Interpolation Methods of Equal Intervals

Finites Differences

Assume that we have a table of values (x_i, y_i), i = 0, 1, 2, …, n of any function $y = f(x)$, the values of x being *equally spaced*, i.e. $x_i = x_0 + ih$, i = 0,1,2, ...,n. Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x, or to obtain the derivative of $f(x)$ for some x in the range $x_0 \le x \le x_n$. The methods for solution of these problems are based on the concept of the *differences* of a function which is known as Finite Differences. Finite difference operators include:

1. forward difference operator,
2. backward difference operator
3. central difference operator and
4. shift operator,
5. mean operator.

Forward Differences (Δ)

If $y_0, y_1, y_2, \ldots \ldots, y_n$ denote a set of values of $y = f(x)$, for equidistant value of $x_0, x_1, x_2, \ldots x_n$ where $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, …..$x_n = x_0 + nh$, then $y_1 - y_0$, $y_2 - y_1, \ldots \ldots \ldots, y_n - y_{n-1}$ are called the *differences* of y. Denoting these differences by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots \ldots \ldots, \Delta y_{n-1}$ respectively, we have,

$$\Delta y_0 = y_1 - y_0,$$
$$\Delta y_1 = y_2 - y_1,$$
$$\ldots\ldots,$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

***General form*:**

$$\mathbf{\Delta y_i = y_{i+1} - y_i,}$$

where i = 0, 1, 2, 3, ….., n-1, and Δ is called the *forward difference operator* and $\Delta y_0, \Delta y_1, \Delta y_2, \ldots, \Delta y_{n-1}$ are called *first forward differences*.

The difference of the first forward differences are called *second forward differences* and denoted by $\Delta^2 y_0, \Delta^2 y_1, \Delta^2 y_2, \ldots \ldots$ Similarly, one can define *third forward differences*, *fourth forward difference*s, etc.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1$$
$$\Delta^2 y_2 = \Delta y_3 - \Delta y_2 = (y_4 - y_3) - (y_3 - y_2) = y_4 - 2y_3 + y_2$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1 = (y_4 - 2y_3 + y_2) - (y_3 - 2y_2 + y_1) = y_4 - 3y_3 + 3y_2 - y_1$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

- Any higher order difference can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients.

$$\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i$$
$$\Delta^n y_o = \sum_{k=0}^{n} c_k \, y_{n-k}$$

$$\Delta^n y_0 = y_n - {}^nc_1\, y_{n-1} + {}^nc_2\, y_{n-2} - {}^nc_3\, y_{n-3} \;\ldots\;\ldots + (-1)^n\, y_o$$

Forward Difference Table: Following table shows how the forward differences of all orders can be formed:

x	*y*	Δ	Δ^2	Δ^3	Δ^4
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

The above table is called a *diagonal difference table*. The first term in the table is y_0. It is called the *leading term*. The difference Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$... are called the *leading differences*.

Example:

x	*y*	Δ	Δ^2	Δ^3	Δ^4
0	*7*				
		2			
1	*9*		*32*		
		34		*60*	
2	*43*		*92*		*24*
		126		*84*	
3	*169*		*176*		
		302			
4	*471*				

- Here, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, $y_0 = 7$, $y_1 = 9$, $y_2 = 43$, $y_3 = 169$, $y_4 = 471$
  $\Delta y_0=2$, $\Delta y_1=34$, $\Delta y_2=126$, $\Delta y_3=302$
  $\Delta^2 y_0=32$, $\Delta^2 y_1=92$, $\Delta^2 y_2=176$ {just a crosscheck: $\Delta^2 y_0 = y_2 - 2y_1 + y_0$}
  $\Delta^3 y_0=60$, $\Delta^3 y_1=84$ {just a crosscheck: $\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$}
  $\Delta^4 y_0=24$

- A convenient check may be obtained by noting the sum of the entries in any column equals the differences between the first and last entries in preceding column.

Newton Gregory Forward Polynomial:
Usually, for forward differences the functions are tabulated at equal intervals, i.e. $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 \ldots.\ x_n - x_{n-1} = na$. Here, with tabulation at equal intervals a difference table for 'n' points may be expressed as shown below:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	....
x_0	f_0					
		Δf_0				
$x_0 + h$	f_1		$\Delta^2 f_0$			
		Δf_1		$\Delta^3 f_0$		
$x_0 + 2h$	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$	....
		Δf_2		$\Delta^3 f_1$		
$x_0 + 3h$	f_3		$\Delta^2 f_2$	....	....	
		Δf_3	....	....	....	
$x_0 + 4h$	f_4	....	....	....	....	
....	....	....	....	....	....	

Figure 3: Tabular form of Newton Gregory Forward Polynomial [1]

Here 'h' is the uniform difference in the value of 'x'.
In general:
$\Delta f_0 = f_1 - f_0$,
$\Delta f_1 = f_2 - f_1$,
....,
$\Delta f_{n-1} = f_n - f_{n-1}$ which is called first forward difference.

Similarly: In general: $\Delta^2 f_i = f_{i+2} - 2\ f_{i+1} + f_i$, which is called second forward difference and so on.
For the polynomial that passes through a group of equidistant point i.e. Newton Gregory Forward Polynomial, we write in terms of the index 's', such that $s = (x-x_0)/h$.

$$P_n(x) = f_n + s\frac{\Delta f_n}{1!} + s(s-1)\frac{\Delta^2 f_n}{2!} + s(s-1)(s-2)\frac{\Delta^3 f_n}{3!} + \cdots$$

This equation for polynomial is called Newton Gregory Forward Interpolation Formula. It is applied when the required point is close to the start of the table.

Example:
For the given table, find the value of f(0.16).

X	0.1	0.2	0.3	0.4
f(x)	1.005	1.020	1.045	1.081

Solution:
$x_0 = 0.1$, $x_1 = 0.2$ … i.e. $h = x_1 - x_0 = 0.2 - 0.1 = 0.1$
We have given, $x = 0.16$. So: $s = (x - x_0)/h = (0.16 - 0.1)/0.1 = 0.6$
Now, generating the Forward Difference Table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0.1	**1.005**			
		0.015		
0.2	1.020		**0.01**	
		0.025		**0.001**
0.3	1.045		0.011	
		0.036		
0.4	1.081			

Now,

$$P_3(x) = f_n + s\frac{\Delta f_n}{1!} + s(s-1)\frac{\Delta^2 f_n}{2!} + s(s-1)(s-2)\frac{\Delta^3 f_n}{3!}$$

$$= 1.005 + 0.6 * 0.015 + 0.6(0.6-1)\frac{0.01}{2!} + 0.6(0.6-1)(0.6-2)\frac{0.001}{3!}$$

$= 1.0128$

Example: By constructing a difference table and taking the second order differences as constant find the fifth term of the series 8, 12, 19, 29,…….

Solution:

The second differences are constant.

$k - 39 - 3 = 0$

$k - 39 = 3$

⇨ $k = 42$

x	y	Δ	Δ^2	Δ^3	Δ^4
1	8				
		4			
2	12		3		
		7		0	
3	19		3		0
		10		0	
4	29		k-39		
		k-29			
5	k				

Backward differences (∇)

The differences $y_1 - y_0$, $y_2 - y_1$ ...., $y_n - y_{n-1}$ are called *first backward differences* if they are denoted by $\nabla y_1, \nabla y_2, \ldots \ldots, \nabla y_n$ respectively, so that $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1, \ldots \ldots$, $\nabla y_n = y_n - y_{n-1}$, where ∇ is called the *backward difference operator*.

The difference of the first backward differences are called *second backward differences* and denoted by $\nabla^2 y_0, \nabla^2 y_1, \nabla^2 y_2, \ldots \ldots$. Similarly, one can define *third backward differences*, *fourth backward differences*, etc.

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0 \text{ etc.}$$

General form:

$$\boldsymbol{\nabla y_i = y_i - y_{i-1}}, \text{ where } i = 1, 2, 3, \ldots, n-1$$
$$\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}$$

Backward Difference Table

The following table shows how the backward differences of all orders can be formed. The following table is called a *horizontal difference table.*

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		
		∇y_3			$\nabla^4 y_4$
x_3	y_3			$\nabla^3 y_4$	
			$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Figure 5: Horizontal Difference Table

Newton Gregory Backward Difference Formula:

If the required point is close to the end of the table, we can use another formula known as Newton Gregory Backward Difference Formula. Here, the reference point is: 'x_n' instead of x0. Therefore, $s = (x - x_n)/h$.

So, the Newton Gregory Formula is given by:

$$P_3(x) = f_n + s\frac{\Delta f_n}{1!} + s(s+1)\frac{\Delta^2 f_n}{2!} + s(s+1)(s+2)\frac{\Delta^3 f_n}{3!} \ldots.$$

The table for the backward difference will be identical to the forward difference table. However, the reference point will be below the point for which the estimate is required. So, the value of 's' will be negative for backward interpolation.

Example:

Estimate the value of sin(45°) using backward difference method with the following set of data.

X	10	20	30	40
f(x)=sin(x)	0.1736	0.342	0.5	0.768

Solution:

$x_0 = 10$, $x_1 = 20$ ... i.e. $h = x_1 - x_0 = 20 - 10 = 10$

We have given, x = 45. So: $s = (x - x_n)/h = (45 - 50)/10 = -0.5$

$$P_4(x) = f_4 + s\frac{\Delta f_n}{1!} + s(s+1)\frac{\Delta^2 f_n}{2!} + s(s+1)(s+2)\frac{\Delta^3 f_n}{3!} + s(s+1)(s+2)(s+3)\frac{\Delta^4 f_n}{4!}$$

Where, the necessary data are fetched from the following Backward Difference Table.

Backward Difference Table:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
10	0.1736				
		0.1684			
20	0.342		-0.0104		
		0.158		0.0048	
30	0.5		-0.0152		0.0024
		0.1428		0.0024	
40	0.6428		-0.0176		
		0.1252			
50	0.768				

$$P_4(x) = 0.768 + (-0.5) * 0.1252 + (-0.5)(-0.5+1)\frac{(-0.0176)}{2!}$$
$$+ (-0.5)(-0.5+1)(-0.5+2)\frac{0.0024}{3!}$$
$$+ (-0.5)(-0.5+1)(-0.5+2)(-0.5+3)\frac{0.0024}{4!}$$
$$P_4(x) = 0.768 - 0.0626 + 0.0022 - 0.00015 - 0.00009375 = 0.70736$$

Central Differences

The *central difference operator* δ is defined by the relations -

$\delta y_{1/2} = y_1 - y_0$, $\delta y_{3/2} = y_2 - y_1$ .... $\delta y_{n-1/2} = y_n - y_{n-1}$

Similarly,

$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$

$\delta^3 y_{3/2} = \delta^2 y_2 - \delta^2 y_1$

In general,

$\delta y_i = y_{i+1/2} - y_{i-1/2}$

$\delta^n y_i = \delta^{n-1} y_{i+1} - \delta^{n-1} y_{i-½}$

- It is clear from the three tables that in a definite numerical case, the same number occurs in the same position whether we use forward, backward or central differences.

x	y	δ	δ^2	δ^3	δ^4
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_1$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

The Averaging Operator (μ)

The averaging operator μ is defined as

$$\mu y_i = ½(y_{i+1/2} + y_{i-1/2})$$

Shift Operator (E)

The shift operator (also called *displacement operator*) E is defined by the equation $Ey_i = y_{i+1}$ or $E f(x) = f(x+h)$, which shows that the effect of E is to shift the functional value y_i to the next higher value y_{i+1}.

$E^2 y_i = E(Ey_i)$

$Ey_{i+1} = y_{i+2}$
So,
$E^n y_i = y_{i+n}$
$E^n y_0 = y_n$
$E^n f(x) = f(x+nh)$
$E^{-n} f(x) = f(x-nh)$

D Operator
The D operator is defined as
$Dy(x) = d/dx\ y(x)$
$Df(x) = d/dx\ f(x) = f'(x)$
$D^2 f(x) = d^2/dx^2\ f(x) = f''(x)$

Some Important Relations between different operators

1. $E \equiv 1 + \Delta$

$\Delta y_0 = y_1 - y_0 = Ey_0 - y_0 = (E-1)y_0$
$\therefore\ \Delta \equiv E - 1$

2. $\Delta E \equiv E\,\Delta$

$E\Delta y_n = E(y_{n+1} - y_n) = Ey_{n+1} - Ey_n = y_{n+2} - y_{n+1} = \Delta y_{n+1} = \Delta E y_n$
$\therefore \Delta E \equiv E\Delta$

3. $\nabla \equiv 1 - E^{-1}$

$\nabla y_n = y_n - y_{n-1} = y_n - E^{-1}y_n = (1 - E^{-1})y_n$
$\therefore \nabla \equiv 1 - E^{-1}$

4. $\Delta - \nabla \equiv \Delta\nabla$

$\Delta - \nabla \equiv (E - 1) - (1-E^{-1}) \equiv E - 2 + E^{-1}$
$\Delta\nabla \equiv (E - 1)(1-E^{-1}) \equiv E - 2 + E^{-1}$
$\therefore\ \Delta - \nabla \equiv \Delta\nabla \equiv \nabla\Delta$

5. $\Delta \equiv \nabla E \equiv E\,\nabla \equiv \delta\, E^{1/2}$

$\nabla \equiv (1-E^{-1})$
$\therefore\ E\nabla \equiv E(1-E^{-1}) \equiv E - 1 \equiv \Delta$
$\therefore\ \nabla E \equiv (1-E^{-1})E \equiv E - 1 \equiv \Delta$
$\therefore\ \delta \equiv E^{1/2} - E^{-1/2}$
$\delta E^{1/2} \equiv E^{1/2}\,(E^{1/2} - E^{-1/2}) \equiv E - 1 \equiv \Delta$
$\therefore\ \Delta \equiv \nabla E \equiv E\nabla \equiv \delta\, E^{1/2}$

6. $\delta \equiv E^{1/2} - E^{-1/2}$

$\delta y_r = y_{r+1/2} - y_{r-1/2}$
$= E^{1/2}y_r - E^{-1/2}y_r = (E^{1/2} - E^{-1/2})y_r$
$\therefore \delta \equiv E^{1/2} - E^{-1/2}$

7. $\mu \equiv ½(E^{1/2} + E^{-1/2})$

$\mu y_r = \frac{1}{2}(y_{r+1/2} + y_{r-1/2})$

$= \frac{1}{2}(E^{1/2} y_r + E^{-1/2} y_r) = \frac{1}{2}(E^{1/2} + E^{-1/2})y_r$

$\therefore \boldsymbol{\mu \equiv \frac{1}{2}(E^{1/2} + E^{-1/2})}$

8. $\boldsymbol{\mu^2 = 1 + 1/4\, \delta^2}$

$\mu \equiv \frac{1}{2}(E^{1/2} + E^{-1/2})$

$\mu^2 \equiv \frac{1}{2}(E^{1/2} + E^{-1/2})^2 \equiv \frac{1}{4}\{ (E^{1/2} - E^{-1/2})^2 + 4\} \equiv \frac{1}{4}(\delta^2 + 4) \equiv \frac{1}{4}\delta^2 + 1$

$\therefore \mu^2 = 1 + 1/4\, \delta^2$

Home Task:

9. $\boldsymbol{\Delta - \nabla \equiv \delta^2}$

10. $\boldsymbol{E^{1/2} \equiv \mu + \frac{1}{2}\delta}$

11. $\boldsymbol{(1+ \Delta)(1-\nabla) \equiv 1}$

12. $\boldsymbol{\nabla \equiv E^{-1}\Delta}$

13. $\boldsymbol{\delta \equiv E^{-1/2}\, \Delta}$

Error Propagation in a Difference Table

Let the function y(x), defined by the (n+1) points (xi, yi), i = 0, 1, 2 … n, be the continuous and differentiable (n+1) times and let y(x) be approximated by a polynomial of degree not exceeding 'n' such that

$$\phi_n(x_i) = y_i, i = 0,1, \ldots .n$$

Then error is given by: $e = y(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!} y^{n+1}(\gamma), x_0 < \pi < x_n$

Where, $\pi_{n+1}(x) = (x - x0)(x - x1)(x - x2) \ \ldots . (x - xn)$and if L = $\frac{y^{n+1}}{(n+1)!}(\gamma)\ then$:

$$e = y(x) - \phi_n(x) = L\pi_{n+1}(x)$$

Since, y(x) is generally unknown and hence, we do not have any information concerning $y^{(n+1)}(x)$. It is almost useless in practical computation. But, on the other hand, it is extremely useful in theoretical work in different branches of numerical analysis. It is useful in determining errors in Newton Interpolating Formula.

Example: The following is a table of values of a polynomial of degree 5. It is given that *f*(3) is in error. Correct the error.

x	0	1	2	3	4	5	6
y	1	2	33	254	1054	3126	7777

Solution:

It is given that $y = f(x)$ is a polynomial of degree 5

$\therefore \Delta^5 y$ must be constant, $f(3)$ is in error.

Consider the forward difference table and let $f(3) = 254 + \in$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		1				
1	2		30			
		31		$160 + \in$		
2	33		$190 + \in$		$200 - 4\in$	
		$221 + \in$		$360 - 3\in$		$220 + 10\in$
3	$254 + \in$		$550 - 2\in$		$420 + 6\in$	
		$1771 - \in$		$1780 + 3\in$		$20 - 10\in$
4	1.50		$1330 + \in$		$440 - 4\in$	
		2101		$1220 - \in$		
5	3126		12550			
		4651				
6	7777					

Figure 6: Image of example [1]

Since, fifth differences of y are constant so

$(220 + 10 \in) - (20 - 10 \in) = 0$

$(220 + 10 \in) = (20 - 10 \in)$

$20 \in = -200$

$\in = -10$

$f(3) = 254 - 10 = 244$

So the corrected value of $f(3) = 244$

Missing Values:

Let a function $y = f(x)$ be given for equally spaced values $x_0, x_1, x_2, \ldots. x_n$ of the argument and $y_0, y_1, y_2, \ldots\ldots y_n$ denote the corresponding values of the function. If one or more values of $y = f(x)$ are missing we can find the missing values of using the relation between the operators E and **Δ.**

Example: Find the missing entry in the following table

X	*0*	*1*	*2*	*3*	*4*
y_x	*1*	*3*	*9*	--	*81*

Solution:

Given $y_0 = 1$; $y_1 = 3$; $y_2 = 9$ and $y_4 = 81$ and we have find $y_3 = ?$

$\therefore\ \Delta^4 y_0 = 0$

$(E-1)^4 y_0 = 0\ ;\ \Delta \equiv E - 1$

$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1)y_0 = 0;$

$\Rightarrow E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$

$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

$\Rightarrow 81 - 4y_3 + 6 * 9 - 4 * 3 + 1 = 0$

$\therefore\ y_3 = 31$

References:

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