Exam Presentation
Life Insurance Mathematics

Sami Zweidler

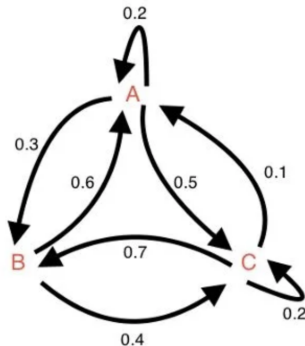ETH Zurich

20/01/2026

Markov Chains

What is a Markov Chain?
It is a Stochastic model that describes sequence of transitions from one state to another according to certain probabilistic rules.

Markow Chains

Definition

$(X_t)_{t \in \mathbb{N}} : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{S} = \{1, 2, 3, \ldots\}$, is called a **Markov chain** if and only if,

$$\mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_1} = i_1, \ldots, X_{t_m} = i_m] = \mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_m} = i_m]$$

for $t_1 < t_2 < \cdots < t_m < t_{m+1}$ and $i_1, i_2, \ldots, i_{m+1} \in \mathcal{S}$.

We say that such a stochastic process $(X_t)_{t \in \mathbb{N}}$ has no memory.

Markov Chains

Chapman-Kolmogorov Theorem

Let $p_{ij}(s, t) = P(X_t = j | X_s = i)$ be the transition probabilities of a Markov chain. Then, for any $0 \leq s < u < t$,

$$p_{ij}(s, t) = \sum_k p_{ik}(s, u) p_{kj}(u, t).$$

Or written in matrix form, $P(s, t) = P(s, u)P(u, t)$.

Idea: What is the probbility of being in state j at time t, given that at time s we are in state i?

Markov Chains

> **Proof**
>
> $$p_{ij}(s,t) = \mathbb{P}[X_t = j | X_s = i] = \mathbb{P}[X_t = j \cap \bigcup_{k \in \mathcal{S}} \{X_u = k\} | X_s = i]$$
>
> $$= \sum_{k \in \mathcal{S}} \mathbb{P}[X_t = j, X_u = k | X_s = i]$$
>
> $$= \sum_{k \in \mathcal{S}} \frac{\mathbb{P}[X_t = j, X_u = k, X_s = i]}{\mathbb{P}[X_s = i]} \cdot \frac{\mathbb{P}[X_u = k, X_s = i]}{\mathbb{P}[X_u = k, X_s = i]}$$
>
> $$= \sum_{k \in \mathcal{S}} \underbrace{\mathbb{P}[X_t = j | X_u = k, X_s = i]}_{\mathbb{P}[X_t = j | X_u = k]} \mathbb{P}[X_u = k | X_s = i]$$
>
> $$= \sum_{k \in \mathcal{S}} p_{ik}(s,u) p_{kj}(u,t)$$

In the above we used $\mathbb{P}[A \cap B | C] = \mathbb{P}[A | B \cap C] \cdot \mathbb{P}[B | C]$ as well as the Markov porperty as well as assuming that $\mathbb{P}[X_u = k, X_s = i] \neq 0$.

Markov Model

To model a life Insurance we need three ingredients:

- a Markov chain $(X_t)_{t in \mathbb{N}}$
- a one-year discount factor $v = \frac{1}{1+i}$
- contract functions $a_i^{pre}(t)$ and $a_{ij}^{post}(t)$

The starting point of an Markov model are the various possible conditions for an insured person, building the state space $\mathcal{S}$. E.g. $\mathcal{S} = \{'living', 'death'\}$.

Induced Cashflow & Mathemtical Reserve

Thiele Equation

Theorem (Thiele's differennce equation)

The mathematical reserve between two subsequent periods are related by:

$$V_i(t) = a_i^{pre}(t) + \sum_{j \in \mathcal{S}} v \cdot p_{ij}(t, t+1) \cdot (a_{ij}^{post}(t) + V_j(t+1))$$

Mathemical Reserves

Mathematical Reserves at given time = PV(future benefits) - PV(future premiums)

Expressed in commutation functions:

$$_tV_x = \frac{M_{x+t} - M_{x+n} + D_{x+n} - \Pi \cdot (N_{x+t} - N_{x+n})}{D_{x+t}}$$

Problem