Exam Presentation
Life Insurance Mathematics

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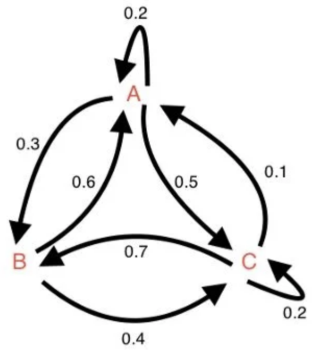
1. **Task 1: Markov Model**
2. **Task 2 : Stopping to pay Premium**
3. **Task 3: Disability Insurance on two lives**

Task 1: Markov Model

Markov Chains

What is a Markov Chain?
It is a Stochastic model that describes sequence of transitions/ possible events in which the probability of each event depends only on the state attained in the previous event.



Markow Chains

Definition

$(X_t)_{t\in\mathbb{N}} : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{S} = \{1, 2, 3, \ldots\}$, is called a **Markov chain** if and only if,

$$\mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_1} = i_1, \ldots, X_{t_m} = i_m] = \mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_m} = i_m]$$

for $t_1 < t_2 < \cdots < t_m < t_{m+1}$ and $i_1, i_2, \ldots, i_{m+1} \in \mathcal{S}$.

We say that such a stochastic process $(X_t)_{t\in\mathbb{N}}$ has no memory.

Markov Chains

Chapman-Kolmogorov Theorem

Let $p_{ij}(s,t) = P(X_t = j | X_s = i)$ be the transition probabilities of a Markov chain. Then, for any $0 \leq s < u < t$,

$$p_{ij}(s,t) = \sum_k p_{ik}(s,u) p_{kj}(u,t).$$

Or written in matrix form, $P(s,t) = P(s,u)P(u,t)$.

Idea: What is the probbility of being in state j at time t, given that at time s we are in state i?

Markov Chains

Proof

$$
\begin{aligned}
p_{ij}(s,t) = \mathbb{P}[X_t = j | X_s = i] &= \mathbb{P}[X_t = j \cap \bigcup_{k \in \mathcal{S}} \{X_u = k\} | X_s = i] \\
&= \sum_{k \in \mathcal{S}} \mathbb{P}\left[X_t = j, X_u = k | X_s = i\right] \\
&= \sum_{k \in \mathcal{S}} \frac{\mathbb{P}[X_t = j, X_u = k, X_s = i]}{\mathbb{P}[X_s = i]} \cdot \frac{\mathbb{P}[X_u = k, X_s = i]}{\mathbb{P}[X_u = k, X_s = i]} \\
&= \sum_{k \in \mathcal{S}} \underbrace{\mathbb{P}[X_t = j | X_u = k, X_s = i]}_{\mathbb{P}[X_t = j | X_u = k]} \mathbb{P}[X_u = k | X_s = i] \\
&= \sum_{k \in \mathcal{S}} p_{ik}(s,u) p_{kj}(u,t)
\end{aligned}
$$

In the above we used $\mathbb{P}[A \cap B | C] = \mathbb{P}[A | B \cap C] \cdot \mathbb{P}[B | C]$ as well as the Markov porperty as well as assuming that $\mathbb{P}[X_u = k, X_s = i] \neq 0$.

Markov Model

To model a life Insurance we need three ingredients:

- a Markov chain $(X_t)_{tin\mathbb{N}}$
- a one-year discount factor $v = \frac{1}{1+i}$
- contract functions $a_i^{pre}(t)$ and $a_{ij}^{post}(t)$

The starting point of an Markov model are the various possible conditions for an insured person, building the state space $\mathcal{S}$. E.g. $\mathcal{S} = \{\text{'living'}, \text{'death'}\}$.

Induced Cashflow & Mathemtical Reserve

A central task in life insurance is the determination of the actuarial reserve, i.e., the amount of money which has to be set aside at a given time t to be able to meet all future obligations/benefits towards each policy.
We denote by A_t the payments that are due for a policy at time t.
$(A_t)_{t\in\mathbb{N}}$ is a stochastic process.

$$A_t = a_{X_t}^{Pre}(t) + a_{X_{t-1}X_t}^{Post}(t)$$

where $a_{ij}^{Post}(-1) = 0$ for all i,j $\in \mathcal{S}$, $t \in \mathbb{N}$.

Induced Cashflow & Mathemtical Reserve

We set $I_i(t) = \mathbb{1}_{\{X_t = i\}}$. Then we can compute the **induced cash flows** as follows:

$$A(t) = \underbrace{\sum_{i \in \mathcal{S}} I_i(t) \cdot a_i^{pre}(t)}_{\text{annuity}} + \underbrace{\sum_{i,j \in \mathcal{S}} I_i(t) \cdot I_j(t+1) \cdot a_{ij}^{post}(t)}_{\text{captial/lump sum paid at time t+1}}$$

Idea: $A(t)$ are the payments are due at time t for a given policy. We can also compute the present value (PV) of A(t) which is given by:

$$\tilde{A}(t) = \sum_{i \in \mathcal{S}} I_i(t) \cdot a_i^{pre}(t) + v \cdot \sum_{i,j \in \mathcal{S}} I_i(t) \cdot I_j(t+1) \cdot a_{ij}^{post}(t)$$

Finally we can define the mathematical reserve at time t as:

$$V_j(t) = \mathbb{E}[\text{PV of future cash flows} |\ X_t = j] = \mathbb{E}[\sum_{\tau=0}^{\infty} \tilde{A}(t+\tau) | X_t = j]$$

Mathematical Reserve

We can compute the mathematical reserves with the following results:

$$\mathbb{E}[I_i(t+\tau)|\ X_t = j] = p_{ji}(t, t+\tau)$$
$$\mathbb{E}[I_i(t+\tau)\ I_k(t+\tau+1)|\ X_t = j] = p_{ji}(t, t+\tau) p_{ik}(t+\tau, t+\tau+1)$$

Hence the reserve is given as:

$$V_j(t) = \sum_{\tau=0}^{\infty} v^{\tau} \left(\sum_{i \in \mathcal{S}} a_i^{Pre}(i+\tau) p_{ji}(t, t+\tau) \right.$$
$$\left. + \ v \sum_{i,k \in \mathcal{S}} a_{ik}^{Post}(t+\tau) p_{ji}(t, t+\tau) p_{ik}(t+\tau, t+\tau+1) \right) \quad (1)$$

Hence $V_j(t)$ is the current value of the future actuarial reserve cash flow (A) based on todays information, i.e. $X_t = j$.

Thiele Equation

We can relate the mathematical reserves at two subsequent time points t and t+1 via the following equation:

Theorem (Thiele's difference equation)

The mathematical reserve between two subsequent periods are related by:

$$V_j(t) = a_j^{pre}(t) + \sum_{i \in \mathcal{S}} v \cdot p_{ji}(t, t+1) \cdot (a_{ji}^{post}(t) + V_i(t+1))$$

Remarks:

- Calculates the expected reserve directly using transition probabilities, while simulation estimates it by averaging over many random trajectories
- To solve Thiele's equation we need the boundary condition $V_j(T) = 0$ for all $j \in \mathcal{S}$.
- The Thiele equation is leads to the same results as for the classical insurance model (using commutation functions)
- Forward computation of reserves is possible, too. but numerically less stable.

Proof Thiele Equation

We start the prove by separating the above sum into $\tau = 0$ and the rest:
Lets start with $\tau = 0$:

$$\sum_{i \in \mathcal{S}} a_i^{Pre}(t) \underbrace{p_{ji}(t,t)}_{\delta_{ij}} + v \sum_{i,k \in \mathcal{S}} a_{ik}^{Post}(t) p_{ji}(t,t) p_{ik}(t,t+1)$$

$$= a_j^{Pre}(t) + v \sum_{k \in \mathcal{S}} a_{jk}^{Post}(t) p_{jk}(t,t+1)$$

Continute with $\tau \geq 1$ and using Chapman-Kolmogorov:

$$\sum_{\tau \geq 1} v^\tau \Bigg(\sum_i a_i^{Pre}(t+\tau) \cdot \underbrace{p_{ji}(t,t+\tau)}_{\sum_l p_{jl}(t,t+1) p_{li}(t+1,t+\tau)}$$

$$+ \sum_{i,k} a_{ik}^{Post}(t+\tau) \cdot \underbrace{p_{ji}(t,t+\tau)}_{\sum_l p_{jl}(t,t+1) p_{li}(t+1,t+\tau)} \cdot p_{ik}(t+\tau, t+\tau+1) \Bigg)$$

Proof Thiele Equation

By factor out terms which are independent of τ as well as rearranging the τ-sum $(\tau - 1 \rightarrow \tau)$ we get:

$$\sum_{l} p_{jl}(t, t+1) \cdot v \cdot \left(\sum_{\tau \geq 0} v^{\tau} p_{li}(t+1, t+1+\tau) \right.$$
$$\left. \left[\sum_{i} a_i^{Pre}(t+1+\tau) + \sum_{i,k} a_{ik}^{Post}(t+1+\tau) p_{ik}(t+1+\tau, t+\tau+1+1) \cdot v \right] \right)$$

Comparing the expression in () with equation (**??**) we recognize that this is just $V_l(t+1)$. Now combing both parts we get:

$$V_j(t) = a_j^{Pre}(t) + v \sum_{k \in \mathcal{S}} p_{jk}(t, t+1) \left(a_{jk}^{Post}(t) + V_k(t+1) \right)$$

This concludes the proof.

How to simulate a trajectory of a discrete time, finite State space Markov Chain

To simulate a trajectory of a discrete time, finite State space Markov Chain sith finite state space $\mathcal{S} = \{1, 2, 3, \ldots, n\}$ and translation probability matrix $P = (p_{ij})$ we can use the **Inverse Transformation Method**:

1. Start with initial state $X_0 = i_0$
2. At each time step t, given $X_t = i$
   - Generate $U \sim \text{Uniform}(0, 1)$
   - Find j such that $\sum_{k=1}^{j-1} p_{ik} \leq U < \sum_{k=1}^{j} p_{ik}$
   - Set $X_{t+1} = j$

This algortihm fulfills the Markove no-memory property by construction since only the current state X_t is used to determine the next state X_{t+1}.

Simulate mathematical reserve

The expression $V(X(\omega))[0]$ denotes the PV of future cash flows generated by $X(\omega)$ at time $t = t_0$ for a given trajectory of the Markov chain. Recall that the mathematical reserve is defined as $V_j(t) = \mathbb{E}[\text{PV of future cash flows}|X_{t_0} = j]$.
Now given n independent trajectories $X(\omega_1), X(\omega_2), \ldots, X(\omega_n)$ we know that by the **Law of Large Numbers**:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} V(\omega_k)[0] = V(t = t_0),$$

i.e., the average of the PV of future chash flows over many independent trajectories converges to the true mathematical reserve at time $t = t_0$ almos surely.

Determination of cumulative Probability Density of the reserves

Let $\{V(\omega_k)[0]\}_{k=1}^{n}$ be independent trajectoires. The empirical ccumulatice distribution function is given by:
$F_n(v) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{[-\infty,v]}(V(\omega_k)[0])$

Then define the random variable $D_n = \sup_{v\in\mathbb{R}} |F_n(v) - F(v)|$. By the **Glivenko-Cantelli Theorem** we know that $D_n \xrightarrow{a.s.} 0$ for $n \to \infty$.

Simulation of individual cash flows

Task 2 : Stopping to pay Premium

Mixed Endowment Insurance

What is a mixed endowment?

- a mix of a term (temporary death) and a pure endowment insurance
- a lump sum is payable on death or reachinga certain age

Let us look at these random variables individually

- term insurance : $Z_1 = v^{k+1}\mathbb{1}_{k<n}$

  $A_{x:\overline{n}|}^{\;1} = \mathbb{E}[Z_1] = \sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x \cdot q_{x+k} = \sum_{k=0}^{n-1} v^{k+1} \cdot \frac{l_{x+k}}{l_x} \cdot \frac{d_{x+k}}{l_{x+k}}$

  $= \sum_{k=0}^{n-1} \frac{C_{x+k}}{D_x} = \frac{M_x - M_{x+n}}{D_x}$
- pure endowment: $Z_2 = v^{k+1}\mathbb{1}_{k\geq n}$
- $A^{1}_{x:\overline{n}|} = \mathbb{E}[Z_2] = \sum_{k=n}^{\infty} v^n\mathbb{P}[K=k] = v^n\mathbb{P}[K \geq n] = v^n \cdot {}_np_x = \frac{D_{x+n}}{D_x}$
- Thus the epxectation value for the mixed enovment is given by

$$A_{x:\overline{n}|} = \mathbb{E}[Z] = \mathbb{E}[Z_1] + \mathbb{E}[Z_2] = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

Equivalence Principle

In order to determine the premium one typically uses the Equivalence Principle

1. $\mathbb{E}[L] = 0$ where L denotes the loss, or equivalently,
2. The expected value of premiums is equal to the expected value of benefits.

Remark:

- The Equivalece Principle is equivalent to the requirement of the mathematical reserve at inception to be zero.

As an example consider a termn insurance where we define the loss as

$$L = C \cdot v^{k+1} \mathbb{1}_{k<n} - \Pi \cdot \ddot{a}_{\min(k,n)}$$

By the Equivalence Pricinple we have $\mathbb{E}[L] = 0$ which leads to

$$\Pi \cdot \ddot{a}_{x:\overline{n}|} = \Pi \cdot A_{x:\overline{n}|}^{\,1}.$$

Mathematical Reserves

The Mathematical Reserves V at given time t are defined as
V = PV(future benefits) - PV(future premiums)

For a mixed endowment insurance with n annual premiums Π the mathematical reserve at time t is given by (expressed in commutation functions)

$$ {}_tV_x = \frac{M_{x+t} - M_{x+n} + D_{x+n} - \Pi \cdot (N_{x+t} - N_{x+n})}{D_{x+t}} $$

2.1 Premium for Product at Inception

- Compute the premium by means of the equivalence principle.
- For the mixes endowment with benefit L we have

$$\Pi = L\frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = L\left(\frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}\right)$$

- Plugging in the values we get $\Pi \approx 12'302.98$

2.2 Benefit $\tilde{L}$ after policyholder stops after one premium.

- Combining the formulas for the mixed endowment insurance $A_{x+k:\overline{n-k}|}$ and the mathematical reserve ${}_kV_x$ we obtain the following expression for the reduced benefit using commutation functions

$$
{}_1\tilde{L} = \frac{{}_kV_x}{A_{x+k:\overline{n-k}|}} = {}_kV_x \cdot \frac{D_{x+k}}{M_{x+k} - M_{x+n} + D_{x+n}}
$$

- Result of the python code is: ${}_1\tilde{L} \approx 9228.77$.

2.3 Benefit Level $\tilde{L}$ as a function of the number of paid premiums

- Let k be the number of paid premiums then by the previous exercise

$$ {}_1\tilde{L} = \frac{{}_kV_x}{A_{x+k:\overline{n-k}|}} = {}_kV_x \cdot \frac{D_{x+k}}{M_{x+k} - M_{x+n} + D_{x+n}} $$

k	Reserve ${}_jV_x$	Benefit Level ${}_k\tilde{L}$
1	8062.41	9228.77
2	16260.21	18375.48
3	24650.21	27498.51
4	33308.23	36670.15
5	42335.99	45980.83
6	51870.01	55545.00
7	62095.67	65509.06
8	73266.94	76062.14
9	85736.24	87450.96
10	100000.0	100000.0

2.4 Which equivalence principle is fulfilled for the first premium assuming only one premium is paid.

- $t = 0$ and premium P
- Payout/Benefits:
  i.) if $t \in [0, 1)$ then $L = 100'000$ ii.) if $t \in [1, 10)$ then ${}_t\tilde{L}$
- by the **Equivalence Principle**: $\mathbb{E}(premium) = \mathbb{E}(benefits)$
- the benefits are computed as

$$\begin{aligned}\mathbb{E}(benefits) &= v \cdot q_x \cdot L + v^2 \cdot {}_1\tilde{L} \cdot p_x \cdot q_{x+1} + \cdots + v^{10} \cdot {}_1\tilde{L} \cdot {}_{10}p_x \\ &= v \cdot q_x \cdot L + v \cdot p_x \cdot {}_1\tilde{L} \cdot A_{x+1:\bar{9}|}\end{aligned}$$

- A short check in the jupyter notebook shows that $\mathbb{E}(premium) = \mathbb{E}(benefits)$

2.5 Which equivalence principle is fulfilled for the second premium assuming only one premium is paid.

- t=1
- Assuming one premium is paid and compute the second one
- Assuming that insured person survives the first year
- Insurer is at risk to pay L if the death occurs in $[1,2)$
- From time $t=2$ on until maturity the insruer is at risk of ${}_2\tilde{L}_x$
- By the Equivalence Principle $\mathbb{E}(premium) = \mathbb{E}(benefits)$
- $\mathbb{E}(premium) = P + {}_1V_x$

2.5 Which equivalence principle is fulfilled for the second premium assuming only one premium is paid.

- Similar to before we compute the benefits:

$$\begin{aligned}\mathbb{E}(benefits) &= v \cdot q_{x+1} \cdot L + v^2 \cdot {}_2\tilde{L} \cdot p_{x+1} \cdot q_{x+2} + \cdots + v^9 \cdot {}_2\tilde{L} \cdot {}_9p_x \\ &= v \cdot q_{x+1} \cdot L + v \cdot p_{x+1} \cdot {}_2\tilde{L} \cdot A_{x+2:\overline{8}|}\end{aligned}$$

- A short check in the jupyter notebook shows that $\mathbb{E}(premium) = \mathbb{E}(benefits)$.

Task 3: Disability Insurance on two lives

Problem

3.1

Two random variables X and Y are **stochastically independent** if and only if their joint probability distribution factorizes:

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y) \quad \forall x, y$$

Or equivalently:

$$P(Y = y \mid X = x) = P(Y = y) \quad \forall x, y.$$

Intuitively: Knowing X provides no information about Y

Two random variables X and Y are **uncorrelated** if their covariance is zero:

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = 0.$$

Or equivalently:

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

3.1

Independence $\Rightarrow$ Uncorrelated, Proof

Let X and Y be discrete and independent random variables.

$$\begin{aligned}\mathbb{E}[XY] &= \sum_x \sum_y xy \cdot P(X = x, Y = y) \\ &= \sum_x \sum_y xy \cdot P(X = x) \cdot P(Y = y) = \mathbb{E}[X] \cdot \mathbb{E}[Y].\end{aligned}$$

This concludes the proof.

Uncorrelated $\not\Rightarrow$ Independence

Two variables can be uncorrelated yet still be dependent!

3.1 Counterexample: Uncorrelated but Dependent

Let $X \sim \text{Uniform}(-1, 1)$ and $Y = X^2$

- $\mathbb{E}[XY] = \mathbb{E}[X^3] = 0$ (by symmetry)
- $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y] = \frac{1}{3}$
- Therefore: $\mathbb{E}[X] \cdot \mathbb{E}[Y] = 0$
- So $\text{Cov}(X, Y) = 0 \Rightarrow$ **uncorrelated**
- But Y is completely determined by $X \Rightarrow$ **dependent!**

3.3

3.4

3.5

3.6