Homework 6

CpE 4903 Neural Networks and Machine Learning

Learning XOR Function Using a 2-Layer Neural Network

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1. 8 points **Forward Propagation Matrices**
With the 2-layer NN and the initial weights, please write down the following matrices.

   (a) Sample matrix, X, that consists of the 4 input training samples. What is the shape?

   Solution:

$$X = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

   The shape of the matrix will be `(2, 4)`, where the number of rows represents the number of features, and the columns represent the number of samples.

   (b) Output target vector, Y, that consists of the 4 labeled output values.

   Solution:

$$Y = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

   (c) Weight matrix, $W^{[1]}$, and the bias vector, $b^{[1]}$, for the hidden layer. What are their shapes

   Solution: $W^{[1]}$ is a `(2, 2)` matrix.

$$W^{[1]} = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$$

$$b^{[1]} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$b^{[1]}$ is a `(2, 1)` matrix/vector.

(d) Weight matrix, $W^{[2]}$, and the bias vector, $b^{[2]}$, for the hidden layer. What are their shapes

Solution: $W^{[2]}$ is a `(1, 2)` matrix.

$$W^{[2]} = \begin{bmatrix} 7 & -6 \end{bmatrix}$$

$$b^{[2]} = \begin{bmatrix} -3 \end{bmatrix}$$

$b^{[2]}$ is a `(1, 1)` matrix/vector.

2. $\boxed{\text{12 points}}$ **Forward Propagation**. Weighted Sum and Activation Output for Each Neuron

(a) Calculate $Z^{[1]}$, $A^{[1]}$, $Z^{[2]}$, and $A^{[2]}$.

Solution: Weighted sum for the first layer:

$$Z^{[1]} = W^{[1]}X + b^{[1]}$$

$$Z^{[1]} = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$Z^{[1]} = \begin{bmatrix} 0 & 2 & 3 & 5 \\ 0 & 5 & 4 & 9 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$Z^{[1]} = \begin{bmatrix} -1 & 1 & 2 & 4 \\ -2 & 3 & 2 & 7 \end{bmatrix}$$

Activation output of the first layer:

$$A^{[1]} = \begin{bmatrix} 0.2689 & 0.7311 & 0.8808 & 0.9820 \\ 0.1192 & 0.9526 & 0.8808 & 0.9991 \end{bmatrix}$$

Weighted sum for the second layer:

$$Z^{[2]} = W^{[2]}A^{[1]} + b^{[2]}$$

$$Z^{[2]} = \begin{bmatrix} 7 & -6 \end{bmatrix} \begin{bmatrix} 0.2689 & 0.7311 & 0.8808 & 0.9820 \\ 0.1192 & 0.9526 & 0.8808 & 0.9991 \end{bmatrix} + \begin{bmatrix} -3 \end{bmatrix}$$

$$Z^{[2]} = \begin{bmatrix} 1.1671 & -0.5979 & 0.8808 & 0.8794 \end{bmatrix} + \begin{bmatrix} -3 \end{bmatrix}$$

$$Z^{[2]} = \begin{bmatrix} -1.8329 & -3.5979 & -2.1192 & -2.1206 \end{bmatrix}$$

Activation output for the second layer:

$$A^{[2]} = \begin{bmatrix} 0.1379 & 0.0267 & 0.1072 & 0.1071 \end{bmatrix}$$

(b) Calculate the cost function, $J(W, b)$.

Solution:

$$J(W,b) = -\frac{1}{4} \times \left(Y^T \cdot \log(A^{[2]}) + (1 - Y^T) \cdot \log(1 - A^{[2]}) \right)$$

$$J(W,b) = -\frac{1}{4} \times \left(\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1.9813 \\ -3.623 \\ -2.233 \\ 2.234 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.1484 \\ -0.0271 \\ -0.113 \\ -0.113 \end{bmatrix} \right)$$

$$J(W,b) = -\frac{1}{4} \times \left(-3.623 - 2.233 - 0.1484 - 0.113 \right) = 1.5294$$

3. $\boxed{\text{10 points}}$ **Back Propagation: Calculating Gradient Parameters**.
Compute one iteration of gradient descent.

Solution: Equation 1:

$$dZ^{[2]} = A^{[2]} - Y$$

$$dZ^{[2]} = \begin{bmatrix} 0.1379 & 0.0267 & 0.1072 & 0.1071 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}$$

$$dZ^{[2]} = \begin{bmatrix} 0.1379 & -0.9733 & -0.8928 & 0.1071 \end{bmatrix}$$

Equation 2:

$$dW^{[2]} = \frac{1}{m} dZ^{[2]} A^{[1]^T}$$

$$dW^{[2]} = \frac{1}{4} \left(\begin{bmatrix} 0.1379 & -0.9733 & -0.8928 & 0.1071 \end{bmatrix} \begin{bmatrix} 0.2689 & 0.1192 \\ 0.7311 & 0.9526 \\ 0.8808 & 0.8808 \\ 0.9820 & 0.9991 \end{bmatrix} \right)$$

$$dW^{[2]} = \frac{1}{4} \left(\begin{bmatrix} -1.3557 & -1.5901 \end{bmatrix} \right) = \begin{bmatrix} -0.3339 & -0.3975 \end{bmatrix}$$

Equation 3:

$$db^{[2]} = \frac{1}{m} \left(\texttt{np.sum(}dZ^{[2]}\texttt{, axis=1, keepdims=True)} \right) = -0.40523$$

Equation 4:

$$dZ^{[1]} = W^{[2]^T} dZ^{[2]} * f^{[1]\prime}(Z^{[1]})$$

$$dZ^{[1]} = \begin{bmatrix} 7 \\ -6 \end{bmatrix} \begin{bmatrix} 0.1379 & -0.9733 & -0.8928 & 0.1071 \end{bmatrix} * f^{[1]\prime}(Z^{[1]})$$

$$f^{[1]\prime}(Z^{[1]}) = Z^{[1]}(1 - Z^{[1]})$$

$$f^{[1]\prime}(Z^{[1]}) = \begin{bmatrix} 0.2689 & 0.7311 & 0.8808 & 0.9820 \\ 0.1192 & 0.9523 & 0.8808 & 0.9991 \end{bmatrix} * \begin{bmatrix} 0.7311 & 0.2689 & 0.1192 & 0.0180 \\ 0.8808 & 0.0477 & 0.1192 & 9.111 \times 10^{-4} \end{bmatrix}$$

$$f^{[1]\prime}(Z^{[1]}) = \begin{bmatrix} 0.1966 & 0.1966 & 0.1050 & 0.0177 \\ 0.1050 & 0.0454 & 0.1050 & 9.1028 \times 10^{-4} \end{bmatrix}$$

$$dZ^{[1]} = \begin{bmatrix} 0.9653 & -6.8131 & -6.2469 & 0.7497 \\ -0.8274 & 5.8398 & 5.3568 & -0.6426 \end{bmatrix} * \begin{bmatrix} 0.1966 & 0.1966 & 0.1050 & 0.0177 \\ 0.1050 & 0.0454 & 0.1050 & 9.1028 \times 10^{-4} \end{bmatrix}$$

$$dZ^{[1]} = \begin{bmatrix} 0.1898 & -1.3395 & -0.6559 & 0.01327 \\ -0.08688 & 0.2651 & 0.5625 & -5.8495 \times 10^{-4} \end{bmatrix}$$

Equation 5:

$$dW^{[1]} = \frac{1}{m} dZ^{[1]} X^T$$

$$dW^{[1]} = \frac{1}{4}\left(\begin{bmatrix} 0.1898 & -1.3395 & -0.6559 & 0.01327 \\ -0.08688 & 0.2651 & 0.5625 & -5.8495 \times 10^{-4} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \right)$$

$$dW^{[1]} = \frac{1}{4}\left(\begin{bmatrix} -0.6426 & -1.3262 \\ 0.5619 & 0.2645 \end{bmatrix} \right) = \begin{bmatrix} -0.1607 & -0.3316 \\ 0.1405 & 0.0661 \end{bmatrix}$$

Equation 6:

$$db^{[1]} = \frac{1}{m}\big(\texttt{np.sum(}dZ^{[1]}\texttt{, axis=1, keepdims=True)}\big) = \frac{1}{4}\left(\begin{bmatrix} -1.7923 \\ 0.7401 \end{bmatrix} \right)$$

$$db^{[1]} = \begin{bmatrix} -0.4481 \\ 0.1850 \end{bmatrix}$$