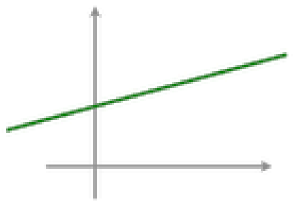
CSE4203: Computer Graphics

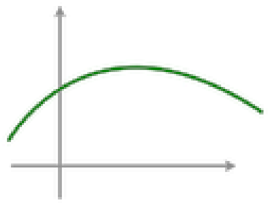
Bézier Curves

Polynomials

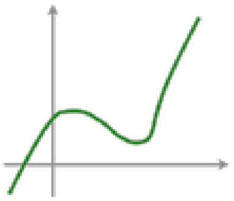
1st degree polynomial: $y = ax^0 + bx^1$



2nd degree polynomial: $y = ax^0 + bx^1 + cx^2$



3rd degree polynomial: $y = ax^0 + bx^1 + cx^2 + dx^3$

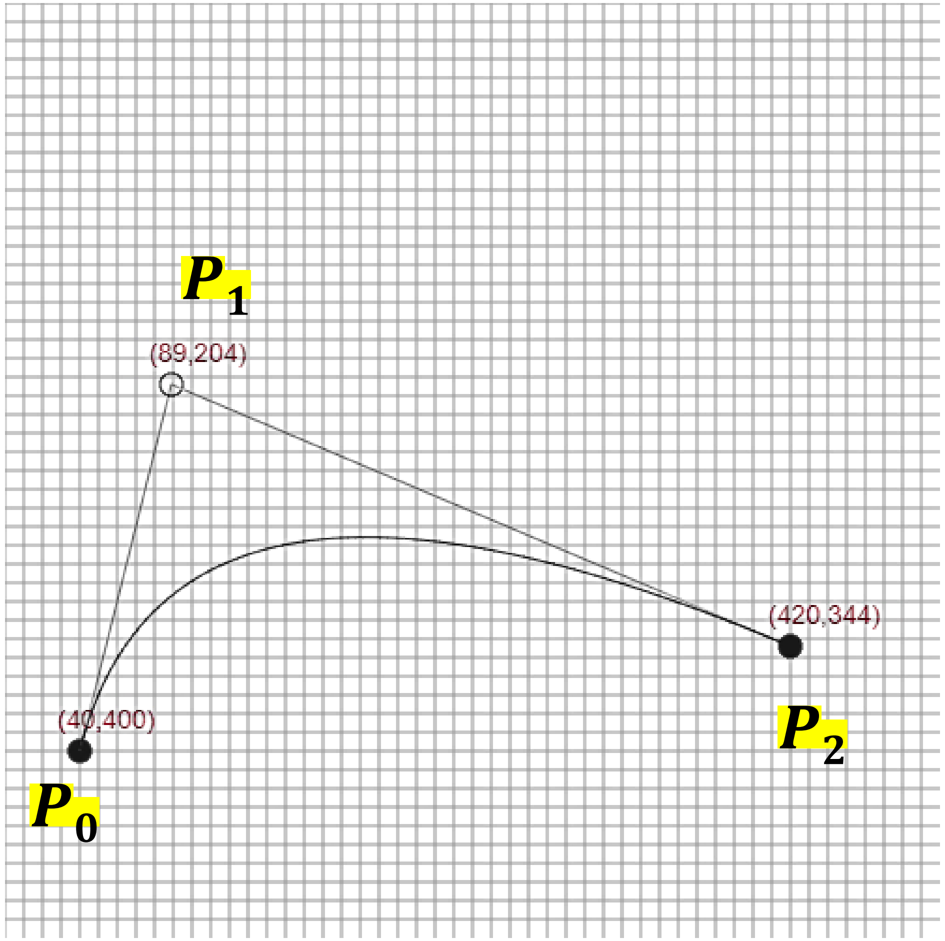


Credit: CPSC 589/689 Course Notes, University of Calgary, Faramarz Samavati, Fall 2013

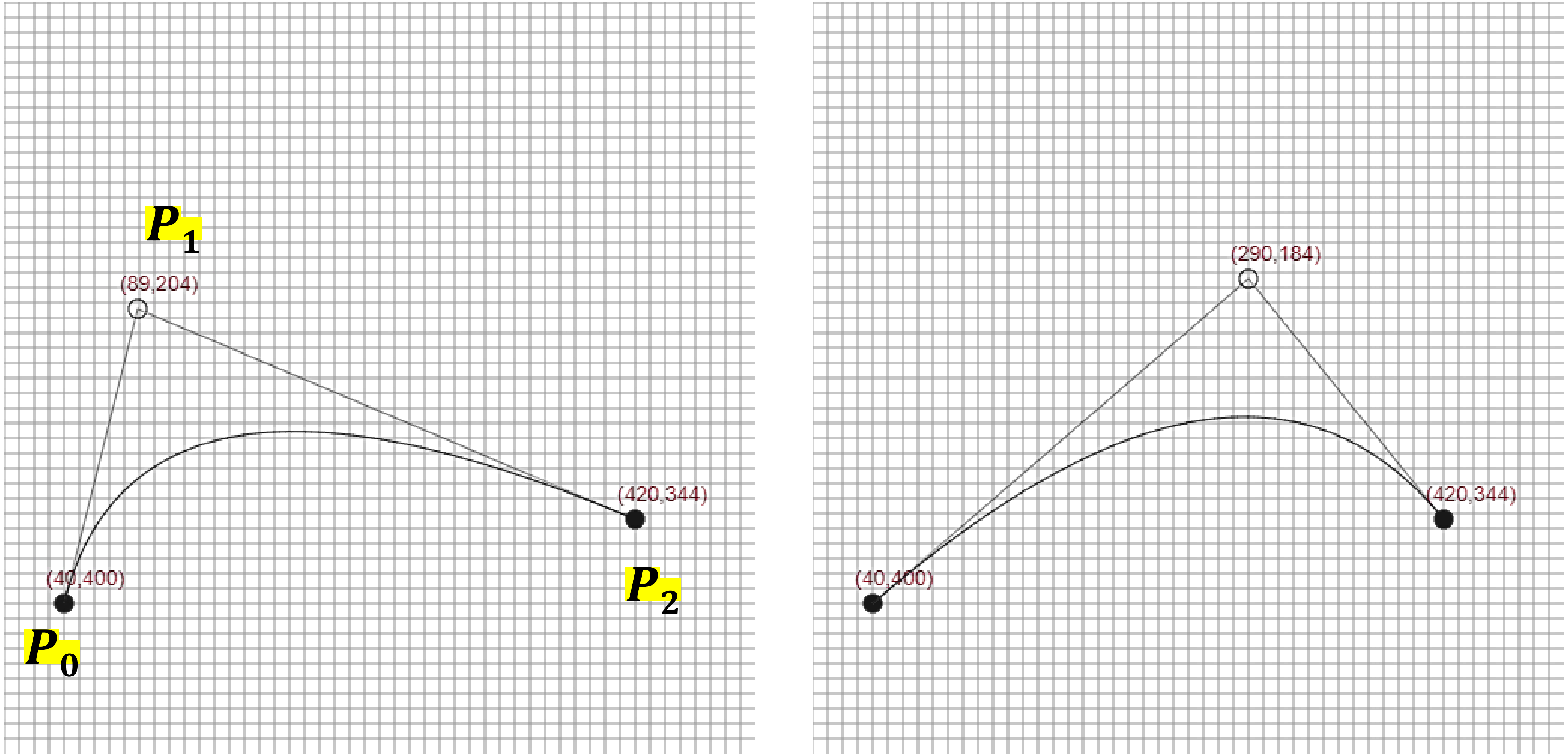
Bézier Curves

- First described in 1972 by Pierre Bézier

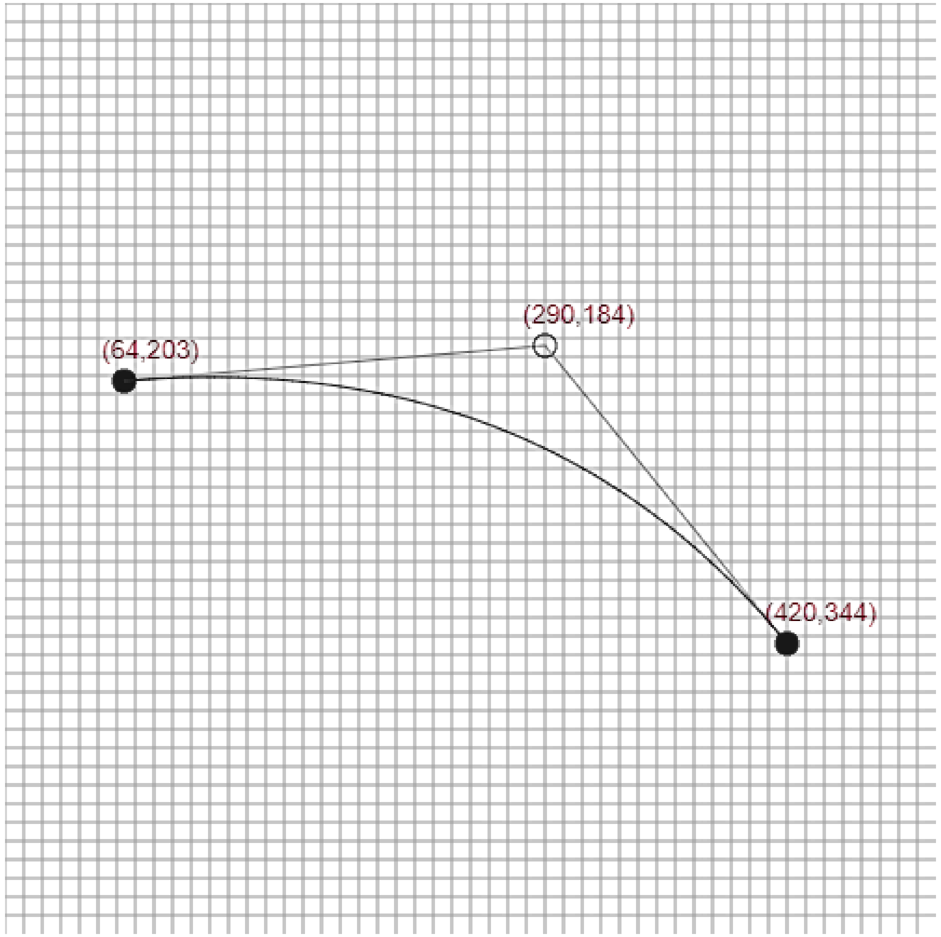
Control Points

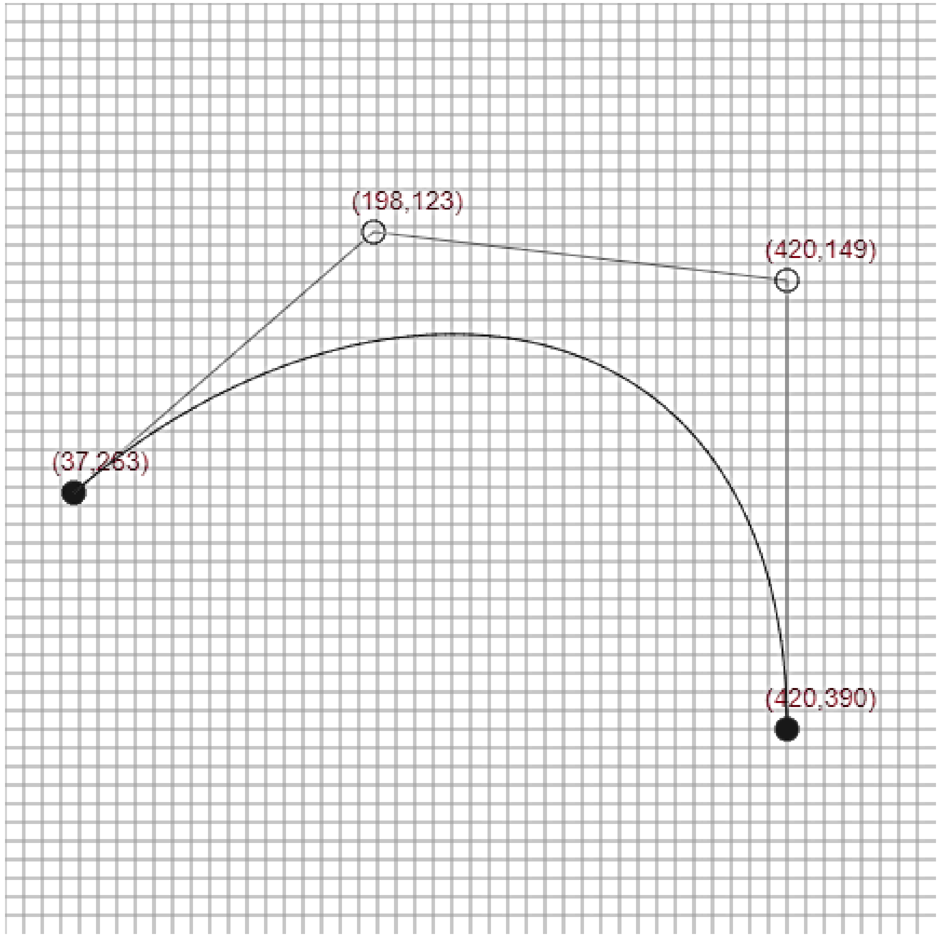


Control Points



Control Points



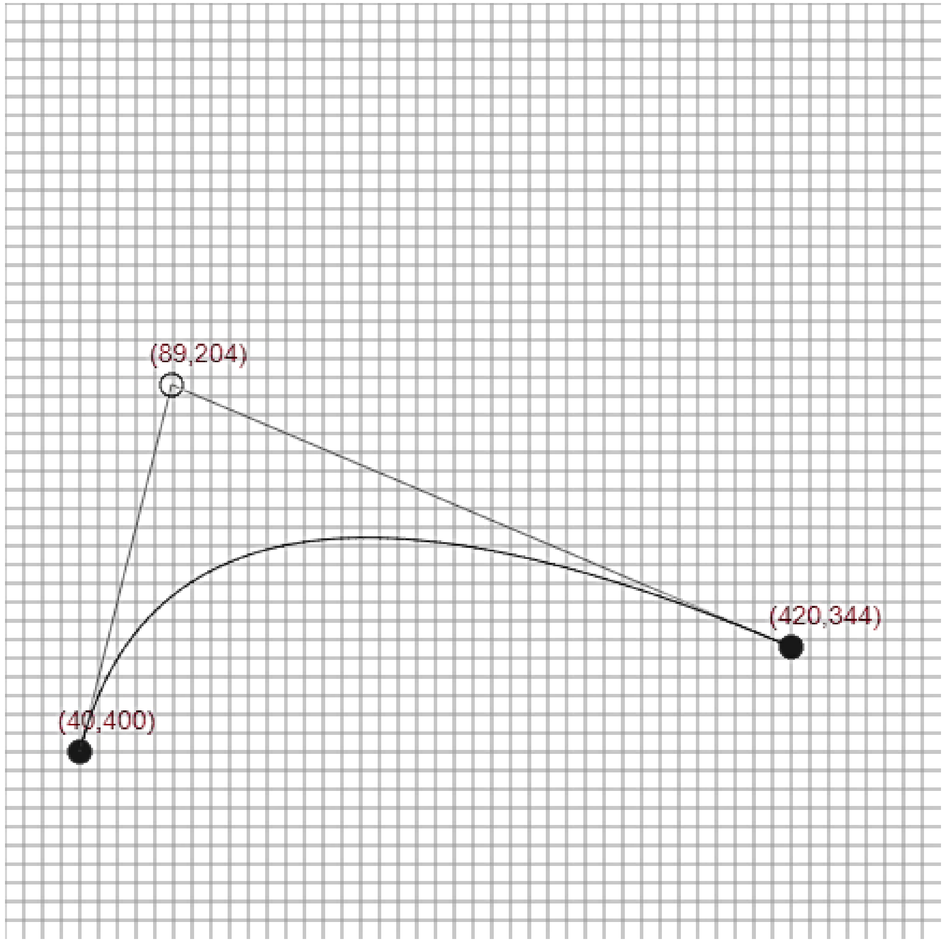


Inputs

- N number of control points
- Degree, $d = N - 1$

For example,

For 3 Control Points, $d = 2$



Inputs

- N number of control points
- Degree, $d = N - 1$

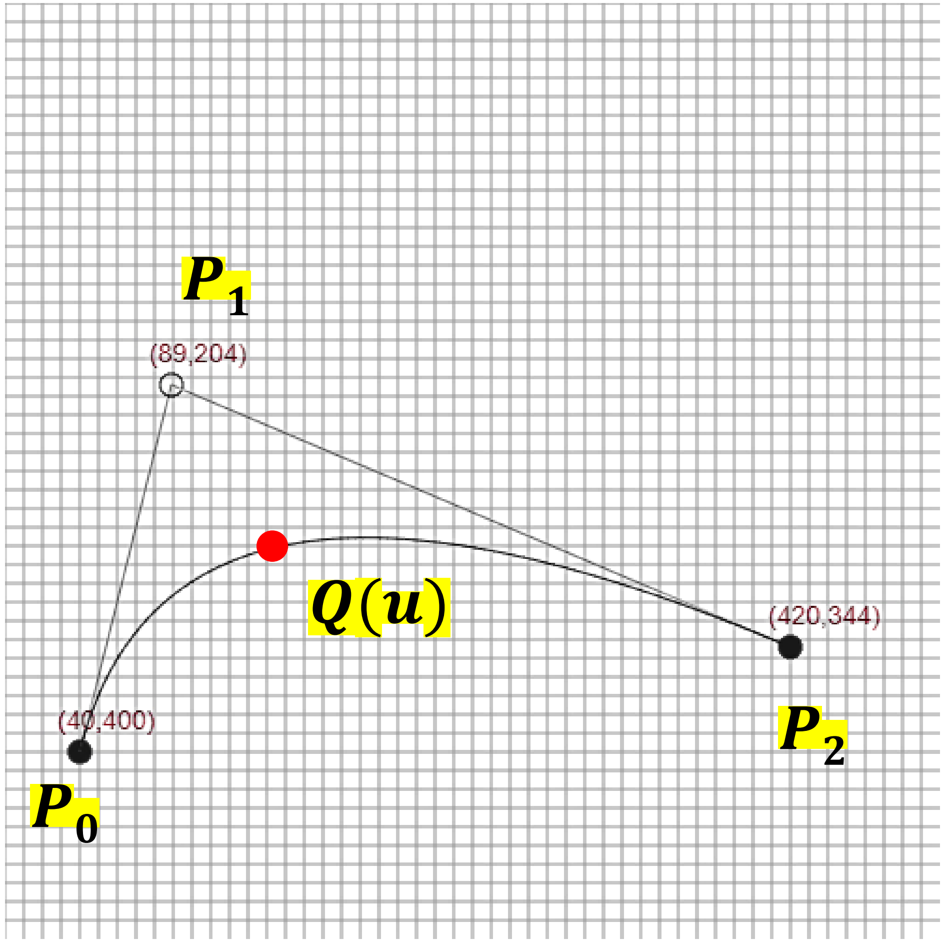
- N number of control points
- Degree, $d = N - 1$

What is the d here?

Bézier Curves

$$Q(u) = \sum_{i=0}^{d} B_{i,d}(u) P_i \qquad 0 \leq u \leq 1$$

Example: $\sum_{i=0}^{2} B_{i,2}(u) P_i = B_{0,2}(u) P_0 + B_{1,2}(u) P_1 + B_{2,2}(u) P_2$



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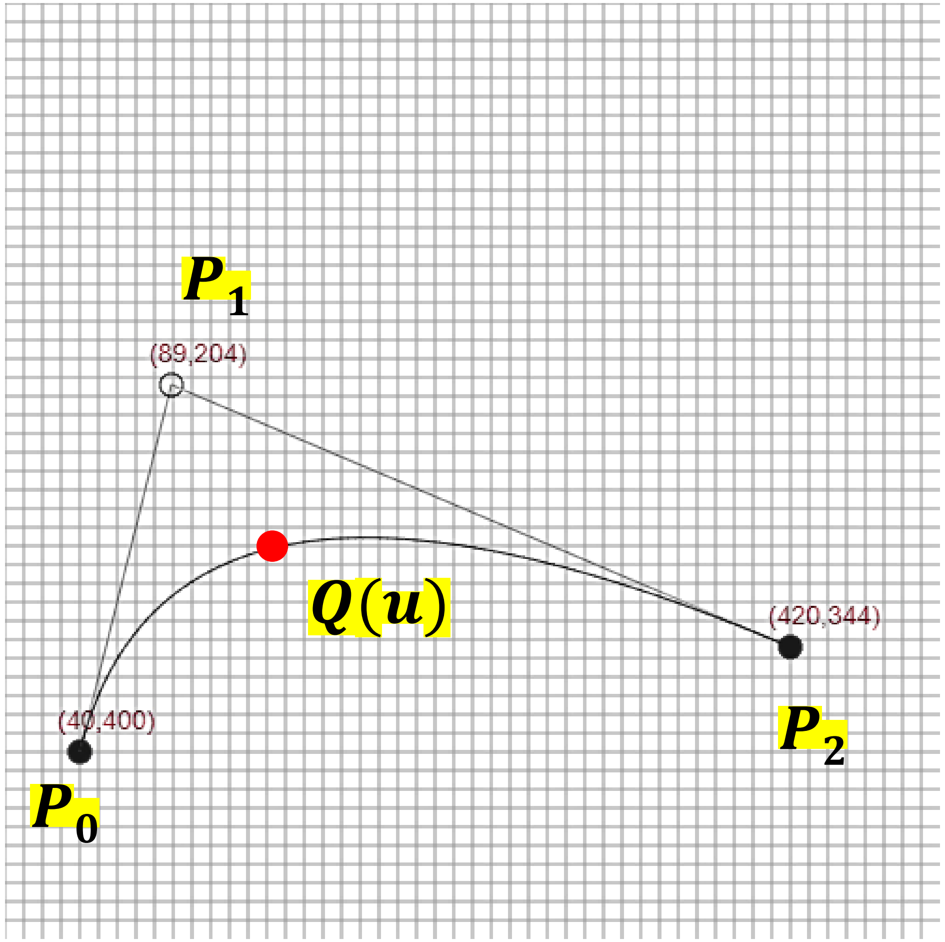
Bézier Curves

$$Q(u) = \sum_{i=0}^{d} B_{i,d}(u) P_i \qquad 0 \leq u \leq 1$$

$$B_{i,d}(u) = \left(\begin{array}{c} d \\ i \end{array} \right) u^i (1-u)^{d-i} \qquad \left(\begin{array}{c} d \\ i \end{array} \right) = \frac{d!}{i!(d-i)!}$$



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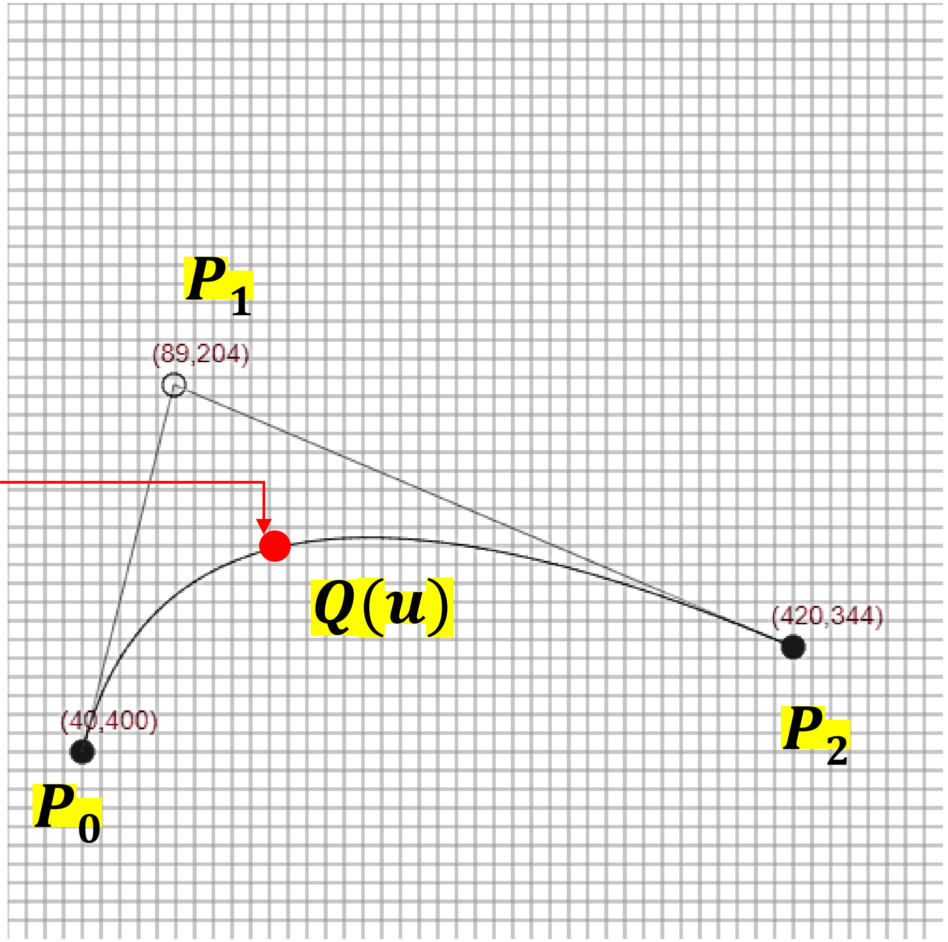
Bézier Curves

$$\boxed{Q(u)} = \sum_{i=0}^{d} B_{i,d}(u) P_i \qquad 0 \le u \le 1$$

$$B_{i,d}(u) = \binom{d}{i} u^i (1-u)^{d-i} \qquad \binom{d}{i} = \frac{d!}{i!(d-i)!}$$

A point on the curve



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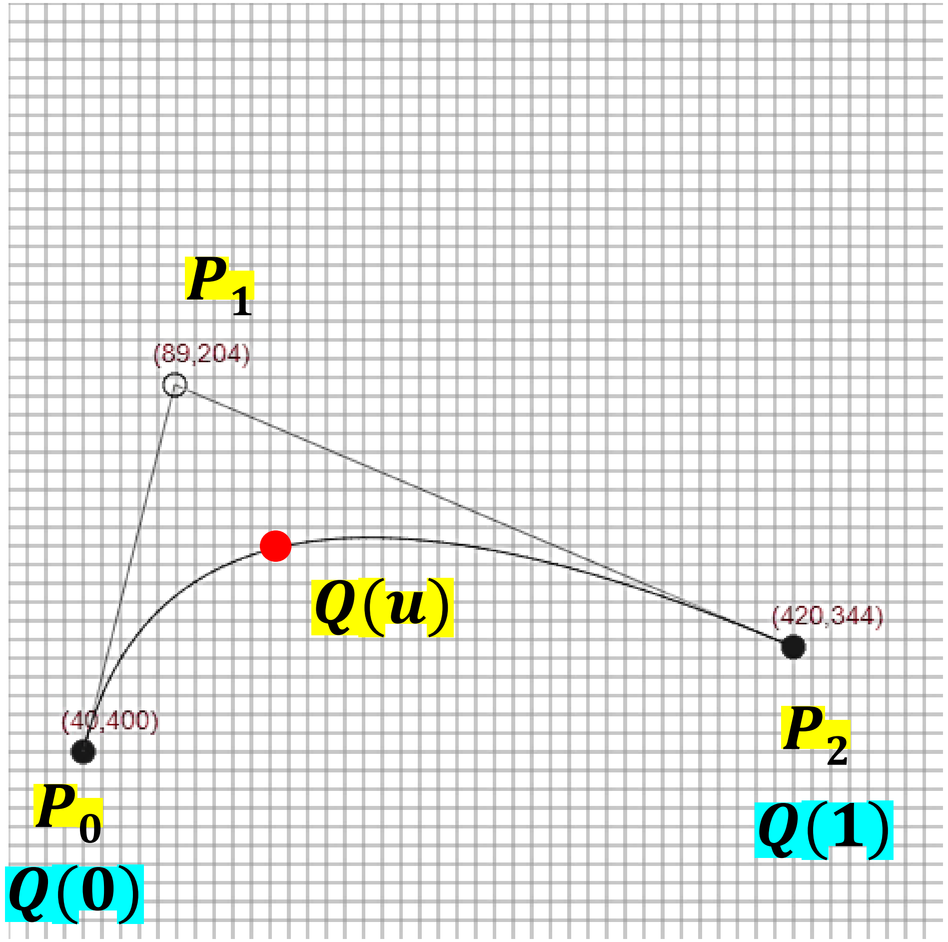
Bézier Curves

$$Q(u) = \sum_{i=0}^{d} B_{i,d}(u) P_i \qquad 0 \leq u \leq 1$$

$$B_{i,d}(u) = \begin{pmatrix} d \\ i \end{pmatrix} u^i (1-u)^{d-i} \qquad \begin{pmatrix} d \\ i \end{pmatrix} = \frac{d!}{i!(d-i)!}$$

Denoted with $Q_d(u)$



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Bézier Curves

$$Q(u) = \sum_{i=0}^{d} B_{i,d}(u) P_i \qquad 0 \leq u \leq 1$$

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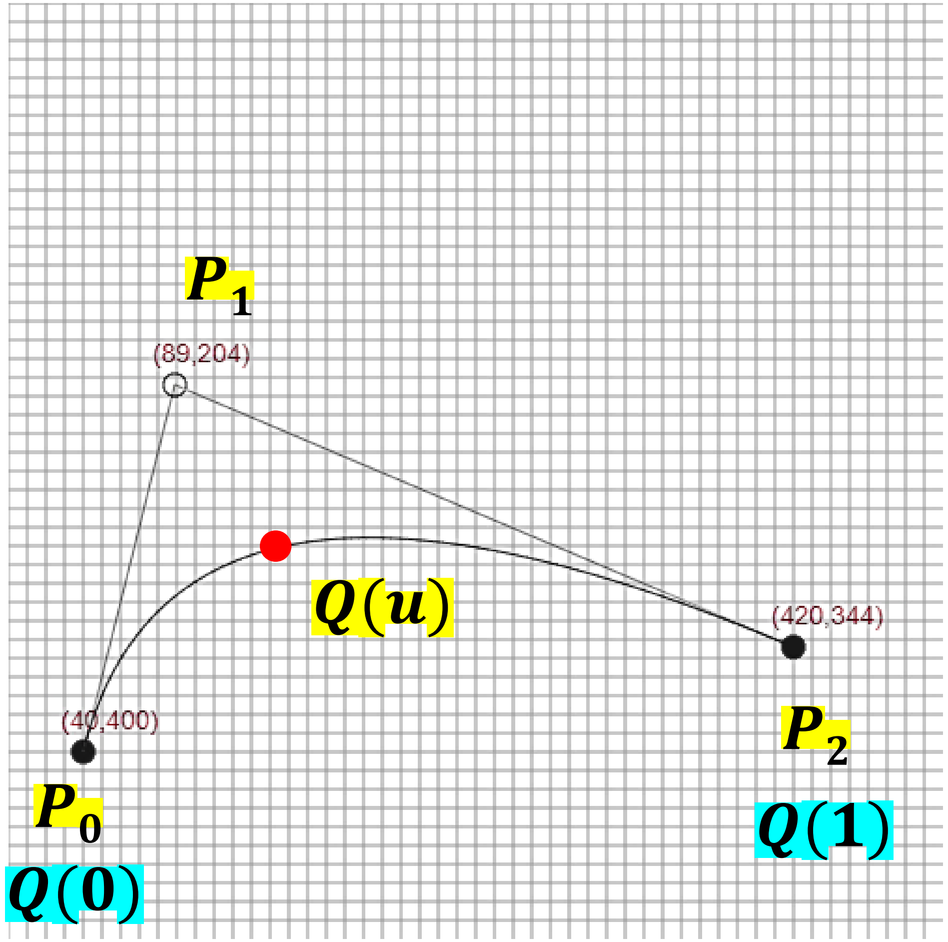
Where is $Q_d(0.5)$ situated?

Where is $Q_d(0)$ situated?

Where is $Q_d(1)$ situated?



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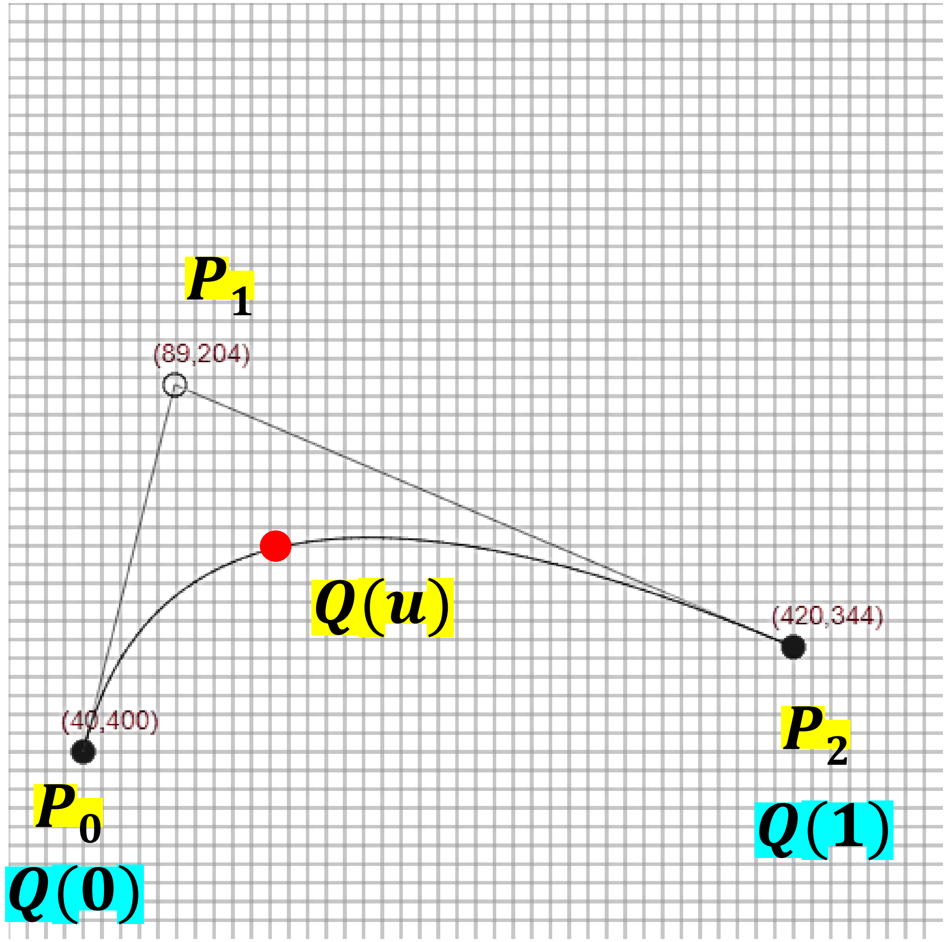
Bézier Curves

$$Q(u) = \sum_{i=0}^{d} B_{i,d}(u) P_i \qquad 0 \leq u \leq 1$$

$$B_{i,d}(u) = \left(\begin{array}{c} d \\ i \end{array} \right) u^i (1-u)^{d-i} \qquad \left(\begin{array}{c} d \\ i \end{array} \right) = \frac{d!}{i!(d-i)!}$$

Online simulator: https://ytyt.github.io/siiiimple-bezier/



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Bézier Curves

$$Q(u) = \sum_{i=0}^{d} B_{i,d}(u) P_i \qquad 0 \leq u \leq 1$$

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These polynomials are called "Bernstein polynomials" and denoted by $B_{i,d}(u)$

$$\begin{array}{ll} B_{0,2}(u) = (1-u)^2 & B_{0,3}(u) = (1-u)^3 \\ B_{1,2}(u) = 2u(1-u) & B_{1,3}(u) = 3u(1-u)^2 \\ B_{2,2}(u) = u^2 & B_{2,3}(u) = 3u^2(1-u) \\ & B_{3,3}(u) = u^3 \end{array}$$

$$Q_2(u) = P_0(1-u) + P_1[2u(1-u)] + P_2(u^2)$$

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Example

Given control points $P_0 = (0,0), P_1 = (4,2), P_2 = (8,0)$, find the Bézier curve values $Q_2(0)$, $Q_2(\frac{1}{2})$ and $Q_2(1)$.

Why subscript 2 for $Q_2(u)$?

Example

Given control points $P_0 = (0,0), P_1 = (4,2), P_2 = (8,0)$, find the Bézier curve values $Q_2(0)$, $Q_2(\frac{1}{2})$ and $Q_2(1)$.

$$Q_2(u) = \sum_{i=0}^{n} B_{i,2}(u)P_i \qquad 0 \leq u \leq 1$$

$$B_{i,d}(u) = \begin{pmatrix} d \\ i \end{pmatrix} u^i(1-u)^{d-i} \qquad \begin{pmatrix} d \\ i \end{pmatrix} = \frac{d!}{i!(d-i)!}$$

$$Q_2(u) = B_{0,2}(u)P_0 + B_{1,2}(u)P_1 + B_{2,2}(u)P_2$$

$$Q_2(u) = (1-u)^2P_0 + 2(1-u)uP_1 + u^2P_2$$

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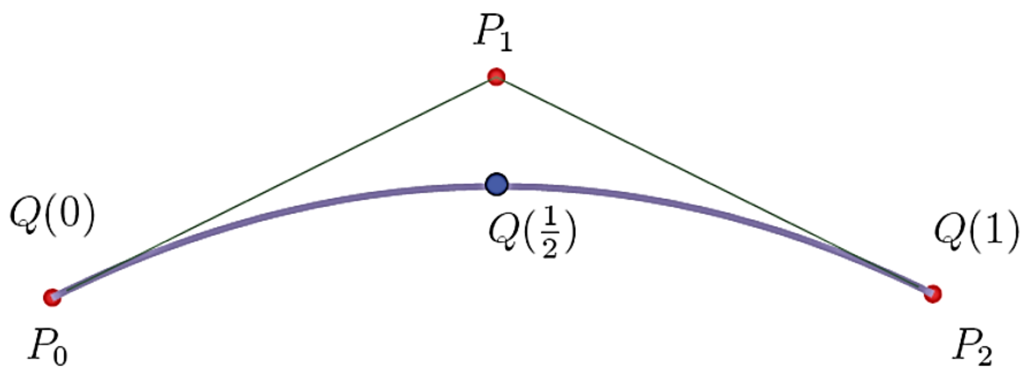
Example

Given control points $P_0 = (0,0), P_1 = (4,2), P_2 = (8,0)$, find the Bézier curve values $Q_2(0)$, $Q_2(\frac{1}{2})$ and $Q_2(1)$.

$$Q_2(u) = (1-u)^2 P_0 + 2(1-u)uP_1 + u^2 P_2$$

- $Q_2(0) = (1-0)^2 P_0 + 2(1-0)0P_1 + 0^2 P_2 = P_0 = (0,0)$
- $Q_2(\frac{1}{2}) =$ ….. *Do calculations ….* $= (4,1)$
- $Q(1) =$ … *Do calculations …* $= (8,0)$



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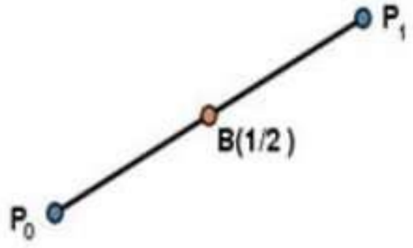
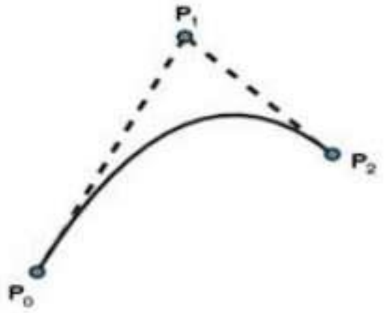
Properties of Bezier Curves

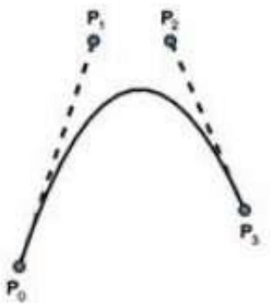
- They generally follow the shape of the control polygon, which consists of the segments joining the control points
- They always pass through the first and last control points
- They are contained in the convex hull of their defining control points
- The degree of the polynomial defining the curve segment (d) is one less than that the number of defining polygon point (n) i.e. n = d+1



Simple Bezier Curve



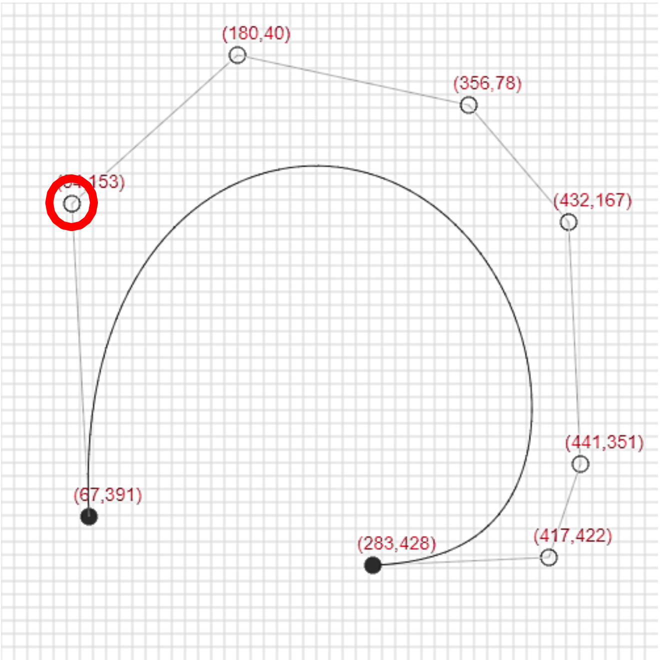
Quadratic Bazier Curve

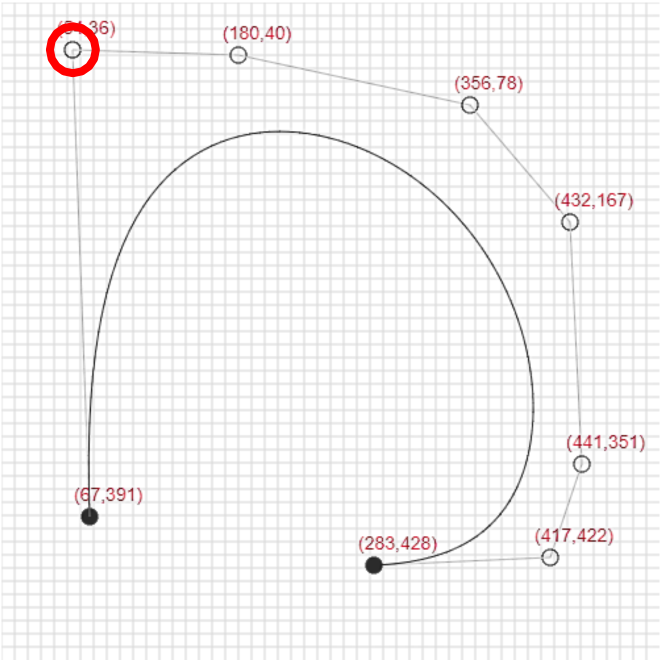


Cubic Bazier Curve

Disadvantages

- A change to any of the control point alters the entire curve.
- Having a large number of control points requires high polynomials to be evaluated. This is expensive to compute.





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