

# Exam Presentation

## Life Insurance Mathematics

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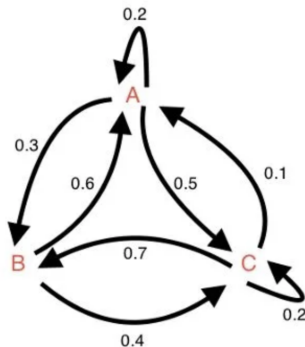
- 1 Task 1: Markov Model
- 2 Task 2 : Stopping to pay Premium
- 3 Task 3: Disability Insurance on two lives

# Task 1: Markov Model

# Markov Chains

What is a Markov Chain?

It is a Stochastic model that describes sequence of transitions/ possible events in which the probability of each event depends only on the state attained in the previous event.



# Markov Chains

## Definition

$(X_t)_{t \in \mathbb{N}} : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathcal{S} = \{1, 2, 3, \dots\}$ , is called a **Markov chain** if and only if,

$$\mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_1} = i_1, \dots, X_{t_m} = i_m] = \mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_m} = i_m]$$

for  $t_1 < t_2 < \dots < t_m < t_{m+1}$  and  $i_1, i_2, \dots, i_{m+1} \in \mathcal{S}$ .

We say that such a stochastic process  $(X_t)_{t \in \mathbb{N}}$  has no memory.

# Markov Chains

## Chapman-Kolmogorov Theorem

Let  $p_{ij}(s, t) = P(X_t = j | X_s = i)$  be the transition probabilities of a Markov chain. Then, for any  $0 \leq s < u < t$ ,

$$p_{ij}(s, t) = \sum_k p_{ik}(s, u) p_{kj}(u, t).$$

Or written in matrix form,  $P(s, t) = P(s, u)P(u, t)$ .

Idea: What is the probability of being in state  $j$  at time  $t$ , given that at time  $s$  we are in state  $i$ ?

# Markov Chains

## Proof

$$\begin{aligned} p_{ij}(s, t) &= \mathbb{P}[X_t = j | X_s = i] = \mathbb{P}[X_t = j \cap \bigcup_{k \in \mathcal{S}} \{X_u = k\} | X_s = i] \\ &= \sum_{k \in \mathcal{S}} \mathbb{P}[X_t = j, X_u = k | X_s = i] \\ &= \sum_{k \in \mathcal{S}} \frac{\mathbb{P}[X_t = j, X_u = k, X_s = i]}{\mathbb{P}[X_s = i]} \cdot \frac{\mathbb{P}[X_u = k, X_s = i]}{\mathbb{P}[X_u = k, X_s = i]} \\ &= \sum_{k \in \mathcal{S}} \underbrace{\mathbb{P}[X_t = j | X_u = k, X_s = i]}_{\mathbb{P}[X_t = j | X_u = k]} \mathbb{P}[X_u = k | X_s = i] \\ &= \sum_{k \in \mathcal{S}} p_{ik}(s, u) p_{kj}(u, t) \end{aligned}$$

In the above we used  $\mathbb{P}[A \cap B | C] = \mathbb{P}[A | B \cap C] \cdot \mathbb{P}[B | C]$  as well as the Markov property as well as assuming that  $\mathbb{P}[X_u = k, X_s = i] \neq 0$ .

# Markov Model

To model a life Insurance we need three ingredients:

- a Markov chain  $(X_t)_{t \in \mathbb{N}}$
- a one-year discount factor  $v = \frac{1}{1+i}$
- contract functions  $a_i^{pre}(t)$  and  $a_{ij}^{post}(t)$

The starting point of an Markov model are the various possible conditions for an insured person, building the state space  $\mathcal{S}$ . E.g.  $\mathcal{S} = \{\text{'living'}, \text{'death'}\}$ .



# Induced Cashflow & Mathematical Reserve

A central task in life insurance is the determination of the actuarial reserve, i.e., the amount of money which has to be set aside at a given time  $t$  to be able to meet all future obligations/benefits towards each policy.

We denote by  $A_t$  the payments that are due for a policy at time  $t$ .  $(A_t)_{t \in \mathbb{N}}$  is a stochastic process.

$$A_t = a_{X_t}^{Pre}(t) + a_{X_{t-1}X_t}^{Post}(t)$$

where  $a_{ij}^{Post}(-1) = 0$  for all  $i, j \in \mathcal{S}$ ,  $t \in \mathbb{N}$ .

# Induced Cashflow & Mathematical Reserve

We set  $l_i(t) = \mathbb{1}_{\{X_t=i\}}$ . Then we can compute the **induced cash flows** as follows:

$$A(t) = \underbrace{\sum_{i \in \mathcal{S}} l_i(t) \cdot a_i^{pre}(t)}_{\text{annuity}} + \underbrace{\sum_{i,j \in \mathcal{S}} l_i(t) \cdot l_j(t+1) \cdot a_{ij}^{post}(t)}_{\text{capital/lump sum paid at time } t+1}$$

Idea:  $A(t)$  are the payments are due at time  $t$  for a given policy. We can also compute the present value (PV) of  $A(t)$  which is given by:

$$\tilde{A}(t) = \sum_{i \in \mathcal{S}} l_i(t) \cdot a_i^{pre}(t) + v \cdot \sum_{i,j \in \mathcal{S}} l_i(t) \cdot l_j(t+1) \cdot a_{ij}^{post}(t)$$

Finally we can define the mathematical reserve at time  $t$  as:

$$V_j(t) = \mathbb{E}[\text{PV of future cash flows} | X_t = j] = \mathbb{E}\left[\sum_{\tau=0}^{\infty} v^{\tau} \tilde{A}(t+\tau) | X_t = j\right]$$

# Mathematical Reserve

We can compute the mathematical reserves with the following results:

$$\begin{aligned}\mathbb{E}[I_i(t + \tau) | X_t = j] &= p_{ji}(t, t + \tau) \\ \mathbb{E}[I_i(t + \tau) I_k(t + \tau + 1) | X_t = j] &= p_{ji}(t, t + \tau) p_{ik}(t + \tau, t + \tau + 1)\end{aligned}$$

Hence the reserve is given as:

$$\begin{aligned}V_j(t) &= \sum_{\tau=0}^{\infty} v^{\tau} \left( \sum_{i \in \mathcal{S}} a_i^{Pre}(i + \tau) p_{ji}(t, t + \tau) \right. \\ &\quad \left. + v \sum_{i, k \in \mathcal{S}} a_{ik}^{Post}(t + \tau) p_{ji}(t, t + \tau) p_{ik}(t + \tau, t + \tau + 1) \right) \quad (1)\end{aligned}$$

Hence  $V_j(t)$  is the current value of the future actuarial reserve cash flow ( $A$ ) based on today's information, i.e.  $X_t = j$ .

# Thiele Equation

We can relate the mathematical reserves at two subsequent time points  $t$  and  $t+1$  via the following equation:

## Theorem (Thiele's difference equation)

The mathematical reserve between two subsequent periods are related by:

$$V_j(t) = a_j^{pre}(t) + \sum_{i \in \mathcal{S}} v \cdot p_{ji}(t, t+1) \cdot (a_{ji}^{post}(t) + V_i(t+1))$$

## Remarks:

- Calculates the expected reserve directly using transition probabilities, while simulation estimates it by averaging over many random trajectories
- To solve Thiele's equation we need the boundary condition  $V_j(T) = 0$  for all  $j \in \mathcal{S}$ .
- The Thiele equation is leads to the same results as for the classical insurance model (using commutation functions)
- Forward computation of reserves is possible, too. but numerically less stable.

# Proof Thiele Equation

We start the prove by separating the above sum into  $\tau = 0$  and the rest:  
 Lets start with  $\tau = 0$ :

$$\begin{aligned} & \sum_{i \in S} a_i^{Pre}(t) \underbrace{p_{ji}(t, t)}_{\delta_{ij}} + v \sum_{i, k \in S} a_{ik}^{Post}(t) p_{ji}(t, t) p_{ik}(t, t + 1) \\ &= a_j^{Pre}(t) + v \sum_{k \in S} a_{jk}^{Post}(t) p_{jk}(t, t + 1) \end{aligned}$$

Continue with  $\tau \geq 1$  and using Chapman-Kolmogorov:

$$\begin{aligned} & \sum_{\tau \geq 1} v^\tau \left( \sum_i a_i^{Pre}(t + \tau) \cdot \underbrace{p_{ji}(t, t + \tau)}_{\sum_l p_{jl}(t, t+1) p_{li}(t+1, t+\tau)} \right. \\ & \left. + \sum_{i, k} a_{ik}^{Post}(t + \tau) \cdot \underbrace{p_{ji}(t, t + \tau)}_{\sum_l p_{jl}(t, t+1) p_{li}(t+1, t+\tau)} \cdot p_{ik}(t + \tau, t + \tau + 1) \right) \end{aligned}$$

# Proof Thiele Equation

By factor out terms which are independent of  $\tau$  as well as rearranging the  $\tau$ -sum ( $\tau - 1 \rightarrow \tau$ ) we get:

$$\sum_l p_{jl}(t, t+1) \cdot v \cdot \left( \sum_{\tau \geq 0} v^\tau p_{li}(t+1, t+1+\tau) \left[ \sum_i a_i^{Pre}(t+1+\tau) + \sum_{i,k} a_{ik}^{Post}(t+1+\tau) p_{ik}(t+1+\tau, t+\tau+1+1) \cdot v \right] \right)$$

Comparing the expression in ( ) with equation (??) we recognize that this is just  $V_l(t+1)$ . Now combining both parts we get:

$$V_j(t) = a_j^{Pre}(t) + v \sum_{k \in S} p_{jk}(t, t+1) (a_{jk}^{Post}(t) + V_k(t+1))$$

This concludes the proof.

# How to simulate a trajectory of a discrete time, finite State space Markov Chain

To simulate a trajectory of a discrete time, finite State space Markov Chain with finite state space  $\mathcal{S} = \{1, 2, 3, \dots, n\}$  and transition probability matrix  $P = (p_{ij})$  we can use the **Inverse Transformation Method**:

- ① Start with initial state  $X_0 = i_0$
- ② At each time step  $t$ , given  $X_t = i$ 
  - Generate  $U \sim \text{Uniform}(0, 1)$
  - Find  $j$  such that  $\sum_{k=1}^{j-1} p_{ik} \leq U < \sum_{k=1}^j p_{ik}$
  - Set  $X_{t+1} = j$

This algorithm fulfills the Markov no-memory property by construction since only the current state  $X_t$  is used to determine the next state  $X_{t+1}$ .

# Simulate mathematical reserve

The expression  $V(X(\omega))[0]$  denotes the PV of future cash flows generated by  $X(\omega)$  at time  $t = t_0$  for a given trajectory of the Markov chain. Recall that the mathematical reserve is defined as

$V_j(t) = \mathbb{E}[\text{PV of future cash flows} | X_{t_0} = j]$ .

Now given  $n$  independent trajectories  $X(\omega_1), X(\omega_2), \dots, X(\omega_n)$  we know that by the **Law of Large Numbers**:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n V(\omega_k)[0] = V(t = t_0),$$

i.e., the average of the PV of future cash flows over many independent trajectories converges to the true mathematical reserve at time  $t = t_0$  almost surely.



# Determination of cumulative Probability Density of the reserves

Let  $\{V(\omega_k)[0]\}_{k=1}^n$  be independent trajectoires. The empirical cumulative distribution function is given by:

$$F_n(v) = \frac{1}{n} \sum_{k=1}^n \mathbb{1}_{[-\infty, v]}(V(\omega_k)[0])$$

Then define the random variable  $D_n = \sup_{v \in \mathbb{R}} |F_n(v) - F(v)|$ . By the **Glivenko-Cantelli Theorem** we know that  $D_n \xrightarrow{a.s.} 0$  for  $n \rightarrow \infty$ .

Hence if we have enough samples/trajectories we can approximate the cumulative distribution function of the mathematical reserves.

## Task 2 : Stopping to pay Premium

# Mixed Endowment Insurance

What is a mixed endowment?

- a mix of a term (temporary death) and a pure endowment insurance
- a lump sum is payable on death or reaching a certain age

Let us look at these random variables individually

- term insurance :  $Z_1 = v^{k+1} \mathbb{1}_{k < n}$

$$\begin{aligned} A_{x:\overline{n}|}^1 &= \mathbb{E}[Z_1] = \sum_{k=0}^{n-1} v^{k+1} {}_k p_x \cdot q_{x+k} = \sum_{k=0}^{n-1} v^{k+1} \cdot \frac{l_{x+k}}{l_x} \cdot \frac{d_{x+k}}{l_{x+k}} \\ &= \sum_{k=0}^{n-1} \frac{C_{x+k}}{D_x} = \frac{M_x - M_{x+n}}{D_x} \end{aligned}$$

- pure endowment:  $Z_2 = v^n \mathbb{1}_{K \geq n}$
- $A_{x:\overline{n}|}^1 = \mathbb{E}[Z_2] = \sum_{k=n}^{\infty} v^n \mathbb{P}[K = k] = v^n \mathbb{P}[K \geq n] = v^n \cdot {}_n p_x = \frac{D_{x+n}}{D_x}$
- Thus the expectation value for the mixed endowment is given by

$$A_{x:\overline{n}|} = \mathbb{E}[Z] = \mathbb{E}[Z_1] + \mathbb{E}[Z_2] = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

# Equivalence Principle

In order to determine the premium one typically uses the Equivalence Principle

- ①  $\mathbb{E}[L] = 0$  where  $L$  denotes the loss, or equivalently,
- ② The expected value of premiums is equal to the expected value of benefits.

Remark:

- The Equivalence Principle is equivalent to the requirement of the mathematical reserve at inception to be zero.

As an example consider a termn insurance where we define the loss as

$$L = C \cdot v^{k+1} \mathbb{1}_{k < n} - \Pi \cdot \ddot{a}_{\min(k,n)}$$

By the Equivalence Principle we have  $\mathbb{E}[L] = 0$  which leads to

$$\Pi \cdot \ddot{a}_{x:\overline{n}|} = \Pi \cdot A_{x:\overline{n}|}^1.$$

# Mathematical Reserves

The Mathematical Reserves  $V$  at given time  $t$  are defined as

$$V = PV(\text{future benefits}) - PV(\text{future premiums})$$

For a mixed endowment insurance with  $n$  annual premiums  $\Pi$  the mathematical reserve at time  $t$  is given by (expressed in commutation functions)

$${}_tV_x = \frac{M_{x+t} - M_{x+n} + D_{x+n} - \Pi \cdot (N_{x+t} - N_{x+n})}{D_{x+t}}$$

## 2.1 Premium for Product at Inception

- Compute the premium by means of the equivalence principle.
- For the mixed endowment with benefit  $L$  we have

$$\Pi = L \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = L \left( \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}} \right)$$

- Plugging in the values we get  $\Pi \approx 12'302.98$

## 2.2 Benefit $\tilde{L}$ after policyholder stops after one premium.

- Combining the formulas for the mixed endowment insurance  $A_{x+k:\overline{n-k}|}$  and the mathematical reserve  ${}_kV_x$  we obtain the following expression for the reduced benefit using commutation functions

$${}_1\tilde{L} = \frac{{}_kV_x}{A_{x+k:\overline{n-k}|}} = {}_kV_x \cdot \frac{D_{x+k}}{M_{x+k} - M_{x+n} + D_{x+n}}$$

- Result of the python code is:  ${}_1\tilde{L} \approx 9228.77$ .

## 2.3 Benefit Level $\tilde{L}$ as a function of the number of paid premiums

- Let  $k$  be the number of paid premiums then by the previous exercise

$${}_k\tilde{L} = \frac{{}_kV_x}{A_{x+k:\overline{n-k}|}} = {}_kV_x \cdot \frac{D_{x+k}}{M_{x+k} - M_{x+n} + D_{x+n}}$$

$k$	Reserve ${}_jV_x$	Benefit Level ${}_k\tilde{L}$
1	8062.41	9228.77
2	16260.21	18375.48
3	24650.21	27498.51
4	33308.23	36670.15
5	42335.99	45980.83
6	51870.01	55545.00
7	62095.67	65509.06
8	73266.94	76062.14
9	85736.24	87450.96
10	100000.0	100000.0



## 2.4 Which equivalence principle is fulfilled for the first premium assuming only one premium is paid.

- $t = 0$  and premium  $P$
- Payout/Benefits:
  - i.) if  $t \in [0, 1)$  then  $L = 100'000$  ii.) if  $t \in [1, 10)$  then  ${}_t\tilde{L}$
- by the **Equivalence Principle**:  $\mathbb{E}(\text{premium}) = \mathbb{E}(\text{benefits})$
- the benefits are computed as

$$\begin{aligned}\mathbb{E}(\text{benefits}) &= v \cdot q_x \cdot L + v^2 \cdot {}_1\tilde{L} \cdot p_x \cdot q_{x+1} + \cdots + v^{10} \cdot {}_1\tilde{L} \cdot {}_{10}p_x \\ &= v \cdot q_x \cdot L + v \cdot p_x \cdot {}_1\tilde{L} \cdot A_{x+1:\overline{9}|}\end{aligned}$$

- A short check in the jupyter notebook shows that  $\mathbb{E}(\text{premium}) = \mathbb{E}(\text{benefits})$

## 2.5 Which equivalence principle is fulfilled for the second premium assuming only one premium is paid.

- Consider  $t = 1$
- Assuming one premium is paid at  $t = 0$
- Assuming that insured person survives the first year
- Insurer is at risk to pay  $L$  if the death occurs in  $[0, 1)$
- From time  $t = 1$  on until maturity the insurer is at risk of  ${}_1\tilde{L}_x$
- By the Equivalence Principle  $\mathbb{E}(\text{premium}) = \mathbb{E}(\text{benefits})$
- $\mathbb{E}(\text{premium}) = {}_1V_x$

## 2.5 Which equivalence principle is fulfilled for the second premium assuming only one premium is paid.

- Similar to before we compute the benefits:

$$\begin{aligned}\mathbb{E}(\text{benefits}) &= v \cdot q_{x+1} \cdot {}_1\tilde{L} + v^2 \cdot {}_1\tilde{L} \cdot p_{x+1} \cdot q_{x+2} + \cdots + v^9 \cdot {}_1\tilde{L} \cdot {}_9p_x \\ &= {}_1\tilde{L} \cdot A_{x+1:\overline{9}|}\end{aligned}$$

- A short check in the jupyter notebook shows that  $\mathbb{E}(\text{premium}) = \mathbb{E}(\text{benefits})$  as well as the equality is exactly the definition for the reduced benefit  ${}_1\tilde{L}$ .

## Task 3: Disability Insurance on two lives

## 3.1

Two random variables  $X$  and  $Y$  are **stochastically independent** if and only if their joint probability distribution factorizes:

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y) \quad \forall x, y$$

Or equivalently:

$$P(Y = y \mid X = x) = P(Y = y) \quad \forall x, y.$$

*Intuitively:* Knowing  $X$  provides no information about  $Y$

Two random variables  $X$  and  $Y$  are **uncorrelated** if their covariance is zero:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = 0.$$

Or equivalently:

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

# 3.1

## Independence $\Rightarrow$ Uncorrelated, Proof

Let  $X$  and  $Y$  be discrete and independent random variables.

$$\begin{aligned}\mathbb{E}[XY] &= \sum_x \sum_y xy \cdot P(X = x, Y = y) \\ &= \sum_x \sum_y xy \cdot P(X = x) \cdot P(Y = y) = \mathbb{E}[X] \cdot \mathbb{E}[Y].\end{aligned}$$

This concludes the proof.

## Uncorrelated $\nRightarrow$ Independence

Two variables can be uncorrelated yet still be dependent!

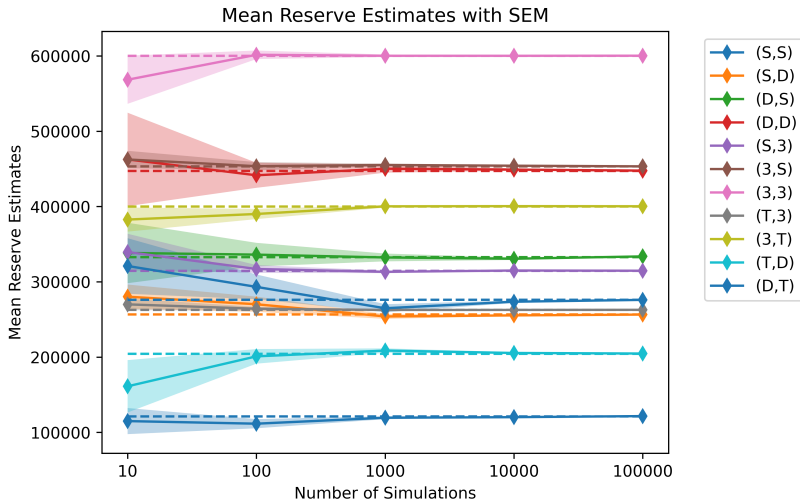
## 3.1 Counterexample: Uncorrelated but Dependent

Let  $X \sim \text{Uniform}(-1, 1)$  and  $Y = X^2$

- $\mathbb{E}[XY] = \mathbb{E}[X^3] = 0$  (by symmetry)
- $\mathbb{E}[X] = 0$  and  $\mathbb{E}[Y] = \frac{1}{3}$
- Therefore:  $\mathbb{E}[X] \cdot \mathbb{E}[Y] = 0$
- So  $\text{Cov}(X, Y) = 0 \Rightarrow$  **uncorrelated**
- But  $Y$  is completely determined by  $X \Rightarrow$  **dependent!**

## 3.4 MC Averages Analysis

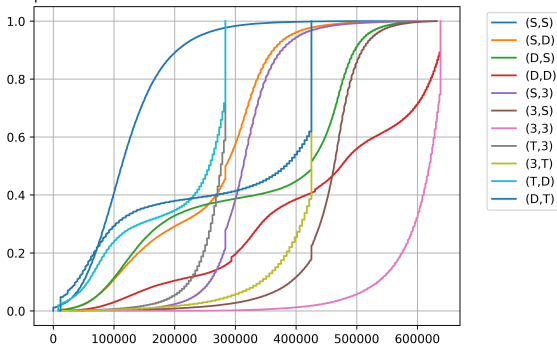
Here we show how the Monte Carlo estimates move with increasing number of simulated trajectories.





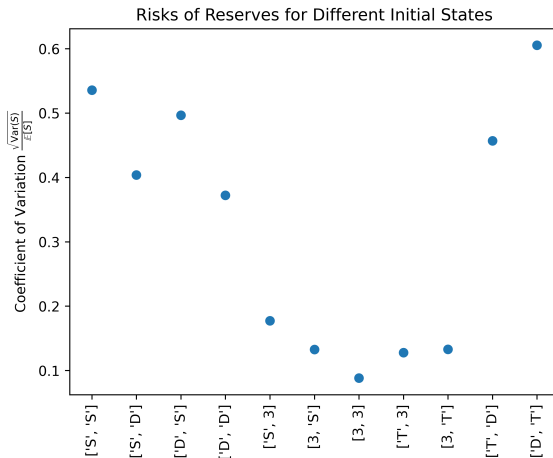
## 3.4 CDF for 100'000 Simulations

Empirical CDF for different intial states with 100'0000 simulations



## 3.5 Discussion of Risk Levels of States

In order to discuss the riskiness of the states we look at the coefficient of variation of the reserve in each state



## 3.5 Discussion of Risk Levels of States

- The state  $(D, T)$  is the most dangerous state (has highest CoV) since the man is disabled and rather young (30y). Hence he has three possibilities: reactivate, remain disabled for extended periods.
- The diversification from having two people in similar but independent states reduces relative variability
- The state  $(3, 3)$  is the safest one (although highest annuity to be paid) since both persons are disabled and either remain disabled or die.
- So ranking the states according to their risk level (from riskiest to safest) we get:

$$\begin{aligned}
 (D, T) &> (D, D) > (3, T) > (D, S) > (3, 3) > (3, S) \\
 &> (T, 3) > (S, D) > (S, 3) > (T, D) > (S, S)
 \end{aligned}$$

## 3.6 Discussion of Changes in Risk when changing from independent to Dependent Random Variables

## 3.7

## 3.8: 8-policy portfolio simulation

- Assuming the 8 policies are independent, we compute the total loss as the sum of the individual mathematical reserves.

$$\mathbb{E}[L] = \sum_{i=1}^8 \mathbb{E}[L_i] = 1'816'458.10$$

- Now we want to compute the stop loss premium  $\Pi(\beta)$  for the deductibles  $\beta \in \{1.5 \cdot \mathbb{E}[L], 3 \cdot \mathbb{E}[L]\}$
- The stop loss premium is defined as

$$\Pi(\beta) = \mathbb{E}[(L - \beta)_+] = \int_{\beta}^{\infty} (1 - F(x)) dx$$

- In order to create a histogram-based approximation of the cumulative distribution function  $F(x)$  we use 1'000'000 simulated portfolio losses (in total 8'000'000 simulations)
-

## 3.8: 8-policy portfolio simulation

Inser pciture of histogram here

## 3.8: 8-policy portfolio simulation

Now we shortly describe how to compute the stop loss premium  $\Pi(\beta)$  using a histogram-based approximation of the cumulative distribution function  $F(x)$ .

- Let the bins of the histogram be  $[a_1, a_2)$ ,  $[a_2, a_3)$ ,  $\dots$ ,  $[a_N, a_{N+1})$ .
- Assume the survival function is constant on each bin:  
 $\bar{F}(x) = 1 - F(x) \approx \bar{F}_i$  where  $\bar{F}_i$  denotes the value of the survival function on bin  $i$ .
- The integral representation approximated by the Riemann sum:

$$\Pi(\beta) \approx \sum_{i: a_i \geq \beta} \bar{F}_i (a_{i+1} - a_i)$$

- Plugging in the values for  $\beta$  we get:

$$\Pi(1.5 \cdot \mathbb{E}[L]) =$$

$$\Pi(3 \cdot \mathbb{E}[L]) =$$