Task 1: Markov Model
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Task 2 : Stopping to pay Premium
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Task 3: Disability Insurance on two lives
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Exam Presentation
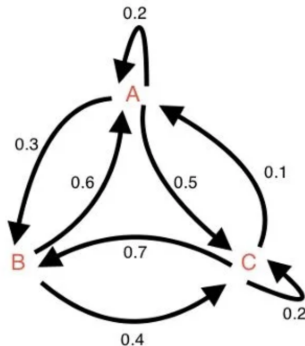# Life Insurance Mathematics

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Task 1: Markov Model

Markov Chains

What is a Markov Chain?
It is a Stochastic model that describes sequence of transitions from one state to another according to certain probabilistic rules.

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Markow Chains

Definition

$(X_t)_{t\in\mathbb{N}} : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{S} = \{1, 2, 3, \dots\}$, is called a **Markov chain** if and only if,

$$\mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_1} = i_1, \dots, X_{t_m} = i_m] = \mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_m} = i_m]$$

for $t_1 < t_2 < \cdots < t_m < t_{m+1}$ and $i_1, i_2, \dots, i_{m+1} \in \mathcal{S}$.

We say that such a stochastic process $(X_t)_{t\in\mathbb{N}}$ has no memory.

Markov Chains

Chapman-Kolmogorov Theorem

Let $p_{ij}(s,t) = P(X_t = j | X_s = i)$ be the transition probabilities of a Markov chain. Then, for any $0 \leq s < u < t$,

$$p_{ij}(s,t) = \sum_k p_{ik}(s,u)p_{kj}(u,t).$$

Or written in matrix form, $P(s,t) = P(s,u)P(u,t)$.

Idea: What is the probbility of being in state j at time t, given that at time s we are in state i?

Markov Chains

Proof

$$
\begin{aligned}
p_{ij}(s,t) &= \mathbb{P}[X_t = j | X_s = i] = \mathbb{P}[X_t = j \cap \bigcup_{k \in \mathcal{S}} \{X_u = k\} | X_s = i] \\
&= \sum_{k \in \mathcal{S}} \mathbb{P}[X_t = j, X_u = k | X_s = i] \\
&= \sum_{k \in \mathcal{S}} \frac{\mathbb{P}[X_t = j, X_u = k, X_s = i]}{\mathbb{P}[X_s = i]} \cdot \frac{\mathbb{P}[X_u = k, X_s = i]}{\mathbb{P}[X_u = k, X_s = i]} \\
&= \sum_{k \in \mathcal{S}} \underbrace{\mathbb{P}[X_t = j | X_u = k, X_s = i]}_{\mathbb{P}[X_t = j | X_u = k]} \mathbb{P}[X_u = k | X_s = i] \\
&= \sum_{k \in \mathcal{S}} p_{ik}(s,u) p_{kj}(u,t)
\end{aligned}
$$

In the above we used $\mathbb{P}[A \cap B | C] = \mathbb{P}[A | B \cap C] \cdot \mathbb{P}[B | C]$ as well as the Markov porperty as well as assuming that $\mathbb{P}[X_u = k, X_s = i] \neq 0$.

Markov Model

To model a life Insurance we need three ingredients:

- a Markov chain $(X_t)_{t in \mathbb{N}}$
- a one-year discount factor $v = \frac{1}{1+i}$
- contract functions $a_i^{pre}(t)$ and $a_{ij}^{post}(t)$

The starting point of an Markov model are the various possible conditions for an insured person, building the state space $\mathcal{S}$. E.g. $\mathcal{S} = \{'living', 'death'\}$.

Induced Cashflow & Mathemtical Reserve

A central task in life insurance is the determination of the actuarial reserve, i.e., the amount of money which has to be set aside at a given time t to be able to meet all future obligations/benefits towards each policy.

We denote by A_t the payments that are due for a policy at time t. $(A_t)_{t \in \mathbb{N}}$ is a stochastic process.

$$A_t = a_{X_t}^{Pre}(t) + a_{X_{t-1}X_t}^{Post}(t)$$

where $a_{ij}^{Post}(-1) = 0$ for all i,j $\in \mathcal{S}$, $t \in \mathbb{N}$.

Induced Cashflow & Mathemtical Reserve

We set $I_i(t) = \mathbb{1}_{\{X_t=i\}}$. Then we can compute the **induced cash flows** as follows:

$$A(t) = \underbrace{\sum_{i \in \mathcal{S}} I_i(t) \cdot a_i^{pre}(t)}_{\text{annuity}} + \underbrace{\sum_{i,j \in \mathcal{S}} I_i(t) \cdot I_j(t+1) \cdot a_{ij}^{post}(t)}_{\text{captial/lump sum paid at time t+1}}$$

Idea: $A(t)$ are the payments are due at time t for a given policy. We can also compute the present value (PV) of A(t) which is given by:

$$\tilde{A}(t) = \sum_{i \in \mathcal{S}} I_i(t) \cdot a_i^{pre}(t) + v \cdot \sum_{i,j \in \mathcal{S}} I_i(t) \cdot I_j(t+1) \cdot a_{ij}^{post}(t)$$

Finally we can define the mathematical reserve at time t as:

$$V_j(t) = \mathbb{E}[\text{PV of future cash flows}|\ X_t = j] = \mathbb{E}[\sum_{\tau=0}^{\infty} \tilde{A}(t+\tau)|X_t = j]$$

Mathematical Reserve

We can compute the mathematical reserves with the following results:

$$\mathbb{E}[I_i(t+\tau)|\ X_t = j] = p_{ji}(t, t+\tau)$$
$$\mathbb{E}[I_i(t+\tau)\ I_k(t+\tau+1)|\ X_t = j] = p_{ji}(t, t+\tau)p_{ik}(t+\tau, t+\tau+1)$$

Hence the reserve is given as:

$$V_j(t) = \sum_{\tau=0}^{\infty} v^\tau \left(\sum_{i \in \mathcal{S}} a_i^{Pre}(i+\tau)p_{ji}(t, t+\tau) \right.$$
$$\left. + v \sum_{i,k \in \mathcal{S}} a_{ik}^{Post}(t+\tau)p_{ji}(t, t+\tau)p_{ik}(t+\tau, t+\tau+1) \right) \quad (1)$$

Hence $V_j(t)$ is the current value of the future actuarial reserve cash flow (A) based on todays information, i.e. $X_t = j$.

Thiele Equation

We can relate the mathematical reserves at two subsequent time points t and t+1 via the following equation:

Theorem (Thiele's difference equation)

The mathematical reserve between two subsequent periods are related by:

$$V_j(t) = a_j^{pre}(t) + \sum_{i \in \mathcal{S}} v \cdot p_{ji}(t, t+1) \cdot (a_{ji}^{post}(t) + V_i(t+1))$$

Remarks:

- Calculates the expected reserve directly using transition probabilities, while simulation estimates it by averaging over many random trajectories
- To solve Thiele's equation we need the boundary condition $V_j(T) = 0$ for all $j \in \mathcal{S}$.
- The Thiele equation is leads to the same results as for the classical insurance model (using commutation functions)
- Forward computation of reserves is possible, too. but numerically less stable.

Proof Thiele Equation

We start the prove by separating the above sum into $\tau = 0$ and the rest:
Lets start with $\tau = 0$:

$$\sum_{i \in \mathcal{S}} a_i^{Pre}(t) \underbrace{p_{ji}(t,t)}_{\delta_{ij}} + v \sum_{i,k \in \mathcal{S}} a_{ik}^{Post}(t) p_{ji}(t,t) p_{ik}(t,t+1)$$

$$= a_j^{Pre}(t) + v \sum_{k \in \mathcal{S}} a_{jk}^{Post}(t) p_{jk}(t,t+1)$$

Continue with $\tau \geq 1$ and using Chapman-Kolmogorov:

$$\sum_{\tau \geq 1} v^\tau \left(\sum_i a_i^{Pre}(t+\tau) \cdot \underbrace{p_{ji}(t,t+\tau)}_{\sum_l p_{jl}(t,t+1) p_{li}(t+1,t+\tau)} \right.$$

$$\left. + \sum_{i,k} a_{ik}^{Post}(t+\tau) \cdot \underbrace{p_{ji}(t,t+\tau)}_{\sum_l p_{jl}(t,t+1) p_{li}(t+1,t+\tau)} \cdot p_{ik}(t+\tau,t+\tau+1) \right)$$

Proof Thiele Equation

By factor out terms which are independent of τ as well as rearranging the τ-sum $(\tau - 1 \rightarrow \tau)$ we get:

$$\sum_{l} p_{jl}(t, t+1) \cdot v \cdot \left(\sum_{\tau \geq 0} v^{\tau} p_{li}(t+1, t+1+\tau) \right.$$

$$\left. \left[\sum_{i} a_{i}^{Pre}(t+1+\tau) + \sum_{i,k} a_{ik}^{Post}(t+1+\tau) p_{ik}(t+1+\tau, t+\tau+1+1) \cdot v \right] \right)$$

Comparing the expression in () with equation (**??**) we recognize that this is just $V_l(t+1)$. Now combing both parts we get:

$$V_j(t) = a_j^{Pre}(t) + v \sum_{k \in \mathcal{S}} p_{jk}(t, t+1) \left(a_{jk}^{Post}(t) + V_k(t+1) \right)$$

This concludes the proof.

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How to simulate a trajectory of a discrete time, finite State space Markov Chain

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Simulate mathematical reserve

Determination of cumulative Probability Density of the reserves

Simulation of individual cash flows

Task 2 : Stopping to pay Premium

Set Up

- Consider a mixed endowment $A_{80:\overline{10}|}$ maturing at age 90 with benefit insured $L = 100'000$.
- Techinical interest rate $i = 2\%$ and Preimum is paid anually prenumerando

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Equivalence Principle

Death & Pure Endowment Insurance

Mathemical Reserves

Mathematical Reserves at given time = PV(future benefits) - PV(future premiums)

Expressed in commutation functions:

$$_tV_x = \frac{M_{x+t} - M_{x+n} + D_{x+n} - \Pi \cdot (N_{x+t} - N_{x+n})}{D_{x+t}}$$

2.1 Premium for Product at Inception

- Compute the premium by means of the equivalence principle.
- For the mixes endowment with benefit L we have

$$\Pi = L \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = L \left(\frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}} \right)$$

- Plugging in the values we get $\Pi \approx 12'302.98$

2.2 Benefit $\tilde{L}$ after policyholder stops after one premium.

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Task 3: Disability Insurance on two lives

Problem

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3.1

Two random variables X and Y are **stochastically independent** if and only if their joint probability distribution factorizes:

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y) \quad \forall x, y$$

Or equivalently:

$$P(Y = y \mid X = x) = P(Y = y) \quad \forall x, y.$$

Intuitively: Knowing X provides no information about Y

Two random variables X and Y are **uncorrelated** if their covariance is zero:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = 0.$$

Or equivalently:

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

3.1

Independence $\Rightarrow$ Uncorrelated

If X and Y are independent, then they are uncorrelated.
Proof:

$$\mathbb{E}[XY] = \sum_x \sum_y xy \cdot P(X = x) \cdot P(Y = y) = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Uncorrelated $\not\Rightarrow$ Independence

Two variables can be uncorrelated yet still be dependent!

3.1 Counterexample: Uncorrelated but Dependent

Let $X \sim \text{Uniform}(-1, 1)$ and $Y = X^2$

- $\mathbb{E}[XY] = \mathbb{E}[X^3] = 0$ (by symmetry)

- $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y] = \frac{1}{3}$

- Therefore: $\mathbb{E}[X] \cdot \mathbb{E}[Y] = 0$

- So $\text{Cov}(X, Y) = 0 \Rightarrow$ **uncorrelated**

- But Y is completely determined by $X \Rightarrow$ **dependent!**

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