Exam Presentation
Life Insurance Mathematics

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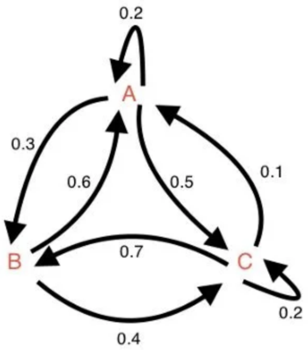
Task 1: Markov Model

Markov Chains

What is a Markov Chain?
It is a Stochastic model that describes sequence of transitions/ possible events in which the probability of each event depends only on the state attained in the previous event.



Markow Chains

Definition

$(X_t)_{t\in\mathbb{N}} : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathcal{S} = \{1, 2, 3, \ldots\}$, is called a **Markov chain** if and only if,

$$\mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_1} = i_1, \ldots, X_{t_m} = i_m] = \mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_m} = i_m]$$

for $t_1 < t_2 < \cdots < t_m < t_{m+1}$ and $i_1, i_2, \ldots, i_{m+1} \in \mathcal{S}$.

We say that such a stochastic process $(X_t)_{t\in\mathbb{N}}$ has no memory.

Markov Chains

Chapman-Kolmogorov Theorem

Let $p_{ij}(s,t) = P(X_t = j | X_s = i)$ be the transition probabilities of a Markov chain. Then, for any $0 \leq s < u < t$,

$$p_{ij}(s,t) = \sum_k p_{ik}(s,u) p_{kj}(u,t).$$

Or written in matrix form, $P(s,t) = P(s,u)P(u,t)$.

Idea: What is the probbility of being in state j at time t, given that at time s we are in state i?

Markov Chains

Proof

$$p_{ij}(s,t) = \mathbb{P}[X_t = j | X_s = i] = \mathbb{P}[X_t = j \cap \bigcup_{k \in \mathcal{S}} \{X_u = k\} | X_s = i]$$

$$= \sum_{k \in \mathcal{S}} \mathbb{P}\left[X_t = j, X_u = k | X_s = i\right]$$

$$= \sum_{k \in \mathcal{S}} \frac{\mathbb{P}[X_t = j, X_u = k, X_s = i]}{\mathbb{P}[X_s = i]} \cdot \frac{\mathbb{P}[X_u = k, X_s = i]}{\mathbb{P}[X_u = k, X_s = i]}$$

$$= \sum_{k \in \mathcal{S}} \underbrace{\mathbb{P}[X_t = j | X_u = k, X_s = i]}_{\mathbb{P}[X_t = j | X_u = k]} \mathbb{P}[X_u = k | X_s = i]$$

$$= \sum_{k \in \mathcal{S}} p_{ik}(s,u) p_{kj}(u,t)$$

In the above we used $\mathbb{P}[A \cap B | C] = \mathbb{P}[A | B \cap C] \cdot \mathbb{P}[B | C]$ as well as the Markov porperty as well as assuming that $\mathbb{P}[X_u = k, X_s = i] \neq 0$.

Markov Model

To model a life Insurance we need three ingredients:

- a Markov chain $(X_t)_{tin\mathbb{N}}$
- a one-year discount factor $v = \frac{1}{1+i}$
- contract functions $a_i^{pre}(t)$ and $a_{ij}^{post}(t)$

The starting point of an Markov model are the various possible conditions for an insured person, building the state space $\mathcal{S}$. E.g. $\mathcal{S} = \{\text{'living'}, \text{'death'}\}$.

Induced Cashflow & Mathemtical Reserve

A central task in life insurance is the determination of the actuarial reserve, i.e., the amount of money which has to be set aside at a given time t to be able to meet all future obligations/benefits towards each policy.
We denote by A_t the payments that are due for a policy at time t.
$(A_t)_{t \in \mathbb{N}}$ is a stochastic process.

$$A_t = a_{X_t}^{Pre}(t) + a_{X_{t-1}X_t}^{Post}(t)$$

where $a_{ij}^{Post}(-1) = 0$ for all i,j $\in \mathcal{S},\ t \in \mathbb{N}$.

Induced Cashflow & Mathemtical Reserve

We set $I_i(t) = \mathbb{1}_{\{X_t=i\}}$. Then we can compute the **induced cash flows** as follows:

$$A(t) = \underbrace{\sum_{i\in\mathcal{S}} I_i(t)\cdot a_i^{pre}(t)}_{\text{annuity}} + \underbrace{\sum_{i,j\in\mathcal{S}} I_i(t)\cdot I_j(t+1)\cdot a_{ij}^{post}(t)}_{\text{captial/lump sum paid at time t+1}}$$

Idea: $A(t)$ are the payments are due at time t for a given policy. We can also compute the present value (PV) of A(t) which is given by:

$$\tilde{A}(t) = \sum_{i\in\mathcal{S}} I_i(t)\cdot a_i^{pre}(t) + v\cdot \sum_{i,j\in\mathcal{S}} I_i(t)\cdot I_j(t+1)\cdot a_{ij}^{post}(t)$$

Finally we can define the mathematical reserve at time t as:

$$V_j(t) = \mathbb{E}[\text{PV of future cash flows}|\ X_t = j] = \mathbb{E}[\sum_{\tau=0}^{\infty} v^\tau \tilde{A}(t+\tau)|X_t = j]$$

Mathematical Reserve

We can compute the mathematical reserves with the following results:

$$\mathbb{E}[I_i(t+\tau)|\ X_t = j] = p_{ji}(t, t+\tau)$$
$$\mathbb{E}[I_i(t+\tau)\ I_k(t+\tau+1)|\ X_t = j] = p_{ji}(t, t+\tau) p_{ik}(t+\tau, t+\tau+1)$$

Hence the reserve is given as:

$$V_j(t) = \sum_{\tau=0}^{\infty} v^{\tau} \left(\sum_{i \in \mathcal{S}} a_i^{Pre}(i+\tau) p_{ji}(t, t+\tau) \right.$$
$$\left. + \ v \sum_{i,k \in \mathcal{S}} a_{ik}^{Post}(t+\tau) p_{ji}(t, t+\tau) p_{ik}(t+\tau, t+\tau+1) \right) \quad (1)$$

Hence $V_j(t)$ is the current value of the future actuarial reserve cash flow (A) based on todays information, i.e. $X_t = j$.

Thiele Equation

We can relate the mathematical reserves at two subsequent time points t and t+1 via the following equation:

Theorem (Thiele's difference equation)

The mathematical reserve between two subsequent periods are related by:

$$V_j(t) = a_j^{pre}(t) + \sum_{i \in \mathcal{S}} v \cdot p_{ji}(t, t+1) \cdot (a_{ji}^{post}(t) + V_i(t+1))$$

Remarks:

- Calculates the expected reserve directly using transition probabilities, while simulation estimates it by averaging over many random trajectories
- To solve Thiele's equation we need the boundary condition $V_j(T) = 0$ for all $j \in \mathcal{S}$.
- The Thiele equation is leads to the same results as for the classical insurance model (using commutation functions)
- Forward computation of reserves is possible, too. but numerically less stable.

Proof Thiele Equation

We start the prove by separating the above sum into $\tau = 0$ and the rest:
Lets start with $\tau = 0$:

$$\sum_{i\in\mathcal{S}} a_i^{Pre}(t) \underbrace{p_{ji}(t,t)}_{\delta_{ij}} + v \sum_{i,k\in\mathcal{S}} a_{ik}^{Post}(t) p_{ji}(t,t) p_{ik}(t,t+1)$$

$$= a_j^{Pre}(t) + v \sum_{k\in\mathcal{S}} a_{jk}^{Post}(t) p_{jk}(t,t+1)$$

Continute with $\tau \geq 1$ and using Chapman-Kolmogorov:

$$\sum_{\tau\geq 1} v^{\tau} \left(\sum_i a_i^{Pre}(t+\tau) \cdot \underbrace{p_{ji}(t,t+\tau)}_{\sum_l p_{jl}(t,t+1)p_{li}(t+1,t+\tau)} \right.$$

$$\left. + \sum_{i,k} a_{ik}^{Post}(t+\tau) \cdot \underbrace{p_{ji}(t,t+\tau)}_{\sum_l p_{jl}(t,t+1)p_{li}(t+1,t+\tau)} \cdot p_{ik}(t+\tau, t+\tau+1) \right)$$

Proof Thiele Equation

By factor out terms which are independent of τ as well as rearranging the τ-sum $(\tau - 1 \to \tau)$ we get:

$$\sum_{l} p_{jl}(t, t+1) \cdot v \cdot \left(\sum_{\tau \geq 0} v^{\tau} p_{li}(t+1, t+1+\tau) \right.$$
$$\left. \left[\sum_{i} a_i^{Pre}(t+1+\tau) + \sum_{i,k} a_{ik}^{Post}(t+1+\tau) p_{ik}(t+1+\tau, t+\tau+1+1) \cdot v \right] \right)$$

Comparing the expression in () with equation (**??**) we recognize that this is just $V_l(t+1)$. Now combing both parts we get:

$$V_j(t) = a_j^{Pre}(t) + v \sum_{k \in \mathcal{S}} p_{jk}(t, t+1) \left(a_{jk}^{Post}(t) + V_k(t+1) \right)$$

This concludes the proof.

How to simulate a trajectory of a discrete time, finite State space Markov Chain

To simulate a trajectory of a discrete time, finite State space Markov Chain sith finite state space $\mathcal{S} = \{1, 2, 3, \dots, n\}$ and translation probability matrix $P = (p_{ij})$ we can use the **Inverse Transformation Method**:

1. Start with initial state $X_0 = i_0$
2. At each time step t, given $X_t = i$
   - Generate $U \sim \text{Uniform}(0, 1)$
   - Find j such that $\sum_{k=1}^{j-1} p_{ik} \leq U < \sum_{k=1}^{j} p_{ik}$
   - Set $X_{t+1} = j$

This algortihm fulfills the Markove no-memory property by construction since only the current state X_t is used to determine the next state X_{t+1}.

Simulate mathematical reserve

The expression $V(X(\omega))[0]$ denotes the PV of future cash flows generated by $X(\omega)$ at time $t = t_0$ for a given trajectory of the Markov chain. Recall that the mathematical reserve is defined as $V_j(t) = \mathbb{E}[\text{PV of future cash flows}|X_{t_0} = j]$.
Now given n independent trajectories $X(\omega_1), X(\omega_2), \ldots, X(\omega_n)$ we know that by the **Law of Large Numbers**:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} V(\omega_k)[0] = V(t = t_0),$$

i.e., the average of the PV of future chash flows over many independent trajectories converges to the true mathematical reserve at time $t = t_0$ almos surely.

Determination of cumulative Probability Density of the reserves

Let $\{V(\omega_k)[0]\}_{k=1}^{n}$ be independent trajectoires. The empirical ccumulatice distribution function is given by:

$$F_n(v) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{[-\infty,v]}(V(\omega_k)[0])$$

Then define the random variable $D_n = \sup_{v \in \mathbb{R}} |F_n(v) - F(v)|$. By the **Glivenko-Cantelli Theorem** we know that $D_n \xrightarrow{a.s.} 0$ for $n \to \infty$.

Hence if we have enough samples/trajectories we can approximate the cumulative distribution function of the mathematical reserves.

Task 2 : Stopping to pay Premium

Mixed Endowment Insurance

What is a mixed endowment?

- a mix of a term (temporary death) and a pure endowment insurance
- a lump sum is payable on death or reachinga certain age

Let us look at these random variables individually

- term insurance : $Z_1 = v^{k+1}\mathbb{1}_{k<n}$

  $A_{x:\overline{n}|}^{\ 1} = \mathbb{E}[Z_1] = \sum_{k=0}^{n-1} v^{k+1}\ {}_kp_x \cdot q_{x+k} = \sum_{k=0}^{n-1} v^{k+1} \cdot \frac{l_{x+k}}{l_x} \cdot \frac{d_{x+k}}{l_{x+k}}$

  $= \sum_{k=0}^{n-1} \frac{C_{x+k}}{D_x} = \frac{M_x - M_{x+n}}{D_x}$

- pure endowment: $Z_2 = v^{k+1}\mathbb{1}_{k\geq n}$
- $A_{x:\overline{n}|}^{1} = \mathbb{E}[Z_2] = \sum_{k=n}^{\infty} v^n \mathbb{P}[K=k] = v^n\mathbb{P}[K \geq n] = v^n \cdot {}_np_x = \frac{D_{x+n}}{D_x}$
- Thus the epxectation value for the mixed enovment is given by

$$A_{x:\overline{n}|} = \mathbb{E}[Z] = \mathbb{E}[Z_1] + \mathbb{E}[Z_2] = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

Equivalence Principle

In order to determine the premium one typically uses the Equivalence Principle

1. $\mathbb{E}[L] = 0$ where L denotes the loss, or equivalently,
2. The expected value of premiums is equal to the expected value of benefits.

Remark:

- The Equivalece Principle is equivalent to the requirement of the mathematical reserve at inception to be zero.

As an example consider a termn insurance where we define the loss as

$$L = C \cdot v^{k+1} \mathbb{1}_{k<n} - \Pi \cdot \ddot{a}_{\min(k,n)}$$

By the Equivalence Pricinple we have $\mathbb{E}[L] = 0$ which leads to

$$\Pi \cdot \ddot{a}_{x:\overline{n}|} = \Pi \cdot A_{x:\overline{n}|}^{\;1}.$$

Mathematical Reserves

The Mathematical Reserves V at given time t are defined as
V = PV(future benefits) - PV(future premiums)

For a mixed endowment insurance with n annual premiums Π the mathematical reserve at time t is given by (expressed in commutation functions)

$$_tV_x = \frac{M_{x+t} - M_{x+n} + D_{x+n} - \Pi \cdot (N_{x+t} - N_{x+n})}{D_{x+t}}$$

2.1 Premium for Product at Inception

- Compute the premium by means of the equivalence principle.
- For the mixes endowment with benefit L we have

$$\Pi = L\frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = L\left(\frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}\right)$$

- Plugging in the values we get $\Pi \approx 12'302.98$

2.2 Benefit $\tilde{L}$ after policyholder stops after one premium.

- Combining the formulas for the mixed endowment insurance $A_{x+k:\overline{n-k}|}$ and the mathematical reserve ${}_kV_x$ we obtain the following expression for the reduced benefit using commutation functions

$$ {}_1\tilde{L} = \frac{{}_kV_x}{A_{x+k:\overline{n-k}|}} = {}_kV_x \cdot \frac{D_{x+k}}{M_{x+k} - M_{x+n} + D_{x+n}} $$

- Result of the python code is: ${}_1\tilde{L} \approx 9228.77$.

2.3 Benefit Level $\tilde{L}$ as a function of the number of paid premiums

- Let k be the number of paid premiums then by the previous exercise

$$
{}_k\tilde{L} = \frac{{}_kV_x}{A_{x+k:\overline{n-k}|}} = {}_kV_x \cdot \frac{D_{x+k}}{M_{x+k} - M_{x+n} + D_{x+n}}
$$

k	Reserve ${}_jV_x$	Benefit Level ${}_k\tilde{L}$
1	8062.41	9228.77
2	16260.21	18375.48
3	24650.21	27498.51
4	33308.23	36670.15
5	42335.99	45980.83
6	51870.01	55545.00
7	62095.67	65509.06
8	73266.94	76062.14
9	85736.24	87450.96
10	100000.0	100000.0

2.4 Which equivalence principle is fulfilled for the first premium assuming only one premium is paid.

- $t = 0$ and premium P
- Payout/Benefits:
  i.) if $t \in [0, 1)$ then $L = 100'000$ ii.) if $t \in [1, 10)$ then ${}_t\tilde{L}$
- by the **Equivalence Principle**: $\mathbb{E}(\textit{premium}) = \mathbb{E}(\textit{benefits})$
- the benefits are computed as

$$
\begin{aligned}
\mathbb{E}(\textit{benefits}) &= v \cdot q_x \cdot L + v^2 \cdot {}_1\tilde{L} \cdot p_x \cdot q_{x+1} + \cdots + v^{10} \cdot {}_1\tilde{L} \cdot {}_{10}p_x \\
&= v \cdot q_x \cdot L + v \cdot p_x \cdot {}_1\tilde{L} \cdot A_{x+1:\overline{9}|}
\end{aligned}
$$

- A short check in the jupyter notebook shows that
  $\mathbb{E}(\textit{premium}) = \mathbb{E}(\textit{benefits})$

2.5 Which equivalence principle is fulfilled for the second premium assuming only one premium is paid.

- Consider $t = 1$
- Assuming one premium is paid at $t = 0$
- Assuming that insured person survives the first year
- Insurer is at risk to pay L if the death occurs in $[0, 1)$
- From time $t = 1$ on until maturity the insruer is at risk of ${}_2\tilde{L}_x$
- By the Equivalence Principle $\mathbb{E}(\mathit{premium}) = \mathbb{E}(\mathit{benefits})$
- $\mathbb{E}(\mathit{premium}) = {}_1V_x$

2.5 Which equivalence principle is fulfilled for the second premium assuming only one premium is paid.

- Similar to before we compute the benefits:

$$\begin{aligned}\mathbb{E}(benefits) &= v \cdot q_{x+1} \cdot {}_1\tilde{L} + v^2 \cdot {}_1\tilde{L} \cdot p_{x+1} \cdot q_{x+2} + \cdots + v^9 \cdot {}_1\tilde{L} \cdot {}_9p_x \\ &= v\, {}_1\tilde{L} \cdot A_{x+1:\overline{9}|}\end{aligned}$$

- A short check in the jupyter notebook shows that $\mathbb{E}(premium) = \mathbb{E}(benefits)$ as well as the euqality is exactly the defintion for the reduced benefit ${}_1\tilde{L}$.

Task 3: Disability Insurance on two lives

3.1

Two random variables X and Y are **stochastically independent** if and only if their joint probability distribution factorizes:

$$P(X = x, Y = y) = P(X = x) \cdot P(Y = y) \quad \forall x, y$$

Or equivalently:

$$P(Y = y \mid X = x) = P(Y = y) \quad \forall x, y.$$

Intuitively: Knowing X provides no information about Y

Two random variables X and Y are **uncorrelated** if their covariance is zero:

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = 0.$$

Or equivalently:

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

3.1

Independence $\Rightarrow$ Uncorrelated, Proof

Let X and Y be discrete and independent random variables.

$$\begin{aligned}\mathbb{E}[XY] &= \sum_x \sum_y xy \cdot P(X = x, Y = y) \\ &= \sum_x \sum_y xy \cdot P(X = x) \cdot P(Y = y) = \mathbb{E}[X] \cdot \mathbb{E}[Y].\end{aligned}$$

This concludes the proof.

Uncorrelated $\not\Rightarrow$ Independence

Two variables can be uncorrelated yet still be dependent!

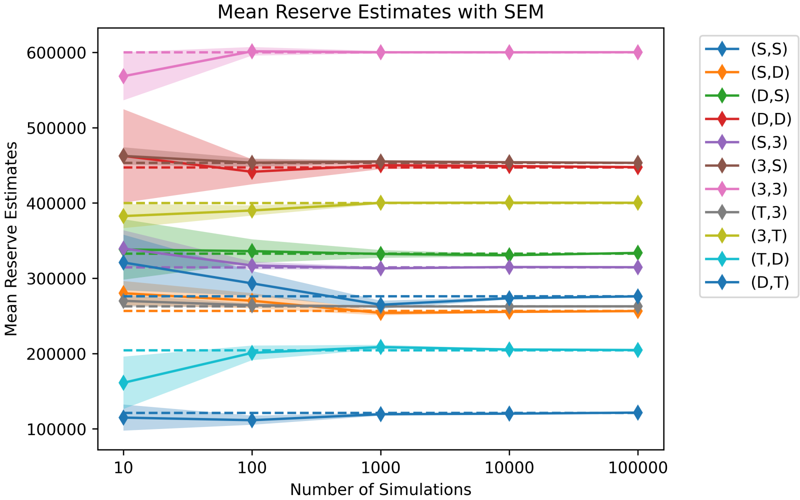
3.1 Counterexample: Uncorrelated but Dependent

Let $X \sim \text{Uniform}(-1, 1)$ and $Y = X^2$

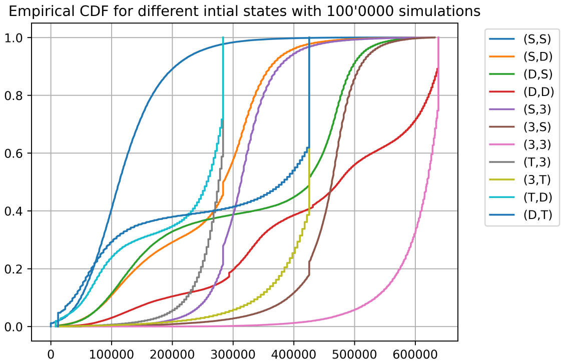
- $\mathbb{E}[XY] = \mathbb{E}[X^3] = 0$ (by symmetry)
- $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y] = \frac{1}{3}$
- Therefore: $\mathbb{E}[X] \cdot \mathbb{E}[Y] = 0$
- So $\text{Cov}(X, Y) = 0 \Rightarrow$ **uncorrelated**
- But Y is completely determined by $X \Rightarrow$ **dependent!**

3.4 MC Averages Analysis

Here we show how the Monte Carlo estimates move with increasing number of simulated trajectories.

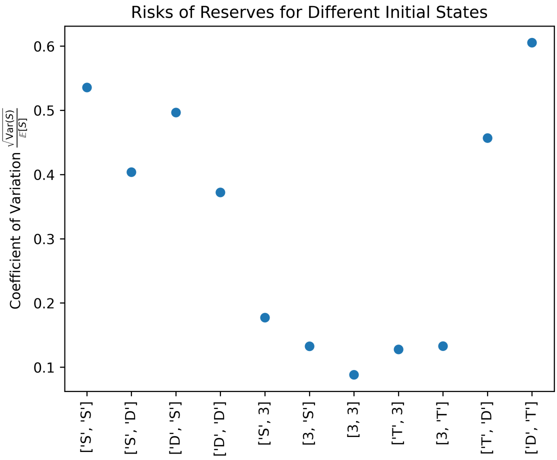


3.4 CDF for 100'000 Simulations



3.5 Discussion of Risk Levels of States

In order to dicuss the riskiness of the states we look at the coefficient of variation of the reserve in each state



3.5 Discussion of Risk Levels of States

- The state (D, T) is the most dangerous state (has highest CoV) since the man is disabled and rather young (30y). Hence he is has three possibilities: reactivate, remain disabled for extended periods.
- The diversification from having two people in similar but independent states reduces relative variability
- The state $(3, 3)$ is the safest one (although highest annuity to be paid) since both persons are disabled and either remain disabled or die.
- So ranking the states according to their risk level (from riskiest to safest) we get:

$$\begin{aligned}(D, T) &> (D, D) > (3, T) > (D, S) > (3, 3) > (3, S) \\ &> (T, 3) > (S, D) > (S, 3) > (T, D) > (S, S)\end{aligned}$$

3.6 Dicussion of Changes in Risk when changing from independent to Dependent Random Variables

3.7

3.8: 8-policy portfolio simulation

- Assuming the 8 policies are independent, we compute the total loss as the sum of the indiviudal mathematical reserves.

$$\mathbb{E}[L] = \sum_{i=1}^{8} \mathbb{E}[L_i] = 1'816'458.10$$

- Now we want to compute the stop loss premium $\Pi(\beta)$ for the deductibles $\beta \in \{1.5 \cdot \mathbb{E}[L], 3 \cdot \mathbb{E}[L]\}$
- The stop loss premium is defined as

$$\Pi(\beta) = \mathbb{E}[(L-\beta)_+] = \int_{\beta}^{\infty} (1 - F(x))\, dx$$

- In order to create a histogram-based approximation of the cumulative distribution function $F(x)$ we use 1'000'000 simulated portfolio losses (in total 8'000'000 simulations)
-

3.8: 8-policy portfolio simulation

Inser pciture of histogram here

3.8: 8-policy portfolio simulation

Now we shortly describe how to compute the stop loss premium $\Pi(\beta)$ using a histogram-based approximation of the cumulative distribution function $F(x)$.

- Let the bins of the histogram be $[a_1, a_2), [a_2, a_3), \ldots, [a_N, a_{N+1})$.
- Assume the survival function is constant on each bin: $\bar{F}(x) = 1 - F(x) \approx \bar{F}_i$ where $\bar{F}_i$ denotes the value of the survival function on bin i.
- The integral representation approximated by the Riemann sum:

$$\Pi(\beta) \approx \sum_{i:\, a_i \geq \beta} \bar{F}_i \,(a_{i+1} - a_i)$$

- Plugging in the values for β we get:

$$\Pi(1.5 \cdot \mathbb{E}[L]) =$$
$$\Pi(3 \cdot \mathbb{E}[L]) =$$