Since $f(z)$ is analytic, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ (Problem 12), and so the above integrals are zero. Then $\oint_C f(z)\,dz = 0$. We are assuming in this derivation that $f'(z)$ [and thus the partial derivatives] are continuous. This restriction can be removed.

(b) Consider any two paths joining points P_1 and P_2 (see Fig. 5-10). By Cauchy's theorem,

$$\int_{P_1AP_2BP_1} f(z)\,dz = 0$$

Then

$$\int_{P_1AP_2} f(z)\,dz + \int_{P_2BP_1} f(z)\,dz = 0$$

or

$$\int_{P_1AP_2} f(z)\,dz = -\int_{P_2BP_1} f(z)\,dz = \int_{P_1BP_2} f(z)\,dz$$

i.e. the integral along P_1AP_2 (path 1) = integral along P_1BP_2 (path 2), and so the integral is independent of the path joining P_1 and P_2.

This explains the results of Problem 18, since $f(z) = z^2$ is analytic.

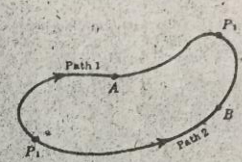

Fig. 5-10

20. If $f(z)$ is analytic within and on the boundary of a region bounded by two closed curves C_1 and C_2 (see Fig. 5-11), prove that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

As in Fig. 5-11, construct line AB (called a *cross-cut*) connecting any point on C_2 and a point on C_1. By Cauchy's theorem (Problem 19),

$$\int_{AQPABRSTBA} f(z)\,dz = 0$$

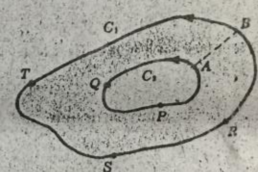

Fig. 5-11

since $f(z)$ is analytic within the region shaded and also on the boundary. Then

$$\int_{AQPA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BRSTB} f(z)\,dz + \int_{BA} f(z)\,dz = 0 \qquad (1)$$

But $\int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz$. Hence (1) gives

$$\int_{AQPA} f(z)\,dz = -\int_{BRSTB} f(z)\,dz = \int_{BTSRB} f(z)\,dz$$

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Note that $f(z)$ need not be analytic *within* curve C_2.

A

21. (a) Prove that $\oint_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & \text{if } n=1 \\ 0 & \text{if } n = 2,3,4,\ldots \end{cases}$ where C is a simple closed curve bounding a region having $z = a$ as interior point.

(b) What is the value of the integral if $n = 0, -1, -2, -3, \ldots$?

(a) Let C_1 be a circle of radius ϵ having center at $z = a$ (see Fig. 5-12). Since $(z-a)^{-n}$ is analytic within and on the boundary of the region bounded by C and C_1, we have by Problem 20,

$$\oint_C \frac{dz}{(z-a)^n} = \oint_{C_1} \frac{dz}{(z-a)^n}$$

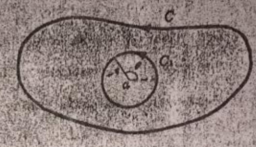

Fig. 5-12

To evaluate this last integral, note that on C_1, $|z-a| = \epsilon$ or $z - a = \epsilon e^{i\theta}$ and $dz = i\epsilon e^{i\theta}\,d\theta$. The integral equals

$$\int_0^{2\pi} \frac{i\epsilon e^{i\theta}\,d\theta}{\epsilon^n e^{in\theta}} = \frac{i}{\epsilon^{n-1}}\int_0^{2\pi} e^{(1-n)i\theta}\,d\theta = \frac{i}{\epsilon^{n-1}} \frac{e^{(1-n)i\theta}}{(1-n)i}\bigg|_0^{2\pi} = 0 \quad \text{if } n \neq 1$$

If $n = 1$, the integral equals $i\int_0^{2\pi} d\theta = 2\pi i$.

(b) For $n = 0, -1, -2, \ldots$, the integrand is $1, (z-a), (z-a)^2, \ldots$ and is analytic everywhere inside C_1 including $z = a$. Hence by Cauchy's theorem the integral is zero.

A 22. Evaluate $\oint_C \frac{dz}{z-3}$ where C is (a) the circle $|z| = 1$, (b) the circle $|z+i| = 4$.

(a) Since $z = 3$ is not interior to $|z| = 1$, the integral equals zero (Problem 19).

(b) Since $z = 3$ is interior to $|z+i| = 4$, the integral equals $2\pi i$ (Problem 21).

23. If $f(z)$ is analytic inside and on a simple closed curve C, and a is any point within C, prove that

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Referring to Problem 20 and the figure of Problem 21, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{C_1} \frac{f(z)}{z-a}\,dz$$

Letting $z - a = \epsilon e^{i\theta}$, the last integral becomes $i\int_0^{2\pi} f(a + \epsilon e^{i\theta})\,d\theta$. But since $f(z)$ is analytic it is continuous. Hence

$$\lim_{\epsilon\to 0} i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta = i\int_0^{2\pi} \lim_{\epsilon\to 0} f(a+\epsilon e^{i\theta})\,d\theta = i\int_0^{2\pi} f(a)\,d\theta = 2\pi i\, f(a)$$

and the required result follows.

A 24. Evaluate (a) $\oint_C \frac{\cos z}{z-\pi}\,dz$, (b) $\oint_C \frac{e^z}{z(z+1)}\,dz$ where C is the circle $|z-1| = 3$.

(a) Since $z = \pi$ lies within C, $\frac{1}{2\pi i}\oint_C \frac{\cos z}{z-\pi}\,dz = \cos\pi = -1$ by Problem 23 with $f(z) = \cos z$, $a = \pi$. Then $\oint_C \frac{\cos z}{z-\pi}\,dz = -2\pi i$.

(b)
$$\oint_C \frac{e^z}{z(z+1)}\,dz = \oint_C e^z\left(\frac{1}{z} - \frac{1}{z+1}\right)dz = \oint_C \frac{e^z}{z}\,dz - \oint_C \frac{e^z}{z+1}\,dz$$
$$= 2\pi i e^0 - 2\pi i e^{-1} = 2\pi i(1 - e^{-1})$$

by Problem 23, since $z = 0$ and $z = -1$ are both interior to C.

A **25.** Evaluate $\oint_C \frac{5z^2-3z+2}{(z-1)^3}\,dz$ where C is any simple closed curve enclosing $z=1$.

Method 1. By Cauchy's integral formula, $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$

If $n=2$ and $f(z) = 5z^2-3z+2$, then $f''(1) = 10$. Hence

$$10 = \frac{2!}{2\pi i}\oint_C \frac{5z^2-3z+2}{(z-1)^3}\,dz \quad \text{or} \quad \oint_C \frac{5z^2-3z+2}{(z-1)^3}\,dz = 10\pi i$$

Method 2. $5z^2-3z+2 = 5(z-1)^2+7(z-1)+4$. Then

$$\oint_C \frac{5z^2-3z+2}{(z-1)^3}\,dz = \oint_C \frac{5(z-1)^2+7(z-1)+4}{(z-1)^3}\,dz$$

$$= 5\oint_C \frac{dz}{z-1} + 7\oint_C \frac{dz}{(z-1)^2} + 4\oint_C \frac{dz}{(z-1)^3} = 5(2\pi i) + 7(0) + 4(0)$$

$$= 10\pi i$$

By Problem 21.

SERIES AND SINGULARITIES

26. For what values of z does each series converge?

(a) $\sum_{n=1}^{\infty} \frac{z^n}{n^2 2^n}$. The nth term $= u_n = \frac{z^n}{n^2 2^n}$. Then

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)^2 2^{n+1}}\cdot\frac{n^2 2^n}{z^n}\right| = \frac{|z|}{2}$$

By the ratio test the series converges if $|z|<2$ and diverges if $|z|>2$. If $|z|=2$ the ratio test fails.

However, the series of absolute values $\sum_{n=1}^{\infty}\left|\frac{z^n}{n^2 2^n}\right| = \sum_{n=1}^{\infty}\frac{|z|^n}{n^2 2^n}$ converges if $|z|=2$, since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Thus the series converges (absolutely) for $|z|\le 2$, i.e. at all points inside and on the circle $|z|=2$.

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$. We have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n z^{2n+1}}{(2n+1)!}\cdot\frac{(2n-1)!}{(-1)^{n-1}z^{2n-1}}\right| = \lim_{n\to\infty}\left|\frac{-z^2}{2n(2n+1)}\right| = 0$$

Then the series, which represents $\sin z$, converges for all values of z.

(c) $\sum_{n=1}^{\infty}\frac{(z-i)^n}{3^n}$. We have $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(z-i)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(z-i)^n}\right| = \frac{|z-i|}{3}$.

The series converges if $|z-i|<3$, and diverges if $|z-i|>3$.

If $|z-i|=3$, then $z-i = 3e^{i\theta}$ and the series becomes $\sum_{n=1}^{\infty} e^{in\theta}$. This series diverges since the nth term does not approach zero as $n\to\infty$.

Thus the series converges within the circle $|z-i|=3$ but not on the boundary.

27. If $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent for $|z| \leqq R$, show that it is uniformly converg for these values of z.

The definitions, theorems and proofs for series of complex numbers and functions are analog to those for real series.

In particular, a series $\sum_{n=0}^{\infty} u_n(z)$ is said to be *absolutely convergent* in a region $\mathcal{R}$ if $\sum_{n=0}^{\infty} |u_n($ converges in $\mathcal{R}$. We can also show that if $\sum_{n=0}^{\infty} |u_n(z)|$ converges in $\mathcal{R}$, then so also does $\sum_{n=0}^{\infty} u_n($ i.e. an absolutely convergent series is convergent.

Also, a series $\sum_{n=0}^{\infty} u_n(z)$ convergent to a sum function $S(z)$ in a region $\mathcal{R}$ is said to be *uniform convergent* in $\mathcal{R}$ if for any $\epsilon > 0$, we can find N such that

$$|S_n(z) - S(z)| < \epsilon \qquad \text{for all } n > N$$

where N depends only on ϵ and not on the particular z in $\mathcal{R}$, and where

$$S_n(z) = u_0(z) + u_1(z) + \cdots + u_n(z)$$

An important test for uniform convergence is the following. *If for all z in $\mathcal{R}$ we can fi constants M_n such that*

$$|u_n(z)| \leqq M_n, \quad n = 0, 1, 2, \ldots \qquad \text{and} \qquad \sum_{n=0}^{\infty} M_n \text{ converges}$$

then $\sum_{n=0}^{\infty} u_n(z)$ converges uniformly in $\mathcal{R}$. This is called the *Weierstrass M test*.

For this particular problem, we have

$$|a_n z^n| \leqq |a_n| R^n = M_n \qquad n = 0, 1, 2, \ldots$$

Since by hypothesis $\sum_{n=0}^{\infty} M_n$ converges, it follows by the Weierstrass M test that $\sum_{n=0}^{\infty} a_n z^n$ converg uniformly for $|z| \leqq R$.

A **28. Locate in the finite z plane all the singularities, if any, of each function and name them**

(a) $\dfrac{z^2}{(z+1)^3}$. $\quad z = -1$ is a pole of order 3.

(b) $\dfrac{2z^3 - z + 1}{(z-4)^2(z-i)(z-1+2i)}$. $\quad z = 4$ is a pole of order 2 (double pole); $z = i$ and $z = 1 - 2i$ a poles of order 1 (simple poles).

$\dfrac{\sin mz}{z^2 + 2z + 2}$, $m \neq 0$. Since $z^2 + 2z + 2 = 0$ when $z = \dfrac{-2 \pm \sqrt{4-8}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$, w can write $z^2 + 2z + 2 = (z - (-1+i))(z - (-1-i)) = (z + 1 - i)(z + 1 + i)$.

The function has the two simple poles: $z = -1 + i$ and $z = -1 - i$.

(d) $\dfrac{1 - \cos z}{z}$. $\quad z = 0$ appears to be a singularity. However, since $\lim_{z \to 0} \dfrac{1 - \cos z}{z} = 0$, it is removable singularity.

Another method.

Since $\dfrac{1-\cos z}{z} = \dfrac{1}{z}\left\{1 - \left(1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots\right)\right\} = \dfrac{z}{2!} - \dfrac{z^3}{4!} + \cdots$, we see that $z = 0$ is a removable singularity.

(e) $e^{-1/(z-1)^2} = 1 - \dfrac{1}{(z-1)^2} + \dfrac{1}{2!\,(z-1)^4} - \cdots$.

This is a Laurent series where the principal part has an infinite number of non-zero terms. Then $z = 1$ is an *essential singularity*.

(f) e^z.

This function has no finite singularity. However, letting $z = 1/u$, we obtain $e^{1/u}$ which has an essential singularity at $u = 0$. We conclude that $z = \infty$ is an essential singularity of e^z.

In general, to determine the nature of a possible singularity of $f(z)$ at $z = \infty$, we let $z = 1/u$ and then examine the behavior of the new function at $u = 0$.

29. If $f(z)$ is analytic at all points inside and on a circle of radius R with center at a, and if $a+h$ is any point inside C, prove *Taylor's theorem* that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$$

By Cauchy's integral formula (Problem 23), we have

$$f(a+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-a-h} \tag{1}$$

By division,

$$\frac{1}{z-a-h} = \frac{1}{(z-a)\,[1-h/(z-a)]}$$

$$= \frac{1}{(z-a)}\left\{1 + \frac{h}{(z-a)} + \frac{h^2}{(z-a)^2} + \cdots + \frac{h^n}{(z-a)^n} + \frac{h^{n+1}}{(z-a)^n\,(z-a-h)}\right\} \tag{2}$$

Substituting (2) in (1) and using Cauchy's integral formulas, we have

$$f(a+h) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-a} + \frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2} + \cdots + \frac{h^n}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}} + R_n$$

$$= f(a) + h f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + R_n$$

where
$$R_n = \frac{h^{n+1}}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}\,(z-a-h)}$$

Now when z is on C, $\left|\dfrac{f(z)}{z-a-h}\right| \leqq M$ and $|z-a| = R$, so that by (14), Page 140, we have, since $2\pi R$ is the length of C,

$$|R_n| \leqq \frac{|h|^{n+1}M}{2\pi R^{n+1}}\cdot 2\pi R$$

As $n \to \infty$, $|R_n| \to 0$. Then $R_n \to 0$ and the required result follows.

If $f(z)$ is analytic in an annular region $r_1 \leqq |z-a| \leqq r_2$, we can generalize the Taylor series to a Laurent series (see Problem 119). In some cases, as shown in Problem 30, the Laurent series can be obtained by use of known Taylor series.

30. Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(a) $\dfrac{e^z}{(z-1)^2}$; $z=1$. Let $z-1=u$. Then $z=1+u$ and

$$\frac{e^z}{(z-1)^2} = \frac{e^{1+u}}{u^2} = e\cdot\frac{e^u}{u^2} = \frac{e}{u^2}\left\{1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\frac{u^4}{4!}+\cdots\right\}$$

$$= \frac{e}{(z-1)^2}+\frac{e}{z-1}+\frac{e}{2!}+\frac{e(z-1)}{3!}+\frac{e(z-1)^2}{4!}+\cdots$$

$z=1$ is a *pole of order 2, or double pole.*

The series converges for all values of $z \neq 1$.

(b) $z\cos\frac{1}{z}$; $z=0$.

$$z\cos\frac{1}{z} = z\left(1-\frac{1}{2!\,z^2}+\frac{1}{4!\,z^4}-\frac{1}{6!\,z^6}+\cdots\right) = z-\frac{1}{2!\,z}+\frac{1}{4!\,z^3}-\frac{1}{6!\,z^5}+\cdots$$

$z=0$ is an *essential singularity.*

The series converges for all values of $z\neq 0$.

(c) $\dfrac{\sin z}{z-\pi}$; $z=\pi$. Let $z-\pi=u$. Then $z=u+\pi$ and

$$\frac{\sin z}{z-\pi} = \frac{\sin(u+\pi)}{u} = -\frac{\sin u}{u} = -\frac{1}{u}\left(u-\frac{u^3}{3!}+\frac{u^5}{5!}-\cdots\right)$$

$$= -1+\frac{u^2}{3!}-\frac{u^4}{5!}+\cdots = -1+\frac{(z-\pi)^2}{3!}-\frac{(z-\pi)^4}{5!}+\cdots$$

$z=\pi$ is a *removable singularity.*

The series converges for all values of z.

(d) $\dfrac{z}{(z+1)(z+2)}$; $z=-1$. Let $z+1=u$. Then

$$\frac{z}{(z+1)(z+2)} = \frac{u-1}{u(u+1)} = \frac{u-1}{u}(1-u+u^2-u^3+u^4-\cdots)$$

$$= -\frac{1}{u}+2-2u+2u^2-2u^3+\cdots$$

$$= -\frac{1}{z+1}+2-2(z+1)+2(z+1)^2-\cdots$$

$z=-1$ is a *pole of order 1, or simple pole.*

The series converges for values of z such that $0<|z+1|<1$.

(e) $\dfrac{1}{z(z+2)^3}$; $z=0,-2$.

Case 1, $z=0$. Using the binomial theorem,

$$\frac{1}{z(z+2)^3} = \frac{1}{8z(1+z/2)^3} = \frac{1}{8z}\left\{1+(-3)\left(\frac{z}{2}\right)+\frac{(-3)(-4)}{2!}\left(\frac{z}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(\frac{z}{2}\right)^3+\cdots\right\}$$

$$= \frac{1}{8z}-\frac{3}{16}+\frac{3}{32}z-\frac{5}{64}z^2+\cdots$$

$z=0$ is a *pole of order 1, or simple pole.*

The series converges for $0<|z|<2$.

Case 2, $z=-2$. Let $z+2=u$. Then

$$\frac{1}{z(z+2)^3} = \frac{1}{(u-2)u^3} = \frac{1}{-2u^3(1-u/2)} = -\frac{1}{2u^3}\left\{1+\frac{u}{2}+\left(\frac{u}{2}\right)^2+\left(\frac{u}{2}\right)^3+\left(\frac{u}{2}\right)^4+\cdots\right\}$$

$$= -\frac{1}{2u^3}-\frac{1}{4u^2}-\frac{1}{8u}-\frac{1}{16}-\frac{1}{32}u-\cdots$$

$$= -\frac{1}{2(z+2)^3}-\frac{1}{4(z+2)^2}-\frac{1}{8(z+2)}-\frac{1}{16}-\frac{1}{32}(z+2)-\cdots$$

$z=-2$ is a *pole of order 3.*

The series converges for $0<|z+2|<2$.

RESIDUES AND THE RESIDUE THEOREM

31. If $f(z)$ is analytic everywhere inside and on a simple closed curve C except at $z=a$ which is a pole of order n so that

$$f(z) = \frac{a_{-n}}{(z-a)^n}+\frac{a_{-n+1}}{(z-a)^{n-1}}+\cdots+a_0+a_1(z-a)+a_2(z-a)^2+\cdots$$

where $a_{-n}\neq 0$, prove that

(a) $\oint_C f(z)\,dz = 2\pi i\,a_{-1}$

(b) $a_{-1} = \lim_{z\to a}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\{(z-a)^n f(z)\}$.

(a) By integration, we have on using Problem 21

$$\oint_C f(z)\,dz = \oint_C \frac{a_{-n}}{(z-a)^n}dz+\cdots+\oint_C\frac{a_{-1}}{z-a}dz+\oint_C\{a_0+a_1(z-a)+a_2(z-a)^2+\cdots\}\,dz$$

$$= 2\pi i\,a_{-1}$$

Since only the term involving a_{-1} remains, we call a_{-1} the *residue* of $f(z)$ at the pole $z=a$.

(b) Multiplication by $(z-a)^n$ gives the Taylor series

$$(z-a)^n f(z) = a_{-n}+a_{-n+1}(z-a)+\cdots+a_{-1}(z-a)^{n-1}+\cdots$$

Taking the $(n-1)$st derivative of both sides and letting $z\to a$, we find

$$(n-1)!\,a_{-1} = \lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\{(z-a)^n f(z)\}$$

from which the required result follows.

A 32. Determine the residues of each function at the indicated poles.

A (a) $\dfrac{z^2}{(z-2)(z^2+1)}$; $z=2,i,-i$. These are simple poles. Then:

Residue at $z=2$ is $\lim_{z\to 2}(z-2)\left\{\frac{z^2}{(z-2)(z^2+1)}\right\} = \frac{4}{5}$.

Residue at $z=i$ is $\lim_{z\to i}(z-i)\left\{\frac{z^2}{(z-2)(z-i)(z+i)}\right\} = \frac{i^2}{(i-2)(2i)} = \frac{1-2i}{10}$.

Residue at $z=-i$ is $\lim_{z\to -i}(z+i)\left\{\frac{z^2}{(z-2)(z-i)(z+i)}\right\} = \frac{i^2}{(-i-2)(-2i)} = \frac{1+2i}{10}$.

A (b) $\dfrac{1}{z(z+2)^3}$; $z = 0, -2$. $z = 0$ is a simple pole, $z = -2$ is a pole of order 3. Then:

$$\text{Residue at } z = 0 \text{ is } \quad \lim_{z\to 0} z \cdot \frac{1}{z(z+2)^3} = \frac{1}{8}$$

$$\text{Residue at } z = -2 \text{ is } \quad \lim_{z\to -2} \frac{1}{2!}\frac{d^2}{dz^2}\left\{(z+2)^3 \cdot \frac{1}{z(z+2)^3}\right\}$$

$$= \lim_{z\to -2} \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{z}\right) = \lim_{z\to -2} \frac{1}{2}\left(\frac{2}{z^3}\right) = -\frac{1}{8}$$

Note that these residues can also be obtained from the coefficients of $1/z$ and $1/(z+2)$ in the respective Laurent series [see Problem 30(c)].

(c) $\dfrac{ze^{zt}}{(z-3)^2}$; $z = 3$, a pole of order 2 or double pole. Then:

A

$$\text{Residue is } \quad \lim_{z\to 3} \frac{d}{dz}\left\{(z-3)^2 \cdot \frac{ze^{zt}}{(z-3)^2}\right\} = \lim_{z\to 3} \frac{d}{dz}(ze^{zt}) = \lim_{z\to 3} (e^{zt} + zte^{zt})$$

$$= e^{3t} + 3te^{3t}$$

(d) $\cot z$; $z = 5\pi$, a pole of order 1. Then:

$$\text{Residue is } \quad \lim_{z\to 5\pi} (z-5\pi)\cdot\frac{\cos z}{\sin z} = \left(\lim_{z\to 5\pi}\frac{z-5\pi}{\sin z}\right)\left(\lim_{z\to 5\pi}\cos z\right) = \left(\lim_{z\to 5\pi}\frac{1}{\cos z}\right)(-1)$$

$$= (-1)(-1) = 1$$

where we have used L'Hospital's rule, which can be shown applicable for functions of a complex variable.

33. If $f(z)$ is analytic within and on a simple closed curve C except at a number of poles $a, b, c, \ldots$ interior to C, prove that

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues of } f(z) \text{ at poles } a, b, c, \text{ etc.)}$$

Refer to Fig. 5-13.

By reasoning similar to that of Problem 20 (i.e. by constructing cross cuts from C to C_1, C_2, C_3, etc.), we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots$$

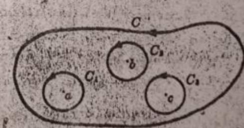

Fig. 5-13

For pole a,

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \cdots + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + \cdots$$

hence, as in Problem 31, $\displaystyle\oint_{C_1} f(z)\,dz = 2\pi i a_{-1}$.

Similarly for pole b, $\displaystyle f(z) = \frac{b_{-n}}{(z-b)^n} + \cdots + \frac{b_{-1}}{(z-b)} + b_0 + b_1(z-b) + \cdots$

so that

$$\oint_{C_2} f(z)\,dz = 2\pi i b_{-1}$$

Continuing in this manner, we see that

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + \cdots) = 2\pi i \text{ (sum of residues)}$$

A 34. Evaluate $\oint_C \frac{e^z\,dz}{(z-1)(z+3)^2}$ where C is given by (a) $|z| = 3/2$, (b) $|z| = 10$.

Residue at simple pole $z = 1$ is $\lim_{z\to 1}\left\{(z-1)\frac{e^z}{(z-1)(z+3)^2}\right\} = \frac{e}{16}$

Residue at double pole $z = -3$ is

$$\lim_{z\to -3}\frac{d}{dz}\left\{(z+3)^2\frac{e^z}{(z-1)(z+3)^2}\right\} = \lim_{z\to -3}\frac{(z-1)e^z - e^z}{(z-1)^2} = \frac{-5e^{-3}}{16}$$

(a) Since $|z| = 3/2$ encloses only the pole $z = 1$,

$$\text{the required integral} = 2\pi i\left(\frac{e}{16}\right) = \frac{\pi i e}{8}$$

(b) Since $|z| = 10$ encloses both poles $z = 1$ and $z = -3$,

$$\text{the required integral} = 2\pi i\left(\frac{e}{16} - \frac{5e^{-3}}{16}\right) = \frac{\pi i(e - 5e^{-3})}{8}$$

EVALUATION OF DEFINITE INTEGRALS

35. If $|f(z)| \leq \frac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 1$ and M are constants, prove that $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ where Γ is the semi-circular arc of radius R shown in Fig. 5-14.

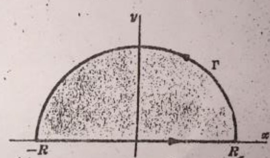

Fig. 5-14

By the result (14), Page 140, we have

$$\left|\int_\Gamma f(z)\,dz\right| \leq \int_\Gamma |f(z)|\,|dz| \leq \frac{M}{R^k}\cdot\pi R = \frac{\pi M}{R^{k-1}}$$

since the length of arc $L = \pi R$. Then

$$\lim_{R\to\infty}\left|\int_\Gamma f(z)\,dz\right| = 0 \quad\text{and so}\quad \lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

36. Show that for $z = Re^{i\theta}$, $|f(z)| \leq \frac{M}{R^k}$, $k > 1$ if $f(z) = \frac{1}{1+z^4}$.

If $z = Re^{i\theta}$, $|f(z)| = \left|\frac{1}{1+R^4e^{4i\theta}}\right| \leq \frac{1}{|R^4e^{4i\theta}| - 1} = \frac{1}{R^4-1} \leq \frac{2}{R^4}$ if R is large enough (say $R > 2$, for example) so that $M = 2$, $k = 4$.

Note that we have made use of the inequality $|z_1 + z_2| \geq |z_1| - |z_2|$ with $z_1 = R^4e^{4i\theta}$ and $z_2 = 1$.

A 37. Evaluate $\int_0^\infty \frac{dx}{x^4+1}$.

Consider $\oint_C \frac{dz}{z^4+1}$, where C is the closed contour of Problem 35 consisting of the line from $-R$ to R and the semi-circle Γ, traversed in the positive (counterclockwise) sense.

Since $z^4 + 1 = 0$ when $z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}$, these are simple poles of $1/(z^4+1)$. Only the poles $e^{\pi i/4}$ and $e^{3\pi i/4}$ lie within C. Then using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/4} = \lim_{z\to e^{\pi i/4}} \left\{(z - e^{\pi i/4})\frac{1}{z^4+1}\right\}$$
$$= \lim_{z\to e^{\pi i/4}} \frac{1}{4z^3} = \frac{1}{4}e^{-3\pi i/4}$$

$$\text{Residue at } e^{3\pi i/4} = \lim_{z\to e^{3\pi i/4}} \left\{(z - e^{3\pi i/4})\frac{1}{z^4+1}\right\}$$
$$= \lim_{z\to e^{3\pi i/4}} \frac{1}{4z^3} = \frac{1}{4}e^{-9\pi i/4}$$

Thus
$$\oint_C \frac{dz}{z^4+1} = 2\pi i\left(\tfrac{1}{4}e^{-3\pi i/4} + \tfrac{1}{4}e^{-9\pi i/4}\right) = \frac{\pi\sqrt{2}}{2} \qquad (1)$$

i.e.
$$\int_{-R}^{R} \frac{dx}{x^4+1} + \int_\Gamma \frac{dz}{z^4+1} = \frac{\pi\sqrt{2}}{2} \qquad (2)$$

Taking the limit of both sides of (2) as $R\to\infty$ and using the results of Problem 36, we have
$$\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{2}$$

Since $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\int_0^\infty \frac{dx}{x^4+1}$, the required integral has the value $\dfrac{\pi\sqrt{2}}{4}$.

38. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

The poles of $\dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$ enclosed by the contour C of Problem 35 are $z = i$ of order 2 and $z = -1+i$ of order 1.

Residue at $z = i$ is $\displaystyle\lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2\frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}$

Residue at $z = -1+i$ is $\displaystyle\lim_{z\to -1+i}(z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$

Then
$$\oint_C \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left\{\frac{9i-12}{100} + \frac{3-4i}{25}\right\} = \frac{7\pi}{50}$$

or
$$\int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R\to\infty$ and noting that the second integral approaches zero by Problem 35, we obtain the required result.

39. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$.

Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i} = \dfrac{z-z^{-1}}{2i}$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ so that
$$\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta} = \oint_C \frac{dz/iz}{5+3\left(\dfrac{z-z^{-1}}{2i}\right)} = \oint_C \frac{2\,dz}{3z^2+10iz-3}$$

where C is the circle of unit radius with center at the origin, as shown in Fig. 5-15 below.

The poles of $\dfrac{2}{3z^2 + 10iz - 3}$ are the simple poles

$$z = \frac{-10i \pm \sqrt{-100 + 36}}{6} = \frac{-10i \pm 8i}{6} = -3i,\ -i/3.$$

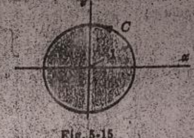

Fig. 5-15

Only $-i/3$ lies inside C.

Residue at $-i/3$ $= \displaystyle\lim_{z \to -i/3} \left(z + \frac{i}{3}\right)\left(\frac{2}{3z^2 + 10iz - 3}\right) = \lim_{z \to -i/3} \frac{2}{6z + 10i} = \frac{1}{4i}$ by L'Hospital's rule.

Then $\displaystyle\oint_C \frac{2\,dz}{3z^2 + 10iz - 3} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}$, the required value.

40. Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta = \frac{\pi}{12}$.

If $z = e^{i\theta}$, $\cos\theta = \dfrac{z + z^{-1}}{2}$, $\cos 3\theta = \dfrac{e^{3i\theta} + e^{-3i\theta}}{2} = \dfrac{z^3 + z^{-3}}{2}$, $dz = iz\,d\theta$.

Then $$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta = \oint_C \frac{(z^3 + z^{-3})/2}{5 - 4\left(\dfrac{z + z^{-1}}{2}\right)}\frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\,dz$$

where C is the contour of Problem 39.

The integrand has a pole of order 3 at $z = 0$ and a simple pole $z = \frac{1}{2}$ within C.

Residue at $z = 0$ is $\displaystyle\lim_{z \to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3 \cdot \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\right\} = \frac{21}{8}$.

Residue at $z = \frac{1}{2}$ is $\displaystyle\lim_{z \to 1/2} \left\{\left(z - \tfrac{1}{2}\right) \cdot \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\right\} = -\frac{65}{24}$.

Then $\displaystyle -\frac{1}{2i}\oint_C \frac{z^6 + 1}{z^3(2z - 1)(z - 2)}\,dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8} - \frac{65}{24}\right\} = \frac{\pi}{12}$ as required.

41. If $|f(z)| \leq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 0$ and M are constants, prove that

$$\lim_{R \to \infty} \int_\Gamma e^{imz} f(z)\,dz = 0$$

where Γ is the semi-circular arc of the contour in Problem 35 and m is a positive constant.

If $z = Re^{i\theta}$, $\displaystyle\int_\Gamma e^{imz} f(z)\,dz = \int_0^\pi e^{imRe^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\,d\theta$.

Then
$$\begin{aligned}
\left|\int_0^\pi e^{imRe^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\,d\theta\right| &\leq \int_0^\pi \left|e^{imRe^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\right| d\theta \\
&= \int_0^\pi \left|e^{imR\cos\theta - mR\sin\theta} f(Re^{i\theta})\, iRe^{i\theta}\right| d\theta \\
&= \int_0^\pi e^{-mR\sin\theta}\, |f(Re^{i\theta})|\, R\,d\theta \\
&\leq \frac{M}{R^{k-1}}\int_0^\pi e^{-mR\sin\theta}\,d\theta = \frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta
\end{aligned}$$

Let $r \to 0$ and $R \to \infty$. By Problem 41, the second integral on the right approaches zero. The first integral on the right approaches

$$-\lim_{r \to 0} \int_{\pi}^{0} \frac{e^{ire^{i\theta}}}{re^{i\theta}} ire^{i\theta}\, d\theta \quad = \quad -\lim_{r \to 0} \int_{\pi}^{0} ie^{ire^{i\theta}}\, d\theta \quad = \quad \pi i$$

since the limit can be taken under the integral sign.

Then we have

$$\lim_{\substack{R \to \infty \\ r \to 0}} 2i \int_{r}^{R} \frac{\sin x}{x}\, dx \;=\; \pi i \qquad \text{or} \qquad \int_{0}^{\infty} \frac{\sin x}{x}\, dx \;=\; \frac{\pi}{2}$$

MISCELLANEOUS PROBLEMS

44. Let $w = z^2$ define a transformation from the z plane (xy plane) to the w plane (uv plane). Consider a triangle in the z plane with vertices at $A(2,1)$, $B(4,1)$, $C(4,3)$. (a) Show that the *image* or *mapping* of this triangle is a curvilinear triangle in the uv plane. (b) Find the angles of this curvilinear triangle and compare with those of the original triangle.

(a) Since $w = z^2$, we have $u = x^2 - y^2$, $v = 2xy$ as the transformation equations. Then point $A(2,1)$ in the xy plane maps into point $A'(3,4)$ of the uv plane (see figures below). Similarly, points B and C map into points B' and C' respectively. The line segments AC, BC, AB of triangle ABC map respectively into parabolic segments $A'C', B'C', A'B'$ of curvilinear triangle $A'B'C'$ with equations as shown in Figures 5-17(a) and (b).

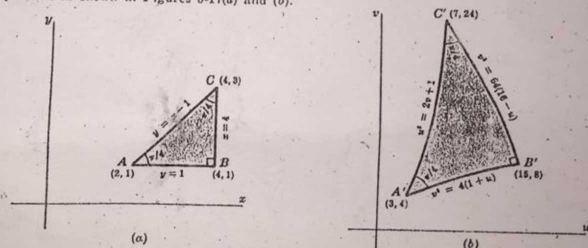

Fig. 5-17

(b) The slope of the tangent to the curve $v^2 = 4(1+u)$ at $(3,4)$ is $m_1 = \left.\dfrac{dv}{du}\right|_{(3,4)} = \left.\dfrac{2}{v}\right|_{(3,4)} = \dfrac{1}{2}$.

The slope of the tangent to the curve $u^2 = 2v + 1$ at $(3,4)$ is $m_2 = \left.\dfrac{dv}{du}\right|_{(3,4)} = u = 3$.

Then the angle θ between the two curves at A' is given by

$$\tan\theta \;=\; \frac{m_2 - m_1}{1 + m_1 m_2} \;=\; \frac{3 - \frac{1}{2}}{1 + (3)(\frac{1}{2})} \;=\; 1, \quad \text{and} \quad \theta = \pi/4$$

Similarly we can show that the angle between $A'C'$ and $B'C'$ is $\pi/4$, while the angle between $A'B'$ and $B'C'$ is $\pi/2$. Therefore the angles of the curvilinear triangle are equal to the corresponding ones of the given triangle. In general, if $w = f(z)$ is a transformation where $f(z)$ is analytic, the angle between two curves in the z plane intersecting at $z = z_0$ has the same magnitude and sense (orientation) as the angle between the images of the two curves, so long as $f'(z_0) \neq 0$. This property is called the conformal property of analytic functions and for this reason the transformation $w = f(z)$ is often called a *conformal transformation* or *conformal mapping function*.

45. Let $w = \sqrt{z}$ define a transformation from the z plane to the w plane. A point moves counterclockwise along the circle $|z| = 1$. Show that when it has returned to its starting position for the first time its image point has not yet returned, but that when it has returned for the second time its image point returns for the first time.

Let $z = e^{i\theta}$. Then $w = \sqrt{z} = e^{i\theta/2}$. Let $\theta = 0$ correspond to the starting position. Then $z = 1$ and $w = 1$ [corresponding to A and P in Figures 5-18(a) and (b)].

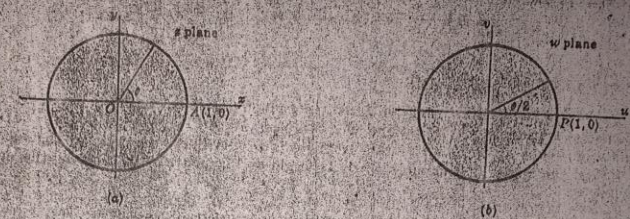

Fig. 5-18

When one complete revolution in the z plane has been made, $\theta = 2\pi$, $z = 1$ but $w = e^{i\theta/2} = e^{i\pi} = -1$ so the image point has not yet returned to its starting position.

However, after two complete revolutions in the z plane have been made, $\theta = 4\pi$, $z = 1$ and $w = e^{i\theta/2} = e^{2\pi i} = 1$ so the image point has returned for the first time.

It follows from the above that w is not a single-valued function of z but is a *double-valued function* of z; i.e. given z, there are two values of w. If we wish to consider it a single-valued function, we must restrict θ. We can, for example, choose $0 \leq \theta < 2\pi$, although other possibilities exist. This represents one branch of the double-valued function $w = \sqrt{z}$. In continuing beyond this interval we are on the second branch, e.g. $2\pi \leq \theta < 4\pi$. The point $z = 0$ about which the rotation is taking place is called a *branch point*. Equivalently, we can insure that $f(z) = \sqrt{z}$ will be single-valued by agreeing not to cross the line Ox, called a *branch line*.

46. Show that $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$, $0 < p < 1$.

Consider $\displaystyle\oint_C \frac{z^{p-1}}{1+z}\,dz$. Since $z = 0$ is a branch point, choose C as the contour of Fig. 5-19 where AB and GH are actually coincident with the x axis but are shown separated for visual purposes.

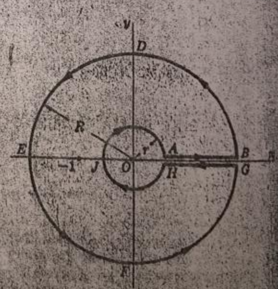

Fig. 5-19

The integrand has the pole $z = -1$ lying within C.

Residue at $z = -1 = e^{\pi i}$ is

$$\lim_{z \to -1} (z+1)\frac{z^{p-1}}{1+z} = (e^{\pi i})^{p-1} = e^{(p-1)\pi i}$$

Then

$$\oint_C \frac{z^{p-1}}{1+z}\,dz = 2\pi i e^{(p-1)\pi i}$$

or, omitting the integrand,

$$\int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i e^{(p-1)\pi i}$$