Exam Presentation
Life Insurance Mathematics

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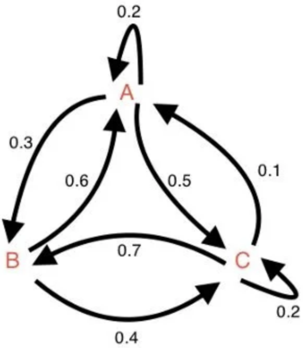
20/01/2026

Markov Chains

What is a Markov Chain?
It is a Stochastic model that describes sequence of transitions from one state to another according to certain probabilistic rules.



Markow Chains

Definition

$(X_t)_{t\in\mathbb{N}} : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{S} = \{1, 2, 3, \ldots\}$, is called a **Markov chain** if and only if,

$$\mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_1} = i_1, \ldots, X_{t_m} = i_m] = \mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_m} = i_m]$$

for $t_1 < t_2 < \cdots < t_m < t_{m+1}$ and $i_1, i_2, \ldots, i_{m+1} \in \mathcal{S}$.

We say that such a stochastic process $(X_t)_{t\in\mathbb{N}}$ has no memory.

Markov Chains

Chapman-Kolmogorov Theorem

Let $p_{ij}(s,t) = P(X_t = j | X_s = i)$ be the transition probabilities of a Markov chain. Then, for any $0 \le s < u < t$,

$$p_{ij}(s,t) = \sum_k p_{ik}(s,u) p_{kj}(u,t).$$

Or written in matrix form, $P(s,t) = P(s,u)P(u,t)$.

Idea: What is the probbility of being in state j at time t, given that at time s we are in state i?

Markov Chains

Proof

$$
\begin{aligned}
p_{ij}(s,t) = \mathbb{P}[X_t = j | X_s = i] &= \mathbb{P}[X_t = j \cap \bigcup_{k \in \mathcal{S}} \{X_u = k\} | X_s = i] \\
&= \sum_{k \in \mathcal{S}} \mathbb{P}\left[X_t = j, X_u = k | X_s = i\right] \\
&= \sum_{k \in \mathcal{S}} \frac{\mathbb{P}[X_t = j, X_u = k, X_s = i]}{\mathbb{P}[X_s = i]} \cdot \frac{\mathbb{P}[X_u = k, X_s = i]}{\mathbb{P}[X_u = k, X_s = i]} \\
&= \sum_{k \in \mathcal{S}} \underbrace{\mathbb{P}[X_t = j | X_u = k, X_s = i]}_{\mathbb{P}[X_t = j | X_u = k]} \mathbb{P}[X_u = k | X_s = i] \\
&= \sum_{k \in \mathcal{S}} p_{ik}(s,u) p_{kj}(u,t)
\end{aligned}
$$

In the above we used $\mathbb{P}[A \cap B | C] = \mathbb{P}[A | B \cap C] \cdot \mathbb{P}[B | C]$ as well as the Markov porperty as well as assuming that $\mathbb{P}[X_u = k, X_s = i] \neq 0$.

Markov Model

To model a life Insurance we need three ingredients:

- a Markov chain $(X_t)_{tin\mathbb{N}}$
- a one-year discount factor $v = \frac{1}{1+i}$
- contract functions $a_i^{pre}(t)$ and $a_{ij}^{post}(t)$

The starting point of an Markov model are the various possible conditions for an insured person, building the state space $\mathcal{S}$. E.g. $\mathcal{S} = \{\text{'living'}, \text{'death'}\}$.

Induced Cashflow & Mathemtical Reserve

A central task in life insurance is the determination of the actuarial reserve, i.e., the amount of money which has to be set aside at a given time t to be able to meet all future obligations/benefits towards each policy.

We denote by A_t the payments that are due for a policy at time t.

$(A_t)_{t \in \mathbb{N}}$ is a stochastic process.

$$A_t = a_{X_t}^{Pre}(t) + a_{X_{t-1}X_t}^{Post}(t)$$

where $a_{ij}^{Post}(-1) = 0$ for all i,j $\in \mathcal{S}$, $t \in \mathbb{N}$.

Induced Cashflow & Mathemtical Reserve

We set $I_i(t) = \mathbb{1}_{\{X_t = i\}}$. Then we can compute the **induced cash flows** as follows:

$$A(t) = \underbrace{\sum_{i \in \mathcal{S}} I_i(t) \cdot a_i^{pre}(t)}_{\text{annuity}} + \underbrace{\sum_{i,j \in \mathcal{S}} I_i(t) \cdot I_j(t+1) \cdot a_{ij}^{post}(t)}_{\text{captial/lump sum paid at time t+1}}$$

Idea: $A(t)$ are the payments are due at time t for a given policy. We can also compute the present value (PV) of A(t) which is given by:

$$\tilde{A}(t) = \sum_{i \in \mathcal{S}} I_i(t) \cdot a_i^{pre}(t) + v \cdot \sum_{i,j \in \mathcal{S}} I_i(t) \cdot I_j(t+1) \cdot a_{ij}^{post}(t)$$

Finally we can define the mathematical reserve at time t as:

$$V_j(t) = \mathbb{E}[\text{PV of future cash flows} |\ X_t = j] = \mathbb{E}[\sum_{\tau=0}^{\infty} \tilde{A}(t+\tau) | X_t = j]$$

Mathematical Reserve

We can compute the mathematical reserves with the following results:

$$\mathbb{E}[I_i(t+\tau)|\ X_t = j] = p_{ji}(t, t+\tau)$$
$$\mathbb{E}[I_i(t+\tau)\ I_k(t+\tau+1)|\ X_t = j] = p_{ji}(t, t+\tau) p_{ik}(t+\tau, t+\tau+1)$$

Hence the reserve is given as:

$$V_j(t) = \sum_{\tau=0}^{\infty}$$

Thiele Equation

Theorem (Thiele's difference equation)

The mathematical reserve between two subsequent periods are related by:

$$V_i(t) = a_i^{pre}(t) + \sum_{j \in \mathcal{S}} v \cdot p_{ij}(t, t+1) \cdot (a_{ij}^{post}(t) + V_j(t+1))$$

Thiele Equation

Proof.

We start

Equivalence Principle

Death & Pure Endowment Insurance

Mathematical Reserves

Mathematical Reserves at given time = PV(future benefits) - PV(future premiums)

Expressed in commutation functions:

$$ {}_tV_x = \frac{M_{x+t} - M_{x+n} + D_{x+n} - \Pi \cdot (N_{x+t} - N_{x+n})}{D_{x+t}} $$

Problem