

# Exam Presentation

## Life Insurance Mathematics

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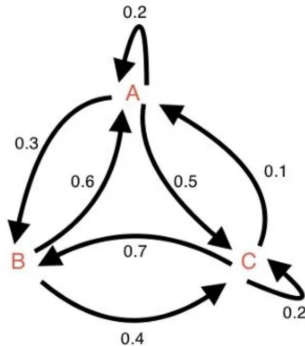
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# Task 1: Markov Model

# Markov Chains

What is a Markov Chain?

It is a Stochastic model that describes sequence of transitions from one state to another according to certain probabilistic rules.



# Markov Chains

## Definition

$(X_t)_{t \in \mathbb{N}} : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathcal{S} = \{1, 2, 3, \dots\}$ , is called a **Markov chain** if and only if,

$$\mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_1} = i_1, \dots, X_{t_m} = i_m] = \mathbb{P}[X_{t_{m+1}} = i_{m+1} | X_{t_m} = i_m]$$

for  $t_1 < t_2 < \dots < t_m < t_{m+1}$  and  $i_1, i_2, \dots, i_{m+1} \in \mathcal{S}$ .

We say that such a stochastic process  $(X_t)_{t \in \mathbb{N}}$  has no memory.

# Markov Chains

## Chapman-Kolmogorov Theorem

Let  $p_{ij}(s, t) = P(X_t = j | X_s = i)$  be the transition probabilities of a Markov chain. Then, for any  $0 \leq s < u < t$ ,

$$p_{ij}(s, t) = \sum_k p_{ik}(s, u) p_{kj}(u, t).$$

Or written in matrix form,  $P(s, t) = P(s, u)P(u, t)$ .

Idea: What is the probability of being in state  $j$  at time  $t$ , given that at time  $s$  we are in state  $i$ ?

# Markov Chains

## Proof

$$\begin{aligned} p_{ij}(s, t) &= \mathbb{P}[X_t = j | X_s = i] = \mathbb{P}[X_t = j \cap \bigcup_{k \in S} \{X_u = k\} | X_s = i] \\ &= \sum_{k \in S} \mathbb{P}[X_t = j, X_u = k | X_s = i] \\ &= \sum_{k \in S} \frac{\mathbb{P}[X_t = j, X_u = k, X_s = i]}{\mathbb{P}[X_s = i]} \cdot \frac{\mathbb{P}[X_u = k, X_s = i]}{\mathbb{P}[X_u = k, X_s = i]} \\ &= \sum_{k \in S} \underbrace{\mathbb{P}[X_t = j | X_u = k, X_s = i]}_{\mathbb{P}[X_t = j | X_u = k]} \mathbb{P}[X_u = k | X_s = i] \\ &= \sum_{k \in S} p_{ik}(s, u) p_{kj}(u, t) \end{aligned}$$

In the above we used  $\mathbb{P}[A \cap B | C] = \mathbb{P}[A | B \cap C] \cdot \mathbb{P}[B | C]$  as well as the Markov property as well as assuming that  $\mathbb{P}[X_u = k, X_s = i] \neq 0$ .

# Markov Model

To model a life Insurance we need three ingredients:

- a Markov chain  $(X_t)_{t \in \mathbb{N}}$
- a one-year discount factor  $v = \frac{1}{1+i}$
- contract functions  $a_i^{pre}(t)$  and  $a_{ij}^{post}(t)$

The starting point of an Markov model are the various possible conditions for an insured person, building the state space  $\mathcal{S}$ . E.g.  $\mathcal{S} = \{\text{'living'}, \text{'death'}\}$ .



# Induced Cashflow & Mathematical Reserve

A central task in life insurance is the determination of the actuarial reserve, i.e., the amount of money which has to be set aside at a given time  $t$  to be able to meet all future obligations/benefits towards each policy.

We denote by  $A_t$  the payments that are due for a policy at time  $t$ .  $(A_t)_{t \in \mathbb{N}}$  is a stochastic process.

$$A_t = a_{X_t}^{Pre}(t) + a_{X_{t-1}X_t}^{Post}(t)$$

where  $a_{ij}^{Post}(-1) = 0$  for all  $i, j \in \mathcal{S}$ ,  $t \in \mathbb{N}$ .

# Induced Cashflow & Mathematical Reserve

We set  $I_i(t) = \mathbb{1}_{\{X_t=i\}}$ . Then we can compute the **induced cash flows** as follows:

$$A(t) = \underbrace{\sum_{i \in \mathcal{S}} I_i(t) \cdot a_i^{pre}(t)}_{\text{annuity}} + \underbrace{\sum_{i,j \in \mathcal{S}} I_i(t) \cdot I_j(t+1) \cdot a_{ij}^{post}(t)}_{\text{capital/lump sum paid at time } t+1}$$

Idea:  $A(t)$  are the payments are due at time  $t$  for a given policy. We can also compute the present value (PV) of  $A(t)$  which is given by:

$$\tilde{A}(t) = \sum_{i \in \mathcal{S}} I_i(t) \cdot a_i^{pre}(t) + v \cdot \sum_{i,j \in \mathcal{S}} I_i(t) \cdot I_j(t+1) \cdot a_{ij}^{post}(t)$$

Finally we can define the mathematical reserve at time  $t$  as:

$$V_j(t) = \mathbb{E}[\text{PV of future cash flows} | X_t = j] = \mathbb{E}\left[\sum_{\tau=0}^{\infty} \tilde{A}(t+\tau) | X_t = j\right]$$

# Mathematical Reserve

We can compute the mathematical reserves with the following results:

$$\begin{aligned}\mathbb{E}[I_i(t + \tau) | X_t = j] &= p_{ji}(t, t + \tau) \\ \mathbb{E}[I_i(t + \tau) I_k(t + \tau + 1) | X_t = j] &= p_{ji}(t, t + \tau) p_{ik}(t + \tau, t + \tau + 1)\end{aligned}$$

Hence the reserve is given as:

$$\begin{aligned}V_j(t) = \sum_{\tau=0}^{\infty} v^{\tau} \left( \sum_{i \in \mathcal{S}} a_i^{Pre}(i + \tau) p_{ji}(t, t + \tau) \right. \\ \left. + v \sum_{i, k \in \mathcal{S}} a_{ik}^{Post}(t + \tau) p_{ji}(t, t + \tau) p_{ik}(t + \tau, t + \tau + 1) \right) \quad (1)\end{aligned}$$

Hence  $V_j(t)$  is the current value of the future actuarial reserve cash flow ( $A$ ) based on today's information, i.e.  $X_t = j$ .

# Thiele Equation

We can relate the mathematical reserves at two subsequent time points  $t$  and  $t+1$  via the following equation:

## Theorem (Thiele's difference equation)

The mathematical reserve between two subsequent periods are related by:

$$V_j(t) = a_j^{pre}(t) + \sum_{i \in S} v \cdot p_{ji}(t, t+1) \cdot (a_{ji}^{post}(t) + V_i(t+1))$$

### Remarks:

- To solve Thiele's equation we need the boundary condition  $V_j(T) = 0$  for all  $j \in S$ .
- The Thiele equation leads to the same results as for the classical insurance model (using commutation functions)
- Forward computation of reserves is possible, too. but numerically less stable.

# Proof Thiele Equation

We start the prove by separating the above sum into  $\tau = 0$  and the rest:  
Lets start with  $\tau = 0$ :

$$\begin{aligned} & \sum_{i \in S} a_i^{Pre}(t) \underbrace{p_{ji}(t, t)}_{\delta_{ij}} + v \sum_{i, k \in S} a_{ik}^{Post}(t) p_{ji}(t, t) p_{ik}(t, t+1) \\ &= a_j^{Pre}(t) + v \sum_{k \in S} a_{jk}^{Post}(t) p_{jk}(t, t+1) \end{aligned}$$

Continue with  $\tau \geq 1$  and using Chapman-Kolmogorov:

$$\begin{aligned} & \sum_{\tau \geq 1} v^\tau \left( \sum_i a_i^{Pre}(t+\tau) \cdot \underbrace{p_{ji}(t, t+\tau)}_{\sum_l p_{jl}(t, t+1) p_{li}(t+1, t+\tau)} \right. \\ & \left. + \sum_{i, k} a_{ik}^{Post}(t+\tau) \cdot \underbrace{p_{ji}(t, t+\tau)}_{\sum_l p_{jl}(t, t+1) p_{li}(t+1, t+\tau)} \cdot p_{ik}(t+\tau, t+\tau+1) \right) \end{aligned}$$

# Proof Thiele Equation

By factor out terms which are independent of  $\tau$  as well as rearranging the  $\tau$ -sum ( $\tau - 1 \rightarrow \tau$ ) we get:

$$\sum_l p_{jl}(t, t+1) \cdot v \cdot \left( \sum_{\tau \geq 0} v^\tau p_{li}(t+1, t+1+\tau) \left[ \sum_i a_i^{Pre}(t+1+\tau) + \sum_{i,k} a_{ik}^{Post}(t+1+\tau) p_{ik}(t+1+\tau, t+\tau+1+1) \cdot v \right] \right)$$

Comparing the expression in ( ) with equation (??) we recognize that this is just  $V_l(t+1)$ . Now combining both parts we get:

$$V_j(t) = a_j^{Pre}(t) + v \sum_{k \in S} p_{jk}(t, t+1) (a_{jk}^{Post}(t) + V_k(t+1))$$

This concludes the proof.

## Task 2 : Stopping to pay Premium

# Set Up



# Equivalence Principle

# Death & Pure Endowment Insurance

# Mathematical Reserves

Mathematical Reserves at given time = PV(future benefits) - PV(future premiums)

Expressed in commutation functions:

$${}_tV_x = \frac{M_{x+t} - M_{x+n} + D_{x+n} - \Pi \cdot (N_{x+t} - N_{x+n})}{D_{x+t}}$$



## Task 3: Disability Insurance on two lives

# Problem

# Appendix