Interpolation Methods of Unequal Interval

LINEAR INTERPOLATION

The simplex form of interpolation is to approximate two data points by straight line. Suppose, we are given two points, $[x_1, f(x_1)]$ and $[x_2, f(x_2)]$, these two points can be connected linearly as shown in the figure below.

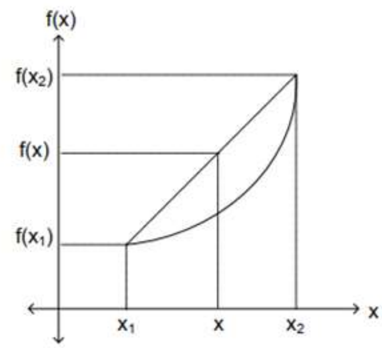

Figure 1: Representation of Linear Interpolation [1]

Using the concept of similar triangle:

$$\frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\therefore f(x) = f(x_1) + (x - x_1)\frac{f(x_2) - f(x_1)}{(x_2 - x_1)}$$

Mathematical equation for a straight line: $y = f(x) = a_0 + a_1x$

$a_0 = f(x_1)$ $a_1 = (x - x_1)\frac{f(x_2)-f(x_1)}{(x_2 - x_1)}$

Hence, the co-efficient 'a_1' represent the first derivative of the given function.

Example [1]:

Given the table of data:

x	1	2	3	4	5
f(x)	1	1.4142	1.7321	2	2.2362

Determine the square root of 2.5.

Solution:

Here x = 2.5, hence 2.5 is in between 2 and 3 so $x_1 = 2$ and $x_2 = 3$.

So,

$$f(x) = f(x_1) + (x - x_1)\frac{f(x_2) - f(x_1)}{(x_2 - x_2)}$$

$$f(2.5) = f(2) + (2.5 - 2)\frac{f(3) - f(2)}{(3 - 2)}$$

$$f(2.5) = 1.4142 + (0.5)\frac{1.7321 - 1.4142}{1} = 1.5732$$

For the interval 2 and 4 $f(2.5) = 1.5607$
True value of $f(2.5) = 1.5811$
So error each interval:
Error 1 = $|1.5811 - 1.5732| = 0.0079$
Error 2 = $|1.5811 - 1.5607| = 0.0204$
Here Error 2>Error 1, hence smaller the interval between the interpolation data points, the better be the approximation.

Interpolation with Unequal Intervals

The Newton's forward and backward interpolation formula and central difference formula are applicable only when values of x are given at equal intervals. When the values of independent variable x are given at unequal intervals then we can use the following formula.

LAGRANGE'S INTERPOLATION POLYNOMIAL FORMULA:

Let $x_0, x_1, x_2, \ldots\ldots, x_n$ be n distinct independent variables and $f_0, f_1, f_2, \ldots, f_n$ be dependent variables. Consider the data points (x_0, f_0), (x_1, f_1), (x_2, f_2) … (x_n, f_n) are connected by a curve. Let $p(x)$ is the interpolating function such that: $p(x_k) = f_k$ for $k = 0, 1, 2, \ldots n$

We can find interpolating function $p(x)$ using Lagrange interpolation formula that passes through all the data points (x_0, f_0), (x_1, f_1), (x_2, f_2) … (x_n, f_n).

Derivation of Lagrange Interpolation Formula:

Let (x_0, f_0), (x_1, f_1), (x_2, f_2) … (x_n, f_n) be the n +1 points. To derive the formula for the polynomial of degree 'n', let us consider a second order polynomial of the form:
$P_2(x) = b_1(x - x_1)(x - x_2) + b_2(x - x_0)(x - x_2) + b_3(x - x_0)(x - x_1)$…………(i)

$$if\ (x_0, f_0), (x_1, f_1), (x_2, f_2) are\ three\ interpolating\ points\ then:$$

$$P_2(x_0) = b_1(x_0 - x_1)(x_0 - x_2) = f_0$$
$$P_2(x_1) = b_2(x_1 - x_2)(x_1 - x_0) = f_1$$
$$P_2(x_2) = b_3(x_2 - x_0)(x_2 - x_1) = f_2$$

$$\left.\begin{aligned} b_1 &= \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} \\ b_2 &= \frac{f_1}{(x_1 - x_2)(x_1 - x_0)} \\ b_3 &= \frac{f_2}{(x_2 - x_0)(x_2 - x_1)} \end{aligned}\right\} \qquad \text{(ii)}$$

Substituting the values of b_1, b_2 and b_3 from equation (ii) to equation (i), we get:

$$P_2(x) = \frac{f_0}{(x_0 - x_1)(x_0 - x_2)}(x - x_1)(x - x_2) + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)}(x - x_0)(x - x_2) + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}(x - x_1)(x - x_0)$$

$$P_2(x) = f_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + f_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + f_2 \frac{(x - x_1)(x - x_0)}{(x_2 - x_0)(x_2 - x_1)} \dots\dots\dots\dots\dots (iii)$$

Where,

$$I_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$I_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$I_2(x) = \frac{(x - x_1)(x - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

Equation (iii) second degree Lagrange's Polynomial. For Lagrange's Polynomial of nth degree, we should have given (n+1) interpolating points. Then the polynomial will be:

$$P_n(x) = f_0 I_0(x) + f_1 I_1(x) + f_2 I_2(x) + \cdots \dots\dots + f_{n-1} I_{n-1}(x) + f_n I_n(x) \dots\dots\dots\dots (iv)$$

This can be generally expressed as:

$$P_n(x) = \sum_{i=0}^{n} f_i I_i(x) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (v)$$

Where,

$$I_i(x) = \prod_{j=0, j\neq i}^{n} \left(\frac{x - x_i}{x_i - x_j}\right) \dots\dots\dots\dots\dots\dots\dots\dots\dots (vi)$$

Here, equation (v) is called Lagrange's Interpolation Polynomial of degree 'n' and then equation (vi) is called Lagrange's Basic Polynomial.

Example:[1]

Given the points below in the table, obtain a cubic polynomial using the Lagrange formula.

	x_0	x_1	x_2	x_3
x	0	1	2	3
f(x)	1	-1	-1	0

Solution:
The equation of Lagrange formula:

$$p_3(x) = f_0 I_0(x) + f_1 I_1(x) + f_2 I_2(x) + f_3 I_3(x)$$

Where,

$$I_0(x) = \frac{(x-x1)(x-x2)(x-x3)}{(x0-x1)(x0-x2)(x0-x3)} = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = \frac{x^3-6x^2+11x-}{-6}$$

$$I_1(x) = \frac{(x-x0)(x-x2)(x-x3)}{(x1-x0)(x1-x2)(x1-x3)} = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{x^3-5x^2+6x}{2}$$

$$I_2(x) = \frac{(x-x0)(x-x1)(x-x3)}{(x2-x0)(x2-x1)(x2-x3)} = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = \frac{x^3-4x^2+3x}{-2}$$

$$I_3(x) = \frac{(x-x0)(x-x1)(x-x2)}{(x3-x0)(x3-x1)(x3-x2)} = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{x^3-x^2+2x}{6}$$

$$f_0 = 1, f_1 = -1, f_2 = -1 \; and \; f_3 = 0$$

So,

$$p_3(x) = 1 * \frac{x^3-6x^2+11x-6}{-6} + (-1) * \frac{x^3-5x^2+6x}{2} + (-1)\frac{x^3-4\,x^2+3x}{-2} + (0)\frac{x^3-x^2+2x}{6}$$

After evaluating the equation we get:

$$p_3(x) = \frac{x^3-3x^2+2x-6}{-6} = -0.16667x^3 + 0.5x^2 - 0.333x + 1$$

So, the cubic polynomial by using Lagrange formula is:
$p_3(x) = -0.16667x^3 + 0.5x^2 - 0.333x + 1$-----------(vii)

Using equation (vii), we can find any value of f for any intermediate value of x.

Notes on Lagrange Interpolation

1. It requires 2(n+1) multiplication/divisions and 2n+1 additions and subtractions.
2. If we want to add one more data point, we have to compute the polynomial from the beginning. It does not use the polynomial already computed. That is, $p_{k+1}(x)$ does not use $p_k(x)$.

Newton Divided Difference

One of the major disadvantages of the Lagrange's Interpolation Formula is if we add another interpolating point to get the better accuracy, then the interpolation coefficients $I_i(x)$ need to recalculate. Therefore we have to seek an interpolation polynomial which has the property that a polynomial of higher degree may be derived from it by simply adding new terms.

Newton general interpolation formula is one such formula, which employs what are called divided differences. It is our principle purpose to define such differences and discuss certain of their properties to obtain the basic formula.
We have Newton's form of polynomial as:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots$$
$$+ a_{n-1}(x - x_0)(x - x_1) \dots (x - x_{n-1}) \dots\dots\dots\text{(viii)}$$

In this polynomial, coefficients can be calculated using divided difference table as given below: [1]

Divide Difference Table

i	x_i	$f(x_i)$	*First Difference* $\blacktriangle f(x)$	*Second Difference* $\blacktriangle^2 f(x)$	*Third Difference* $\blacktriangle^3 f(x)$	*Fourth Difference* $\blacktriangle^4 f(x)$
0	x_0	$f(x_0)$				
1	x_1		$f(x_0, x_1)$			
		$f(x_1)$		$f(x_0, x_1, x_2)$		
			$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$	
2	x_2	$f(x_2)$		$f(x_1, x_2, x_3)$		$f(x_0, x_1, x_2, x_3, x_4)$
			$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$	
3	x_3	$f(x_3)$		$f(x_2, x_3, x_4)$		
			$f(x_3, x_4)$			
4	x_4	$f(x_4)$				

In divide difference table higher order divide difference is obtained using lower order divide difference.

First Oder Difference

$f(x_0, x_1) = \frac{f(x1) - f(x0)}{x1 - x0}$

$f(x_1, x_2) = \frac{f(x2) - f(x1)}{x2 - x1}$

$f(x_2, x_3) = \frac{f(x3) - f(x2)}{x3 - x2}$

$f(x_3, x_4) = \frac{f(x4) - f(x3)}{x4 - x3}$

Second Oder Difference

$f(x_0, x_1, x_2) = \frac{f(x1, x2) - f(x0, x1)}{x2 - x0}$

$f(x_1, x_2, x_3) = \frac{f(x2, x3) - f(x1, x2)}{x3 - x1}$

$f(x_2, x_3, x_4) = \frac{f(x3, x4) - f(x2, x3)}{x4 - x2}$

Third Order Difference

$f(x_0, x_1, x_2, x_3) = \frac{f(x1,x2,x3) - f(x0,x1,x2)}{x3-x0}$

$f(x_1, x_2, x_3, x_4) = \frac{f(x2,x3,x4) - f(x1,x2,x3)}{x4-x1}$

Fourth Order Difference

$f(x_0, x_1, x_2, x_3, x_4) = \frac{f(x1,x2,x3,x4) - f(x0,x1,x2,x3)}{x4-x0}$

The $(k+1)^{th}$ divided difference

$$f[x_0,x_1,\ldots\ldots x_{k+1}] = \frac{f[x0,x1\ldots\ldots xk] - f[x1,x2\ldots\ldots x_{k+1}]}{x0 - x_{k+1}}$$

Note: The entries at the top of each column represented the divide difference coefficients.

To construct the interpolating polynomial, we need to determine the coefficients a_0, a_1, $a_2,\ldots,a_n$ of equation (viii).

Let (x_0, f_0), (x_1, f_1), (x_2, f_2) … (x_n, f_n) are the interpolating points. So we have to find interpolating function $p(x)$ such that, $p_k(x_k) = f_k$ $\quad k = 0, 1, 2,\ldots,n$ and passes through (x_0, f_0), (x_1, f_1), (x_2, f_2) … (x_n, f_n)

Now, $x = x_0$ we have (using equation viii)
$p_0(x_0) = f(x_0) = a_0$

Similarly $x = x_1$ we have
$p_1(x_1) = a_0 + a_1(x_1 - x_0) = f_1(x_1)$
Substituting the value of a_0, we get
$a_1 = \frac{f_1(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1)$ = first divide difference

At $x = x_2$ we get,
$p_2(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2(x_2)$

Now substituting value of a_0 and a_1 and rearranging the terms, we get
$a_2 = \frac{\frac{f_2(x_2) - f_1(x_1)}{x_2 - x_1} - \frac{f_1(x_1) - f_0(x_0)}{x_1 - x_0}}{x_2 - x_0} = \frac{f(x1,x2) - f(x0,x1)}{x2-x0} = f(x_0, x_1, x_2)$ = second divide difference

So,
$a_n = f(x_0, x_1, x_2, \ldots x_n)$

Now substituting the values of coefficients a_i at equation viii, we get

$$p_n(x_n) = f(x_0) + f(x_0, x_1)(x - x_0) + f(x_0, x_1, x_2)\ (x - x_0)\ (x - x_1) + \cdots .. + f(x_0, x_1, x_2, \ldots x_n)\ (x - x_0)\ (x - x_1) \ldots \ldots (x - x_n) \ldots\ldots\ldots \text{(ix)}$$

Equation (ix) is known as Newton divide difference interpolation formula.

Properties of Divided differences

1. Divided differences are *symmetric* function of their arguments.

$$f[x_0,x_1] = \frac{f(x0) - f(x1)}{x0 - x1} = \frac{f(x1) - f(x0)}{x1 - x0} = f[x_1,x_0]$$

2. The divided difference of the product of a constant and a function is equal to the product of the constant and the divided difference of the function i.e.

$k\, f[x_0,x_1] = f\, k[x_1,x_0]$ where k is a constant.

3. The divided difference of the sum (or difference) of two functions is equal to the sum (or difference) of the corresponding separate divided differences.

If $f(x) = g(x) + h(x)$ then, $f[x_0,x_1] = g[x_0,x_1] + h[x_0,x_1]$

4. The n^{th} order divided differences of a polynomial of n^{th} degree are constants.
5. *The $(n+1)^{th}$ order divided differences of a polynomial of n^{th} degree will be zero.*

Example: Use Newton divided difference formula to find $f(6)$ for the following table.

x	5	7	11	13	21
f(x)	150	392	1452	2366	9702

***Solution*:**

x	f(x)	▲f(x)	▲$^2 f$(x)	▲$^3 f$(x)	▲$^4 f$(x)
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		46		
		917			
21	9702				

We have $f(x_0) = 150$, $f(x_0, x_1) = 121$, $f(x_0, x_1, x_2) = 24$, $f(x_0, x_1, x_2, x_3) = 1$, $f(x_0, x_1, x_2, x_3, x_4) = 0$

$$p_4(x) = f(x_0) + f(x_0, x_1)(x - x_0) + f(x_0, x_1, x_2)\ (x - x_0)\ (x - x_1)$$

$+ f(x_0, x_1, x_2, x_3)\,(x - x_0)\,(x - x_1)(x - x_2) + f(x_0, x_1, x_2, x_3, x_4)\,(x - x_0)\,(x - x_1)(x - x_2)(x - x_3)$

$= 150 + (121)\,(x - 5) + (x - 5)\,(x - 7)\,(24) + (x - 5)\,(x - 7)\,(x - 11)\,(1) + 0$
$= 150 + 121\,(x - 5) + (x^2 - 12x + 35)24 + (x^3 - 23x^2 + 167x - 385)\,1$
$= 150 + 121x - 605 + (24x^2 - 288x + 840) + (x^3 - 23x^2 + 167x - 385)\,1$
$= x^3 + x^2$; Interpolating polynomial using Newton divide difference formula

$p_4(6) = 6^3 + 6^2 = 216 + 36 = 252$

References:

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