$

$$f(x) = F_s^{-1} \quad \text{(R.H.S.)}$$

$$= \frac{2}{\pi}\int_0^1 \sin sx\, ds + \frac{4}{\pi}\int_1^2 \sin sx ds$$

$$= \frac{2}{\pi}\left(\frac{1-\cos x}{x}\right) + \frac{4}{\pi}\left(\frac{\cos x - \cos 2x}{x}\right)$$

$$= \frac{2}{\pi x}[1 - \cos x + 2\cos x - 2\cos 2x]$$

$$f(x) = \frac{2}{\pi x}(1 + \cos x - 2\cos 2x)$$ **Ans.**

Example 24. *Find the function if its sine transform is* $\dfrac{e^{-s}}{s}$.

Let $$F_s(f(x)) = \frac{e^{-as}}{s}$$

Then, $$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds \qquad \text{... (1)}$$

$$\frac{df}{dx} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\cos sx\, ds$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+x^2}$$

$$f(x) = \sqrt{\frac{2}{\pi}}\cdot a\int \frac{dx}{a^2+x^2}$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{x}{a} + c \qquad \text{... (2)}$$

At $x = 0$, $f(0) = 0$ using (1)

Using this in (2), $c = 0$

Hence, $$f(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\frac{x}{a}.$$

Setting $\qquad a = 0,$

$$F_s^{-1}\left(\frac{1}{s}\right) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}} \quad \textbf{Ans.} \qquad \text{... (3)}$$

Example 25. *Prove (1)* $F\{x^n\,(x)\} = (-i)^n \dfrac{d^n F(s)}{ds^n}$ *and (ii)* $F\{f^n\,(x)\} = (-is)^n F(s)$

(iii) Hence solve for f (x) if $\int_{-\infty}^{\infty} f(t)\, e^{|x-r|} dt = \phi(x)$ *is known.*

Proof. (*i*)
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ix)^n e^{isx} f(x)\, dx$$

$$(-i)^n \frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{isx} f(x)\, dx$$

$$= F\{x^n f(x)\}$$

(*ii*) Similarly,

$$F\{f^n(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} \frac{d^n}{dx^n} f(x) \cdot dx$$

$$= (-is)^n F(s).$$

Using integration by parts successively and making assumptions that $f,\ f',\ \ldots.\ f^{(n-1)} \to 0$ as $f(x) \to \pm\infty$.

(*iii*)
$$\frac{1}{\sqrt{2\pi}} \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{|x-t|} dt, \text{ from the given equation}$$

$$= f(x) * e^{-|x|}$$

By convolution theorem,

$$\frac{1}{\sqrt{2\pi}} \bar{\phi}(s) = F(s) \cdot \sqrt{\frac{2}{\pi}} \frac{1}{1+s^2}$$

$$F(s) = \frac{1}{2}(1+s^2)\bar{\phi}(s)$$

$$= \frac{1}{2}[\bar{\phi}(s) - (-is)^2 \bar{\phi}(s)]$$

$\therefore$
$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2}\phi''(x) \text{ using the result derived in } (ii)$$

EXERCISE 14.2

Using Parseval's identity

1. Prove that $\displaystyle\int_0^{\infty} \frac{\sin at}{t(a^2+t^2)} dt = \frac{\pi}{2} \frac{1-e^{-a^2}}{a^2}$

2. Evaluate $\displaystyle\int_0^{\infty} \left(\frac{1-\cos x}{x}\right)^2 dx$ **Ans.** $\dfrac{\pi}{2}$

14.13. FOURIER TRANSFORM OF DERIVATIVES

We have already seen that,

$$F\{f^n(x)\} = (-is)^n \; F(s)$$

(*i*) $\therefore \; F\left(\dfrac{\partial^2 u}{\partial x^2}\right) = (-is)^2 \, F\{u(x)\} = -s^2 \bar{u}$ where $\bar{u}$ is Fourier transform of u w.r.t. x.

(*ii*)
$$F_c\{f'(x)\} = -\sqrt{\frac{2}{\pi}} f(0) + s\, F_s(s)$$

$$\text{L.H.S.} = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cdot \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \, d\{f(x)\}$$

$$= \sqrt{\frac{2}{\pi}} \left[\{f(x) \cos sx\}_0^\infty + s \int_0^\infty f(x) \sin sx \, dx \right]$$

$$= s\, F_s(s) - \sqrt{\frac{2}{\pi}} f(0) \;\; \text{assuming } \; f(x) \to 0 \text{ as } x \to \infty$$

(*iii*)
$$F_s\{f'(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin sx \, d\,[f(x)]$$

$$= \sqrt{\frac{2}{\pi}} \left[\{f(x) \sin sx\}_0^\infty - s \int_0^\infty f(x) \cos sx \, dx \right]$$

$$= -s\, F_c(s)$$

(*iv*)
$$F_c\{f''(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos sx \, d\,[f'(x)]$$

$$= \sqrt{\frac{2}{\pi}} \left[\{f'(x) \cos sx\}_0^\infty + s \int_0^\infty f'(x) \sin sx \, dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} f'(0) + s\, F_s\{f'(x)\}$$

$$= -s^2 \, F_c(s) - \sqrt{\frac{2}{\pi}} f'(0) \;\; \text{assuming } f(x),\; f'(x) \to 0 \text{ as } x \to \infty$$

(*v*)
$$F_s\{f''\}(x) = \sqrt{\frac{2}{\pi}} \left[\int_0^\infty \sin sx \, d[f'(x)] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(f'(x) \sin sx\right)_0^\infty - s \int_0^\infty f'(x) \cos sx \, dx \right]$$

$$= -s\,F_c\{f'(x)\} = -s\left[s\,F_s\,(s) - \sqrt{\frac{2}{\pi}} f(0)\right]$$

$$= -s^2\,F_s(s) + \sqrt{\frac{2}{\pi}}\,s\,f(0) \text{ assuming } f(x),\ f'(x) \to 0 \text{ as } x \to \infty.$$

14.14. RELATIONSHIP BETWEEN FOURIER AND LAPLACE TRANSFORMS

Consider

$$f(t) = \begin{cases} e^{-st}\,g(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases} \qquad \text{... (1)}$$

Then the Fourier transform of $f(t)$ is given by

$$F\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{(is-x)t} g(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-pt} g(t)\,dt \quad \text{where } p = x - is$$

$$= \frac{1}{\sqrt{2\pi}} L\{g(t)\}$$

$\therefore$ Fourier transform of $f(t) = \dfrac{1}{\sqrt{2\pi}} \times$ Laplace transform of $g(t)$ defined by (1).

14.15. SOLUTION OF BOUNDARY VALUE PROBLEMS BY USING INTEGRAL TRANSFORM

Solution of heat conduction problems by Laplace transform.

Example 26. *A semi-infinite solid $x > 0$ is initially at temperature zero. At time $t = 0$, a constant temperature u_0 is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid and at any time $t > 0$.*

Solution. Let $u(x, t)$ be the temperature at any point x and at any time t. The equation governing the flow of heat in the solid is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x > 0,\ t > 0 \qquad \text{... (1)}$$

The initial and boundary conditions are

$u = 0$ when $t = 0$ for all $x (x \geq 0)$ ... (2)

$u = u_0$ when $x = 0$ for all t, ... (3)

u is finite for all x and for all t, ... (4)

Multiplying equation (1) by e^{-st} and integrate w.r.t. 't' from 0 to ∞,

$$\int_0^{\infty} \frac{\partial u}{\partial t} e^{-st}\,dt = c^2 \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-st}\,dt \quad \text{or} \quad s\bar{u} = c^2 \frac{d^2\bar{u}}{dx^2} \qquad \text{... (5)}$$

($\bar{u}$ = Laplace transform of u)

Similarly Laplace transformation of equation (3) gives

$$s\,\bar{u} = u_0 \quad \text{when} \quad x = 0 \quad \text{or} \quad \bar{u} = \frac{u_0}{s} \qquad \text{... (6)}$$

Equation (5) is an ordinary differential equation and its solution is given by

$$\bar{u} = Ae^{\frac{\sqrt{s}}{c}x} + Be^{-\frac{\sqrt{s}}{c}x} \qquad \text{... (7)}$$

According to condition (4), u is finite at $x \to \infty$.

$\therefore \qquad A = 0$

So (7) becomes $\qquad \bar{u} = Be^{-\frac{\sqrt{s}}{c}x}$

Using (6) equation $\qquad \bar{u} = \dfrac{u_0}{s}$ when $x = 0$, $u_0 / s = B$

Thus (8) becomes $\qquad \bar{u} = \dfrac{u_0}{s} e^{-\frac{\sqrt{s}}{c}x}$

To get u from $\bar{u}$, we invert the transformation.

$$u = u_0\left(1 - erf\,\frac{x}{2c\sqrt{t}}\right) \quad \textbf{Ans.}$$

Solution of wave equation by Laplace transform

Example 27. *An infinitely long string having one end at x = 0 is initialy at rest along x-axis. The end x = 0 is given a transverse displacement f(t), when t > 0. Find the displacement of any point of the string at any time.*

Solution. Let $y\,(x, t)$ be the displacement, then wave equation is

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2} \qquad \text{... (1)}$$

subject to the conditions

$y\,(x, 0) = 0 \qquad \text{... (2)} \qquad \dfrac{\partial y}{\partial t}(x, 0) = 0 \qquad \text{... (3)}$

$y\,(0, t) = f\,(t) \qquad \text{... (4)} \qquad y\,(x, t)$ is bounded $\qquad \text{... (5)}$

On taking Laplace transform of (1), we have

$$L\left(\frac{\partial^2 y}{\partial x^2}\right) = c^2\, L\frac{\partial^2 y}{\partial t^2}$$

$$s^2\bar{y} - sy(x,0) - \frac{\partial y}{\partial t}(x,0) = c^2 \frac{d^2\bar{y}}{dx^2} \qquad \text{... (6)}$$

On putting $y\,(x, 0) = 0$, $\dfrac{\partial y}{\partial t}(x, 0) = 0$ in (6) get

$$s^2\,\bar{y} = c^2 \frac{d^2\bar{y}}{dx^2} \quad \text{or} \quad \frac{d^2\bar{y}}{dx^2} = \left(\frac{s}{c}\right)^2 \bar{y} \qquad \text{... (7)}$$

Laplace transform of (4), $\bar{y}(0, s) = \bar{f}(s)$ at $x = 0 \qquad \text{... (8)}$

On solving (7), we get $\bar{y} = Ae^{\frac{sx}{c}} + Be^{-\frac{sx}{c}} \qquad \text{... (9)}$

According to condition (5), y is finite at $x \to \infty$, this gives $A = 0$ sol (9) becomes

$$\bar{y} = Be^{-\frac{sx}{c}} \qquad \text{... (10)}$$

Putting the value of $\bar{y}(0, s) = \bar{f}(s)$ at $x = 0$ in (10), we get $\bar{f}(s) = B$

Thus (10) becomes $$y = \bar{f}(s) \cdot e^{-\frac{sx}{c}}$$

To get y from $\bar{y}$, we use complex inversion formula

$$y = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\left(t-\frac{x}{c}\right)s} - f(s)\, ds$$

Hence $$y = f\left(t - \frac{x}{c}\right) \quad \textbf{Ans.}$$

Example 28. *A uniform rod of length l is at rest in its equilibrium position with the end $x = 0$ fixed. At $t = 0$, a constant force F_0 per unit area is applied at the free end. Determine the motion of the rod for $t > 0$.*

Solution. Let $y(x, t)$ be the displacement in the rod. Equation of motion is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad l > x > 0, \quad t > 0 \qquad \text{... (1)}$$

subject to the conditions

$$y(x, 0) = 0 \quad \text{... (2)} \qquad \frac{\partial y}{\partial t}(x, 0) = 0 \qquad \text{... (3)}$$

$$y(0, t) = 0 \quad \text{... (4)} \qquad \frac{\partial y}{\partial x}(l, t) = \frac{F_0}{E} \qquad \text{... (5)}$$

where E = Young's modulus.

Applying Laplace transform on (1) $$c^2 \frac{d^2 \bar{y}}{dx^2} = s^2 \bar{y} \qquad \text{... (6)}$$

Eq. (6) is an ordinary differential equation and its solution is

$$\bar{y} = Ae^{\frac{sx}{c}} + Be^{-\frac{sx}{c}} \qquad \text{... (7)}$$

Putting $y = 0$, $x = 0$ from (2) in (7), we get

$0 = A + B$ or $B = -A$, then (7) becomes

$$\bar{y} = A\left(e^{\frac{sx}{c}} - e^{-\frac{sx}{c}}\right) \qquad \text{... (8)}$$

Laplace transform of (3) $$\frac{d\bar{y}}{dx} = \frac{F_0}{Es} \quad \text{at } x = l \qquad \text{... (9)}$$

Differentiating (8) w.r.t. 'x' we get

$$\frac{d\bar{y}}{dx} = A\left(\frac{s}{c}e^{\frac{sx}{c}} + \frac{s}{c}e^{-\frac{sx}{c}}\right) \qquad \text{... (10)}$$

Putting the value of $\frac{d\bar{y}}{dx}$ from (9) in (10), we have

$$\frac{F_0}{ES} = A\frac{s}{c}\left(e^{\frac{s}{c}l} + e^{-\frac{s}{c}l}\right) \qquad \text{or} \qquad A = \frac{F_0}{Es}\frac{c}{s}\frac{1}{e^{\frac{s}{c}l} + e^{-\frac{s}{c}l}}$$

Putting the value of A in (8) we obtain

$$\bar{y} = \frac{cF_0}{Es^2}\,\frac{e^{\frac{s}{c}x} - e^{-\frac{s}{c}x}}{e^{\frac{sl}{c}} + e^{-\frac{sl}{c}}} = \frac{cF_0}{E.s^2}\,\frac{1-e^{-\frac{2s}{c}x}}{1+e^{-\frac{2sl}{c}}}\,\frac{e^{\frac{s}{c}x}}{e^{\frac{s}{c}l}}$$

$$= \frac{cF_0}{Es^2}\left[\left(1-e^{\frac{-2s}{c}x}\right)\left(1+e^{-\frac{2sl}{c}}\right)^{-1}\right]e^{\frac{s}{c}(x-l)}$$

$$= \frac{c\,F_0}{Es^2}\left[\left(1-e^{-\frac{2s}{c}x}\right)\left(1-e^{-\frac{2sl}{c}}+\ldots\right)\right]\times e^{\frac{s}{c}(x-l)}$$

$$\bar{y} = \frac{c\,F_0}{Es^2}\left[1-e^{-\frac{2s}{c}x} - e^{-\frac{2sl}{c}} + e^{-\frac{2s}{c}(x+l)} + \ldots\right]e^{\frac{s}{c}(x-l)} \qquad \ldots (11)$$

Putting $x = l$ in (11) we get

$$x = l\ \bar{y} = \frac{cF_0}{Es^2}\left[1-e^{-\frac{2s}{c}l} - e^{-\frac{2sl}{c}} + e^{-\frac{2s(l+l)}{c}}\ \ldots\right] \qquad \ldots (12)$$

Applying inversion transformation on (12) we get

$$y = \frac{F_0c}{E}t, \qquad 0 < t < \frac{2l}{c} \qquad \ldots (13)$$

$$y = \frac{F_0c}{E}t - \frac{2\,F_0c}{E}\left(t-\frac{2l}{c}\right); \qquad \frac{2l}{c} < t < \frac{4l}{c}$$

Putting $\frac{2l}{c} = \lambda$ in (13), we have

$$y = \begin{cases} At & 0 < t < \lambda \\ At - 2A\,(t-\lambda), \text{ where } A = \dfrac{F_0C}{E}, & \lambda < t < 2\lambda \end{cases} \quad \textbf{Ans.}$$

Solution of Transmission Lines equations by Laplace Transformations.

Example 29. *A semi-infinite transmission line, of negligible inductance and leakage per unit length has its voltage and current equal to zero. A constant voltage v_0 is applied at the sending end ($x = 0$) at $t = 0$. Find the voltage and current at any point ($x > 0$) at any instant.*

Solution. Let v and i be the voltage and current at any point x and at any time t.

$$\left.\begin{aligned} -\frac{\partial v}{\partial x} &= Ri + L\frac{\partial i}{\partial t} \\ -\frac{\partial i}{\partial x} &= c\frac{\partial v}{\partial t} + GV \end{aligned}\right]$$

On putting $L = 0$, $G = 0$ in above equations we get

$$\frac{\partial v}{\partial x} = -Ri \qquad \ldots (1)$$

$$\frac{\partial i}{\partial x} = -c\frac{\partial v}{\partial t} \qquad \text{... (2)}$$

Conditions are $v\ (x, 0) = 0$ ... (3)

$i\ (x, 0) = 0$ ... (4)

$v\ (0, t) = v_0$ ... (5)

$v\ (x, t)$ finite for all x and t. ... (6)

Applying Laplace transform of (1) and (2), we get

$$\frac{d\bar{v}}{dx} = -R\bar{i} \qquad \text{... (7)}$$

$$\frac{d\,\bar{i}}{dx} = -c(s\bar{v} - v) \text{ and } v\ (x, 0) = 0 \text{ or } \frac{d\,\bar{i}}{dx} = -c\,s\,\bar{v} \qquad \text{... (8)}$$

Differentiating (7) w.r.t. 'x' we get $\dfrac{d^2\bar{v}}{dx^2} = -R\dfrac{d\,\bar{i}}{dx}$

or $$\frac{d^2\bar{v}}{dx^2} = -R(-cs\bar{v}) \qquad \left(\frac{d\bar{v}}{dx} = -cs\bar{v}\right)$$

$$\frac{d^2\bar{v}}{dx^2} = Rcs\bar{v} \text{ or } \frac{d^2\bar{v}}{dx^2} - Rcs\bar{v} = 0 \qquad \text{... (9)}$$

Laplace transform of (5) is $\bar{v}(0,s) = \dfrac{v_0}{s}$ ... (10)

And Laplace transform of (6) is $\bar{v}(x,s)$ remains finite as $x \to \infty$. ... (11)

Equation (9) is an ordinary differential equation and its solution is

$$\bar{v} = Ae^{\sqrt{Rcs}\,x} + Be^{-\sqrt{Rcs}\,x} \qquad \text{.. (12)}$$

As $x \to \infty, \bar{v}$ remains finite only when $A = 0$.

So (12) becomes $$\bar{v} = B\,e^{-\sqrt{Rcs}\,x} \qquad \text{... (13)}$$

Putting $\bar{v} = \dfrac{v_0}{s}$ and $x = 0$ in (13) we get $\dfrac{v_0}{s} = B$

Substituting the value of B in (13) we have

$$\bar{v} = \frac{v_0}{s}e^{\sqrt{Rcs}\,x}$$

On applying inversion transform we get

$$v = v_0\ L^{-1}\left[\frac{e^{-\sqrt{Rcs}\,x}}{s}\right] = v_0\,erf\,c\left[\frac{x\sqrt{Rc}}{2\sqrt{t}}\right]$$

$$v = v_0\frac{x\sqrt{Rc}}{2\sqrt{\pi}}\int_0^t u^{-\frac{3}{2}}e^{-\left(\frac{Rcx^2}{4u}\right)}\cdot du \qquad \text{... (14)}$$

From (1) $$i = \frac{-1}{R}\frac{\partial v}{\partial x} \qquad \text{... (15)}$$

On differentiating (14) we get

$$\frac{\partial v}{\partial x} = \frac{v_0\,x\sqrt{Rc}}{2\sqrt{\pi}}t^{-\frac{3}{2}}e^{-\frac{Rcx^2}{4t}}$$

Putting the value of $\frac{\partial v}{\partial x}$ in (15), we obtain

$$i = \frac{v_0 x}{2\sqrt{\pi}}\sqrt{\frac{c}{R}}\cdot t^{-\frac{3}{2}} e^{-\frac{Rcx^2}{4t}}$$ **Ans.**

Solution of partial differential Equations by Fourier Transform

Example 30. *Solve* $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t \geq 0$ *with conditions u (x, 0) = f(x),* $\frac{\partial u}{\partial t}$ *(x, 0) = g (x) and assuming u,* $\frac{\partial u}{\partial x} \to 0$ *as x* $\to \pm\infty$.

Solution. Taking Fourier transform on both sides of the differential equation,

$\frac{d^2\overline{u}}{dt^2} = \alpha^2(-s^2\overline{u})$ where $\overline{u}$ is Fourier transform of u with respect to x,.

$$\frac{d^2\overline{u}}{dt^2} + \alpha^2 s^2 \overline{u} = 0$$

Auxiliary equation is $m^2 + \alpha^2 s^2 = 0 \quad \Rightarrow \quad m = \pm i\alpha s$

$$\therefore \qquad \overline{u}(s,t) = Ae^{i\alpha st} + Be^{-i\alpha st} \qquad \text{... (1)}$$

Since $\qquad u\ (x,\ 0) = f\ (x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$,

$\overline{u}(s,0) = F(s)$ and $\frac{d\overline{u}}{dt}(s,0) = G(s)$ on taking transform.

Using these condition in (1),

$$\overline{u}(s,0) = A + B = F(s) \qquad \text{... (2)}$$

$$\frac{d\overline{u}}{dt}(s,\ 0) = i\alpha s\,(A - B) = G(s) \qquad \text{...(3)}$$

Solving
$$A = \frac{1}{2}\left[F(s) + \frac{G(s)}{i\alpha s}\right]$$

$$B = \frac{1}{2}\left[F(s) - \frac{G(s)}{i\alpha s}\right]$$

Using these values in (1),

$$\overline{u}(s,t) = \frac{1}{2}\left[F(s) + \frac{G(s)}{i\alpha s}\right]e^{i\alpha st} + \frac{1}{2}\left[f(s) - \frac{G(s)}{i\alpha s}\right]e^{-i\alpha st} \qquad \text{... (4)}$$

By inversion theorem, (4) reduces to,

$$u(x,\ t) = \frac{1}{2}\left[f(x - \alpha t) - \frac{1}{\alpha}\int_{\alpha}^{x-\alpha t} g(\theta)\,d\theta\right] + \frac{1}{2}\left[f(x + \alpha t) + \frac{1}{\alpha}\int_{\alpha}^{x+\alpha t} g(\theta)\,d\theta\right]$$

Using the result

$$F\left(\int_{\alpha}^{x} f(t)\,dt\right) = \frac{F(s)}{(-is)}$$ **Ans.**

14.16 FOURIER TRANSFORMS OF PARTIAL DERIVATIVE OF A FUNCTION

$$F_f\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2\, F(u) \quad \text{where } F(u) \text{ is Fourier transform of } u \text{ w.r.t. } x.$$

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = s\,(u)_{x=0} - s^2\, F_s(u) \qquad \text{(sine transform)}$$

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - s^2\, F_c(u) \qquad \text{(cosine transform)}$$

Proof. Let $F\,[u\,(x,\,t)]$ be the Fourier transform of the function $u\,(x,\,t)$, *i.e.*

$$F[u(x,t)] = \int_{-\infty}^{\infty} e^{isx} u(x,t)\,dx$$

The Fourier transform of $\dfrac{\partial^2 u}{\partial x^2}$ is given by

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} e^{isx}\,\frac{\partial^2 u}{\partial x^2}\,dx.$$

Integrating by parts, we have

$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \left[e^{isx}\,\frac{\partial u}{\partial x} - \int ise^{isx}\,\frac{\partial u}{\partial x}\,dx\right]_{-\infty}^{\infty}$$

$$= \left[e^{isx}\,\frac{\partial u}{\partial x} - i\,s\,e^{isx} u + \int (i\,s)^2\, e^{isx} u\,dx\right]_{-\infty}^{\infty} \qquad \text{Again integrating}$$

$$= \left[0 - 0 - s^2 \int_{-\infty}^{\infty} e^{isx}.u\,dx\right] \qquad \left[\begin{array}{l} u = 0,\ \dfrac{\partial u}{\partial x} = 0 \\ \text{when } \ x \to \infty \end{array}\right]$$

Thus $$F\left[\frac{\partial^2 u}{\partial x^2}\right] = -s^2\, F[u(x,t)]$$

Similarly the Fourier sine transform of $\dfrac{\partial^2 u}{\partial x^2}$ is given by

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin sx\,dx$$

or $$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = s\,[u(x,t)]_{x=0} - s^2\, F_s\,[u(s,t)] \qquad \text{(sine transform)}$$

and $$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - s^2\, F_c\,[u(s,t)] \qquad \text{(cosine transform)}$$

Solution of heat conduction problems by Fourier sine Transforms

Example 31. *Solve the equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x > 0,\ t > 0$$

subject to the conditions

(i) $u = 0$ when $x = 0,\ t > 0$

(ii) $u = \begin{cases} 1 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$ when $t = 0$

(iii) $u\ (x, t)$ is bounded.

(**Note.** If u at $x = 0$ is given, take Fourier sine transform and if $\frac{\partial u}{\partial x}$ at $x = 0$ is given, use Fourier cosine transform.)

Solution. In view of the initial conditions, we apply Fourier sine transform

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx\, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx\, dx$$

$$\frac{\partial}{\partial t}\int_0^\infty u \sin sx\, dx = -s^2\, \overline{u}(s) + s\, u(0) \qquad u = 0 \text{ when } x = 0$$

$$\frac{\partial \overline{u}}{\partial t} = -s^2\, \overline{u} \quad \text{or} \quad \frac{\partial \overline{u}}{\partial t} + s^2\, \overline{u} = 0$$

$$\therefore \qquad \overline{u} = Ae^{-s^2 t} \qquad \text{... (1)}$$

$$\overline{u} = \overline{u}\,(s,t) = \int_0^\infty u(x,t) \sin sx\, dx$$

$$\overline{u} = \overline{u}\,(s,\ 0) = \int_0^\infty u(x,\ 0) \sin sx\, dx$$

$$\overline{u}(s,\ 0) = \int_0^\infty 1 \cdot \sin sx\, dx = \left[\frac{-\cos sx}{s}\right]_0^1 = \frac{1 - \cos s}{s} \qquad \text{... (2)}$$

From (2) putting the value of $\overline{u}$ $(s, 0)$ in (1) we get $\frac{1-\cos s}{s} = A$

$$\therefore \qquad \overline{u} = \frac{1-\cos s}{s} e^{-s^2 t} \qquad \text{or} \qquad u = \frac{2}{\pi}\int_0^\infty \left(\frac{1-\cos s}{s}\right) e^{-s^2 t} \sin xs\, ds$$

Example 32. *Solve* $\frac{\partial u}{dt} = k\frac{\partial^2 u}{\partial x^2}$ *for* $x \geq 0,\ t \geq 0$ *under the given conditions* $u = u_0$ *at* $x = 0$, $t > 0$ *with initial condition* $u\ (x,\ 0) = 0,\ x \geq 0$

Solution. Taking Fourier sine transforms

$$F_s\left(\frac{\partial u}{\partial t}\right) = F_s\left(k\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{d}{dt}\overline{u} = k\left[-s^2 \overline{u} + \sqrt{\frac{2}{\pi}}\, s\, u(0,t)\right]$$

$$= -ks^2\bar{u} + \sqrt{\frac{2}{\pi}}\, k\, s\, u_0$$ where $\bar{u}$ is the Fourier sine transform of u.

$$\frac{d\bar{u}}{dt} + sk^2\bar{u} = \sqrt{\frac{2}{\pi}}\, s\, k\, u_0$$

This is linear in $\bar{u}$.

$$\therefore \qquad \bar{u}\, e^{ks^2t} = \sqrt{\frac{2}{\pi}} ku_0 \int s\, e^{ks^2t} dt = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{s} e^{ks^2t} + c \qquad \ldots (1)$$

Since, $u\ (x, 0) = 0$, $\bar{u}\ (s, 0) = 0$. Using this in (1)

$$0 = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{s} + c \qquad \Rightarrow \qquad c = -\sqrt{\frac{2}{\pi}}\, \frac{u_0}{s}$$

$$e^{ks^2t} = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{s}\left(e^{ks^2t} - 1\right) \qquad \Rightarrow \qquad \bar{u} = \sqrt{\frac{2}{\pi}}\, \frac{u_0}{s}\left(1 - e^{ks^2t}\right)$$

By inversion theorem,

$$u(x,t) = \frac{2u_0}{\pi} \int_0^\infty \left(\frac{1 - e^{ks^2t}}{s}\right) \sin sx\, ds.$$ **Ans.**

Example 33. *Solve* $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ *for* $0 \le x < \infty,\ t > 0$ *given the conditions*

(i) $u\ (x,\ 0) = 0$ *for* $x \ge 0$ *(ii)* $\dfrac{\partial u}{\partial x}(0,t) = -a$ *(constant)*

(iii) $u(x,\ t)$ *is bounded.*

Solution. In this problem, $\dfrac{\partial u}{\partial x}$ at $x = 0$ is given. Hence, take Fourier cosine transform on both sides of the given equation.

$$F_c\left(\frac{\partial u}{\partial t}\right) = F_c\left(k\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{d\bar{u}}{dt} = k\left(-s^2\bar{u} - \sqrt{\frac{2}{\pi}} \cdot \frac{\partial u}{\partial x}(0,\ t)\right)$$

$$= -ks^2\bar{u} + \sqrt{\frac{2}{\pi}}\, ka \qquad \text{using condition } (ii)$$

$$\frac{d\bar{u}}{dt} + ks^2\bar{u} = \sqrt{\frac{2}{\pi}}\, ka$$

This is linear in $\bar{u}$. Therefore, solving

$$\bar{u}e^{ks^2t} = \int \sqrt{\frac{2}{\pi}}\, ka\, e^{ks^2t} dt = \sqrt{\frac{2}{\pi}} ka \frac{e^{ks^2t}}{ks^2} + c$$

$$\bar{u}(s,t) = \sqrt{\frac{2}{\pi}}\, \frac{a}{s^2} + c\, e^{-ks^2t} \qquad \ldots (1)$$

Since $u(x, 0) = 0$ for $x \geq 0$.

$\bar{u}(s, 0) = 0$.

Using this in (1), we get

$$\bar{u}(s,0) = c + \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2} = 0$$

$$\therefore \qquad c = -\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2}$$

Substituting this in (1)

$$\bar{u}(s, t) = \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2}\left(1 - e^{-ks^2 t}\right)$$

By inversion theorem,

$$u(x,t) = \frac{2}{\pi}\cdot a\int_0^\infty \frac{1-e^{-ks^2 t}}{s^2}\cos sx\, ds. \qquad \textbf{Ans.}$$

EXERCISE 14.3

1. Use Fourier sine transform to solve the equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

Under the conditions $u(0, t) = 0$, $u(x, 0) = e^{-x}$, $u(x, t)$ is bounded.

Ans. $u(x,t) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{s}{s^2+1} e^{-s^2 t}\sin sx\, ds$

2. A tightly stretched string with fixed end points $x = b$ and $x = c$ is initially in a position given by $y = b \sin\left(\dfrac{\pi x}{c}\right)$. It is released from rest in this position. Show by the method of Laplace transform that the displacement y at any distance x from one end and at any time t is given by

$$y = b\sin\frac{\pi x}{c}\cos\frac{\pi q}{c}t.$$

and y satisfies the equation $\dfrac{\partial^2 y}{\partial x^2} = a^2\dfrac{\partial^2 y}{\partial t^2}$

3. A string is stretched tightly between $x = 0$ and $x = l$ and both ends are given displacement $y = a \sin pt$ perpendicular to the string. If the string satisfies the differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Show that the oscillations of the string are given by

$$y = a\sec\frac{Pl}{2c}\cos\left(\frac{Px}{c} - \frac{Pl}{2c}\right)\sin pt.$$

4. An infinite cable with resistance R ohms/km, capacitance C Farads/km, and negligible inductance and leakage is subjected to constant E.M.F. E_0 at the home end at time $t = 0$. Using the operational method show that the entering current at any subsequent time t is

$$I(t) = E_0 \left(\frac{C}{\pi Rt} \right)^{1/2}$$

5. Solve the equation for high voltage semi-infinite line with the following initial and boundary conditions
$v(x, t) = 0$ and $i(x, 0) = 0$, $v(0, t) = v_0\, u(t)$, $v(x, t)$ is finite as $x \to \infty$.

Ans. $v = v_0\, u[t - x\sqrt{LC}]$, for $x \le \dfrac{t}{\sqrt{LC}}$ and

$v = 0$ for $x > \dfrac{t}{\sqrt{LC}}$

6. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ if

(*i*) $\dfrac{\partial u}{\partial x}(0,t) = 0$ for $t > 0$. (*ii*) $u(x, 0) = \begin{cases} x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$

(*iii*) and $u(x, t)$ is bounded for $x > 0$, $t > 0$

Ans. $\left[u(x,t) = \dfrac{2}{\pi} \displaystyle\int_0^\infty \left(\frac{\sin s}{s} + \frac{\cos s - 1}{s^2} \right) e^{-s^2 t} \cos sx \, ds \right]$

14.17. FINITE FOURIER TRANSFORMS

Let $f(x)$ denote a function which is sectionally continuous over the range $(0, l)$. Then the **finite Fourier sine transform** of $f(x)$ on this interval is defined as

$$F_s(p) = \bar{f}_s(P) = \int_0^l f(x) \sin \frac{p\pi x}{l} dx$$

where p is an integer (Instead of s, we take p as a parameter)

Inversion formula for sine transform

If $f_s(p) = F_s(p)$ is the finite Fourier sine transform of $f(x)$ in $(0, l)$ then the inversion formula for sine transform is

$$f(x) = \frac{2}{l} \sum_{p=l}^{\infty} \bar{f}_s(p) \sin \frac{p\pi x}{l}$$

Proof. For the given function $f(x)$ in $(0, l)$, if we find the half range Fourier sine series, we get.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{... (1)}$$

where
$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$\therefore \quad b_p = \frac{2}{l} \int_0^l f(x) \sin \frac{p\pi x}{l} dx = \frac{2}{l} \bar{f}_s(p) \text{ by definition}$$

Substituting in (1), we get

$\therefore$ $$f(x) = \frac{2}{l}\sum_{p=1}^{\infty} f_s(p)\sin\frac{p\pi x}{l}$$ **Ans.**

Finite Fourier Cosine Transform

Let $f(x)$ denote a sectionally continuous function in $(0, l)$.

Then the Finite Fourier cosine transform of $f(x)$ over $(0, l)$ is defined as

$$F_c(p) = \bar{f}_c(p) = \int_0^l f(x)\cos\frac{p\pi x}{l}dx \text{ where } p \text{ is an integer.}$$

Inversion formula for cosine transform

If $\bar{f}_c$ (P) is the finite Fourier cosine transform of $f(x)$ in $(0, l)$. then the inversion formula for cosine transform is

$$f(x) = \frac{1}{l}\bar{f}_c(0) + \frac{2}{l}\sum_{p=l}^{\infty}\bar{f}_c(p)\cos\frac{p\pi x}{l}$$

where $$\bar{f}_c(0) = \int_0^l f(x)\,dx.$$

Proof. If we find half range Fourier cosine series for $f(x)$ in $(0, l)$, we obtain.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} \qquad \text{... (2)}$$

where $$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$\therefore$ $$a_p = \frac{2}{l}\bar{f}_c(p)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\bar{f}_c(0).$$

Substituting in (2), we get,

$$f(x) = \frac{1}{l}f_c(0) + \frac{2}{l}\sum_{p=1}^{\infty} f_c(p)\cos\frac{p\pi x}{l}$$

Example. 34. *Find the finite Fourier sine and cosine transforms of*

(*i*) $f(x) = 1$ in $(0, \pi)$

(*ii*) $f(x) = x$ in $(0, l)$

(*iii*) $f(x) = x^2$ in $(0, l)$

(*iv*) $f(x) = 1$ in $0 < x < \pi/2$

$= -1$ in $\pi/2 < x < \pi$

(*v*) $f(x) = x^3$ in $(0, l)$

(*vi*) $f(x) = e^{ax}$ in $(0, l)$

$$(i)\quad \bar{f}_s\ (p)\ =\ F_s\ (1)\ = \int_0^{\pi} 1.\sin \frac{p\pi x}{\pi} dx = \left(-\frac{\cos px}{p}\right)_0^{\pi} = \frac{1-\cos p\pi}{p} \qquad \text{if} \quad p \neq 0$$

$$\bar{f}_c(p)\ = \int_0^{\pi} 1\cdot \cos px\, dx = \left(\frac{\sin px}{p}\right)_0^{\pi} = \frac{1}{p}(0-0) = 0$$

$$(ii)\quad \bar{f}_s\,(p) = F_s\,(p) = \int_0^{l} x\ \sin\frac{p\pi x}{l} dx$$

$$= \left[(x)\left(\frac{-\cos\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (1)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right)\right]_0^l = \frac{-l}{p\pi}(l\ \cos\ p\,\pi)$$

$$= \frac{-l^2}{p\pi}\,(-1)^P \qquad \text{if} \quad p \neq 0$$

$$\bar{f}_c\,(p) = F_c\,(x) = \int_0^{l} x\cos\frac{p\pi x}{l} dx$$

$$= \left[(x)\left(\frac{\sin\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (1)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right)\right]_0^1 = \frac{l^2}{p^2\pi^2}[(-1)^p - 1] \quad \text{if} \quad p \neq 0$$

$$(iii)\quad \bar{f}_s\,(p) = F_s\,(x^2) = \int_0^{l} x^2\ \sin\frac{p\pi x}{l} dx$$

$$= \left[(x^2)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (2x)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right) + (2)\left(\frac{\cos\frac{p\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{-l^3}{p\pi}\,(-1)^p + \frac{2l^3}{p^3\pi^3}\left[(-1)^p - 1\right] \qquad \text{if} \quad p \neq 0$$

$$\bar{f}_c(p) = \int_0^{l} (x^2)\cos\frac{p\pi x}{l} dx$$

$$= \left[(x^2)\left(\frac{\sin\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (2x)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right) + (2)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2l^3}{p^2\pi^2}(-1)^p \qquad \text{if } p \neq 0$$

(iv) $$F_s\{f(x)\} = \int_0^{\pi/2} \sin px\,dx + \int_{\pi/2}^{\pi} (-1)\sin px\,dx$$

$$= \left(-\frac{\cos px}{p}\right)_0^{\pi/2} + \left(\frac{\cos px}{p}\right)_{\pi/2}^{\pi}$$

$$= -\frac{1}{p}\left(\cos\frac{p\pi}{2} - 1\right) + \frac{1}{p}\left(\cos p\pi - \cos\frac{p\pi}{2}\right)$$

$$= -\frac{1}{p}\left(\cos p\pi - 2\cos\frac{p\pi}{2} - 1\right) \qquad \text{if } p \neq 0$$

$$F_c(f(x)) = \int_0^{\pi/2} \cos px\,dx - \int_{\pi/2}^{\pi} \cos px\,dx$$

$$= \left(\frac{\sin px}{p}\right)_0^{\pi/2} - \left(\frac{\sin px}{p}\right)_{\pi/2}^{\pi} = \frac{2}{p}\sin\frac{p\pi}{2} \qquad \text{If } p \neq 0$$

(v) $$F_s(x^3) = \int_0^l x^3 \sin\frac{p\pi x}{l}\,dx$$

$$= \left[(x^3)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (3x^2)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right) + (6x)\left(\frac{\cos\frac{p\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right) - (6)\left(-\frac{\sin\frac{p\pi x}{l}}{\frac{p^4\pi^4}{l^4}}\right)\right]_0^l$$

$$= -\frac{l^p}{p\pi}(-1)^p + \frac{6\,l^4}{p^3\pi^3}(-1)^P \qquad \text{if } p \neq 0$$

$$F_c(x^3) = \int_0^l x^3 \cos\frac{p\pi x}{l}\,dx$$

$$= \left[(x^3)\left(\frac{\sin\frac{p\pi x}{l}}{\frac{p\pi}{l}}\right) - (3x^2)\left(-\frac{\cos\frac{p\pi x}{l}}{\frac{p^2\pi^2}{l^2}}\right) + (6x)\left(\frac{-\sin\frac{p\pi x}{l}}{\frac{p^3\pi^3}{l^3}}\right) - (6)\left(\frac{\cos\frac{p\pi x}{l}}{\frac{p^4\pi^4}{l^4}}\right)\right]_0^l$$

$$= \frac{3l^4}{\pi^2 p^2}(-1)^p - \frac{6l^4}{p^4\pi^4}[(-1)^p - 1] \qquad \text{if } p \neq 0$$

(vi) $$F_s(e^{ax}) = \int_0^l e^{ax}\sin\frac{p\pi x}{l}\,dx$$

$$= \left\{ \frac{e^{ax}}{a^2 + \frac{p^2\pi^2}{l^2}} \left[a \sin \frac{p\pi x}{l} - \frac{p\pi}{l} \cos \frac{p\pi x}{l} \right] \right\}_0^l$$

$$= \frac{e^{ax}}{a^2 + \frac{p^2 a^2}{l^2}} . \left(-\frac{p\pi}{l}(-1)^p \right) + \frac{1}{a^2 + \frac{p^2\pi^2}{l^2}} \left(\frac{p\pi}{l} \right)$$

$$F_c(e^{ax}) = \left\{ \frac{e^{ax}}{a^2 + \frac{p^2\pi^2}{l^2}} \left[a \cos \frac{p\pi x}{l} + \frac{p\pi}{l} \sin \frac{p\pi x}{l} \right] \right\}_0^l$$

$$= \frac{e^{al}}{a^2 + \frac{p^2\pi^2}{l^2}} a(-1)^p - \frac{1}{a^2 + \frac{p^2\pi^2}{l^2}} (a)$$

Example 35. *Find f (x) if its finite Fourier sine transform is* $\frac{2\pi}{p^3}(-1)^{p-1}$ *for p = 1, 2, ...,* $0 < x < \pi$.

Solution. By inversion Theorem,

$$f(x) = \frac{2}{\pi} \sum_{p=1}^{\infty} \frac{2\pi}{p^3}(-1)^{p-1} \sin px \quad = 4 \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p^3} \sin px$$

Example 36. *Find f (x) if its finite Fourier sine transform is given by*

(*i*) $F_s(p) = \dfrac{1 - \cos p\pi}{p^2 \pi^2}$ *for p = 1, 2, 3, ... and* $0 < x < \pi$

(*ii*) $F_s(p) = \dfrac{16(-1)^{p-1}}{p^3}$ *for p = 1, 2, 3, ... and* $0 < x < 8$

(*iii*) $F_s(p) = \dfrac{\cos \frac{2\pi p}{3}}{(2p+1)^2}$ *for p = 1, 2, 3, ... and* $0 < x < 1$.

Solution. By inversion theorem

(*i*) $$f(x) = \frac{2}{\pi} \sum_{p=1}^{\infty} \left(\frac{1 - \cos p\pi}{p^2\pi^2} \right) . \sin px \quad = \frac{2}{\pi^3} \sum_{p=1}^{\infty} \left(\frac{1 - \cos p\pi}{p^2} \right) . \sin px$$

(*ii*) $$f(x) = \frac{2}{l} \sum_{p=1}^{\infty} F_s(p) \sin\left(\frac{p\pi x}{l} \right)$$

$$= \frac{2}{8} \sum_{p=1}^{\infty} \frac{16(-1)^{p-1}}{p^3} \sin\left(\frac{p\pi x}{8} \right) \quad \text{since } l = 8 \quad = 4 \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p^3} \sin\left(\frac{p\pi x}{8} \right)$$

(*iii*) $f(x) = \frac{2}{l}\sum_{p=1}^{\infty} F_s(p)\sin\left(\frac{p\pi x}{l}\right) = 2\sum_{p=1}^{\infty} \frac{\cos\left(\frac{2\pi p}{3}\right)}{(2p+1)^2}\sin(p\pi x)$ since $l = 1$ **Ans.**

Example 37. *Find f (x) if its finite Fourier cosine transform is*

(*i*) $F_c(p) = \frac{1}{2p}\left(\frac{p\pi}{2}\right)$ *for p = 1, 2, 3, ...*

$= \frac{\pi}{4}$ *for p = 0 given* $0 < x < 2\pi$

(*ii*) $F_c(p) = \frac{6\sin\frac{p\pi}{2} - \cos p\pi}{(2p+1)\pi}$ *for p = 1, 2, 3, ...*

$= \frac{2}{\pi}$ *for p = 0 given 0 < x < 4*

(*iii*) $F_c(p) = \frac{\cos\left(\frac{2p\pi}{3}\right)}{(2p+1)^2}$ *for p = 1, 2, 3, ...*

= 1 *for p = 0 given 0 < x < 1*

Solution: By inversion theorem,

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{p=1}^{\infty} F_c(p).\cos\frac{p\pi x}{l}.$$

(*i*) Here $F_c(0) = \pi/4$ and $l = 2\pi$

$$\therefore \quad f(x) = \frac{1}{2\pi}\left(\frac{\pi}{4}\right) + \frac{2}{2\pi}\sum_{p=1}^{\infty} \frac{1}{2p}\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{p\pi x}{2\pi}\right)$$

$$= \frac{1}{8} + \frac{1}{2\pi}\sum_{p=1}^{\infty} \frac{1}{p}\sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{px}{2}\right)$$

(*ii*) Here $F_c(0) = \frac{2}{\pi}$ and $l = 4$

$$f(x) = \frac{1}{4}\left(\frac{2}{\pi}\right) + \frac{2}{4}\sum_{p=1}^{\infty} \frac{\left(6\sin\frac{p\pi}{2} - \cos p\pi\right)}{(2p+1)\pi}\cos\left(\frac{p\pi x}{4}\right)$$

$$= \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{p=1}^{\infty} \frac{\left(6\sin\frac{p\pi}{2} - \cos p\pi\right)}{(2p+1)}.\cos\left(\frac{p\pi x}{4}\right)$$

(*iii*) Here $F_c(0) = 1$, $l = 1$

$$\therefore \quad f(x) = \frac{1}{1} + \frac{2}{1}\sum_{p=1}^{\infty} \frac{1}{(2p+1)^2}\cos\left(\frac{2p\pi}{3}\right).\cos(p\pi x)$$

$$= 1 + 2\sum_{p=1}^{\infty} \frac{\cos\left(\frac{2p\pi}{3}\right)}{(2p+1)^2}\cos(p\pi x)$$ **Ans.**

Example 38. *Find the finite Fourier sine transform of f (x) = 1 in (0, π). Use the inversion theorem and find Fourier series for f (x) = 1 in (0, π). Hence prove*

(*i*) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....=\pi/4$ (*ii*) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....=\pi^2/8$

Solution. $$F_s(1)=\int_0^{\pi} 1.\sin\left(\frac{p\pi x}{\pi}\right)dx$$

$$\bar{f}_s(p)=\frac{1-\cos p\pi}{p} \text{ if } p\neq 0$$

By inversion theorem,

$$f(x)=\frac{2}{l}\sum_{p=1}^{\infty} F_s\ (p)\sin\frac{p\pi x}{l}$$

$$1=\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1-(-1)^p}{p}.\sin px \qquad \text{since } l=\pi$$

$$1=\frac{4}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right] \qquad .... (1)$$

This is the half range Fourier sine series for $f(x) = 1$ in $(0, \pi)$ getting $x=\pi/2$.

On putting $x=\frac{\pi}{2}$ in (1) we get

$$\frac{4}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....\right]=1$$

$$\therefore \qquad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....=\pi/4$$

In the half Fourier sine series $l_n=\frac{4}{\pi}.\frac{1}{n}$ for n odd

By using Parseval's Theorem

$$\text{(Range)}\left[\frac{1}{2}\sum b_n^2\right]=\int_0^{\pi}(1)^2\ dx$$

$$\pi\left[\frac{1}{2}.\frac{16}{\pi^2}\sum_{n=1.3.5....}^{\pi}\frac{1}{n^2}\right]=\pi$$

i.e., $$\sum_{n=1.3.5}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$ **Ans.**

EXERCISE 14.4

Find the finite Fourier sine and cosine transforms of

1. $f(x)=2x$ in (0, 4) **Ans.** $F_{s(s)}=\begin{cases}\frac{32}{s\pi}(1-ws\,s\pi), & s\neq 0\\ 0, & s=0\end{cases}$

2. $f(x) = x$ in $(0, \pi)$ **Ans.** $F_s(s) = \begin{cases} \dfrac{\pi}{s}(-1)^{s+1}, & s \neq 0 \\ 0, & s = 0 \end{cases}$, $F_c(s) = \begin{cases} \dfrac{(-1)^s - 1}{s^2}, & s \neq 0 \\ \dfrac{\pi^2}{2}, & s = 0 \end{cases}$

3. $f(x) = \cos ax$ in $(0, \pi)$ **Ans.** $F_s(s) = \dfrac{s}{s^2 - a^2}[1 - (-1)^s \cos a\pi]$, $F_c(s) = 0$

4. $f(x) = 1 - \dfrac{x}{\pi}$ in $(0, \pi)$ **Ans.** $F_s(s) = \dfrac{1}{s}$, $F_c(s) = \dfrac{-1}{s^2\pi}[1 - \cos s\pi]$

5. $f(x) = \begin{cases} x \text{ in } (0, \pi/2) \\ \pi - x \text{ in } (\pi/2, \pi) \end{cases}$ **Ans.** $F_s(s) = \dfrac{2}{s^2} \sin \dfrac{s\pi}{2}$, $F_c(s) = \dfrac{1}{s^2}[1 + \cos s\pi]$

6. Find finite Fourier cosine transform of $\left(1 - \dfrac{x}{\pi}\right)^2 . 0 < x < \pi$. **Ans.** $F_c(s) = \begin{cases} \dfrac{2}{\pi s^2}, & s \neq 0 \\ \dfrac{\pi}{3}, & s = 0 \end{cases}$

7. Find $f(x)$ if $\bar{f}_c(p) = \dfrac{\sin\left(\dfrac{p\pi}{2}\right)}{2p}$, $p = 1, 2, 3 \ldots$ and

$= \dfrac{\pi}{4}$ if $p = 0$ given $0 < x < 2\pi$. **Ans.** $\dfrac{1}{8} + \dfrac{1}{\pi} \displaystyle\sum_{p=1}^{\infty} \dfrac{\sin \dfrac{p\pi}{2}}{2p} \cos \dfrac{px}{2}$

8. $f(x) = \dfrac{\pi}{3} - x + \dfrac{x^2}{2\pi}$ in $[0, \pi]$ **Ans.** $F_s(s) = \dfrac{1}{6s^3\pi}[\pi^2 \cos p\pi + 6 \cos p\pi, + 2p^2\pi - 6]$, $F_c(s) = \dfrac{1}{s^2}$

14.18 FINITE FOURIER SINE AND COSINE TRANSFORMS OF DERIVATIVES

Using the definition and the integration by parts, we can easily prove the following results. For $0 \leq x \leq l$,

(i) $$F_s(f'(x)) = -\frac{p\pi}{l}\bar{f}_c(p)$$

(ii) $$F_c\{f'(x)\} = f(l)(-1)^p - f(0) + \frac{p\pi}{l}\bar{f}_s(p)$$

(iii) $$F_s\{f''(x)\} = -\frac{p^2\pi^2}{l^2}\bar{f}_s(p) + \frac{p\pi}{l}\left[f(0) - (-1)^p f(l)\right]$$

(iv) $$F_c\{f''(x)\} = -\frac{p^2\pi^2}{l^2}\bar{f}_c(p) + f'(l)(-1)^p - f'(0)$$

Proof: (*i*) $$F_s(f'(x)) = \int_0^l f'(x) \sin \frac{p\pi x}{l} dx = \int_0^l \sin \frac{p\pi x}{l} \cdot d\{f(x)\}$$

$$= \left(f(x) \sin \frac{p\pi x}{l}\right)_0^l - \int_0^l f(x) . \cos \frac{p\pi x}{l} \frac{p\pi}{l} dx$$

$$= -\frac{p\pi}{l}\bar{f}_c(p) \qquad \text{... (1)}$$

(*ii*) $$F_c\{f'(x)\} = \int_0^l f'(x)\cos\frac{p\pi x}{l}dx = \left(f(x)\cos\frac{p\pi x}{l}\right)_0^l - \int_0^l f(x).\frac{p\pi}{l}\sin\frac{p\pi x}{l}dx$$

$$= (-1)^P \bar{f}(l) - f(0) + \frac{p\pi}{l}\bar{f}_s(p) \qquad \text{... (2)}$$

(*iii*) $$F_s[f''(x)] = \int_0^l \sin\frac{p\pi x}{l}\, d[f'(x)]$$

$$= \left(f'(x)\sin\frac{p\pi x}{l}\right)_0^l - \frac{p\pi}{l}\int_0^l f'(x)\cos\frac{p\pi x}{l}ds$$

$$= -\frac{p\pi}{l}\left[(-1)^p f'(l) - f(0) + \frac{p\pi}{l}\bar{f}_s(p)\right] \qquad \text{[Using (2)]}$$

$$= -\frac{p^2\pi^2}{l^2}\bar{f}_s(p) + \frac{p\pi}{l}\left[f(0) - (-1)^p f(l)\right] \qquad \text{... (3)}$$

(*iv*) $$F_c\{f''(x)\} = \int_0^l \cos\frac{p\pi x}{l}d[f'(x)]$$

$$= \left[f'(x)\cos\frac{p\pi x}{l}\right]_0^l + \frac{p\pi}{l}\int_0^l f'(x)\sin\frac{p\pi x}{l}dx$$

$$= (-1)^p f'(l) - f'(0) + \frac{p\pi}{l}\left[-\frac{p\pi}{l}\bar{f}_c(p)\right] \qquad \text{[Using (1)]}$$

$$= -\frac{p^2\pi^2}{l^2}\bar{f}_c(p) + f'(l)(-1)^p - f'(0) \qquad \text{...}$$

(4)

Note. If $u = u(x, t)$, then

$$F_s\left[\frac{\partial u}{\partial x}\right] = \frac{-p\pi}{l}F_c(u)$$

$$F_c\left[\frac{\partial u}{\partial x}\right] = \frac{p\pi}{l}F_s(u) - u(0, t) + (-1)^p u(l, t)$$

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{p^2\pi^2}{l^2}F_s(u) + \frac{p\pi}{l}[u(0, t) - (-1)^p u(l, t)]$$

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\frac{p^2\pi^2}{l^2}F_c(u) + \frac{\partial u}{\partial x}(l, t)\cos p\pi - \frac{\partial u}{\partial x}(0, t)$$

Example 39. *Using finite Fourier transform, solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ given } u(0, t) = 0 \text{ and } u(4, t) = 0$$

and $u(x, 0) = 2x$ *where* $0 < x < 4$, $t > 0$

Solution. Since u (0, t) given, take finite Fourier sine transform.

$$\int_0^4 \frac{\partial u}{\partial t} \sin\frac{p\pi x}{4} dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin\frac{p\pi x}{4} dx$$

$$\frac{d}{dt}\bar{u}_s = F_s\left(\frac{\partial^2 u}{\partial x^2}\right) = -\frac{p^2\pi^2}{16}\bar{u}_s + \frac{p\pi}{4}[u\,(0,\,t) - (-1)^p u\,(4,\,t)]$$

$$= -\frac{p^2\pi^2}{16}\bar{u}_s \ \text{[using } u(0, t) = 0,\ u(4, t) = 0]$$

$$\frac{d\,\bar{u}_s}{\bar{u}_s} = -\frac{p^2\pi^2}{16} dt$$

Integrating
$$\log\,\bar{u}_s = -\frac{p^2\pi^2}{16} t + c$$

$$\bar{u}_s = A\,e^{-\frac{\pi^2 p^2}{16}t} \qquad \text{... (1)}$$

Since $u(x, 0) = 2x$

$$\bar{u}_s\,(p,\,0) = \int_0^4 (2x)\,\sin\left(\frac{p\pi x}{4}\right) dx = -\frac{32}{p\pi}\cos p\pi \qquad \text{... (2)}$$

Using (2) in (1),

$$\bar{u}_s\ (p,\,0) = A = -\frac{32}{p\pi}\cos p\pi\,.$$

Substituting in (1),

$$\therefore \qquad \bar{u}_s = -\frac{32}{p\pi}(-1)^p\ e^{-\frac{p^2\pi^2}{16}t}$$

By inversion Theorem,

$$u(x,\,t) = \frac{2}{4}\sum_{p=1}^{\infty} \frac{32}{p\pi}(-1)^{p+1} e^{-\frac{p^2\pi^2}{16}} \sin\left(\frac{p\pi x}{4}\right)$$ **Ans.**

Example 40. *Solve* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, *0 < x < 6, t > 0*

given $\frac{\partial u}{\partial x}(0,\,t) = 0,\ \frac{\partial u}{\partial x}(6,\,t) = 0$ *and* $u\,(x,\,0) = 2x$.

Solution. Since $\frac{\partial u}{\partial x}(0,\,t)$ is given, use finite Fourier cosine transform

$$\int_0^6 \frac{\partial u}{\partial t} \cos\frac{p\pi x}{6} dx = \int_0^6 \frac{\partial^2 u}{\partial x^2} \cos\frac{p\pi x}{6} dx$$

$$\frac{d}{dt}\bar{u}_c = -\frac{p^2\pi^2}{36}\bar{u}_c + \frac{\partial u}{\partial x}(6,\,t)\cos p\pi - \frac{\partial u}{\partial x}\,(0,\,t) = -\frac{p^2\pi^2}{36}\bar{u}_c$$

$$\Rightarrow \qquad \frac{d\,\bar{u}_c}{\bar{u}_c} = -\frac{p^2\pi^2}{36}\ dt$$

$$\Rightarrow \qquad \log \bar{u}_c = -\frac{p^2\pi^2}{36} t + c$$

$$\bar{u}_c = A\, e^{-\frac{p^2\pi^2}{36} t} \qquad \dots (1)$$

$$u(x,\ 0) = 2x.$$

$\therefore$ At $t = 0$

$$\bar{u}_c\ (p,\ 0) = \int_0^6 (2x)\ \cos\frac{p\pi x}{6} dx\ = \frac{72}{p^2\pi^2}(\cos\ p\pi - 1) \qquad \dots (2)$$

Using this in (1), we get $\bar{u}_c(p, 0) = A = \dfrac{72}{p^2\pi^2}(\cos\ p\pi - 1)$

Substituting in (1), we get $\bar{u}_c\ (p,\ t) = \dfrac{72}{p^2\pi^2}(\cos p\pi)$

By inversion theorem,

$$u(x,\ t) = \frac{1}{l}\bar{f}_c(0) + \frac{2}{1}\sum_{p=1}^{\infty} \bar{f}_c(p)\cos\left(\frac{p\pi x}{l}\right)$$

$$= \frac{1}{6}\int_0^6 (2x)\ dx + \frac{2}{6}\sum_{p=1}^{\infty} \frac{72}{p^2\pi^2}(\cos p\pi - 1)\ e^{-\frac{p^2\pi^2}{36}t}.\cos\left(\frac{p\pi x}{6}\right)$$

$$= 6 + \frac{24}{\pi^2}\sum_{p=1}^{\infty} \frac{(\cos p\pi - 1)}{p^2} e^{-\frac{p^2\pi^2}{36}t} \cos\left(\frac{p\pi x}{6}\right). \qquad \textbf{Ans.}$$

Example 41. *Solve* $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2},\ 0 < x < 4,\ t > 0$

given $u\ (0,\ t) = 0$; $\quad u\ (4,\ t) = 0$; $\quad u\ (x,\ 0) = 3\ \sin\ \pi x - 2\sin 5\pi x$.

Solution. sin u (0, t) is given, take finite Fourier sine transform. The equation becomes (as in example 39 on page 977)

$$\frac{d}{dt}\bar{u}_c = 2\left[-\frac{p^2\pi^2}{16}\bar{u}_s + \frac{p\pi}{4}\{u(0,\ t) - (-1)^p\, u(4,\ t)\}\right] = -\frac{p^2\pi^2}{8}\bar{u}_s$$

Solving we get, $\qquad \bar{u}_s = A\, e^{-\frac{p^2\pi^2}{8}t} \qquad \dots (1)$

$$u(x,\ 0) = 3\sin\pi x - 2\sin 5\pi x$$

Taking sine Transform,

$$\bar{u}_c\ (p,\ 0) = \int_0^4 (3\sin\pi x - 2\sin 5\pi x)\sin\frac{p\pi x}{4} dx\ = 0 \text{ if } p \neq 4,\ p \neq 20.$$

If $p = 4$, $\qquad \bar{u}_s\ (4,\ 0) = 6$

If $p = 20$, $\qquad \bar{u}_s\ (20,\ 0) = -4$

$$u(x,\ t) = \frac{2}{4}\sum_{p=1}^{\infty} \bar{u}_s\ (p,\ t)\sin\left(\frac{p\pi x}{4}\right)$$

$$= \frac{1}{2}[6\, e^{-\frac{p^2\pi^2}{8}t} \sin \pi x - 4\, e^{-\frac{p^2\pi^4}{8}t} \sin 5\pi x]$$

where p in the first term is 4 and p in the second term is 20

$$= 3e^{-2\pi^2 t} \sin \pi x - 2e^{-50\pi^2 t} \sin 5\pi x.$$ **Ans.**

EXERCISES 14.5

1. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 6, t > 0$ given that $u(0, t) = 0 = u(6, t)$ and $u(x, 0) = \begin{cases} 1 \text{ for } 0 < x < 3 \\ 0 \text{ for } 3 < x < 6 \end{cases}$

$$\left[\textbf{Ans.}\ u(x,t) = \frac{2}{\pi} \sum_{p=1}^{\pi} \left(\frac{1 - \cos \frac{p\pi}{2}}{p} \right) e^{-\frac{p^2\pi^2 t}{36}} \sin\left(\frac{p\pi x}{6} \right) \right]$$

2. Solve $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ subject to conditions v (0, t) = 1, v (π, t) = 3

$v(x, 0) = 1$ for $0 < x < \pi$, $t > 0$

$$\left[\textbf{Ans.}\ v(x, t) = \frac{4}{\pi} \sum_{p=1}^{\infty} \frac{\cos p\pi}{p} e^{-p^2 t} \sin px + 1 + \frac{2x}{\pi} \right]$$

3. Solve $\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2}$

$$\left[\textbf{Ans.}\ \theta\ (x, t) = \pi \sum_{p=1}^{\infty} (-1)^{p+1}\ e^{-p^2 t} \sin px \right]$$

given $\theta(0, t) = 0$, $\theta(\pi, t) = 0$, $\theta\ (x, 0) = 2x$ for $0 < x < \pi$, $t > 0$

TABLE

Function f (x)	*Fourier Sine Transform F_s (s)*
$\begin{cases} 1 & 0 < x < b \\ 0 & x > b \end{cases}$	$\frac{1 - \cos bs}{s}$
x^{-1}	$\frac{x}{2}$
$\frac{x}{x^2 + b^2}$	$\frac{\pi}{2} e^{-bs}$
e^{-bx}	$\frac{s}{s^2 + b^2}$
$x^{n-1} e^{-bx}$	$\frac{\Gamma(n) \sin(n \tan^{-1} s / b)}{(s^2 + b^2)^{n/2}}$
$x e^{-bx^2}$	$\frac{\sqrt{\pi}}{4b^{3/2}} s e^{-s^2/4b}$
$x^{-1/2}$	$\sqrt{\frac{\pi}{2s}}$
x^{-n}	$\frac{\pi s^{n-1} \sec(n\pi / 2)}{2\Gamma(n)}$, $0 < n < 2$

$\dfrac{\sin bx}{x}$	$\dfrac{1}{2}\ln\left(\dfrac{s+b}{s-b}\right)$
$\dfrac{\cos bx}{x}$	$\begin{cases} 0 & s<b \\ \pi/4 & s=b \\ \pi/2 & s>b \end{cases}$
$\tan^{-1}(x/b)$	$\dfrac{\pi}{2s}e^{-bs}$
$\csc bx$	$\dfrac{\pi}{2b}\tanh\dfrac{\pi s}{2b}$
$\dfrac{1}{e^{2x}-1}$	$\dfrac{\pi}{4}\cot h\left(\dfrac{\pi s}{2}\right)-\dfrac{1}{2s}$
$\begin{cases} 1 & 0<x<b \\ 0 & x>b \end{cases}$	$\dfrac{\sin bs}{s}$
$\dfrac{1}{x^2+b^2}$	$\dfrac{\pi e^{-bs}}{2b}$
x^{-n}	$\dfrac{\pi s^{n-1}\sec(n\pi/2)}{2\Gamma(n)},\ 0<n<1$
$\ln\left(\dfrac{x^2+b^2}{x^2+c^2}\right)$	$\dfrac{e^{-cs}-e^{-bs}}{\pi s}$
$\dfrac{\sin bx}{x^2}$	$\begin{cases} \pi/2 & s<b \\ \pi/4 & s=b \\ 0 & s>b \end{cases}$
$\sin bx^2$	$\dfrac{\sqrt{\pi}}{8b}\left(\cos\dfrac{s^2}{4b}-\sin\dfrac{s^2}{4b}\right)$
$\cos bx^2$	$\sqrt{\dfrac{\pi}{8b}}\left(\cos\dfrac{s^2}{4b}+\sin\dfrac{s^2}{4b}\right)$
$\operatorname{sech} bx$	$\dfrac{\pi}{2b}\operatorname{sec h}\dfrac{s\pi}{2b}$
$\dfrac{\cosh(\sqrt{\pi}x/2)}{\cosh(\sqrt{\pi}\,x)}$	$\dfrac{\sqrt{\pi}}{2}\dfrac{\cosh(\sqrt{\pi}s/2)}{\cosh(\sqrt{\pi}\ s)}$
$\dfrac{e^{-b\sqrt{x}}}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{2s}}\{\cos(2b\sqrt{s})-\sin(2b\sqrt{s})\}$