Homework 2

CpE 4903 Neural Networks and Machine Learning

Univariate Linear Regression - Part 1

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1. $\boxed{\text{20 points}}$ Given the provided dataset, please calculate the Mean Squared Error (MSE) cost function manually for the following sets of weights:

   (a) $a_0 = 0, a_1 = 13$

   Solution: The MSE cost function $J(A)$ is defined as:

   $$J(A) = \frac{1}{2m} \sum_{i=1}^{m} [a_0 + a_1 x^{(i)} - y^{(i)}]^2$$

   From the dataset, we know that $m = 4, x = [8, 10, 12, 14], y = [100, 130, 150, 180]$. Substitute the values in the cost function:

   $$J(A) = \frac{1}{2 \times 4} \sum_{i=1}^{4} [13x^{(i)} - y^{(i)}]^2$$

   Expand the summation:

   $$J(A) = \frac{1}{2 \times 4} \left(16 + 0 + 36 + 4 \right)$$

   $$J(A) = \frac{1}{2 \times 4} \left(56 \right)$$

   $$J(A) = 7$$

   (b) $a_0 = 0, a_1 = 16$

Solution: Substitute the values in the cost function:

$$J(A) = \frac{1}{2 \times 4} \sum_{i=1}^{4} [16x^{(i)} - y^{(i)}]^2$$

Expand the summation:

$$J(A) = \frac{1}{2 \times 4} \Big(784 + 900 + 1764 + 1936 \Big)$$

$$J(A) = \frac{1}{2 \times 4} \Big(5384 \Big)$$

$$J(A) = 673$$

(c) $a_0 = 10, a_1 = 10$

Solution: Substitute the values in the cost function:

$$J(A) = \frac{1}{2 \times 4} \sum_{i=1}^{4} [10 + 10x^{(i)} - y^{(i)}]^2$$

Expand the summation:

$$J(A) = \frac{1}{2 \times 4} \Big(100 + 400 + 400 + 900 \Big)$$

$$J(A) = \frac{1}{2 \times 4} \Big(1800 \Big)$$

$$J(A) = 225$$

2. $\boxed{\text{20 points}}$ Using the Normal Equation, please find the optimal weight values for the univariate linear regression model. In the lecture notes, we mentioned that this is equivalent to solving the following two linear equations for a_0 and a_1. You can use matrix inversion, however, it is not necessary in this simple case.

Solution: The normal equation can be used to calculate the weights:

$$\left(\sum_{i=1}^{4} i \right) a_0 + \left(\sum_{i=1}^{4} x^{(i)} \right) a_1 = \sum_{i=1}^{4} y^{(i)}$$

$$\left(\sum_{i=1}^{4} x^{(i)} \right) a_0 + \left(\sum_{i=1}^{4} (x^{(i)})^2 \right) a_1 = \sum_{i=1}^{4} x^{(i)} y^{(i)}$$

Simplify this system of linear equations by solving the summations:

$$(1+2+3+4)a_0 + (8+10+12+14)a_1 = (100+130+150+180)$$

$$(8+10+12+14)a_0 + (8^2+10^2+12^2+14^2)a_1 = 8(100)+10(130)+12(150)+14(180)$$

This can then be further simplified to:

$$10a_0 + 44a_1 = 560$$

$$44a_0 + 504a_1 = 6420$$

This system can be represented in the form of matrices as well:

$$\begin{bmatrix} 10 & 44 \\ 44 & 504 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 560 \\ 6420 \end{bmatrix}$$

Let the coefficient matrix be denoted by A, the unknown's matrix be X, and the resultant matrix be B. So, this system can be written as $AX = B$.

To find the solution to this system, we need to make sure that A is invertible; we can do it by checking the determinant. If it's not zero, then it's invertible.

$$det(A) = (10 \times 504) - (44 \times 44) = 3104 \neq 0$$

So now, the system can be written as:

$$X = \frac{1}{det(A)} A^{-1} B$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{3104} \begin{bmatrix} 504 & -44 \\ -44 & 10 \end{bmatrix} \begin{bmatrix} 560 \\ 6420 \end{bmatrix}$$

Multiply the matrices:

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{3104} \begin{bmatrix} (504 \times 560) + (-44 \times 6420) \\ (-44 \times 560) + (10 \times 6420) \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{3104} \begin{bmatrix} -240 \\ 39560 \end{bmatrix} = \begin{bmatrix} -\frac{15}{194} \\ \frac{4945}{388} \end{bmatrix}$$

Hence the weights should have the following values:

$$a_0 = -\frac{15}{194},\ a_1 = \frac{4945}{388}$$