Iterative Solution of Linear Equations
Find Approximation Solution Ax = b where A€R, when
1. Number of equations n is large
2. A is sparse very few non zero coefficients.
3. A is structured (meaning product of Ax can be calculated efficiently.)

Main Idea: Avoid computation of A^{-1} and perform "cheap operation" Ax.

- Direct method solution poses some problems when the systems are growing larger or when most coefficients are zero.
- They require prohibitively large number of floating point operation and thus not only time-consuming, but also severely affect the solution's accuracy due to round-off errors.

Iterative approaches provide a solution for this, and this approach can also solve ill-conditioned structures without addressing the issue of round-off errors.

Jacobi's Method
Jacobi method is based on the idea same as fixed point iteration method discussed in the chapter "system of non linear equations". Recalling that equation of the form: f(x) = 0 can be rearranged into a form: x = g(x).
The function g(x) can be evaluated iteratively using an initial approximation x as:

$$x_{i+1} = g(x_i) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

Jacobi method extends this idea to a system of equations. It is direct substitution method where the values of unknowns are improved by substituting directly the previous values.

Jacobi Iteration Method:
Let us consider a system of 3 equations of 3 unknowns.

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_n$$

Step 1:
We rewrite the original system as

$$x_1^{(0)} = \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3]$$
$$x_2^{(0)} = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3]$$
$$x_3^{(0)} = \frac{1}{a_{33}}[b_n - a_{31}x_1 - a_{32}x_2]$$

Step 2:
Compute the first approximation $x_1^{(1)}, x_2^{(1)}), x_3^{(1)}$ using $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right]$$
$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_2^{(0)} - a_{13}x_3^{(0)}\right]$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right]$$

Step 3:
Similarly second approximation $x_1^{(2)},\ x_2^{(2)},\ x_3^{(2)})$ evaluated using $x_1^{(1)}$, $x_2^{(1)}$, $x_3^{(1)}$.

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}\right]$$
$$x_2^{(2)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_2^{(1)} - a_{13}x_3^{(1)}\right]$$
$$x_3^{(2)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right]$$

Step n+1:
In general, if $x_1^{(n)}$, $x_2^{(n)},\ldots,x_n^{(n)}$ are a system of n^{th} approximations, then the next approximation is given by the formula:

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}\,x_2^{(n)} - a_{13}\,x_3^{(n)} - \ldots\ldots - a_{1n}\,x_n^{(n)}\right]$$
$$x_2^{(n+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\,x_1^{(n)} - a_{23}\,x_3^{(n)} - \ldots\ldots - a_{2n}\,x_n^{(n)}\right]$$
…. …. …. …. …. …. …. …. .. … … … …
$$x_n^{(n+1)} = \frac{1}{a_{nn}}\left[b_3 - a_{31}\,x_1^{(n)} - a_{32}\,x_2^{(n)} - \ldots\ldots - a_{nn-1}\,x_n^{(n)}\right]$$

In general, an iteration for xi can be obtained from the ith equation as follows

$$x_i^{(k+1)} = \frac{1}{aii}\left[b_i - a_{i1}\,x_1^{(k)} - \ldots. - a_{ii-1}\,x_{i-1}^{(k)} - a_{ii+1}\,x_{i+1}^{(k)} - \ldots - a_{in}\,x_n^{(k)}\right]$$

Algorithm: Jacobi Iteration Method
Choose start point $X^0 = (x_1^0, x_2^0, \ldots\ldots x_n^0)$
For j=0,1,2….. until the stoping criteria
For i = 1, 2, 3,….n

$$x_i^{(k+1)} = \frac{1}{aii}\left[b_i - a_{i1}\,x_1^{(k)} - \ldots. - a_{ii-1}\,x_{i-1}^{(k)} - a_{ii+1}\,x_{i+1}^{(k)} - \ldots - a_{in}\,x_n^{(k)}\right]$$

end
end

Choosing starting vector:
1. All vector $x_i = 1$
2. Best choice $x_i = b_i/a_{ii}$

Stopping Criteria:
1. Maximum number of iterations reached.
2. Reached the tolerance level.

Example: Solve the following system of linear equations by Jacobi's method

$$5x + 2y + z = 12$$
$$x + 4y + 2z = 15$$
$$x + 2y + 5z = 20$$

Solution:

The above system is diagonally dominant i.e. in each equation the absolute value of the largest coefficient is greater than the sum of the absolute values of the other coefficients. The given equations can be written as

$x = 1/5\,[12 - 2y - z]$
$y = 1/4\,[15 - x - 2z]$
$z = 1/5\,[20 - x - 2y]$

We start the iteration by putting $x = 0,\ y = 0,\ z = 0$

$\therefore$ For the first iteration we get

$x^{(1)} = 1/5\,[12 - 0 - 0] = 2.40$
$y^{(1)} = 1/4\,[15 - 0 - 0] = 3.75$
$z^{(1)} = 1/5\,[20 - 0 - 0] = 4.00$

Substituting the values $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$ we get,

$x^{(2)} = 1/5\,[12 - 2(3.75) - 4.00] = 0.10$
$y^{(2)} = 1/4\,[15 - 2.40 - 2(4.00)] = 1.15$
$z^{(2)} = 1/5\,[20 - 2.40 - 2(3.75)] = 2.02$

The iteration process is continued and results are tabulated as follows

Iteration	1	2	3	4	5	6	7	8
X	2.40	0.10	1.54	0.61	1.41	0.80	1.08	1.084
Y	3.75	1.15	1.72	1.17	2.29	1.69	1.95	1.95
Z	4.00	2.02	3.57	2.60	3.41	3.20	3.16	3.164

The values of x, y, z at the end of the 8^{th} iteration are

$x = 1.084, \quad y = 1.95 \quad$ and $\quad z = 3.164$

Gauss–Seidel method

- Gauss-Seidel method is an improved version of Jacobi iteration method.
- In Jacobi method, we begin with the initial values $x_1^{(0)}, x_2^{(0)}, \ldots \ldots x_n^{(0)}$ and obtain next approximation $x_1^{(1)}, x_2^{(1)}, \ldots \ldots x_n^{(1)}$.
- Note that, in computing $x_2^{(1)}$, we used $x_1^{(0)}$ and not $x_1^{(1)}$ which has just been computed. Since, at that point, both $x_1^{(0)}$ and $x_1^{(1)}$ are available, we can use $x_1^{(1)}$ which is a better approximation for computing $x_2^{(1)}$. Similarly, for computing $x_3^{(1)}$ we can use $x_1^{(1)}$ and $x_2^{(1)}$ along with $x_4^{(0)} \ldots, x_n^{(0)}$. This idea can be extended to all subsequent computations. This approach is called the Gauss-Seidel method.

- The Gauss-Seidel method uses the most recent values of x as soon as they become available at any point of iteration process. During the $(k+1)^{th}$ iteration of Gauss-Seidel method, x_i takes the form

$$x_i^{(k+1)} = \frac{1}{a_{11}} [b_i - a_{i1} x_1^{(k+1)} - \dots - a_{ii-1} x_{i-1}^{(k+1)} - a_{ii+1} x_{i+1}^{(k)} - \dots - a_{in} x_n^{(k)}]$$

Condition for Convergence

Same as Jacobi's method. But the convergence in Gauss-Seidel method is more rapid than in Jacobi's method.

Algorithm: Gauss-Seidel Method

Obtain n, a_{ij} and b_i values
Set $x_0 = b_i/a_{ii}$ for i=1,2,…n
Set key =0
For i=1,2,…,n
    Set sum = b_i
    For j=1,2,…n (j is not equal to i)
    sum = sum - $a_{i*j}x_{0j}$
Set x_i = sum/a_{ii}
If key = 0 then
If abs(x_i–x_{0i}/x_i)>error then
If key = 1 then set $x_{0i} = x_i$

Example: Obtain the solution of the following system using Gauss–Seidel iteration method

$$2x + y + z = 5$$
$$3x + 5y + 2z = 15$$
$$2x + y + 4z = 8$$

Solution:

$$x = [5 - y - z]/2$$
$$y = [15 - 3x - 2z]/5$$
$$z = [8 - 2x - y]/4$$

We start the iteration by putting x = 0, y = 0, z = 0

∴ For the first iteration, we get

$$x^{(1)} = [5 - 0 - 0]/2 = 2.50$$
$$y^{(1)} = [15 - 3\times2.5 - 0]/5 = 1.5$$
$$z^{(1)} = [8 - 2\times2.5 - 1.5]/4 = 0.4$$

Substituting the values $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$ we get,

$$x^{(2)} = [5 - 1.5 - 0.4]/2 = 1.6$$
$$y^{(2)} = [15 - 3\times1.6 - 2\times0.4]/5 = 1.9$$
$$z^{(2)} = [8 - 2\times1.6 - 1.9]/4 = 0.7$$

Condition for Convergence

Sufficient condition for convergence for solving one non-linear equation is $|\, g'(x) \,| < 1$. And for two non-linear equations, $f(x1, x2)$ and $g\ (x1, x2)$, are

$$\left.\begin{array}{l} \left|\frac{\partial F}{\partial x_1}\right|+\left|\frac{\partial G}{\partial x_1}\right| < 1 \\ \left|\frac{\partial F}{\partial x_2}\right|+\left|\frac{\partial G}{\partial x_2}\right| < 1 \end{array}\right\} \qquad (1)$$

We can express the Gauss-Seidel algorithm as follows:

$$x1 = f(x1, x2) = \frac{1}{a_{11}}(b1 - a12x2)$$
$$= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x2$$
$$x2 = g\,(x1, x2) = \frac{1}{a_{22}}(b2 - a21x1)$$
$$= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x1$$

The partial derivatives of these equations are

$$\left|\frac{\partial F}{\partial x_1}\right| = 0,\ \left|\frac{\partial F}{\partial x_2}\right| = -\frac{a_{12}}{a_{11}} \qquad \text{and}$$

$$\left|\frac{\partial G}{\partial x_1}\right| = -\frac{a_{21}}{a_{22}},\ \left|\frac{\partial G}{\partial x_2}\right| = 0$$

Substituting these values in first equation

$$\left|\frac{a_{21}}{a_{22}}\right| < 1 \quad \text{and} \quad \left|\frac{a_{12}}{a_{11}}\right| < 1$$

This means that

$$|a_{11}| > |a_{12}| \qquad \text{and} \quad |a_{22}| > |a_{21}|$$

For each row, the absolute value of the diagonal element should be greater than the sum of absolute values of the other elements in the equation. i.e.

$$|\,a_{ii}\,| > \sum_{i=1}^{n} |a_{ij}|; \qquad \text{for } i \neq j$$

- Remember that this condition is sufficient, but not necessary, for convergence. Some systems may converge even if this condition is not satisfied.
- Systems that satisfy the above condition are called *diagonally dominant* systems. Convergence of such systems is guaranteed.

References:

1. **BalaGurushamy, E.** *Numerical Methods.* New Delhi : Tata McGraw-Hill, 2000.

2. **Steven C.Chapra, Raymon P. Cannale.** *Numerical Methods for Engineers.* New Delhi : Tata McGRAW-HILL, 2003. ISMN 0-07-047437-0.