Unique Paths

Solution

There is a robot on an `m x n` grid. The robot is initially located at the **top-left corner** (i.e., `grid[0][0]`). The robot tries to move to the **bottom-right corner** (i.e., `grid[m - 1][n - 1]`). The robot can only move either down or right at any point in time.

Given the two integers `m` and `n`, return *the number of possible unique paths that the robot can take to reach the bottom-right corner*.

The test cases are generated so that the answer will be less than or equal to $2 * 10^9$.

Example 1:


```
Input: m = 3, n = 7
Output: 28
```

Example 2:

```
Input: m = 3, n = 2
Output: 3
Explanation: From the top-left corner, there are a total of 3 ways to reach the bottom-right co
1. Right -> Down -> Down
2. Down -> Down -> Right
3. Down -> Right -> Down
```

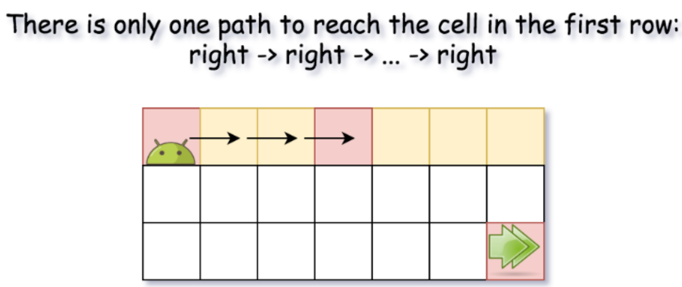
Constraints:

- `1 <= m, n <= 100`

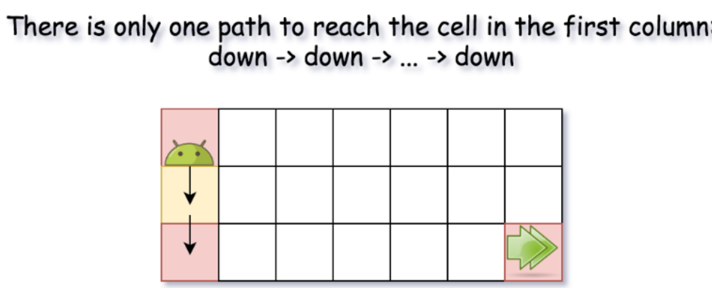
Solution

Overview

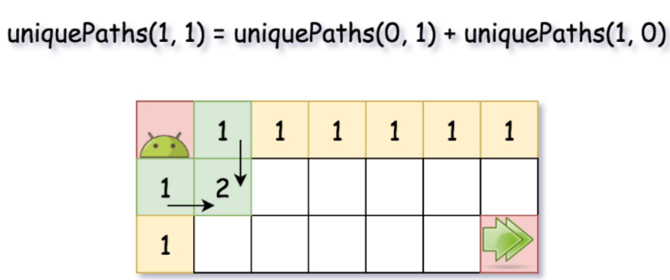
Since robot can move either down or right, there is only one path to reach the cells in the first row: right->right->...->right.



The same is valid for the first column, though the path here is down->down-> ...->down.



What about the "inner" cells `(m, n)`? To such cell one could move either from the cell on the left `(m, n - 1)`, or from the cell above `(m - 1, n)`. That means that the total number of paths to move into `(m, n)` cell is `uniquePaths(m - 1, n) + uniquePaths(m, n - 1)`.



Now, one could transform these ideas into 3-liner recursive solution:

```
class Solution {
 public int uniquePaths(int m, int n) {
  if (m == 1 || n == 1) {
   return 1;
  }
```

```
    return uniquePaths(m - 1, n) + uniquePaths(m, n - 1);
  }
}
```

This solution is not fast enough to pass all the testcases, though it could be used as a starting point for the DP solution.

Approach 1: Dynamic Programming

One could rewrite recursive approach into dynamic programming one.

Algorithm

- Initiate 2D array `d[m][n] = number of paths`. To start, put number of paths equal to 1 for the first row and the first column. For the simplicity, one could initiate the whole 2D array by ones.
- Iterate over all "inner" cells: `d[col][row] = d[col - 1][row] + d[col][row - 1]`.
- Return `d[m - 1][n - 1]`.

Implementation: video

```
class Solution {
  public int uniquePaths(int m, int n) {
    int[][] d = new int[m][n];

    for(int[] arr : d) {
      Arrays.fill(arr, 1);
    }
    for(int col = 1; col < m; ++col) {
      for(int row = 1; row < n; ++row) {
        d[col][row] = d[col - 1][row] + d[col][row - 1];
      }
    }
    return d[m - 1][n - 1];
  }
}
```

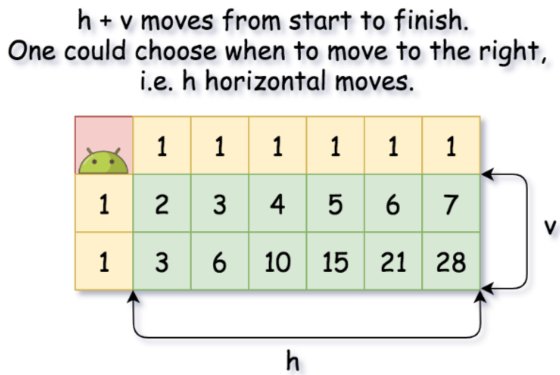
Complexity Analysis

- Time complexity: $\mathcal{O}(N \times M)$.
- Space complexity: $\mathcal{O}(N \times M)$.

Approach 2: Math (Python3 only)

Could one do better than $\mathcal{O}(N \times M)$? The answer is yes.

The problem is a classical combinatorial problem: there are $h + v$ moves to do from start to finish, $h = m - 1$ horizontal moves, and $v = n - 1$ vertical ones. One could choose when to move to the right, i.e. to define h horizontal moves, and that will fix vertical ones. Or, one could choose when to move down, i.e. to define v vertical moves, and that will fix horizontal ones.



In other words, we're asked to compute in how many ways one could choose p elements from $p + k$ elements. In mathematics, that's called binomial coefficients

$C_{h+v}^{h} = C_{h+v}^{v} = \frac{(h+v)!}{h!v!}$

The number of horizontal moves to do is $h = m - 1$, the number of vertical moves is $v = n - 1$. That results in a simple formula

$C_{h+v}^{h} = \frac{(m+n-2)!}{(m-1)!(n-1)!}$

The job is done. Now time complexity will depend on the algorithm to compute factorial function $(m + n - 2)!$. In short, standard computation for $k!$ using the definition requires $\mathcal{O}(k^2 \log k)$ time, and that will be not as good as DP algorithm.

The best known algorithm to compute factorial function is done by Peter Borwein. The idea is to express the factorial as a product of prime powers, so that $k!$ can be computed in $\mathcal{O}(k(\log k \log \log k)^2)$ time. That's better than $\mathcal{O}(k^2)$ and hence beats DP algorithm.

The authors prefer not to discuss here various factorial function implementations, and hence provide Python3 solution only, with built-in divide and conquer factorial algorithm. If you're interested in factorial algorithms, please check out good review on this page.

Implementation

Code:

```
from math import factorial
class Solution:
    def uniquePaths(self, m: int, n: int) -> int:
```

return factorial(m + n - 2) // factorial(n - 1) // factorial(m - 1)

Complexity Analysis

- Time complexity: $\mathcal{O}((M+N)(\log(M+N)\log\log(M+N))^2)$.
- Space complexity: $\mathcal{O}(1)$.

Comment:

In Python 3.8, there's a new function that calculates n-choose-k directly:

```
def uniquePaths(self, m: int, n: int) -> int:
    return math.comb(m + n - 2, n - 1)
```