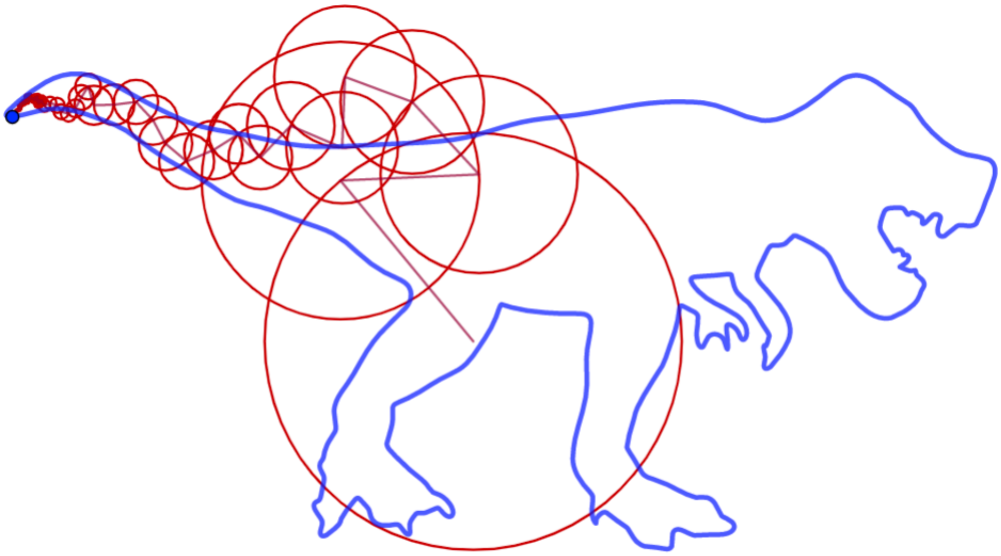
Mathematical Methods of Physics 1

Summary based on

lectures in He20 hold by Prof. T.H. Willwacher
&
teaching classes hold by M. Mohanarangan



type setted by Sami Zweidler
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Preface

The following LaTeX script is based on a personal summary I wrote while preparing for an exam, using lecture notes from the Mathematical Methods I course held by Prof. Dr. T. Willwacher at ETH Zurich in Autumn 2020.

This summary contains the most important theorems and statements from the lectures. Not all proofs are included, some were omitted based on their relevance to the exam (in my opinion).

In addition to the lecture content, the summary includes selected material from the accompanying exercise classes taught by M. Mohanarangan, especially key concepts and exercises.

I've also added a few examples where the mathematical methods from the course are applied to physics, topics typically covered in later physics courses. These sections are marked with a star ($*$). Some of these are newer additions and include cool examples from advanced physics courses, explained as simply and intuitively as possible.

Lastly, a small disclaimer for mathematicians: this summary was written by a physics student, so do not expect full mathematical rigor in every proof or solution.

Contents

1 crash course measure theory

As the title of this section forshadows, this chapter will not rigorously introduce the Lebesgue integral. We rather provide a intuitive understanding of it a give a rough overview of the convergence theorems and L^p spaces.

1.1 Lebesgue Integral

1.1.1 construction

The construction of the Lebesgue integral presupposes a certain foundation in mass theory. One starts the construction on a so-called measure space, i.e. a set equipped with a σ-algebra and a measure. Furthermore, the notion of measurability (of both sets as well as of functions) is central.

1. **indicator function:** We define the integral of an indicator function 1_E of a measurable set E as measure of the set:

$$\int 1_E \, d\mu = \mu(E) \in [0, +\infty].$$

2. **simple functions:** Simple function are non-negative, measurable functions, which only take on finite many values. We can thus express every simple function ϕ as a finite linear combinatipn of indicator functions, i.e.,

$$\phi = \sum_{i=1}^{n} \lambda_i \cdot 1_{E_i}.$$

Thereby $\lambda_i \leq 0$ and E_i are measruable set on which the functions takes on the value λ_i. For such functions we can extend the integral through linearity:

$$\int \phi \, d\mu = \sum_{i=1}^{n} \lambda_i \cdot \mu(E_i).$$

The integral of ϕ is therefore simply the sum of the products of function value of ϕ and measure of the quantity on which the function takes the respective value.

3. **non-negative functions:** Let now f be a non-negative measurable function. One can find a monotonically growing sequence $(\phi_n)_n$ of simple functions, which converge pointwise to f. We define the corresponding integral through:

$$\int f \, d\mu = \lim_{n\to\infty} \int \phi_n \, d\mu.$$

In this part one has to pay attention on well-definiteness. First, one has to find such a func- tion sequence $(\phi_n)_n$ and then one has to show that the integral does not depend on the chosen sequence.

4. **general functions:** Measurable functions which are not necessarily non-negative can be written as $f = f^+ - f^-$. Subsequently we compute the integral of the positive and negative part as in step 3. We extend the integral through:

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

In the case where $\int |f| \, d\mu < \infty$, we say that f is Lebesgue integrable.

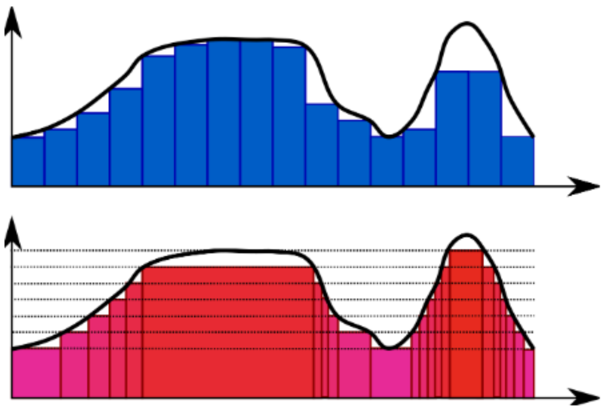

Figure 1: Limit value formation for the Riemann integral (blue) and the Lebesgue integral (red).

1.1.2 intuition

The Riemann integral considers the area under a graph by partitioning the x-axis and approximating the function by rectangles. The Lebesgue integral, on the other hand, partitions the y- axis and then measures the size of the pre-image. This definition thus considers more general surfaces than just rectangles. Therefore, the Lebesgue integral is more flexible, which finally allows us to to integrate a larger class of functions.

The key point of the Lebesgue integral is that function values can be rearranged without changing the value of the integral. value of the integral. This rearrangement allows us to integrate even exotic functions, such as the Dirichlet function, to integrate.

Example.
We consider the Dirichlet function, i.e., the indicator function $1_{\mathbb{Q}}$ restricted to $[0, 1]$.

- No matter how we partition the interval $[0, 1]$, each partition will contain rational as well as irational numbers. Thus the upper sum is always one and the lower sum is always zero. Since the two values differ, the function is not Riemann-integrable.
- We can partition the domain into 0 and 1 and consider the pre-images of the corresponding sets. We obtain:

$$\int_{[0,1]} 1_{\mathbb{Q}}\, d\mu = 1 \cdot \mu(\mathbb{Q} \cap [0, 1]) + 0 \cdot \mu(\mathbb{R}\backslash\mathbb{Q} \cap [0, 1]) = 0.$$

Obviously, we can now integrate on arbitrary measure spaces and are no longer bound to the structure of Rn as in the Riemann integral, but this is not the reason why the Lebesgue integral is already introduced in this lecture. It is also not because now a larger class of functions is integrable. With the introduction of Fourier series and Fourier transforms, we will often see expressions of the form

$$\sum_n \int f_n(x)\, dx \qquad \text{and} \qquad \partial_x \int f(x)\, dx.$$

Many problems in mathematical physics allow elegant solutions provided, that we can interchange integral and sum or derivative. So the question is if and when

$$\sum_n \int f_n(x)\,dx \stackrel{?}{=} \int \left(\sum_n f_n(x)\right) dx \qquad \text{and} \qquad \partial_x \int f(x)\,dx \stackrel{?}{=} \int \partial_x f(x)\,dx.$$

The interchanging of limits and integrals is difficult to describe in the context of Riemann integration. The Lebesgue integral, on the other hand, proves to be clearly more suitable.

The theorem of monotone convergence (MCT), the theorem of dominated convergence (DCT) are two important results that answer the question "When do limit and integral commute with each other?".

These theorems are essentially also the reason why we prefer the Lebesgue integral to the Riemann integral. The MCT and DCT give us certain conditions under which we can we can swap the limit and the boundary value.

1.2 convergence theorems

The following two theorems justify why we can interchange limit and integral in certain cases.

For that let E be a fixed measurable subset of $\mathbb{R}^n$.
All subsequent functions are defined on E.

Theorem 1.1 (Monotone convergence theorem, MCT)**.** Let f_i be a sequence of integrable functions with $0 \leq f_i(x) \leq f_{i+1}(x) \to f(x)$ (for $i \to \infty$) for all x. If the sequence $\int_E f_i(x)\,dx$ is bounded, then f is integrable and it holds that:

$$\lim_{i\to\infty} \int_E f_i(x)\,dx = \int_E f(x)\,dx. \tag{1}$$

Example. The MCT does not hold for the Riemann integral. We consider therefore the Dirichlet function $1_{\mathbb{Q}}$ restrictd to $[0,1]$. Let $(q_n)_n$ be an enumeration of all rational numbers in $[0,1]$ (this works since $\mathbb{Q}$ is countable). We define

$$g_n(x) = \begin{cases} 1, & \text{for } x = q_j \text{ and } j \leq n, \\ 0, & \text{else.} \end{cases}$$

The function g_n is takes on finite many times the value 1 and else 0. Thus the Riemann integral is 0 for all g_n. Each g_n is non-negative and the sequence $(g_n)_n$ is monotonically increasing. Nevertheless this sequence converges to the Dirichlet function $1_{\mathbb{Q}}$ which is, as it is known, not Riemann integrable.

Theorem 1.2 (Dominated convergence theorem, DCT)**.** Let f_i be a sequence of integrable functions with $\lim_{i\to\infty} f_i(x) = f(x)$ and suppose there exists a function $g \in L^1$ with
$\mid f_i(x) \mid \leq g(x) \forall i, x$. Then f is integrable and we have:

$$\lim_{i\to\infty} \int_E f_i(x)\,dx = \int_E f(x)\,dx$$

Example. Compute the integral: $\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n\sin(x/n)}{x(x^2+1)}\,dx$.

Let $x \in \mathbb{R}$ and begin defining the sequence $f_n(x) = \frac{n\sin(x/n)}{x(x^2+1)}$ for $n \in \mathbb{N}$.

The sequence is integrable and converges pointwise to:

$$\lim_{n\to\infty} f_n(x) = \underbrace{\lim_{n\to\infty} \frac{\sin(x/n)}{x/n}}_{=1} \frac{1}{x^2+1} = \frac{1}{x^2+1}$$

From this we also notice that $g(x) = \frac{1}{x^2+1}$ is a suitable dominating function because:

$$\begin{aligned} |f_n(x)| &= \left| \frac{\sin(x/n)}{x/n} \cdot \frac{1}{x^2+1} \right| \\ &= \frac{|\sin(x/n)|}{|x/n|} \frac{1}{x^2+1} \leq \frac{1}{x^2+1} \end{aligned}$$

The inequation comes from the fact, that $\left|\frac{\sin(x/n)}{x/n}\right| \leq 1 \, \forall x \in \mathbb{R}$. Hence we can apply the DCT to conclude:

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}} \frac{n\sin(x/n)}{x(x^2+1)}\,dx &= \int_{\mathbb{R}} \lim_{n\to\infty} \frac{n\sin(x/n)}{x(x^2+1)}\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \tan^{-1}(x)\Big|_{-\infty}^{\infty} = \pi. \end{aligned}$$

The following theorem states under what condition a multidimensional function is integrable. It also says something about how we calculate such multidimensional integrals.

Theorem 1.3 (Fubini's theorem)**.**
Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ measurable sets and let f be a measurable function on $E \times F$. Then:

1. If $f \geq 0$ then:

$$\int_E \left(\int_F f(x,y)\,dx \right) dy = \int_{E\times F} f(x,y)\,dx\,dy = \int_F \left(\int_E f(x,y)\,dy \right) dx \tag{2}$$

2. If f is complex valued and $\int_E \int_F |f(x,y)|\,dx\,dy < \infty$ or $\int_F \int_E |f(x,y)|\,dy\,dx < \infty$, then f is integrable i.e. $\int_{E\times F} |f(x,y)|\,dx\,dy < \infty$.

3. If f is complex valued and integrable over $E \times F$ then $f_x := f(x,\cdot)$ respectively $f_y := f(\cdot,y)$ are integrable for almost all x resp. y and it holds the equation (2).

Corollary 1.4. If $\int_{\mathbb{R}^n} |f(x)|\,dx < \infty$, we can compute $\int_{\mathbb{R}^n} f(x)\,dx$ as a multiple integral, namely:

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x)\,dx_1 \cdots dx_n$$

The calculation of the integral does not depend on the order of the execution.

1.3 L^p spaces

Let $E \subset \mathbb{R}^n$ a measurable set. All functions are complex valued and well defined on E.

Definition 1.5. Two integrable functions $f, g : E \to \mathbb{C}$ are called equivalent, if $f(x) = g(x)$ almost everywhere. This defines a equivalence relation.

The set of equivalence classes is called $L^1(E)$:

$$L^1(E) = \left\{ f \, \text{measurable} \middle| \int_E |f(x)| \; dx < \infty \right\} \Big/ \sim$$

More general for $p \geq 1$:

$$L^p(E) = \left\{ f \, \text{measurable} \middle| \int_E |f(x)|^p \; dx < \infty \right\} \Big/ \sim$$

Lemma 1.6. For $f, g \in L^p(E)$, $\lambda \in \mathbb{C}$ we have $f + \lambda g \in L^p(E)$. Thus $L^p(E)$ is a complex vector space.

Theorem 1.7. $L^p(E)$ with the norm

$$\|f\|_p = \left(\int_E |f(x)|^p \; dx \right)^{\frac{1}{p}}$$

is a normed vector space.

Proof. Positive definiteness and homogeneity are clear. The triangle inequality can also established easily for $L^1(E)$. For $L^p(E)$ it is called the Minkowski inequaltiy and it takes a little bit more effort to show this. $\square$

Theorem 1.8 (Fischer-Riesz)**.** $\forall p \geq 1 \, L^p(E)$ is a Banach space.

Hereinafter we assume that $E \subset \mathbb{R}^n$ is locally compact, i.e. each point of E has a compact neighbourhood in E. Examples are finite intersections and unions of open and closed subsets.

Definition 1.9. The **support** of a function $f : E \to \mathbb{C}$ is the subset
$\text{supp} f = \overline{\{x \in E \,|\, f(x) \neq 0\}} \subset E$.
We can then define $C_0(E) = \{$continuous functions on E with compact support$\}$.

Theorem 1.10. Let $E \subset \mathbb{R}^n$ locally compact and $1 \leq p \leq \infty$. The continuous functions with compact support are dense in $L^p(E)$, i.e. $\forall f \in L^p(E)$, $\forall \epsilon > 0 : \exists g \in C_0(E)$ with $\|f - g\|_p < \epsilon$

2 Fourier Series

2.1 Definition & Representation

Definition 2.1. Let $L > 0$ be some fixed positive number. Fourier series are series of the form

$$f(x) := \sum_{n=-\infty}^{\infty} f_n e^{\frac{2\pi i n}{L} x}, \quad x \in \mathbb{R},\, f_n \in \mathbb{C}$$

If the above series converges $\forall x \in \mathbb{R}$ then $f(x)$ is periodic, with period L:

$$f(x+L) = f(x), \quad \forall x \in \mathbb{R}.$$

The classical Fourier theory considers the question, in which cases it is possible to write/represent a periodic function as a Fourier series. To answer this question we have to think about the convergence of infinite series. This will be discussed in subsection 2.2. We now want to get an intuitive understanding how the Fourier series works geometrically. But first, here is an important computation trick:

Lemma 2.2.

$$\frac{1}{L}\int_0^L e^{\frac{2\pi i n}{L} x}\, dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \in \mathbb{Z}\backslash\{0\} \end{cases}$$

We begin to consider Fourier series from a historical context. Fourier series were introduced by Ch. Fourier with the goal to solve the heat equation.

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= D \frac{\partial^2}{\partial x^2} u(t,x), \qquad && D > 0, \\ u(0,x) &= f(x), && \forall x \in [0, L] \end{aligned} \tag{3}$$

This equation describes for example the change of the temperature distribution on a metal stick during some time progression. The heat equation is a linear partial differential equation (PDE) whose general solution was unknown at this time. However for some simple initial conditions f , the solution takes the form of a wave. Fourier's idea was then to represent the solution as a superposition of sine and cosine waves. These linear combinations of sine and cosine waves are called Fourier series. In the complex case the Fourier series is a linear combination of $e^{i(\dots)}$ terms like in the preceding definition.
The Fourier series can be considered as the analogue of the Taylor series. Instead of expanding analytic functions we try to represent piecewise continuous, periodic functions and instead of monoms we use sine and cosine waves respectively exponential function terms as building blocks.

Proposition 2.3 (real-valued description of Fourier series)**.** Let f be in $C^1(\mathbb{R}/L\mathbb{Z})$. Then the real-valued Fourier Series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} x\right) + b_n \sin\left(\frac{2\pi n}{L} x\right)$$

Proof. With the aid of the identities $\cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$ and $\sin(nx) = \frac{e^{inx}-e^{-inx}}{2i}$ we obtain

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{e^{inx}+e^{-inx}}{2}\right) + b_n \left(\frac{e^{inx}-e^{-inx}}{2i}\right)$$

We define new coefficients $f_{\pm n}$ for $n \in \mathbb{N}_0$ with

$$f_n = \frac{1}{2}\left(a_n - ib_n\right), \quad f_{-n} = \frac{1}{2}\left(a_n + ib_n\right) = \bar{f_n}, \quad \textit{and} \quad f_0 = \frac{a_0}{2}$$

The above equation $f_{-n} = \bar{f_n}$ holds since f_n are real valued. Hence we get:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{\frac{2\pi i n}{L}x}$$

which is the Fourier series of f. □

The coefficients a_n, b_n can be obtained by:

$$a_n = 2 \cdot \Re(f_n) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n}{L}x\right)\,dx,$$
$$b_n = -2 \cdot \Im(f_n) = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi n}{L}x\right)\,dx$$

The advantage of the complex representation is, that the it is in a more compact form. Furthermore this form is much better for the visual understanding, because our sine and cosine waves are now exponential functions with different exponents. We picture ourself the $e^{\frac{2\pi i n}{L}x}$ terms as rotating vectors with constant integer frequency. With the coefficients f_n we can determine size and starting angle of these vectors.
By adding up, we cling the rotating vectors together. Visually this means that the Fourier series "draws" the function f.

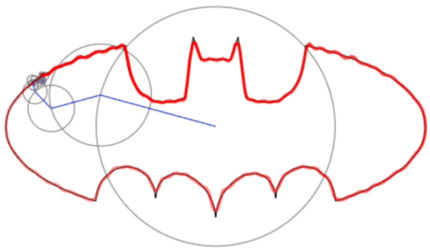

Figure 2: fourier series of "batman function", java applet can be found under this link

In order to determine the coefficients, we first consider the case $n = 0$ and $f_0 e^{\frac{2\pi i 0}{L}x}$. This vector does not rotate with time and was scaled by f_0. We consider it as the "staring" vector and the remaining vectors get attached to it.
Thus we say that f_0 is the centre of mass of the drawing.

$$\frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L \dots + f_{-1}e^{-\frac{2\pi i(-1)}{L}x} + f_0 e^{-\frac{2\pi i(0)}{L}x} + f_1 e^{-\frac{2\pi i(1)}{L}x} + \dots\, dx$$
$$= \dots + \frac{1}{L}\int_0^L f_{-1}e^{-\frac{2\pi i(-1)}{L}x}\,dx + \frac{1}{L}\int_0^L f_0 e^{-\frac{2\pi i(0)}{L}x}\,dx + \frac{1}{L}\int_0^L f_1 e^{-\frac{2\pi i(1)}{L}x}\,dx + \dots = f_0$$

The last equality above follows from Lemma 2.2, since every integral without a factor f_0 in it vanishes. Attentive reader should have noticed that the above derivation of f_0 was not so rigorous, since we did not justified why we can change the integral with the infinite sum.
The argument for this will be discussed in the next section, when we talk about convergence of Fourier series. In conclusion we know how to determine f_0. For the other coefficients we apply a trick by for instance multiplying f with $e^{\frac{-2\cdot 2\pi i}{L}}$. There through,

$$f_2 \cdot e^{\frac{2\cdot 2\pi i}{L}} \cdot e^{\frac{-2\cdot 2\pi i}{L}} = f_2$$

become the non rotating vector. An analogous calculation as before provides:

$$\begin{aligned}
&\frac{1}{L}\int_0^L f(x)e^{\frac{-2\cdot 2\pi i}{L}x}\,dx \\
&= ... + \frac{1}{L}\int_0^L f_0 e^{-\frac{2\cdot 2\pi i}{L}x}\,dx + \frac{1}{L}\int_0^L f_1 e^{-\frac{-1\cdot \pi i 2}{L}x}\,dx + \frac{1}{L}\int_0^L f_2 e^{-\frac{0\cdot 2\pi i}{L}x}\,dx + ... \\
&= f_2
\end{aligned}$$

We can do this $\forall n \in \mathbb{Z}$, and thus obtain the formula for the n-th Fourier coefficient:

$$f_n = \frac{1}{L}\int_0^L f(x)e^{-\frac{2\pi i n}{L}x}\,dx$$

Theorem 2.4 (Riemann Lebesgue Lemma for continuous functions)**.** Let $f : \mathbb{R} \to \mathbb{C}$ continuous and L-periodic. Then:

$$f_n \to 0, \quad | n | \to \infty$$

Proof.

$$\begin{aligned}
f_n &= \frac{1}{L}\int_0^L e^{-\frac{2\pi i n}{L}x} f(x)\,dx = \frac{1}{L}\int_{0+\frac{L}{2n}}^{L+\frac{L}{2n}} e^{-\frac{2\pi i n}{L}x} f(x)\,dx \\
&\overset{x\to x+\frac{L}{2n}}{=} \frac{1}{L}\int_0^L e^{-\frac{2\pi i n}{L}x} e^{-i\pi} f\left(x+\frac{L}{2n}\right)\,dx = -\frac{1}{L}\int_0^L e^{-\frac{2\pi i n}{L}x} f\left(x+\frac{L}{2n}\right)\,dx
\end{aligned}$$

Thus,

$$f_n = \frac{1}{2}(f_n + f_n) = \frac{1}{2L}\int_0^L e^{-\frac{2\pi i n}{L}x}\left[f(x) - f\left(x+\frac{L}{2n}\right)\right]\,dx$$

Since f is continuous on the the compact interval $[-L/2 \mid n \mid n, L + L/2 \mid n \mid]$, f is uniformly continuous. $\forall \epsilon > 0$, $\exists N$ such that for $| n | \leq N$: $| f(x) - f\left(x+\frac{L}{2n}\right) | < \epsilon$, $\forall x \in [0, L]$. From that follows:

$$| f_n | \leq \frac{1}{2L}\int_0^L \left|e^{-\frac{2\pi i n}{L}x}\right|\left|f(x) - f\left(x+\frac{L}{2n}\right)\right|\,dx < \frac{\epsilon}{2}$$

□

Corollary 2.5. Let $f \in C^k(\mathbb{R}/L\mathbb{Z})$ and let f_n be te Fourier coefficients of f.Then:

$$| n |^k f_n \to 0\,, \quad | n | \to \infty$$

2.2 Convergence of Fourier series

There are essentially three propositions/theorems which we use to show that a certain Fourier series converges to a function f.

Theorem 2.6. Let $\{f_n\}_{n\in\mathbb{Z}}$ and $\sum_{n\in\mathbb{Z}} \mid f_n \mid < \infty$. Then the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{\frac{2\pi i n}{L}x}$$

converges absolutely and uniformly towards a L-periodic and continuous function f for all points $x \in \mathbb{R}$.
Furthermore the Fourier coefficients are given by:

$$f_n = \frac{1}{L}\int_0^L f(x)e^{-\frac{2\pi i n}{L}x}\,dx$$

Attention! This theorem says nothing about against which function the Fourier series converges. If we compute a Fourier series of an arbitrary function, it is possible that the Fourier series converges to another function. The follwoing two theorems make a statement about when a Fourier series converges to the function itself.

Theorem 2.7. Let $f \in C^1(\mathbb{R}/L\mathbb{Z})$ and let f_n be the Fourier coefficients of f.
Then $\forall x \in \mathbb{R}$:

$$f(x) = \lim_{N\to\infty}\sum_{n=-N}^{N} f_n e^{\frac{2\pi i n}{L}} x$$

Definition 2.8. A function $f : [a, b] \longrightarrow \mathbb{C}$ is of bounded variation, if

$$\sup_P \sum_i \mid f(x_{i+1}) - f(x_i) \mid \leq V \quad , V = const.$$

where as we form the supremum over all partitions P on $[a, b]$.

This definition guarantees that the length of the graph is well defined (i.e finite). In particular every continuous function is of bounded variation.

Theorem 2.9. Let f be L-periodic and of bounded variation on $[0, L]$ and $s_N f(x) = \sum_{n=-N}^{N} f_n exp(2\pi i n x/L)$ the Nth partial sum of its Fourier series. The following holds:

1. $\lim_{N\to\infty} s_N f(x) = \frac{1}{2}(f(x+0)+f(x-0))$. In particular the Fourier series converges to f(x) point wise whenever f is continuous.
2. The convergence is uniformly on each closed interval $I \in \mathbb{R}$, whenever f is continuous at every point of I.

Theorem 2.10 (Parseval's identity)**.** Let $f(x) = \sum_{n\in\mathbb{Z}} f_n e^{\frac{2\pi i n}{L}x}$ with $f_n \in \mathbb{C}$ and $\sum_n \mid f_n \mid< \infty$. Furthermore let g be a L-periodic function which is integrable on $[0, L]$. Then:

$$\langle f, g\rangle = \frac{1}{L}\int_0^L \overline{f(x)}g(x)\,dx = \sum_{n\in\mathbb{Z}} \bar{f}_n g_n,$$

where g_n are the Fourier coefficients of g. In particular for $g = f$:

$$\frac{1}{L}\int_0^L \mid f(x) \mid^2 \, dx = \sum_{n\in\mathbb{Z}} \mid f_n \mid^2$$

Remark. We can even weaken the conditions to $f, g \in L^2([0, L])$.

Proof. Plugging in the definition of f provides:

$$\langle f, g\rangle = \frac{1}{L}\int_0^L \sum_{n\in\mathbb{Z}} \overline{f_n} e^{\frac{-2\pi i n x}{L}} g(x)\,dx$$

We use the dominated convergence theorem to change position of integral and the infinite sum. We control the conditions:

1. for $N \in \mathbb{N}$ define $s_N(x) = \sum_{|n|\le N} \bar{(f_n)} e^{-\frac{2\pi i n}{L}x} g(x)$. Those partial sums are integrable on $[0, L]$, because g is $L^1[0, L]$ by assumption
2. the function sequence $(s_N)_{n\in\mathbb{Z}}$ converges point wise towards the integrand
3. a majorant is $\forall N \in \mathbb{N}$ given by: $|s_n| \le |g(x)| \sum_{n\in\mathbb{Z}} |f_n|\,, \forall x \in [0, L]$. The majorant is integrable by assumption.

We conclude:

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{L}\int_0^L \sum_{n\in\mathbb{Z}} \overline{f_n} e^{-\frac{2\pi i n}{L}x} g(x)\,dx = \lim_{N\to\infty} \frac{1}{L}\int_0^L \sum_{|n|\le N} \overline{f_n} e^{-\frac{2\pi i n}{L}x} g(x)\,dx \\ &= \lim_{N\to\infty} \sum_{|n|\le N} \frac{1}{L}\int_0^L \overline{f_n} e^{-\frac{2\pi i n}{L}x} g(x)\,dx = \sum_{n\in\mathbb{Z}} \overline{f_n} \frac{1}{L}\int_0^L e^{-\frac{2\pi i n}{L}x} g(x)\,dx = \sum_{n\in\mathbb{Z}} \overline{f_n} g_n\end{aligned}$$

□

Example (Basel problem)**.** We first calculate the Fourier coefficients of the sawtooth function $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}, f(x) = x$ for $x \in (-\pi, \pi]$. We distinguish between the cases $n = 0$ and $n \neq 0$:

$$\begin{aligned} n = 0: &\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0 \\ n \neq 0: &\quad \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx = \frac{i}{2\pi n}\left[xe^{-inx}\right]_{-\pi}^{\pi} - \frac{i}{2\pi n}\int_{-\pi}^{pi} e{-}inx\,dx = \frac{i}{2n}\left(e^{-\pi n} + e^{i\pi n}\right) \\ &\quad = \frac{i(-1)^n}{n} \end{aligned}$$

Thus the Fourier series is given by:

$$f(x) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{i(-1)^n}{n} e^{inx}$$

Obviously $f \in L^2([-\pi,\pi])$. Hence we can apply Parseval's identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|x|^2\,dx = \sum_{n\in\mathbb{Z}}|f_n|^2$$

The left hand side result in:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|x|^2\,dx = \frac{1}{\pi}\int_0^{\pi}x^2\,dx = \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{\pi^2}{3}$$

The right side:

$$\sum_{n\in\mathbb{Z}}|f_n|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}}\left|\frac{i(-1)^n}{n}\right|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Rearranging provides the desired result:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Example. Calculate firstly the Fourier series of $f(x) = |\sin(x)|$ and secondly the value of the series (i)$\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$, (ii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$
f is π periodic so we can determine the Fourier coefficients the following way:

$$\begin{aligned}
f_n &= \frac{1}{\pi}\int_0^{\pi}|\sin(x)|e^{-2inx}\,dx = \frac{1}{2\pi i}\int_0^{\pi}\left(e^{ix}-e^{-ix}\right)e^{-2inx}\,dx \\
&= \frac{1}{2\pi i}\left(\int_0^{\pi}e^{-i(2n-1)x}\,dx - \int_0^{\pi}e^{-i(2n+1)x}\,dx\right) \\
&= \frac{1}{2\pi i}\left[\frac{i}{2n-1}\left(e^{-i(2n-1)\pi}-1\right) - \frac{i}{2n+1}\left(e^{-i(2n+1)\pi}-1\right)\right] = \frac{-2i}{2\pi i}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right] \\
&= -\frac{2}{\pi}\frac{1}{4n^2-1}
\end{aligned}$$

Therefore:

$$|\sin(x)| = \sum_{n=-\infty}^{\infty}-\frac{2}{\pi}\frac{1}{4n^2-1}e^{2inx} = \frac{2}{\pi}-\sum_{n=1}^{\infty}\frac{4}{\pi(4n^2-1)}\cos(2nx)$$

We can use the Fourier series to compute the above to infinite sums:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{4n^2-1} &= \frac{-\pi}{4}\left(|\sin(0)|-\frac{2}{\pi}\right) = \frac{1}{2} \\
\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} &= \frac{-\pi}{4}\left(|\sin(\pi/2)|-\frac{2}{\pi}\right) = \frac{2-\pi}{4}
\end{aligned}$$

2.3 Poisson's summation formula

Let $f \in C^1(\mathbb{R})$ and $|f|, |f'| \leq \frac{C}{1+x^2}$ with $C > 0$ and let $g(x) = \sum_{k\in\mathbb{Z}} f(x+kL)$ with $g \in C^1(\mathbb{R})$, $g(x+L) = g(x)$ (i.e g is L-periodic). The above series converges uniformly on $[0, L]$ because:

$$\left|g(x) - \sum_{|k|\leq N} f(x+kL)\right| \leq \sum_{|k|\geq N} \leq \sum_{|k|\geq N}\frac{C}{1+(x+kL^2)} \leq \sum_{|k|\geq N}\frac{C}{|kL|^2} \stackrel{N\to\infty}{\to} 0$$

Hence g is continuous as limit of continuous functions. We can show the same thing for the derivative of f. $\implies g \in C^1(\mathbb{R}/L\mathbb{Z})$.
The Fourier coefficients of g are given by:

$$g_n = \frac{1}{L}\int_0^L \sum_{k\in\mathbb{Z}} f(x+kL)e^{-\frac{2\pi in}{L}x}\,dx = \sum_{k\in\mathbb{Z}} \frac{1}{L}\int_0^L f(x+kL)e^{-\frac{2\pi in}{L}x}\,dx$$
$$\overset{x\to x+kL}{=} \sum_{k\in\mathbb{Z}} \frac{1}{L}\int_{kL}^{(k+1)L} f(x)e^{-\frac{2\pi in}{L}x}\,dx = \frac{1}{L}\int_{-\infty}^{\infty} f(x)e^{-\frac{2\pi in}{L}x}\,dx = \frac{1}{L}\hat{f}\left(\frac{2\pi n}{L}\right)$$

Definition 2.11. The **Fourier transform** of a function $f \in L^1(\mathbb{R})$ is the function on $\mathbb{R}$:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

Now we can rewrite g the following way:

$$g(x) = \sum_{n\in\mathbb{Z}} f(x+nL) = \sum_{n\in\mathbb{Z}} g_n e^{\frac{2\pi in}{L}x} = \sum_{n\in\mathbb{Z}} \frac{1}{L}\hat{f}\left(\frac{2\pi n}{L}\right) e^{\frac{2\pi in}{L}x}$$

For $x = 0$ we obtain Poisson's summation formula:

$$\sum_{n\in\mathbb{Z}} f(nL) = \frac{1}{L}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{2\pi n}{L}\right)$$

The formula also holds for more general conditions, for instance if f is continuous and of bounded variation and integrable on $\mathbb{R}$.

2.4 Transposition of Integral & Limits

On the next pages we often have to switch derivatives and limits with integrals. To justify that, here are some important and useful results.
First, for a (bounded) function $f : X \to \mathbb{C}$ on a set X, the **uniform norm** is defined as:

$$\|f\|_\infty = \sup_{x\in X} |f(x)|.$$

A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly towards f, iff $\lim_{n\to\infty} \|f - f_n\|_\infty \to 0$.
Furthermore if f_n is the partial sum of a series containing of functions with $f_n = \sum_{k=1}^n h_k$ and $h_k : X \to \mathbb{C}$, then a sufficient criteria for uniform convergence is that the the series is norm convergent, i.e.

$$\sum_{k=1}^{\infty} \|h_k\| < \infty.$$

The following two theorems are the ones which allow us to switch the the position of limits and integrals.

Theorem 2.12.

1. Let $f_n : X \to \mathbb{C}$ be a sequence of continuous functions on a metric space X, which converge uniformly towards a function $f : X \to \mathbb{C}$. Then f is also continuous.

2. Let now $X \subset \mathbb{R}^m$ be open and $f_n : X \to \mathbb{C}$ a sequence of functions of class C^1 (i.e continuously differentiable), which converge point wise towards $f : X \to \mathbb{C}$. We further assume that the partial derivatives $\partial_j f_n$ for each $j = 1, ..., m$ converge uniformly. Then f is of class C^1 and it holds:

$$\partial_j f(x) = \lim_{n\to\infty} \partial_j f_n(x)$$

Regarding the exchangeability of derivatives and integrals we have the following.

Theorem 2.13. Let X be a metric space and $E \subset \mathbb{R}^n$ measurable. Let further $f : X \times E \to \mathbb{C}$ be a function, such that $x \mapsto f(x,y)$ is continuous for all $y \in E$ and $y \mapsto f(x,y)$ is integrable for all $x \in X$. We define:

$$F(x) = \int_E f(x,y)\,dy$$

1. If there exists an integrable function (majorant) $g \geq 0$ on E such that $|f(x,y)| \leq g(y)\,\forall x \in X, \forall y \in E$, then the function F is continuous.
2. Let now $X \subset \mathbb{R}^m$ be open and let the functions $x \mapsto f(x,y)$ for all y continuously differentiable (in the variable x). Furthermore let $g \in L^1(E)$ with $g \geq 0$ on E such that $\left|\partial_j f(x,y)\right| \leq g(y)$ for all x, y and $j = 1, ..., m$
   Then F is continuously differentiable and it holds:
   $$\partial_j F(x) = \int_E \partial_j f(x,y)\,dy$$

For the proof look at the script.

2.5 heat equation on a ring

We consider a heat conducting ring with perimeter L. The temperature distribution for the time t is given by the function $u(x,t)$ where $x \in [0,L]$ and $u(0,t) = u(L,t)$. We extend $u(\cdot,t)$ periodically on $\mathbb{R}$. The function u fulfils the heat equation with given initial condition:

$$\begin{cases} \partial_t u(x,t) = D \cdot \partial_x^2 u(x,t), \\ u(x,0) = f(x) \end{cases}$$

D is some positive constant. We set $D = 1$ and $L = 2\pi$. We further assume that $f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$.

$$\frac{\partial}{\partial t} u(x,t) = D \frac{\partial^2}{\partial x^2} u(x,t)$$

We want to represent u through f.
For that we assume that $u \in C^\infty(\mathbb{R}/2\pi\mathbb{Z} \times (0,\infty)) \cap C(\mathbb{R}/2\pi\mathbb{Z} \times [0,\infty))$ and we evolve u in its Fourier series, i.e:

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{inx}$$

with the Fourier coefficients:

$$u_n(t) = \frac{1}{2\pi} \int_0^{2\pi} u(x,t) e^{-inx}\,dx$$

The Fourier coefficients fulfil the PDE individually. As a consequence from the smoothness of u, $u_n(t)$ is also smooth and since the integration range is compact we can use the DCT

1.2 and theorem 2.13 to drag in the the derivative under the integral:

$$\begin{aligned}
\partial_t u_n(t) &= \partial_t \left(\frac{1}{2\pi} \int_0^{2\pi} u(x,t) e^{-inx}\, dx \right) \overset{2.13}{=} \frac{1}{2\pi} \int_0^{2\pi} \partial_t(u(x,t)) e^{-inx}\, dx \\
&= \frac{1}{2\pi} \int_0^{2\pi} \partial_x^2 (u(x,t)) e^{-inx}\, dx \\
&\overset{\text{part. Int.}}{=} \frac{1}{2\pi} \left(\underbrace{[\partial_x u(x,t) e^{-inx}]_0^{2\pi}}_{=0 \text{ b.c. it's periodic}} - \int_0^{2\pi} \partial_x u(x,t)(-in) e^{-inx}\, dx \right) \\
&= \frac{in}{2\pi} \int_0^{2\pi} \partial_x u(x.t) e^{-inx}\, dx \\
&= \frac{in}{2\pi} \left([u(x,t)e^{-inx}]_0^{2\pi} - \int_0^{2\pi} u(x,t)(-in) e^{-inx}\, dx \right) \\
&= \frac{(in)^2}{2\pi} \int_0^{2\pi} u(x,t) e^{-inx}\, dx = -n^2 u_n(t) \\
\implies \partial_t u_n(t) &= -n^2 u_n(t)
\end{aligned}$$

If $u(x,t)$ for $t \to 0$ converges uniformly, we obtain:

$$u_n(0) = \frac{1}{2\pi} \int_0^{2\pi} \lim_{t\to 0} u(x,t) e^{-nx}\, dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-nx}\, dx = f_n$$

The unique solution of the preceding ODE is given by: $u_n(t) = e^{-tn^2} f_n$ and we obtain for the solution u:

$$u(x,t) = \sum_{n\in\mathbb{Z}} e^{-tn^2+inx} f_n$$

In a last step we have to show that our representation of u fulfil all requested characteristics, i.e. if u is smooth and unique and if u fulfils the PDE (heat equation).

1. we can conclude by theorem 2.12 that the function $u(x,t)$ is infinite times differentiable for all $(x,t) \in \mathbb{R} \times (0,\infty)$. For that we have to show that the partial derivatives of u $\partial_t^j \partial_x^k u(x,t) = \sum_{n\in\mathbb{Z}} (-n^2)^j (in)^k e^{-n^2 t + inx} f_n \ \forall k,j$ converge uniformly. It is enough to show that the series is normal convergent.We have:

$$\sup_{(x,t)\in\mathbb{R}\times[a,b]} |(-n^2)^j (in)^k e^{-n^2 t+inx} f_n| = n^{2j+k} e{-n^2 a} |f_n|$$

   and $|f_n| \to 0$ because of the Riemann Lebesgue Lemma 2.4 and we see that the series converges.

2. our function also fulfils the PDE since:

$$\partial_t u = \sum_{n\in\mathbb{Z}} (-n^2) e^{-n^2 t + inx} f_n = \partial_x^2 u$$

3. our function is also in $C(\mathbb{R}/2\pi\mathbb{Z} \times [0,\infty))$ (meaning u is continuously extendable on the edge). This can be shown by using the first part of theorem 2.12 and we again we will prove normal convergence of u on $\mathbb{R} \times [0,1]$. Since $u \in C^2$, we can estimate f_n by corollary 2.5: $|f_n| \leq \frac{C}{n^2}$ for some positive constant C. We then obtain:

$$\sup_{(x,t)\in\mathbb{R}\times[0,1]} \mid e^{-n^2 t + inx} f_n \mid \leq \frac{C}{n^2}$$

   and the series $\sum_{n\in\mathbb{Z}} \frac{C}{n^2}$ converges.

We can rearrange our solution u to:

$$\begin{aligned} u(x,t) &= \sum_{n\in\mathbb{Z}} e^{-n^2t+inx} f_n = \sum_{n\in\mathbb{Z}} e^{-n^2t+inx} \frac{1}{2\pi}\int_0^{2\pi} f(y)e^{-iny}\,dy \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{Z}} e^{-n^2t+in(x-y)} f(y)\,dy \end{aligned}$$

Hence we have proved the following thing:

Theorem 2.14. Let $f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ and $\forall x \in \mathbb{R}$, $t > 0$:

$$K(x,t) = \sum_{n\in\mathbb{Z}} e^{-n^2t+inx}$$

Then the function $u(x,t)$ given by:

$$u(x,t) = \sum_{n\in\mathbb{Z}} e^{-n^2t+inx} f_n \overset{t\geq 0}{=} \frac{1}{2\pi}\int_0^{2\pi} K(x-y,t)f(y)\,dy$$

with $u \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}\times(0,\infty)) \cup C(\mathbb{R}/2\pi\mathbb{Z}\times[0,\infty))$ solves the heat equation with the initial condition $\lim_{t\to 0} u(x,t) = f(x)$. Besides, u is the unique solution of this IVP.

The function $K(x,t)$ is called **Jacobian Theta function**. The Theta function posses a slightly different which we can deduce with the aid of the Poisson's summation formula. The other representation of the Theta function is given by:

$$K(x,t) = \sum_{n\in\mathbb{Z}} \sqrt{\frac{\pi}{t}} e^{\frac{-(x-2\pi n)^2}{4t}}$$

For the derivation we need the following result:

Lemma 2.15. Let $f(x) = e^{\frac{-x^2}{4t}}$. Then $\hat{f}(k) = \int_{-\infty}^{\infty} e^{\frac{-x^2}{4t}-ikx}\,dx = \sqrt{4\pi t}e^{-k^2t}$

Proof. The lemma follows from an application of Proposition 3.5 with $a = 1/4t$ and $b = -ik$. $\square$

It is not very rigorous to use a proposition for the proof that was not introduced before, but trust me, you can easily go check out proposition 3.5 because it does not require foreknowledge from this course to understand it.

The application of Poisson's summation formula

$$\sum_{n\in\mathbb{Z}} f(x+nL) = \frac{1}{L}\sum_{n\in\mathbb{Z}} \hat{f}\left(\frac{2\pi n}{L}\right) e^{\frac{2\pi n}{L}x}$$

for $f(x) = e^{\frac{-x^2}{4t}}$, $L = 2\pi$ provides:

$$\sum_{n\in\mathbb{Z}} e^{-\frac{(x-2\pi n)^2}{4t}} = \sqrt{\frac{t}{\pi}} \sum_{n\in\mathbb{Z}} e^{-n^2t+inx}$$

3 Fourier transform

The goal of the Fourier transform is to disassembling noises(signals) into their frequency components. Pure tones are described by sine waves with various frequencies. If we now play several tones simultaneously and observe the resulting wave, it will look pretty complicated. A recording device is eventually considering only the sum of the pure sine waves. So the main question, which we are asking ourself, is how to decompose such a signal into pure frequencies. We are looking for a mathematical tool, with which we can transform a signal from the real space into the frequency space to decompose it to certain frequencies.

3.1 Defintion & Properties

Definition 3.1. Let $n \geq 1$. Let $f \in L^1(\mathbb{R}^n)$. The **Fourier transform** of f is the function on $\mathbb{R}^n$.

$$\hat{f}(k) = \int_{\mathbb{R}^n} f(x)e^{-ik\cdot x}\,dx \tag{4}$$

where $k \cdot x = \sum_{i=1}^n k_i x_i$. The inverse Fourier transform of is given by:

$$\check{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(k)e^{ik\cdot x}\,dk \tag{5}$$

There is no unified notation for the Fourier transform. In other books or scripts you will also see $\mathcal{F}[f] = \int_{\mathbb{R}^n} f(x)e^{-ik\cdot x}\,dx$ instead of $\hat{f}(k)$ for a function f. The notation for the inverse transform is then $\mathcal{F}^{-1} = \check{f}$ for a function f.

Remark. Sometimes in other literature we also have the following definitions for the Fourier transform:

$$\hat{f}(k) = \int_{\mathbb{R}^n} f(x)e^{-2\pi ik\cdot x}\,dx \qquad\qquad \hat{f}(k) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-ik\cdot x}\,dx$$

The only difference is the prefactor, so if we calculate some Fourier transform with one of the preceding definitions we have to make sure that we deal correctly with the prefactors.

Let's reconsider the formula:

$$\hat{f}(k) = \overbrace{\int_{\mathbb{R}^n} \underbrace{f(x)\,\underbrace{e^{-ik\cdot x}}_{\text{rotation}}}_{\text{wound up function}}\,dx}^{\text{center of mass}}$$

We know already that $e^{ik\cdot x}$ describes a rotation with frequency k. If we multiply this with the function f, f gets wound up around the origin. With summing up over the real space, using the integral, we consider the centre mass of the curled function f. To make it more clearly, an example of this, programmed in geogebra can be found here.
Thus we can determine the spectrum of frequency of any integrable function. Elements in $\mathbb{C}$ are uniquely described through phase and amplitude. So the Fourier transform provides us information about the phase and amplitude of an frequency, meaning the Fourier transform tells us how strong a wave(with a certain frequency) is contained in a signal. In other words, for an integrable function f, $\hat{f}$ tells us with which amplitude and which phase we need a wave with frequency k, to reconstruct the signal f.

Lemma 3.2. Let $f \in L^1(\mathbb{R}^n)$. Then the functions $\hat{f}, \check{f}$ are uniformly continuous, and for all $x, k \in \mathbb{R}^n$ it holds: $\left|\hat{f}(k)\right| \leq \|f\|_1$, $\left|\check{f}(k)\right| \leq (2\pi)^{-n}\|f\|_1$

Proof. □

Proposition 3.3. Let $f, g \in L^1$ and $\alpha, \beta \in \mathbb{C}$. Then:

1. **linearity**: $\widehat{\alpha f + \beta g} = \alpha \hat{f} + \beta \hat{g}$
2. **time scaling**: $\lambda \in \mathbb{R}\backslash\{0\} \implies \widehat{f(\lambda \cdot)}(k) = |\lambda|^{-n}\, \hat{f}(\frac{k}{\lambda})$
3. **time shift**: $\widehat{f_y}(k) = \hat{f}(k)e^{-iky}$, where $f_y(x) = f(x-y)$
4. **hat switch**: $f\hat{g}, \hat{f}g \in L^1(\mathbb{R}^n) \implies \int_{\mathbb{R}^n} \hat{f}g\, dx = \int_{\mathbb{R}^n} f\hat{g}\, dx$
5. **conjugation**: $\overline{\hat{f}(k)} = \widehat{\overline{f}(-k)}$
6. **differentiation**: $f \in C^1(\mathbb{R}^n)$ and $f, \partial_j f \in L^1(\mathbb{R}^n) \implies \widehat{\partial_j f} = ik_j \hat{f}$ and likewise $x_1 f, ..., x_n f, f \in L^1(\mathbb{R}^n) \implies \hat{f} \in C^1(\mathbb{R}^n)$ and $\partial_j \hat{f} = (-ix_j f)^\wedge$

The proof of the above proposition is left as an exercise.

3.2 Conventions of $\mathcal{F}$

For the Fourier transform there are an irritating number of conventions for it, due to possible changes in the sign of the exponential and the normalization factor which may put a factor of $\frac{1}{2\pi}$ or $\frac{1}{\sqrt{2\pi}}$ in front of the integral. The following convention is mostly commonly found in engineering literature or in communications and signal processing. The forward transform is

$$\mathcal{F}[f(t)] = \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi\omega t}\, dt$$

where ω is the frequency in Hertz, t is the time in seconds. The inverse Fourier transform is then:

$$\mathcal{F}^{-1}[\tilde{f}(\omega)] = \int_{-\infty}^{\infty} f(t)e^{i2\pi\omega t}\, d\omega$$

The next convention is the symmetric one:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-ikt}\, dt \quad \text{and} \quad \check{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikt}\, dk$$

The convention which we use in the lecture is described in equation (4) & (5).

3.3 $\mathcal{F}$ of Gaussian functions

In the following we calculate the Fourier transform of the function $f(x) = e^{-\frac{|x|^2}{2}}$, with $|x|^2 = x_1^2 + ... + x_n^2$. We first consider the case for $n = 1$ and derive under the integral:

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} e^{-x^2/2 - ikx}\, dx \\
\frac{d}{dk}\hat{f}(k) &= \int_{-\infty}^{\infty} e^{-x^2/2}(-ix)e^{-ikx}\, dx = \int_{-\infty}^{\infty} i\left(\frac{d}{dx}e^{-x^2/2}\right) e^{-ikx}\, dx \\
&\overset{part.Int.}{=} -\int_{-\infty}^{\infty} ie^{-x^2/2}(-ik)e^{-ikx}\, dx = -k\hat{f}(k) \\
\hat{f}(0) &= \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}
\end{aligned}$$

Therefore we obtain $\hat{f}(k)$ as a solution of the ordinary differential equation $\hat{f}'(k) = -k\hat{f}(k)$ with the initial condition $\hat{f}(0) = \sqrt{2\pi}$.

$$\hat{f}(k) = \sqrt{2\pi}e^{\frac{-k^2}{2}}$$

Putting the derivative under the integral is justified by the theorem 2.13 since $e^{-x^2/2}$ fulfills its conditions.

Next, we want to generalize the preceding case for an arbitrary n. The integral can be restored to the $n = 1$ case:

$$\begin{aligned}\hat{f}(k) &= \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2} - ik\cdot x}\, dx = \int_{\mathbb{R}^n} \prod_{j=1}^{n} e^{-\frac{x_j^2}{2} - ik_j x}\, dx \\ &= \prod_{j=1}^{n}\left(\int_{-\infty}^{\infty} e^{-\frac{x_j^2}{2} - ik_j x}\, dx\right) = \prod_{j=1}^{n}\sqrt{2\pi}e^{-\frac{k_j^2}{2}} = (2\pi)^{n/2}e^{-\frac{|k|^2}{2}}\end{aligned}$$

Summarized:

$$\left(e^{-\frac{|x|^2}{2}}\right)^{\wedge} = (2\pi)^{n/2}e^{-\frac{|k|^2}{2}}$$

Proposition 3.4.
Let $A = A^T$ a symmetric, positive definite $n \times n$ matrix and $f(x) = e^{-\frac{1}{2}\langle Ax,x\rangle}$ where as $\langle Ax, x\rangle = Ax \cdot x = \sum_{i,j} A_{ij}x_i x_j$.
Then $f \in L^1$ and

$$\hat{f}(k) = \frac{(2\pi)^{n/2}}{\sqrt{detA}}e^{-\frac{1}{2}\langle Ax,x\rangle}.$$

Let us consider now

Proof. Due to the spectral theorem there exists a orthogonal matrix $R \in O(n)$ such that:

$$R^T AR = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} = D$$

With the substitution $x = Ry$ we obtain:

$$\begin{aligned}\hat{f}(k) &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Ax,x\rangle - i\langle k,x\rangle}\, dx = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle ARy,Ry\rangle - i\langle k,Ry\rangle} detR\, dy \\ &\overset{(1)}{=} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Dy,y\rangle - i\langle R^Tk,y\rangle}\, dy = \int_{\mathbb{R}^n} e^{\sum_{s=1}^n \lambda_s {y_s}^2 - i\sum_{s=1}^n (R^Tk)_s y_s}\, dy \\ &= \prod_{s=1}^{n}\int_{-\infty}^{\infty} e^{\lambda_s {y_s}^2 - i(R^Tk)_s y_s}\, dy_s = \prod_{s=1}^{n}\left(\frac{2\pi}{\lambda_s}\right)^{1/2} e^{\frac{1}{2}(R^Tk)_s^2\lambda_s^{-1}} \\ &= \frac{(2\pi)^{n/2}}{\sqrt{detR}}e^{-\frac{1}{2}\langle D^{-1}R^Tk,R^Tk\rangle} \overset{(2)}{=} \frac{(2\pi)^{n/2}}{\sqrt{detA}}e^{-\frac{1}{2}\langle A^{-1}k,k\rangle}\end{aligned}$$

In (1) we made use of the property that: $\langle x, y\rangle = \langle Rx, Ry\rangle \forall R \in= (n)$ and $\forall x, y \in \mathbb{R}^n$. In (2) we used the fact that: $RD^{-1}R^T = R(R^TAR)^{-1}R^T = A^{-1}$. $\square$

Proposition 3.5 (Gaussian integral formula)**.** Let $a > 0$, $b, c \in \mathbb{C}$. Then the following holds:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\, dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}$$

The preceding formula is convenient for computing purposes. You will sometimes have to evaluate a multidimensional Gaussian integral (for instance to solve heat or wave equation) and then the formula can be useful.

Proof.

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\, dx \overset{completing\, square}{=} e^{\frac{b^2}{4a}+c} \int_{-\infty}^{\infty} e^{-a\left(x-\frac{b}{2a}\right)^2}\, dx \overset{x-b/2a \mapsto x}{=} e^{\frac{b^2}{4a}+c} \int_{-\infty}^{\infty} e^{-ax^2}\, dx$$

Now define $I := \int_{-\infty}^{\infty} e^{-ax^2}\, dx$. Squaring I and switching to polar coordinates yields:

$$\begin{aligned} I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x^2+y^2)}\, dx\, dy \overset{(r,\varphi)\to(x,y)}{=} \int_0^{\infty}\int_0^{2\pi} e^{-ar^2} r\, d\varphi dr = \frac{2\pi}{2a}\int_0^{\infty} 2are^{-ar^2}\, dr \\ = \frac{\pi}{a}\left[-e^{-ar^2}\right]_0^{\infty} = \frac{\pi}{a} \end{aligned}$$

Taking the square root and plug in I above, proofs the formula. □

Remark. The paramteres in the above formula 3.5 can be mor gneral namely:

$$a, b, c \in \mathbb{C}, \quad \text{ and } \mathrm{Re}(a) > 0 \text{ or } \mathrm{Re}(a) = 0, \mathrm{Im}(a) \neq 0, \mathrm{Im}(b) \neq 0.$$

Example. Fourier transform of $e^{-m|x|}$:

$$\begin{aligned} \left(e^{-m|x|}\right)^{\wedge} &= \int_{-\infty}^{\infty} e^{-m|x|-ikx}\, dx = \int_{-\infty}^{0} e^{mx-ikx}\, dx + \int_0^{\infty} e^{-mx-ikx}\, dx \\ &= \frac{1}{m+ik} + \frac{1}{m-ik} = \frac{2m}{m^2+k^2} \in L^1(\mathbb{R}) \end{aligned}$$

Example. Let $h_0(x) = e^{-\frac{x^2}{2}}$. We have already seen that $\widehat{h_0}(k) = (2\pi)^{1/2} h_0(k)$ that means h_0 is an eigenvector of the linear operator $\mathcal{F} : f \mapsto \hat{f}$.

3.4 A * is born - Convolutions

Definition 3.6. Let $f, g \in L^1(\mathbb{R}^n)$. The **convolution** $f * g$ of f with g is defined through:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy$$

Proposition 3.7. Let $f, g \in L^1(\mathbb{R}^n)$. Let $f * g$ be the convolution defined as in 3.6. Then the following holds:

1. $\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$
2. the convolution is a bilinear, associative, commutative mapping $L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$

3. $\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$
4. Let $f \in C^l(\mathbb{R}^n)$ for some $l \geq 0$ with bounded partial derivatives. Then $f * g \in C^l(\mathbb{R}^n)$. In particular: $\partial_\alpha(f * g) = (\partial_\alpha f) * g = f * (\partial_\alpha g)$ for some $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$.

Proof. For 1. it holds that

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)g(y)| \, dx \right) dy = \|f\|_1 \int_{\mathbb{R}^n} |g(y)| \, dy = \|f\|_1 \|g\|_1.$$

Due to Fubini 1.3 $f(x-y)g(y) \int L^1(\mathbb{R}^n \times \mathbb{R}^n)$ and the integral for the convolution exists for all x

$$\begin{aligned} \|f * g\| &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y) \, dy \right| dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x.y)g(y)| \, dydx \\ &= \int_{\mathbb{R}^n} |g(y)| \int_{\mathbb{R}^n} |f(x-y)| \, dxdy = \|f\|_1 \int_{\mathbb{R}^n} |g(y)| \, dy = \|f\|_1 \|g\|_1. \end{aligned}$$

Note that we have equality if f and g are non negativ.

For the 2nd point we have that: bilinearity follows from the linearity of the integral. Commutativity follows by substitution $u = x - y$,

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(x-y)g(y) \, dy = \int_{u(y=-\infty)}^{u=(y=\infty)} f(u)g(x-u) \, (-du) \\ &= \int_{\infty}^{-\infty} f(u)g(x-u) \, (-du) = \int_{-\infty}^{\infty} f(u)g(x-u) \, du = g * f(x). \end{aligned}$$

Associativity follows with Fubini and substitution.

For the 3rd point we have

$$\begin{aligned} \widehat{f * g}(k) &= \int_{\mathbb{R}^n} (f * g)(x) e^{-ikx} \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y)g(y) e^{ikx} \, dydx \\ &\overset{u=x-y}{=} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u)g(y) e^{-ik(u+y)} \, dudy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(u) e^{-iku} \, du \, e^{-iky} g(y) \, dy \\ &= \hat{f}(k)\hat{g}(k). \end{aligned}$$

For the last item, let $f \in C^l(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. It holds that

$$|\partial^\alpha f(x-y)g(y)| \leq \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x-y)| \, |g(y)| \in L^1(\mathbb{R}^n)$$

for all $\alpha \in \mathbb{N}_0^n, |\alpha| \leq l$. Due to theorem 2.13 $f * g$ is continuously differntiable and

$$\partial^\alpha (f * g)(x) = \partial^\alpha \int_{\mathbb{R}^n} f(x-y)g(y) \, dy = \int_{\mathbb{R}^n} \partial^\alpha f(x-y)g(y) \, dy = (\partial^\alpha f * g)(x).$$

$\square$

3.5 Inversion rule L^1 functions

Theorem 3.8 (inverse theorem for L^1 functions)**.**

$$f, \hat{f} \in L^1(\mathbb{R}^n) \implies f^{\wedge\vee} = f \qquad\qquad f, \check{f} \in L^1(\mathbb{R}^n) \implies f^{\vee\wedge} = f$$

The proof of the above theorem is rather tedious and it is left as a exercise to read the proof in the MMP 1 script.

Corollary 3.9. $f, \hat{f} \in L^1(\mathbb{R}^n) \implies f^{\wedge\wedge}(x) = (2\pi)^n f(-x)$ almost everywhere

Proof. $f^{\wedge\wedge}(x) = \int_{\mathbb{R}^n} \hat{f}(k) e^{-ikx}\, dk = (2\pi)^n f^{\wedge\vee}(-x) = (2\pi)^n f(-x)$ □

Corollary 3.10. $f, \hat{f} \in L^1(\mathbb{R}^n) \implies \check{f} \in L^1(\mathbb{R}^n)$

Proof. Define $\tilde{f}(x) = f(-x)$. Then $f, \hat{f} \in L^1(\mathbb{R}^n) \implies f^{\wedge\wedge}(x) = (2\pi)^n \tilde{f} \in L^1(\mathbb{R}^n) \implies f^{\wedge\wedge\wedge} \in L^1(\mathbb{R}^n)$
So, $(2\pi)^{2n} \check{f}(x) = f^{\wedge\wedge\wedge} \in L^1(\mathbb{R}^n)$. □

Theorem 3.11 (Plancherel)**.**
$f, \hat{f} \in L^1(\mathbb{R}^n) \implies f, \hat{f} \in L^2(\mathbb{R}^n)$ and it holds: $\|f\|_2 = (2\pi)^{-n/2} \|\hat{f}\|_2$.
Written out:

$$\left(\int_{\mathbb{R}^n} \overline{f(x)} f(x)\, dx \right)^{1/2} = \frac{1}{(2\pi)^{n/2}} \left(\int_{\mathbb{R}^n} \overline{\hat{f}(k)} \hat{f}(k)\, dk \right)^{1/2}.$$

Proof.

$$\|\hat{f}\|_2^2 = \int_{\mathbb{R}^n} \underbrace{|\hat{f}(k)|}\, \le \|f\|_1 |\hat{f}|\, dk \le \|f\|_1 \|\hat{f}\|_1 < \infty$$

$$\|f\|_2^2 = \| \left(f^{\vee}\right)^{\wedge} \|_2^2 \le \|\check{f}\|_1 \|f\|_1 < \infty$$

$$\|f\|_2^2 = \int_{\mathbb{R}^n} \overline{f(x)} f(x)\, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (f^{\wedge})^{\wedge}(-x) \overline{f(x)}\, dx \overset{(1)}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(-x) \hat{\overline{\hat{f}}}(x)\, dx$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(-x) \overline{\hat{f}}(-x)\, dx \overset{-x \to x}{=} \frac{1}{(2\pi)^n} \|\hat{f}\|_2^2$$

In (1) we made use of the 4^{th} property from Proposition 3.3 □

Theorem 3.12 (Parseval)**.**
Let $f, g \in L^1(\mathbb{R})$. Then we have: $\|fg\|_2^2 = \frac{1}{2\pi} \|\hat{f}\hat{g}\|_2^2$, or written out:

$$\int_{-\infty}^{\infty} f(x) \overline{g(x)}\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) \overline{\hat{g}(k)}\, dk.$$

3.6 Schwartz space $\mathscr{S}$

A multi index is a tuple $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$. We use the following notations for multiindices:

$$x^\alpha = x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n} \qquad |\alpha| = \alpha_1 + ... + \alpha_n \qquad \alpha! = \alpha_1! \cdot ... \cdot \alpha_n!$$

$$\partial_i = \frac{\partial}{\partial x_i} \qquad \partial^\alpha = \partial_1^{\alpha_1} ... \partial_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

For $X \subset \mathbb{R}^n$ set $C(X) = C^0(X) = \{f : X \to \mathbb{C}, \text{continuous}\}$.
For $\Omega \subset \mathbb{R}^n$ open set $C^k(\Omega) = \{f : \Omega \to \mathbb{C} \,:\, \partial_k f \in C(\Omega) \forall \alpha \in \mathbb{N}_0^n \text{with } |\alpha| \leq k\}$
With this definition we can rewrite **Taylor's theorem** for $f \in C^k(\Omega)$:

$$f(x+t) = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leq k}} \partial^\alpha f(x) \frac{t^\alpha}{\alpha!} + o(|t|^k)$$

Example. Compute the Taylor series of $f(x,y,z) = e^{2x+yz}$ to the second order around $(x,y,z) = (0,0,0)$.
We first calculate all necessary partial derivatives:

$$\begin{aligned} \partial_x f &= 2e^{2x+yz} & \partial_y f &= ze^{2x+yz} & \partial_z f &= ye^{2x+yz} \\ \partial^2_{xx} f &= 4e^{2x+yz} & \partial^2_{yy} f &= z^2 e^{2x+yz} & \partial^2_{zz} f &= y^2 e^{2x+yz} \\ \partial^2_{xy} f &= 2ze^{2x+yz} & \partial^2_{xz} f &= 2ye^{2x+yz} & \partial^2_{yz} f &= (1+yz)e^{2x+yz} \end{aligned}$$

Then we evaluate them around $(x,y,z) = (0,0,0)$.

$$\begin{aligned} \partial_x f(0,0,0,) &= 2 & \partial_y f(0,0,0) &= 0 & \partial_z f(0,0,0) &= 0 \\ \partial^2_{xx} f(0,0,0) &= 4 & \partial^2_{yy} f(0,0,0) &= 0 & \partial^2_{zz} f(0,0,0) &= 0 \\ \partial^2_{xy} f(0,0,0) &= 0 & \partial^2_{xz} f(0,0,0) &= 0 & \partial^2_{yz} f(0,0,0) &= 1 \end{aligned}$$

We obtain then:

$$Tf(x,y,z) = 1 + 2x + 4\frac{x^2}{2!} + 1\frac{yz}{1! \cdot 1!} = 1 + 2x + 2x^2 + xy$$

Definition 3.13. For $\varphi \in C^\infty(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{N}^n$ we define $\|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi(x) \right|$.
The **Schwartz space** is the complex vector space:

$$\mathscr{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) \,|\, \|\varphi\|_{\alpha,\beta} < \infty \,\forall \alpha, \beta \in \mathbb{N}^n\}$$

The Schwartz space is a function space, which is customized for the Fourier transform. Principally we are dealing here with functions f, such that all derivatives exist on $\mathbb{R}$ and they converge faster to 0 for $x \to \pm\infty$ as every negative power. So we are working with smooth functions, which decay faster than any polynomial. Gaussian functions are a good example for Schwartz functions.

Remark. The quantities $\|\varphi\|_{\alpha,\beta}$ are not norms (it holds for instance $\|\,1\|_{\alpha,\beta} = 0$, if $\beta \neq 0$). We call the quantities $\|\varphi\|_{\alpha,\beta}$ half norms.

Example.

1. $e^{-a|x|^2} \in \mathscr{S}(\mathbb{R}^n) \forall a > 0$ then $x^\alpha \partial^\beta e^{-a|x|^2} = \underbrace{P(x)}_{\text{polynomial}} \cdot\; e^{-a|x|^2}$
2. $e^{-a|x|} \notin \mathscr{S}(\mathbb{R}^n)$ since it is not differentiable at $x = 0$
3. $(1+|x|^2)^{-s} \notin \mathscr{S}(\mathbb{R}^n)$, because the functions does not decrease fast enough
4. $e^{-x_1^2 + x_1 x_2 - x_2^2} \in \mathscr{S}(\mathbb{R}^2)$, because $e^{-x_1^2 + x_1 x_2 - x_2^2} = e^{\frac{1}{2}\langle Ax, x\rangle}$ with $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
5. $C_0^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$

Remark.

- Every arbitrary often differentiable function f with compact support is a Schwartz function. That means $C_0^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$.
- The Schwartz space $\mathscr{S}$ is separable.
- $\mathscr{S}(\mathbb{R}^n) \in L^p(\mathbb{R}^n)$ for every $1 \leq p \leq \infty$.

Remark. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then $\forall \alpha, \beta \in \mathbb{N}_0^n$, $x^\alpha \partial^\beta \varphi \in \mathscr{S}(\mathbb{R}^n)$.
Similarly $P(x)Q(\partial)\varphi \in \mathscr{S}(\mathbb{R}^n)$ for all polynomials P, Q.

Lemma 3.14. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $\beta \in \mathbb{N}^n$, $k \in \mathbb{N}$. Then there exists a constant $C_{\beta,k} = C_{\beta,k}(\varphi)$ such that

$$\|\partial^\beta \varphi(x)\| \leq \frac{C_{\beta,k}}{(1+|x|^2)^k}$$

Proof. $(1+|x|^2)^k \partial^\beta \varphi \in \mathscr{S}(\mathbb{R}^n)$, so in particular $|(1+|x|^2)^k \partial^\beta \varphi(x)| \leq C_{\beta,k}$, $\forall x$. Dividing provides the result. □

Corollary 3.15. $\mathscr{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$, $\forall p \in (1, \infty)$

Proof.

$$\int_{\mathbb{R}^n} \|\varphi(x)\|^p \, dx \leq \int_{\mathbb{R}^n} \frac{C_{\beta,k}^p}{(1+|x|^2)^{kp}} \, dx < \infty$$

for k big enough, for instance $k = \lfloor \frac{n}{2} \rfloor + 1$. □

Definition 3.16. A sequence $\{\varphi_j\}_{j=0}^\infty \in \mathscr{S}(\mathbb{R}^n)$ converges to $\varphi \in \mathscr{S}(\mathbb{R}^n)$ if, $\lim_{j\to\infty} \|\varphi_j - \varphi\|_{k,l} = 0$ for all $k, l \in \mathbb{N}$ We write then:

$$\varphi = \mathscr{S}\!-\lim_{j\to\infty} \varphi_j \quad or \quad \varphi_j \overset{\mathscr{S}}{\to} \varphi, (j \to \infty)$$

An equivalent condition is: $\lim_{j\to\infty} \|\varphi_j - \varphi\|_{\alpha,\beta} = 0$ for all $\alpha, \beta \in \mathbb{N}^n$

Definition 3.17. A linear mapping $F : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is continuous, if the following holds for all convergent sequences:

$$\varphi_j \overset{\mathscr{S}}{\to} \varphi \implies F(\varphi_j) \overset{\mathscr{S}}{\to} F(\varphi).$$

Lemma 3.18. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$. The followings holds:

1. $(\partial_j \varphi)^\wedge(k) = i k_j \widehat{\varphi}(k)$
2. $\partial_j \widehat{\varphi}(k) = \frac{\partial}{\partial k_j} \widehat{\varphi}(k) = (-i x_j \varphi)^\wedge(k)$
3. $(\partial_j \varphi)^\vee(k) = -i k_j \widecheck{\varphi}(k)$
4. $\partial_j \widecheck{\varphi}(k) = (i x_j \varphi)^\vee(k)$

Proof. Here we show the first two identities. The other two can be proven in a similar way as the first ones. For 1. we have

$$\begin{aligned}\left(\partial_j\varphi\right)^\wedge(k) &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}\varphi(x)e^{-ik\cdot x}\,dx \\ &= \int_{\mathbb{R}^{n-1}} \left\{\int_{\mathbb{R}} \left(\frac{\partial}{\partial x_j}\varphi(x)\right) e^{-ik\cdot x}\,dx_j\right\} dx_1 \dots dx_{j-1}dx_{j+1}\dots dx_n \\ &= \dots = \varphi(x)e^{ik\cdot x}\Big|_{x_j=-\infty}^{\infty} - \int_{\mathbb{R}} \varphi(x)\frac{\partial}{\partial x_j}e^{-ik\cdot x}\,dx_j.\end{aligned}$$

Since φ is a test function of the Schwartz space, the boundary terms at $\pm\infty$. Hence

$$(\partial_j\varphi)^\wedge(x) = -\int_{\mathbb{R}^n}\varphi(x)\frac{\partial}{\partial x_j}e^{-ik\cdot x}\,dx = ik_j\hat{\varphi}(x)$$

$\square$

Lemma 3.19. $\varphi \in \mathscr{S}(\mathbb{R}^n) \implies \widehat{\varphi}, \check{\varphi} \in \mathscr{S}(\mathbb{R}^n)$

Proof.

$$k^\alpha\partial^\beta\hat{\varphi}(k) = k^\alpha\left((-ix)^\beta\varphi\right)^\wedge = \left((-i\partial)^\alpha(-ix)^\beta\varphi\right)^\wedge(k),$$
$$\left|k^\alpha\partial^\beta\hat{\varphi}(k)\right| = \left|(\partial^\alpha x^\beta\varphi)^\wedge(k)\right| \le \int_{\mathbb{R}^n}\left|\partial^\alpha(x^\beta\varphi)(x)\right|\,dx,$$

and thus $\sup_k\left|k^\alpha\partial^\beta\hat{\varphi}(k)\right| < \infty$, $\forall\alpha,\beta\in\mathbb{N}^m$, so $\hat{\varphi}\in\mathscr{S}(\mathbb{R}^n)$. We can do the analogous procedure for $\check{\varphi}\in\mathscr{S}(\mathbb{R}^n)$. $\square$

Theorem 3.20. The linear mapping:

$$\mathcal{F}: \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \quad \varphi \mapsto \widehat{\varphi},$$

is bijective and continuous. Its inverse is given by:

$$\mathcal{F}^{-1}: \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \quad \varphi \mapsto \check{\varphi},$$

Proof. Bijectivity follows from the inversion theorem 3.8 and the above lemma 3.19.
We first show continuity at $x = 0$.
Let φ_j be a sequence in $\mathscr{S}$ with $\varphi_j \overset{\mathscr{S}}{\to} 0$. We then have

$$\begin{aligned}\|\hat{\varphi}_j\| &= \max_{\substack{|\alpha|\le k\\|\beta|\le l}}\sup_{k\in\mathbb{R}^n} \le \max_{\substack{|\alpha|\le k\\|\beta|\le l}}\int_{\mathbb{R}^n}\left|\partial^\alpha x^\beta\varphi_j(x)\right|\,dx = \max_{\substack{|\alpha|\le k\\|\beta|\le l}}\int_{\mathbb{R}^n}\frac{(1+|x|^2)^m}{(1+|x|^2)^m}\left|\partial^\alpha x^\beta\varphi_j(x)\right|\,dx \\ &\le \max_{\substack{|\alpha|\le k\\|\beta|\le l}}\sup_{k\in\mathbb{R}^n}(1+|x|^2)^m\left|\partial^\alpha x^\beta\varphi_j(x)\right|\cdot\int_{\mathbb{R}^n}\frac{1}{(1+|y|^2)^m}\,dy \\ &\overset{m>n/2}{\le} \text{const.}\max_{\substack{|\alpha'|\le k\\|\beta'|\le l+2m}}\sup_{x\in\mathbb{R}^n}\left|x^{\beta'}\partial^{\alpha'}\varphi_j(x)\right| \le \text{const.}\,\|\varphi_j\|_{l+2m,k}\overset{j\to\infty}{\to} 0.\end{aligned}$$

Continuity everywhere now follows from

$$\varphi_j \overset{\mathscr{S}}{\to} \varphi \Leftrightarrow \varphi_j - \varphi \equiv \psi_j \overset{\mathscr{S}}{\to} 0 \implies \hat{\psi}_j \overset{\mathscr{S}}{\to} 0$$

Hence

$$\hat{\varphi}_j \xrightarrow{\mathscr{S}} \hat{\varphi}.$$

□

Why do we even consider the Schwartz space? Based on the theorem 3.20 we see that the Fourier transform describes an automorphism on the Schwartz space. On the reference to this we can define the Fourier transform on the dual space of $\mathscr{S}(\mathbb{R}^n)$, the space of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$, by the following:

$$\mathcal{F}(u)(\phi) := u(\mathcal{F}(\phi))$$

where $u \in \mathscr{S}'(\mathbb{R}^n)$ (i.e. $u : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$ linear and continuous) and $\phi \in \mathscr{S}(\mathbb{R}^n)$. You can find more to that in chapter 5.

After all that here is an example how we can use the Fourier transform to compute definite integrals.

Example.
Calculate the following integrals:

$$\text{(i)} \int_0^\infty \frac{\cos(x)}{1+4x^2}\,dx \quad \text{and} \quad \text{(ii)} \int_0^\infty \frac{1}{(1+x^2)^2}\,dx.$$

We know from an preceding example that $\left(e^{-m|x|}\right)^\wedge(k) = \frac{2m}{m^2+k^2} \in L^1(\mathbb{R})$
So we have $\left(e^{-|x|}\right)^\wedge(k) = \frac{2}{1+k^2}$. Setting $f(x) = e^{-|x|}$ and applying the inversion theorem 3.8 on f yields to:

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{2e^{ikx}}{1+k^2}\,dk \\
&= \frac{1}{\pi}\int_{-\infty}^0 \frac{e^{ikx}}{1+k^2}\,dk + \frac{1}{\pi}\int_0^\infty \frac{e^{ikx}}{1+k^2}\,dk \\
&= \frac{1}{\pi}\int_0^\infty \frac{e^{ikx}+e^{-ikx}}{1+k^2}\,dk = \frac{2}{\pi}\int_0^\infty \frac{\cos(kx)}{1+k^2}\,dk \\
&\overset{k\to 2k}{=} \frac{4}{\pi}\int_0^\infty \frac{\cos(2kx)}{1+(2k)^2}\,dk
\end{aligned}$$

Therefore (i) is given by: $\frac{\pi}{4}f(1/2) = \frac{\pi}{4}e^{-\frac{1}{2}}$.
For (ii) we again set $f(x) = e^{-|x|}$ and use the Plancherel theorem 3.11.

$$\begin{aligned}
\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(1+x^2)^2}\,dx &= \frac{1}{8}\|\hat{f}\|_2^2 \overset{\text{plancherel}}{=} \frac{2\pi}{8}\|f\|_2^2 \\
= \frac{\pi}{4}\int_{-\infty}^\infty e^{-2|x|}\,dx &= \frac{2\pi}{4}\int_0^\infty e^{-2x}\,dx = \frac{\pi}{4}
\end{aligned}$$

Let us do another example of computing the Fourier transform of a given function. This time we need some knowledge from complex analysis to solve the following problem.

Example.
Lets calculate the Fourier transform of the function $f(x) = \frac{1}{1+x^2}$. Since $f \in L^1(\mathbb{R})$ and f is real valued we can apply proposition 3.3 and use that $\overline{\hat{f}(k)} = \hat{\bar{f}}(-k) = \hat{f}(-k)$. Therefore it is enough to compute $\hat{f}(-k)$ for $k \geq 0$

$$\hat{f}(-k) = \int_{\mathbb{R}} f(x) e^{ikx} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{ikx}}{1+x^2} \, dx.$$

To solve the above integral we extend it to the complex plane and use the residue theorem A.1. Lets consider the complex integral over the path $\Gamma = \Gamma_R + \gamma$ where Γ_R denotes the path the half circle in the upper half plane and γ the straight line from $-R$ to R.

$$\int_{\Gamma} \frac{e^{ikz}}{1+z^2} \, dz = \int_{\Gamma_R} \frac{e^{ikz}}{1+z^2} \, dz + \int_{\gamma} \frac{e^{ikz}}{1+z^2} \, dz = 2\pi i \ \mathrm{Res}(f, i)$$

The integral over $\Gamma_R(t) = Re^{it}$ for $0 \leq t \leq \pi$ can be estimated by,

$$\begin{aligned}
\left| \int_{\Gamma_R} \frac{e^{ikz}}{1+z^2} \, dz \right| &= \left| \int_0^{\pi} \frac{e^{ikR(\cos(t) + i\sin(t))}}{1+R^2 e^{2it}} Re^{it} \, dt \right| = R \int_0^{\pi} \frac{e^{-Rk\sin(t)}}{|1+R^2 e^{2it}|} \, dt \\
&\overset{reverse\,\Delta}{=} R \int_0^{\pi} \frac{e^{-Rk\sin(t)}}{1-R^2} \, dt \overset{\sin(x)=\sin(\pi-x)}{=} R \int_0^{\pi/2} \frac{e^{-Rk\sin(t)}}{1-R^2} \, dt + R \int_{pi/2}^{\pi} \frac{e^{-Rk\sin(\pi-t)}}{1-R^2} \, dt \\
&= 2R \int_0^{\pi/2} \frac{e^{-Rk\sin(t)}}{1-R^2} \, dt,
\end{aligned}$$

Now we use that $\sin t \geq \frac{2}{\pi} t$ for $t \in [0, \pi/2]$,

$$2R \int_0^{\pi/2} \frac{e^{-Rk\sin(t)}}{1-R^2} \, dt = 2R \int_0^{\pi/2} \frac{e^{-2Rk/\pi t}}{1-R^2} \, dt \overset{s=\frac{2}{\pi}t}{=} 2R \frac{\pi}{2R} \int_0^{R} \frac{e^{-ks}}{1-R^2} \, ds \overset{R\to\infty}{\longrightarrow} 0$$

The residue at the point $z = i$ can be computed with proposition A.2,

$$\mathrm{Res}(f, i) = \lim_{z \to i} \frac{e^{ikz}}{1+z^2}(z - i) = \frac{e^{-k}}{2i}$$

Finally we obtain

$$\hat{f}(-k) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^2} \, dz = 2\pi i \ \mathrm{Res}(f, i) = \pi e^{-k}$$

In conclusion $\hat{f} = \pi e^{-|k|}$.

3.7 regularity & decrease characteristics

First we consider functions which are invariant under rotations and how their Fourier transforms can be computed. In addition we compute the surface of the n-th dimensional unit sphere. Furthermore we give two results which are very useful in order to estimate the Fourier transform of functions under certain assumptions, but we will not proof them. Lets start with the Fourier transformation of functions which are invariant under rotations.
A function $g : \mathbb{R}^n \to \mathbb{C}$, invariant under rotations, is a function with the property,

$$g(Rx) = g(x) \quad \forall R \in O(n).$$

We have the following lemma.

Lemma 3.21. A function g is invariant under rotations if and only if g is of the form:

$$g(x) = f(|x|), \quad |x| = \sqrt{x_1^2 + ... + x_n^2}.$$

Proof.
" $\Rightarrow$ ": If g is of the named form, then the function is invariant under rotation because $|Rx| = |x|$, $\forall R \in= O(n)$.
" $\Leftarrow$ ": Let g be invariant under rotation, and define $f(r) := g(r, 0, ..., 0)$. Since $\forall x \in \mathbb{R}^n$ there exists an orthogonal matrix R such that $Rx = (|x|, 0, ..., 0)$ and therefore $g(x) = g(|x|, 0, ..., 0) = f(|x|)$. □

Let $g \in L^1$ and invariant under rotations. The Fourier transform $\hat{g}(k)$ is also invariant under rotations, since $\forall R \in O(n)$ we can follow:

$$\hat{g}(Rk) = \int_{\mathbb{R}^n} g(x)e^{-i(Rk,x)}\, dx \overset{x \to Rx}{=} |\det R| \int_{\mathbb{R}^n} g(x)e^{-i(Rk,Rx)}\, dx = \hat{g}(k),$$

where as we used that $|\det R| = 1$ and $(Rk, Rx) = (k, x)$ for all orthogonal matrices R.

Lemma 3.22. The "surface" of the (n-1)-dimensional unit sphere $S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ is given by:

$$|S^{n-1}| = \int_{S^{n-1}} d\Omega(y) = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \begin{cases} \frac{2\pi^k}{(k-1)!}, & n = 2k \\ \frac{2^{2k+1}\pi^k k!}{(2k)!}, & n = 2k+1 \end{cases}$$

We recall that the appearing Euler Gamma function has for $\mathrm{Re}(s) > 0$ the form

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\, dt.$$

In addition the function fulfills $\Gamma(s + 1) = s\Gamma(s)$ and for $n \in \mathbb{N}$ it holds that $\Gamma(n + 1) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$.

Proof.

$$\pi^{n/2} = \int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \int_0^\infty e^{-r^2} r^{n-1}\, dr \int_{S^{n-1}} d\Omega(y)$$
$$\overset{s=r^2}{=} |S^{n-1}| \frac{1}{2} \int_0^\infty e^{-s} s^{\frac{n}{2}-1}\, ds = |S^{n-1}| \frac{1}{2}\Gamma(\frac{n}{2}).$$

This concludes the first equation. For the second equality follows by induction and $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$. □

The regularity (continuity, differentiability,...) of a functions corresponds to the decrease characteristics of its Fourier transform.

Theorem 3.23. Let $f \in C_0^s(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that

$$|\hat{f}(k)| \leq \frac{C}{(1+|k|)^s}$$

Proof. This is immediately follows from lemma 3.14 for $\beta = 0$ since the continuous functions with finite support are a non-trivial subset of the test functions in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. □

Theorem 3.24 (Riemann-Lebesgue)**.**

$$f \in L^1(\mathbb{R}^n) \implies \int_{-\infty}^{\infty} \hat{f}(k)\, dk = 0$$

The proof of the above theorem can be found in the script.

3.8 Wave equation

The wave equation is given by:

$$\begin{cases} \frac{1}{c^2}\partial_{t^2}^2 u(x,t) = \Delta u(x,t), & x \in \mathbb{R}^n, t \geq 0\,, \\ u(x,0) = f(x), \\ \partial_t u(x,0) = g(x) \end{cases}$$

This equation is very important in physics, since the components of the B- and E-field in vacuum fulfil this equation.
To find solutions we make the assumptions that our initial conditions and the solution u are regular enough (preferably Schwartz functions) so that all rearrangements are valid. In this way we derive a formal solution and verify in the end if this representation fulfils all desired properties. To solve the equation we transform it into the frequency space via Fourier transform:

$$\begin{cases} \frac{1}{c^2}\partial_{t^2}^2 u(x,t) = \Delta u(x,t), \\ u(x,0) = f(x), \\ \partial_t u(x,0) = g(x) \end{cases} \xrightarrow{\text{Fourier transform } \mathscr{F}} \begin{cases} \frac{1}{c^2}\partial_{t^2}^2 \hat{u}(k,t) = -|k|^2 \hat{u}(k,t), \\ \hat{u}(k,0) = \hat{f}(k), \\ \partial_t \hat{u}(k,0) = \hat{g}(k). \end{cases}$$

A bit more detailed:

$$\frac{1}{c^2}\partial_{t^2}^2 \hat{u}(k,t) \overset{\text{DCT}}{=} (\frac{1}{c^2}\partial_{t^2}^2 u)^\wedge(k,t) \overset{\text{DGL}}{=} (\Delta u)^\wedge(k,t) \overset{p.I}{=} \int_{\mathbb{R}^n} u(x,t)\Delta e^{-ik\cdot x}\, dx = -|k|^2 \hat{u}(k,t).$$

In the frequency space we have an ordinary differential equation, whose solution we already know (for a fixed k). It holds that,

$$\hat{u}(k,t) = A(k)\cos(|k|ct) + B(k)\sin(|k|ct),$$

and with the initial conditions we obtain,

$$\hat{u}(k,t) = \hat{f}(k)\cos(|k|ct) + \frac{\hat{g}(k)}{|k|c}\sin(|k|ct).$$

With the inversion theorem 3.8 we get:

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\hat{f}(k)\cos(|k|ct) + \frac{\hat{g}(k)}{|k|c}\sin(|k|ct)\right)e^{ik\cdot x}\,dk.$$

Let $f, g \in C_0^\infty(\mathbb{R}^n)$. Then$f, g \in \mathscr{S}(\mathbb{R}^n)$ and with Lemma 3.14 we have that,

$$|\partial^\beta f(x)| \le \frac{c_{\beta,k}}{(1+|x|^2)^k},$$

for $\beta \in \mathbb{N}_0^n, k \in \mathbb{N}$. This allows us to swap the derivative and the integral. If we know the dimension we can evaluate the formal solution u more precise. We get:

$$n=1: \quad u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(r)\,dr\right],$$

$$n=2: \quad u(x,t) = \frac{1}{2\pi c}\left[\partial_t\left(\int_{B(x,ct)}\frac{f(y)}{\sqrt{c^2t^2-|y-x|^2}}\,dy\right) + \int_{B(x,ct)}\frac{g(y)}{\sqrt{c^2t^2-|y-x|^2}}\,dy\right],$$

$$n=3: \quad u(x,t) = \frac{1}{4\pi c^2}\left[\partial_t\left(\int_{\partial B(x,ct)} f(y)\,d\Omega(y)\right) + \frac{1}{t}\int_{\partial B(x,ct)} g(y)\,d\Omega(y)\right],$$

where as Ω is the surface measure on the sphere.
The derivation of the solutions for $n = 2, 3$ is pretty challenging. For $n = 1$ the derivation is rather simple, so we will derive it now. We can evaluate the Integral for the formal solution of u term-wise.

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(k)\cos(kct)e^{ikx}\,dk &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(k)\frac{1}{2}(e^{ikct}+e^{-ikct})e^{ikx}\,dk \\
&= \frac{1}{4\pi}\left(\int_{\mathbb{R}}\hat{f}(k)e^{ik(x+ct)}\,dk + \int_{\mathbb{R}}\hat{f}(k)e^{ik(x-ct)}\,dk\right) \\
&= \frac{1}{2\pi}(f(x+ct)+f(x-ct)),
\end{aligned}$$

where as we use the inversion theorem 3.8 in the last line. Let's consider the second summand:

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\hat{g}(k)}{kc}\sin(kct)\,dk &= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{\hat{g}(k)}{kc}\frac{1}{2i}(e^{ikct}-e^{-ikct})\,dk \\
&= \frac{1}{4\pi c}\int_{\mathbb{R}}\hat{g}(k)\underbrace{\frac{1}{ik}(e^{ik(x+ct)}-e^{ik(x-ct)})}_{\text{write as integral}}\,dk = \frac{1}{4\pi c}\int_{\mathbb{R}}\hat{g}(k)\int_{x-ct}^{x+ct}e^{iky}\,dy\,dk \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct}\frac{1}{2\pi}\int_{\mathbb{R}}\hat{g}(k)e^{iky}\,dk\,dy = \frac{1}{2c}\int_{x-ct}^{x+ct}g(y)\,dy,
\end{aligned}$$

where as we implicitly used Fubini's theorem 1.3. In conclusion we obtain:

$$u(x,t) = \frac{1}{2}\left[f(x+ct)+f(x-ct)+\frac{1}{c}\int_{x-ct}^{x+ct}g(y)\,dy\right].$$

3.9 Heat equation

The heat equation $\partial_t u(x,t) = D\nabla^2 u(x,t)$ describes the time- and position-dependence of the temperature $u(x,t)$ at time t at position $x \in \mathbb{R}^n$ in a homogenous heat conducting medium.

The physical constant $D > 0$ can be chosen to be 1 for suitable choices of the time and length scales. The initial value problem (IVP) reads

$$\begin{cases} \partial_t u(x,t) - \nabla^2 u(x,t) &= 0, \qquad t > 0,\, x \in \mathbb{R}^n \\ u(x,0) &= f(x). \end{cases} \tag{6}$$

We are looking for a solution $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times (0,\infty))$ (we want to allow non-differentiable initial conditions $f \in C(\mathbb{R}^n)$). We initially assume that $f \in \mathscr{S}(\mathbb{R}^n)$. The Fourier transform

$$\hat{u}(k,t) = \int_{\mathbb{R}^n} u(x,t) e^{-ik\cdot x}\, dx \qquad \text{and} \qquad \hat{f}(k) = \int_{\mathbb{R}^n} f(x) e^{-ik\cdot x}\, dx$$

yields the ODE

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}(k,t) &= -k^2 \hat{u}(k,t) \\ \hat{u}(k,0) = \hat{f}(k) \end{cases} \tag{7}$$

with solution

$$\hat{u}(k,t) = e^{-k^2 t}\hat{f}(k).$$

The back-transformation provides the formal solution

$$u(x,t) = \int_{\mathbb{R}^n} K_t(x-y) f(y)\, dy$$

where K is the heat kernel:

$$K_t(x) = \left(e^{-k^2 t}\right)^\vee (x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

Theorem 3.25. Let f be continuous and restricted on $\mathbb{R}^n$. Then

$$u(x,t) = \int_{\mathbb{R}^n} K_t(x-y) f(y)\, dy \tag{8}$$

is a solution of the heat equation in $C^\infty(\mathbb{R}^n \times (0,\infty))$ and for all $x \in \mathbb{R}^n$ it holds that $\lim_{t\to 0^+} u(x,t) = f(x)$.

4 Hilbertspaces & Eigenvalueproblems

In this section we introduce the notion of infinite dimensional Hilber spaces and discuss a selection of orthonormal bases on certain Hilber space which play an important role in mathematical and theoretical physics.

4.1 Orthogonal systems & Hilbert spaces

Let V be a complex or real vector space endowed with a scalar product $(\cdot, \cdot)$, i.e. a mapping $V \times V \to \mathbb{C}$ (or $\mathbb{R}$) with,

1. $(f, g) = \overline{(g, f)}$
2. $(f, \lambda g + \mu h) = \lambda(f, g) + \mu(f, h)$
3. $(f, f) \geq 0 \Leftrightarrow f \equiv 0$,

$\forall f, g, h \in V$ and $\lambda, \mu \in \mathbb{C}$ (or $\mathbb{R}$). Then $\|f\| = \sqrt{(f, f)}$ defines a norm on V.

Definition 4.1. A complex or real vector space H with dot product is called **Hilbert space**, if it is with respect to the norm $f \mapsto | f\| = \sqrt{(f, f)}$ is a Banach space, i.e. if all Cauchy sequences converge in H regarding $\| \cdot \|$.

In particular, $L^2(E)$, with $E \subset \mathbb{R}^n$ measurable, is a Hilbert space.

Theorem 4.2. Let H be a Hilbert space and let $(\varphi)_{j=1}^{\infty}$ be an orthonormal system. TFAE:

1. $(\varphi)_{j=1}^{\infty}$ is complete,
2. $\phi = \sum_{j=1}^{\infty} (\varphi_j, \phi)\varphi_j, \quad \forall \phi \in H$
3. $\|\phi\|^2 = \sum_{j=1}^{\infty} |(\varphi_j, \phi)|^2, \quad \forall \phi \in H$ (Parseval identity).

Definition 4.3. A complete orthonormal system on a Hilbert space is called orthonormal basis (ONB).

Definition 4.4. A Hilbert space is called **separable** if it has a countable orthonormal basis.

Remark. These definitions are essential in the formalism of quantum mechanics. The Hilbert space, which is the basis of wave mechanics, is the space of square-integrable functions, the so-called L^2-space. In the simplest case of a 1-dimensional configuration space (the generalization for f-dimensional systems is relatively obvious), the relevant L^2 space is $L^2(\mathbb{R})$, which is the space of real-valued functions $f : \mathbb{R} \to \mathbb{C}$ with

$$\int_{\mathbb{R}} |f(x)|^2 \, dx < \infty.$$

The Hilbert space $L^2(\mathbb{R})$ is infinite dimensional and separable. That means, it has a countable infinite basis $\{f_n\}_n$, i.e., linear independent vectors f_n such that every $f \in L^2(\mathbb{R})$ can be written as a linear combination of f_n's,

$$f = \sum_{i=1}^{\infty} c_n \cdot f_n \qquad c_n \in \mathbb{C}.$$

In quantum mechanics, this f would now describe the system we are considering.

4.2 Hermite polynomials

Definition 4.5. The n-th Hermite-polynomial is:

$$H_n(x) = (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}, \quad n = 0, 1, 2, ... \tag{9}$$

With induction one can show that H_n is a polynomial of degree n. They fulfill the following recursive formula:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} H_n(x) - 2x \frac{\mathrm{d}}{\mathrm{d}x} H_n(x) + 2n H_n(x) = 0.$$

Theorem 4.6. The functions,

$$\psi_n(x) = \frac{1}{\pi^{1/4} 2^{n/2} (n!)^{1/2}} H_n(x) e^{-\frac{x^2}{2}}, \quad n = 0, 1, 2, ...,$$

build a complete orthonormal system (ONB) on $L^2(\mathbb{R})$.

Remark. The hermitian operator $H = A^* A + \frac{1}{2} = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2$ is the Hamilton operator of the quantum mechanical harmonic oscillator. The eigenvalues of H can be interpreted as energies. The above defined functions ψ_n build a complete set of eigenfunctions of H, meaning

$$H\psi_n = E_n \psi_n,$$

(time independent Schroedinger equation) with eigenvalues $E_n = n + \frac{1}{2}$.

We can show that the Hermite functions ψ_n are eigenfunctions of the linear operator the Fourier transform. We show this for a slightly alternative form of the Hermite functions.

Proposition 4.7. The functions $h_n(x) = H_n(x) e^{-\frac{1}{2}x^2} = (-1)^n e^{\frac{1}{2}x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}$ are eigenfunctions of the Fourier transform, namely:

$$\widehat{h_n}(k) = (-i)^n (2\pi)^{1/2} h_(k).$$

Proof.

$$\begin{aligned}
\widehat{h_n}(k) &= \int_{-\infty}^{\infty} (-1)^n e^{\frac{1}{2}x^2} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2} \right) e^{-ikx}\, dx \overset{p.I.}{=} \int_{-\infty}^{\infty} e^{-x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{\overbrace{\frac{1}{2}x^2 - ikx}^{\text{compl. square}}} \right) dx \\
&= e^{\frac{1}{2}k^2} \int_{-\infty}^{\infty} e^{-x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left(e^{\frac{1}{2}(x-ik)^2} \right) dx \overset{(1)}{=} i^n e^{\frac{1}{2}k^2} \int_{-\infty}^{\infty} e^{-x^2} \frac{\mathrm{d}^n}{\mathrm{d}k^n} \left(e^{\frac{1}{2}(x-ik)^2} \right) dx \\
&= i^n e^{\frac{1}{2}k^2} \frac{\mathrm{d}^n}{\mathrm{d}k^n} \int_{-\infty}^{\infty} e^{-x^2 + \frac{1}{2}x^2 - ikx - \frac{1}{2}k^2}\, dx = i^n e^{\frac{1}{2}k^2} \frac{\mathrm{d}^n}{\mathrm{d}k^n} \left(e^{-\frac{1}{2}k^2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2 - ikx}\, dx}_{(e^{-x^2/2})^\wedge(k) = \sqrt{2\pi} e^{-k^2/2}} \right) \\
&= i^n e^{\frac{1}{2}k^2} \frac{\mathrm{d}^n}{\mathrm{d}k^n} \sqrt{2\pi} e^{-k^2} = (-i)^n \sqrt{2\pi} h_n(k)
\end{aligned}$$

In (1) we use the trick that for a differentiable function g it holds that:
$\frac{\mathrm{d}}{\mathrm{d}x} g(x - tk) = \frac{-1}{t} \frac{\mathrm{d}}{\mathrm{d}k} g(x - tk)$. □

Lemma 4.8. $\mathscr{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, i.e. for each $\epsilon > 0$ and $f \in L^2(\mathbb{R})$ there exists a $\phi \in \mathscr{S}(\mathbb{R})$ with $\|f - \phi\|_2 < \epsilon$.

Proof.
This follows form the fact that for $\psi_j \in \mathscr{S}(\mathbb{R})$: $\|f - \sum_{j=1}^{n} (\psi_j, f)\psi_j\| \to 0$, for $n \to \infty$. □

Remark. The above lemma is good to know for future quantum mechanic courses. Namely the QM formalism is defined on L^2 vector spaces with corresponding wave functions. We often get to the point where we have to integrate the wave functions by parts and we obtain a term which has to be evaluated at $\pm\infty$. Now in a physicists way of thinking this term will be zero, since we interpret the wave functions as functions of the Schwartz space(we can do this in a good approximation because $\mathscr{S}$ is dense in L^2.)

4.3 Quantum harmonic oscillator

The harmonic oscillator in quantum mechanics describes analogously to the harmonic oscillator in classical physics describes the behavior of a particle in a potential of the form,

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}\omega^2 x^2. \tag{10}$$

This is one of the few quantum mechanical systems for which an exact analytical solution is known. An arbitrary potential can be approximated in the neighborhood of a stable equilibrium position by a harmonic potential, which provides the motivation to consider the problem in more detail.

The Schroedinger equation is a linear partial differential equation, which describes the quantum mechanical state of non-relativistic system. It is thus the quantum analogon to the classical Newton's second law $\vec{F} = m\vec{a}$. The Schroedinger equation allows us to analyze quantum mechanical systems and to make predictions. As usual in quantum mechanics the state of the system is described by a wave function. The stationary Schroedinger equation describes the variation of standing waves by applying a Hamiltonian operator to the wave. Analogous to the setting in general mechanics, the Hamiltonian operator describes the total energy of a system, i.e. the sum of kinetic and potential energy. However, the Hamiltonian equations of motion are no longer valid in this form but must be generalized by Poisson brackets. The Hamilton operator is obtained through the canonical quantisation of the usual Hamiltonian (with generalised coordinate x and canonical momentum p). Instead of just considering these expressions as scalars, we promote them to operaots, i.e., the position operator $\hat{x} = x$ and the momemtum operator $\hat{p} = -i\hbar\nabla$. Hence the Hamiltonian for our system is given by,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega^2}{2}x^2. \tag{11}$$

We consider now the stationary (time independent) Schroedinger equation

$$\hat{H}\psi = E\psi \tag{12}$$

for a non relativistic particle. This is an eigenvalue problem, where ψ is an eigenfunction to the eigenvalue E. The corresponding states are stationary states (eigenstates) and the eigenvalues just correspond to the possible energy levels of the system (discrete energy, which belongs to a state). The Schroedinger equation allows the calculation of such energy levels. In one dimension, the Schroedinger equation is

$$-\frac{\hbar^2}{2m}\partial_x^2\,\psi(x) + \frac{m\omega^2}{2m}x^2\psi(x) = E\psi(x), \tag{13}$$

with the boundary condition $\lim_{n\to\infty}\psi(x)=0$.

4.3.1 conventional solution

We find the solution of the sationary Schroedinger equation (13) in the one dimesional case. For that we reformulate eq. (13) by introducing $y=\sqrt{\frac{m\omega}{\hbar}}x$. Then it holds that $\mathrm{d}y=\sqrt{\frac{m\omega}{\hbar}}\mathrm{d}x$, hence $\partial_x^2\psi=\frac{m\omega}{\hbar}\partial_y^2\psi$. In additional we define $\epsilon=\frac{2E}{\hbar\omega}$. We get:

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\partial_y^2\psi+\frac{m\omega^2}{2}\frac{\hbar}{m\omega}y^2\psi&=E\psi\\ -\frac{\hbar\omega}{2}\partial_y^2\psi+\frac{\hbar\omega}{2}y^2\psi&=E\psi\\ -\partial_y^2\psi+y^2\psi&=\epsilon\psi\\ \partial_y^2\psi&=(y^2-\epsilon)\psi. \end{aligned}\tag{14}$$

We try now to solve this last equation. For large y ϵ gets neglectable. We thus assume that $y>>\sqrt{\epsilon}$ and we consider the differential equation

$$\partial_y^2\psi\approx y^2\psi.$$

A corresponding solution is given by

$$\psi(y)\approx Ae^{-\frac{1}{2}y^2}+Be^{\frac{1}{2}y^2}$$

where it must hold that $B=0$ to fulfill the boundary conditions $\psi\to 0$ for $y\to\infty$. Otherwise $e^{\frac{1}{2}y^2}$ would blow up for $y\to\infty$. Therefore, $\psi(y)\approx Ae^{-\frac{1}{2}y^2}$ for y sufficiently large. We try now the ansatz

$$\psi(y)=h(y)\cdot e^{-\frac{1}{2}y^2}$$

and insert it into eq. (14) where as,

$$\begin{aligned}\partial_y\psi&=(h'(y)-yh(y))\,e^{-\frac{1}{2}y^2},\\ \partial_y^2\psi&=\left(h''(y)-2yh'(y)-h(y)+y^2h(y)\right)e^{-\frac{1}{2}y^2}.\end{aligned}$$

Therefore

$$\begin{aligned}\left(h''(y)-2yh'(y)-h(y)+y^2h(y)\right)e^{-\frac{1}{2}y^2}&=(y^2-\epsilon)h(y)e^{-\frac{1}{2}y^2}\\ \Longleftrightarrow h''(y)-2yh'(y)+(\epsilon-1)h(y)&=0.\end{aligned}$$

Comparing the latter equation with eq. (4.5) whose soltuion are given by the Hermite polynomials (9), we see that $h(y)=H_n(y)$ for $n\in\mathbb{N}$. This comparison implies not only the solution of eq. (14) also the possible energy levels. Namely it is valid that

$$2n=\epsilon_n-1\Longleftrightarrow\epsilon_n=2n+1\Longleftrightarrow E_n=\hbar\omega\left(n+\frac{1}{2}\right).$$

The corresponding solutions are of the form

$$\psi_n(x)=N_nH_n(y)e^{-\frac{1}{2}y^2}=N_nH_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega}{2\hbar}x^2},$$

where N_n is an normalisation factor such that the scalar product $(\psi_n, \psi_n) = 1$.

The solution has a few remarkable properties. The energy spectrum is represented by a discrete set $\left\{\hbar\omega\left(n+\frac{1}{2}\right)\right\}_{n\in\mathbb{N}}$. This phenomenon is a general property of quantum mechanical systems and is essentially the eponym of quantum physics. This "peculiarity" occurs even in classical physics, when waves are confined in a certain way (e.g. with clamped strings). Besides, the lowest possible energy level is $E_0 = \frac{\hbar\omega}{2}$ (also called the groundstate energy) higher as the minimum of the potential and the gorund state ψ_0 has the form of a Gaussian curve.

If we look at the absolute square $|\psi_n|^2$ instead of ψ_n we can consider the probability distribution for the position of the particle. The ground state is concentrated at the origin, as it is to be expected for a state with little energy. If we increase the energy, we see that the probability that the particle will end at the classical "turning points" becomes larger. These turning points are those where the energy of the state coincides with the potential energy (and thus the particle moves the slowest). That is, the correspondence principle is fulfilled: For large quantum numbers we get the same results in the limit as in classical physics.

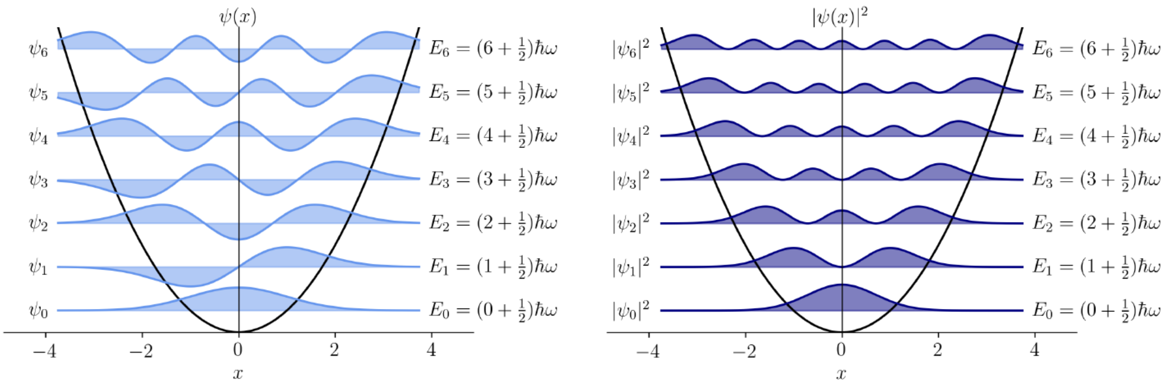

Figure 3: The first seven solutions for the quantum mechanical harmonic oscillator.We plot on the left the wavefunction of ψ_n and on the right the absolut square $|\psi_n|^2$.

4.3.2 more elegeant solution*

This solution is based on an operator technique with **creation** and **annihilation operators**. These operators play an important role in many areas of theoretical physics, for example they appear in the field quantization of free fields. We define

$$a := \frac{1}{\sqrt{2}}(y + \partial_y) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \frac{i}{\sqrt{m\hbar\omega}}p\right),$$
$$a^\dagger := \frac{1}{\sqrt{2}}(y - \partial_y).$$

The inversion is then

$$y = \frac{1}{\sqrt{2}}(a + a^\dagger),$$
$$\partial_y = \frac{1}{\sqrt{2}}(a - a^\dagger).$$

Inserted in the Hamiltonian (11) we get:

$$\begin{aligned} H &= \frac{\hbar\omega}{2}(-\partial_x^2 + x^2) \\ &= \frac{\hbar\omega}{4}\left[-(a - a\dagger)(a - a\dagger) + (a + a\dagger)(a + a\dagger)\right] \\ &= \frac{\hbar\omega}{2}\left(a \dagger a + a\dagger\right) = \hbar\omega\left(a \dagger a + \frac{1}{2}[a, a\dagger]\right). \end{aligned} \tag{15}$$

For the commutator we find that $[a, a^\dagger] = 1$. With the definition of the number operator $N = a^\dagger a$ we find

$$H = \hbar\omega\left(N + \frac{1}{2}\right).$$

The eigenvalueproblem $H\psi = E\psi$ is then reduced to

$$N|n\rangle = n\,|n\rangle$$

with which we pass over to the Dirac notation, i.e., we denote the state ψ_n with $|n\rangle$. We investigate the action of the ladder operators on the eigenstates $|n\rangle$ of N. We obtain that,

$$[N, a^\dagger] = a^\dagger \qquad \text{and} \qquad [N, a] = -a.$$

The states $a^\dagger\,|n\rangle$ and $a\,|n\rangle$ define new eigenstates of N with eigenvalues $n+1$ and $n-1$. Those new states are not correctly normalized. With correct normalisation we have the proper eigenstates

$$|n+1\rangle = \frac{1}{\sqrt{n+1}}a^\dagger\,|n\rangle \qquad \text{and} \qquad \frac{1}{n}a\,|n\rangle\,. \tag{16}$$

We can iteratively find all eigenstates by applying a aribtrarily often to the state $|n\rangle$. This yields

$$a^k\,|n\rangle = \sqrt{n(n-1)(n-2)\ldots(n-k+1)}\,|n-k\rangle$$

where $|n-k\rangle$ is an eigenvector to the eigenvalue $n-k$. On the other hand $n = branN|n\rangle = \langle an|an\rangle \leq 0$. For $k \in \mathbb{N}$ large enough we get a negative eigenvalue which is not possible. That means the iteration has to abort for $k = n$. In other words for $n = 0$ we have

$$a\,|0\rangle = 0.$$

Also,

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,, \qquad n \geq 0.$$

The eigenvalue n determines the energy $E_n = \hbar\omega(N + \frac{1}{2}) = \hbar\omega(n + \frac{1}{2})$. Therefore n counts the available energy-quants $\hbar\omega$. The application of $a^\dagger$ creates another energy-quant which increases the oscillation amplitude. The smallest eigenstate, the groundstate $|0\rangle$, does not posses any energy quants, but $E_0 \neq \hbar\omega/2$, a consequence of Heisenberg's uncertainty principle.

Now we need the wavefunction in the position space. First consider the state $|0\rangle$. We know that $a\,|0\rangle = 0$. Hence,

$$0 = \sqrt{2}\,\langle y|\,a\,|0\rangle = (y + \partial_y)\,\langle y|0\rangle = (y + \partial_y)\psi_0(y) = 0.$$

The solution of this differential equation is simple, $\psi_0(y) \propto e^{-y^2/2}$. With the right normailisation we obtain

$$\psi_0(y) = \frac{1}{\sqrt[4]{\pi}}e^{-y^2/2}. \tag{17}$$

We get the states $\langle y|n\rangle$ through iteratively application of $a^\dagger$,

$$\langle y|n\rangle = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}(y-\partial_y)^n e^{-y^2/2} = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}H_n(y)e^{-y^2/2}.$$

That is the same result as we got before in 4.3.1.

4.4 Legendre polynomials

Definition 4.9 (Rodrigues formula)**.**
The n-th Legendre polynomial is:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad \text{for } n = 0, 1, 2, ...$$

Proposition 4.10.

1. P_l has degree l,
2. $\int_{-1}^1 P_l(x)P_{l'}(x)\,dx = \frac{2}{2l+1}\delta_{ll'}$
3. $P_l(1) = 1$.

This means that $\sqrt{\frac{2l+1}{2}}P_l$ are orthonormal.

Proof.
1.) is clear, since P_l is the l-th derivative of a polynomial of degree $2l$.
2.) for $l' < l$ we obtain through partial integration,

$$\begin{aligned}(P_l, P_{l'}) &= \int_{-1}^1 \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l \frac{1}{2^{l'} l'!}\frac{\mathrm{d}^{l'}}{\mathrm{d}x^{l'}}(x^2-1)^{l'}\,dx \\ &\overset{\text{l-times p.I.}}{=} \frac{(-1)^l}{2^{l+l'}l!\,l'!}\int_{-1}^1 (x^2-1)^l \underbrace{\frac{\mathrm{d}^{l+l'}}{\mathrm{d}x^{l+l'}}\underbrace{(x^2-1)^{l'}}_{\deg = 2l' < l+l'}}_{=0}\,dx = 0.\end{aligned}$$

The boundary terms form the partial integrations steps again vanish. Let now $l = l'$. Then,

$$(P_l, P_l) = \frac{(-1)^l}{2^{2l}(l!)^2}\int_{-1}^1 (x^2-1)^l \frac{\mathrm{d}^{2l}}{\mathrm{d}x^{2l}}(x^2-1)^l\,dx = \frac{(-1)^l}{2^{2l}(l!)^2}(2l)!\underbrace{\int_{-1}^1 (x^2-1)^l\,dx}_{I_l}$$

The integral I_l can be computed recursively. For $l \geq 1$ we have:

$$\begin{aligned}I_l &= \int_{-1}^1 1(x^2-1)^l\,dx \overset{p.I.}{=} -l\int_{-1}^1 2x^2(x^2-1)^{l-1}\,dx = -2l\int_{-1}^1 (x^2-1+1)(x^2-1)^{l-1}\,dx \\ &= -2l(I_l + I_{l-1}) \\ &\Rightarrow I_l = -\frac{2l}{2l+1}I_{l-1}\end{aligned}$$

For $l = 0$ we have $I_0 = 2$. Then we can derive,

$$I_l = (-1)^l \frac{2l\cdot(2l-1)\ldots 4\cdot 2}{(2l+1)\cdot(2l-1)\ldots 3\cdot 1}\cdot 2 = (-1)^l\frac{2^{2l}(l!)^2}{(2l+1)!}2.$$

Therefore,

$$(P_l, P_l) = \frac{(-1)^l}{2^{2l}(l!)^2}(2l)!(-1)^l \frac{2^{2l}(l!)^2}{(2l+1)!}2 = \frac{2}{2l+1}.$$

3.)

$$P_l(1) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left[(x-1)^l(x+1)^l\right]\Big|_{x=1} = \frac{1}{2^l l!}l!\,(x+1)^l\Big|_{x=1} = 1.$$

□

Lets consider now the differential operator on the vector space of polynomials in x, given by,

$$u \mapsto Lu = \frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}\right]u.$$

L is a hermitian operator i.e.,

$$\begin{aligned}(Lu, v) &= \int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}\overline{u}\right)v\,dx = -\int_{-1}^{1}(x^2-1)\left(\frac{\mathrm{d}}{\mathrm{d}x}\overline{u}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}v\right)\,dx + (x^2-1)\overline{u}v\Big|_{-1}^{1}\\ &= \int_{-1}^{1}\overline{u}(Lv)\,dx - (x^2-1)\overline{u}'v\Big|_{-1}^{1} = (u, Lv).\end{aligned}$$

The boundary vanish, since x^2-1 vanishes for 1 and -1.

Theorem 4.11.

1. The Legendre polynomials P_l are eigenfunctions of the operator L with eigenvalue $l(l+1)$. In other words the fulfil the Lengendre differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}\right]P_l(x) = l(l+1)P_l(x).$$

2. The orthonormal polynomials $\sqrt{\frac{2l+1}{2}}P_l$ are a complete orthonormal system (ONB) on $L^2([-1,1])$.

Lemma 4.12. Let $x, y \in \mathbb{R}^n$ with $r := |x| > |y| := r'$. With $\cos(\theta) = \frac{x\cdot y}{rr'}$ the following holds:

$$\frac{1}{|x-y|} = \frac{1}{\sqrt{r^2 - 2rr'\cos(\theta) + (r')^2}} = \sum_{l=0}^{\infty}\frac{(r')^l}{r^{l+1}}P_l(\cos\theta)$$

Proof. Since the Legendre Polynomials on $[-1,1]$, we can write:

$$\frac{1}{|x-y|} = \sum_{l=0}^{\infty} c_l P_l(\cos\theta)$$

and with $y = \delta r$ (with $\delta < 1$) we obtain:

$$\frac{1}{|x-y|} = \frac{1}{\sqrt{r^2 - 2rr'\cos(\theta) + (r')^2}} = \frac{1}{r\sqrt{1+\delta^2 - 2\delta\cos(\theta)}} \tag{18}$$

By setting $\theta = 0$ we obtain $P_l(1) = 1$ and therefore:

$$\frac{1}{r\sqrt{1+\delta^2-2\delta}} = \frac{1}{r}\cdot\frac{1}{\sqrt{(1-\delta)^2}} = \frac{1}{r}\cdot\frac{1}{1-\delta} = \frac{1}{r}\sum_{n=0}^{\infty}\delta^n$$

This yields then to:

$$\frac{1}{|x-y|} = \frac{1}{r} \cdot \sum_{l=0}^{\infty} \frac{(r')^l}{r^l} P_l(\cos\theta)$$

In other words the statement directly follows from the Taylor expansion (in the variable t) of:

$$\frac{1}{\sqrt{1+t^2-2tz}} = \sum_{n=0}^{\infty} P_n(z) t^n \tag{19}$$

which is convegent for $|t| < 1$ and $|z| \leq 1$. □

The above Lemma is important for physical purposes, since the Coulomb potential (in 3 dimensions) of a charge q at position y is proportional to $\frac{1}{|x-y|}$.

4.5 spherical harmonics

We have an ONB on $L^2(S^1)$ given through the functions $e^{im\varphi}$ with $m \in \mathbb{Z}$. The most important property of these functions is that they are eigenfunctions with of the spherical Laplacian Δ_{S^1} meaning,

$$\Delta_{S^1} e^{im\varphi} = \frac{\partial^2}{\partial\varphi^2} e^{im\varphi} = -m^2 e^{im\varphi}.$$

Remark (Reminder)**.**
You should have already seen the Laplacian operator in Cartesian coordinates. It is given by:

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2} = \sum_{l=1}^{n} \frac{\partial^2}{\partial x_l^2}.$$

We now can also define the Laplacian in spherical coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} = \frac{n-1}{2} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}},$$

where as $\Delta_{S^{n-1}}$ is the Laplacian restricted on the n-dimensional unit sphere. Since most of the application happen in 2 and 3 dimensions, we will write out the Laplacian (in spherical coordinates) for these dimensions explicitly:

$$n = 2: \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} \underbrace{\frac{\partial^2}{\partial\varphi^2}}_{\Delta_{S^1}},$$

$$n = 3: \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} \left(\underbrace{\frac{\partial^2}{\partial\theta^2} + \cot(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}}_{\Delta_{S^2}} \right).$$

Definition 4.13. A function f is called harmonic, iff $\Delta f = 0$. A harmonic polynomial $p \in \mathbb{C}[x_1, ...x_n]_l$ is a polynomial with $\Delta p = 0$

Definition 4.14. The spherical harmonics Y (of degree l) are the restrictions of harmonic polynomials (homogenous of degree l) on S^{n-1}. The sperical harmonics are eigenfunctions of the spherical Laplacian:

$$\Delta_{S^{n-1}} Y = -l(l+n-2)Y.$$

Lets verify the eigenvalue equation. Let $p \in \mathbb{C}[x_1, ...x_n]_l$ be harmonic, homogenous and of degree l. Define $X = \frac{x}{|x|}$ and $r = |x|$, so it holds that $x = rX$ with $X \in S^{n-1}$ (so X is an element on the n-dimensional unit sphere). Then, $p(x) = p(rX) = r^l p(X)$. Since p is harmonic we obtain:

$$\begin{aligned} 0 = \Delta p(x) &= \left(\frac{\partial^2}{\partial r^2} = \frac{n-1}{2}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}}\right)(r^l p(X)) \\ &= l(l-1)r^{l-2}p(X) + \frac{n-1}{r} l r^{l-1} p(X) + \frac{1}{r^2} r^l \Delta_{S^{n-1}} p(X) \\ &\iff \\ \Delta_{S^{n-1}} p(X) &= (-(l-1) - (n-1)l)p(X) = -l(l+n-2)p(X) \end{aligned}$$

We just now want to emphasize that for $n = 3$ the eigenvalues are $-l(l+1)$.

Definition 4.15.
We define for $l = 0, 1, 2, ...$ and $m = -l, ..., l-1, l$ the **spherical harmonics**:

$$Y_{l,m}(\theta, \phi) = \frac{(-1)^m}{\sqrt{2\pi}} \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}} P_{l,m}(\cos\theta) e^{im\phi}$$

where as $P_{l,m}(X)$ are the associated Legendre functions defined through:

$$P_{l,m}(X) = \frac{(1-Z^2)^{m/2}}{2^l l!} \frac{d^{m+l}}{dX^{m+l}}(X^2-1)^l \overset{m \geq 0}{=} (1-X)^{m/2} \frac{d^m}{dX^m} \underbrace{P_l(X)}_{\text{Legendre polynomial}}$$

Theorem 4.16. The spherical harmonics $\{Y_{l,m}(\theta, \phi)\}_{\substack{l=0,1,... \\ m=-l,...,l}}$ form a complete orthonormal basis of the Hilbert space $L^2(S^2)$ which consists of eigenvectors of Δ_{S^2}.
The eigenvalue equation is given by:

$$\Delta_{S^2} Y_{l,m}(\theta, \phi) = -l(l+1) Y_{l,m}(\theta, \phi)$$

Proof. We first proof that the spherical harmonics are eigenfunctions of the spherical laplacian. For that recall that $r^l Y_{l,m}$ is a homegenous harmonic polynomial. Therefore,

$$0 = \Delta(r^l Y_{l,m}) = r^{l-2}(l(l-1) + 2l + \Delta_{S^2})Y_{l,m}.$$

Solving after $\Delta_{S^2} Y_{l,m}$ yields:

$$\Delta_{S^2} Y_{l,m} = -l(l+1) Y_{l,m}$$

To show that $\{Y_{l,m}(\theta, \phi)\}_{\substack{l=0,1,... \\ m=-l,...,l}}$ form a ONB we have to verify orthonormality and completeness. The orthonormality follows from the fact that,

$$\int_0^\pi \int_0^{2\pi} \overline{Y_{l,m}(\theta, \varphi)} Y_{l',m'}(\theta, \varphi) \sin\theta \, d\varphi d\theta = \delta_{ll'}\delta_{mm'}.$$

To show that the $Y_{l,m}$ are complete we use the fact continuous functions are dense in L^2. Let $f(\theta, \varphi)$ be continuous and χ a continuous function on $\mathbb{R}_+$ with $\chi(r) = 1$ for $|r-1| < \delta$ and $\chi(r) = 0$ for $|r-1| > 2\delta$. Then $\tilde{f} = \chi(r) f(\theta, \varphi)$ a continuous function on $\mathbb{R}^3$. Due to the Weierstrass's approximation theorem, there exists a polynomial $u(x_1, x_2, x_3)$ with,

$$\max_{\|x\| \leq 1} |u(x) - \tilde{f}(x)| < \epsilon$$

In particular, $|u(x) - f(x)| < \epsilon$ for $|x| = 1$. Due to the remark above u can be written as $u = \sum_{s=0}^{N} r^s u_s$ where as u_s are harmonic polynomials. The restriction of u on S^2 is a sum of spherical harmonics. So there exists coefficients c_{lm} such that

$$\Bigg|\underbrace{\sum_{\substack{l\leq N\\ |m|\leq l}} c_{lm} Y_{l,m}(\theta,\varphi)}_{Y:=} - f(\theta,\varphi)\Bigg| < \epsilon.$$

Then, $\|Y - f\|_2^2 = \int_{S^2} |Y - f|^2 d\Omega < 4\pi\epsilon \implies \|Y - f\|_2 = \sqrt{4\pi}\epsilon$. So for every $g \in L^2(S^2)$ and $\forall \epsilon > 0$ there exists a linear combination of spherical harmonics Y such that $\|Y - g\| < \epsilon$.
Let now $h \in L^2(S^2)$ such that $\langle Y_{l,m}, h\rangle = 0$ for all l, m. Let Y be a liner combination of spherical harmonics $\{Y_{l,m}\}$ such that $\|h - Y\| < \epsilon$. Then,

$$\|h - Y\|_2^2 = \|h\|_2^2 - 2\Re(\underbrace{\langle h, Y\rangle}_{=0}) + \underbrace{\|Y\|_2^2}_{\geq 0} < \epsilon^2.$$

Hence, $\forall \epsilon > 0 : \|h\|_2 < \epsilon \implies h \equiv 0$.

$\square$

Remark. Spherical harmonics are crucial in physics because they provide a natural way to describe functions on the surface of a sphere. They appear in problems with spherical symmetry, like atomic orbitals, gravitational and electric fields.

A perfect example of spherical harmonics in physics is the hydrogen atom. When solving the Schroedinger equation for the electron in a hydrogen atom, the problem naturally has spherical symmetry, since the force between the electron and proton depends only on distance. By switching to spherical coordinates, the equation separates into radial and angular parts. The angular part is exactly solved by spherical harmonics. These describe the shape of the electrons orbitals, like the familiar s, p, d shapes, and explain why electrons form specific patterns around the nucleus. Without spherical harmonics, we couldn't understand atomic structure this clearly.

Lemma 4.17. The functions $\left\{\sqrt{\frac{2}{L}}\sin(\omega_n x)\right\}_{n=0}^{\infty}$ with $\omega_n = \frac{n\pi}{L}$ is a complete orthonormal system of $L^2([0, L])$.

4.6 separation ansatz

A separable partial differential equation is a differential equation, which can be split up into equations of lower dimension(this process is called separation ansatz). In this way we can solve PDEs by solving several more simple PDE (ideally ODEs). This method should not be mixed up with the separation of variables, a method to solve ODE's(read this if you cannot remember it)
Lets go through an example to get an idea how this separation ansatz works. Let's consider the temperature distribution on a stick of length L evolving in time t.Together with initial and boundary condition we have:

$$\begin{cases} \partial_t u(x,t) & = \alpha \partial_x^2 u(x,t), & (x,t) \in [0,L] \times \mathbb{R}_{\geq 0} \\ u(0,t) & = u(L,t) = 0, & \forall t \geq 0 \\ u(x,0) & = f(x), & x \in [0,L] \end{cases}$$

1. **separation of variables**
   In a first step we assume that we can write u as an product of two functions which do not depend on the same variable, i.e $u(x,t) = X(x) \cdot T(t)$. We replace u with $X(x)T(t)$ in our problem.

$$\begin{cases} \partial_t(X(x)T(t))) & = \alpha \partial_x^2 X(x)T(t), & (x,t) \in [0,L] \times \mathbb{R}_{\geq 0} \\ X(0)T(t) & = X(L)T(t) = 0, & \forall t \geq 0 \\ X(x)T(0) & = f(x), & x \in [0,L] \end{cases}$$

   We exclude trivial solutions (i.e. we exclude $T \equiv 0$), therefore the initial condition implies $X(0) = X(L) = 0$. The differential equation is now given by:

$$X(x)\dot{T}(t) = \alpha X''(x)T(t).$$

   Now we arrived at the step which is the most important in this procedure, we separate the terms such that we get:

$$\text{"expressions in t"} = \text{"expressions in x"}.$$

   In our example this yields to:

$$\frac{\dot{T}(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)}.$$

   The lhs only depends on t and the rhs only on x. In order that the equation is true for all (x,t) both sides must be constant, i.e.

$$\frac{\dot{T}(t)}{\alpha T(t)} = \lambda = \frac{X''(x)}{X(x)},$$

   where as λ is unknown. We will just see, that the λ has a problem specific meaning. With the initial conditions of L we get the following ODE's:

$$\left\{ \dot{T}(t) = \alpha\lambda T(t) \right. \qquad \text{and} \qquad \begin{cases} X''(x) = \lambda X(x), \\ X(0) = X(L) = 0. \end{cases}$$

2. **solving the ODE's**
   We show now, that X only solves the equation, if $\lambda < 0$. The characteristic polynomial of the ODE is $x^2 - \lambda$ which has the roots $\pm\sqrt{\lambda}$. Hence, the solution space is spanned by $e^{\sqrt{\lambda}}$ and $e^{-\sqrt{\lambda}}$. This means we try the ansatz: $X(x) = Ce^{\sqrt{\lambda}} + De^{-\sqrt{\lambda}}$ for some

constants C, D. From the initial condition $X(0) = 0$ we get that $D = -C$. If we insert this into $X(L) = 0$ we obtain:

$$0 = X(L) = Ce^{\sqrt{\lambda}L} - Ce^{-\sqrt{\lambda}L} = C(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L})$$

Either $C = 0$ which implies that $X(x) \equiv 0$ which is of no significance for us or

$$e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}) = 0.$$

This equation is only fulfilled, if $\sqrt{\lambda} \in \mathbb{C}$. If $\sqrt{\lambda} \in \mathbb{R}$, the equation cannot be fulfilled for all x, since the the exponential function is strictly monotone on $\mathbb{R}$. For $\sqrt{\lambda} \in \mathbb{C}$, $\lambda < 0$. We can now go on with this ansatz or we begin with a new ansatz in sine and cosine form. We will do both variances.

- The solution above is:
$$X(x) = C(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}).$$
For each negative number z it holds that $\sqrt{z} = i\sqrt{-z}$. Applied on λ we obtain
$$X(x) = C(e^{i\sqrt{-\lambda}x} - e^{-i\sqrt{-\lambda}x}) = 2iV\sin\left(\sqrt{-\lambda}x\right) = b_\lambda \sin(\omega x),$$
where as we defined $b_\lambda := 2iV$ and $\omega := \sqrt{-\lambda}$.

- Alternatively we can use the sine/cosine ansatz to solve the ODE. The general solution X for $\lambda < 0$ is given through:
$$X(x) = a_\lambda \cos \omega x + b_\lambda \sin \omega x,$$
where as a_λ, b_λ are some constants and $\omega = \sqrt{-\lambda}$. The initial condition $X(0) = 0$ implies that $a_\lambda = 0$.

Summarized with both versions we obtain $X(x) = b_\lambda \sin \omega x$. With the second initial condition we $X(L) = 0$ we get that

$$X(L) = b_\lambda \sin \omega L = 0 \implies \omega L = n\pi \quad \text{for } n \in \mathbb{Z}$$

Since $\omega > 0$, we get that $\omega = \frac{n\pi}{L}$ for $n \in \mathbb{N}$. We remark that we got the first restrictions on possible values of λ. Overall we obtain the solutions

$$X_n(x) = b_n \sin\left(\frac{\pi n}{L}x\right) \quad \forall n \in \mathbb{N}.$$

Now lets solve the ODE for T. With separation of variables we obtain the solutions,

$$T_n(t) = C_n e^{\alpha\lambda_n t} = C_n e^{\alpha(-\omega_n)^2 t} = C_n e^{-\alpha\omega_n^2 t} = C_n e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t},$$

whereby we still have to determine the C_n.

3. **superposition of the solutions:**
We got that $\forall n \in \mathbb{N}$ a solution is,
$$u_n(x,t) = X_n(x)T_n(t) = A_n e^{-\alpha\omega_n^2 t} \sin\left(\frac{n\pi}{L}\right),$$
for $A_n = b_n \cdot C_n$. We additionally notice that ω_n (respectively λ_n) are the possible eigen-oscillations of the solutions. Since the heat equation is linear, the general solution is a superposition of all possible solutions. So,
$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha\omega_n^2 t} \sin\left(\frac{n\pi}{L}\right),$$
whereby the A_n are constants which we still have to determine.

4. **dermination of the constants with intitial conditions:**
   With the remaining initial conditions we obtain:
   $$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$
   From the theory of Fourier series we know that under certain regularity conditions of $\tilde{f}$ an odd continuation of f on $[0,L]$ to $\mathbb{R}$, it is possible to express f with the base $\left\{sin\left(\frac{n\pi}{L}x\right)\right\}$. Hence we get that,
   $$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

5. **verify solution:**
   For a given initial condition f we eventually have to check that our superposition solution solves the initial value problem. Since we do not treat an explicit example, we cannot do the verification. Nevertheless here are some tips/central points which you should have in mind during the verification.
   - In order to show that u is continuous, it do not suffice to show that all summands all smooth functions. We also have to show that the series converges absolute and uniformly. If this is the case, u is continuos as uniforme limit of continuous functions (due to theorem 2.12).
   - Afterwards we consider the partial derivatives ∂^α of the single terms. If we again can show that the series of the partial derivatives converges uniformly, we can conclude with theorem 2.12 that u is differentiable, even smooth and we are allowed to interchange limit and derivative.
   - Lastly, since each term of the series solves the PDE, we can conclude that u is a solution of the PDE.

4.7 Recipe how to solve PDEs with spherical harmonics

Here is a recipe to solve partial differential equations (PDE) with spherical harmonics. Just to remember spherical harmonics are eigenfunctions of the spherical Laplace operator Δ_{S^2}. Explicitly: $\Delta_{S^2}Y_{l,m}(\theta,\phi) = \left(\frac{\partial^2}{\partial\theta^2} + \cot(\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin(\theta)^2}\frac{\partial^2}{\partial\phi^2}\right)Y_{l,m}(\theta,\phi) = -l(l+1)Y_{l,m}(\theta,\phi)$ for some $l \in \mathbb{N}$.

1. **Separation:**
   Work always in spherical coordinates when you work on spherical problems. Make the ansatz of separation: $u_{l,m}(r,\theta,\phi) = R_l(r)Y_{l,m}(\theta,\phi)$. Separation of variables provides two ordinary differential equation (ODE). The independent (θ,ϕ)-equation should be always $\Delta_{S^2}Y_{l,m} = -\lambda Y_{l,m}$. We choose $\lambda = l(l+1) implies$ solutions are spherical harmonics.
   Since the spherical harmonics form an ONB on $L^2(S^2)$, the solution has the form of:
   $$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{l,m}R_l(r)Y_{l,m}(\theta,\phi)$$

2. **solve radial equation:**
   The radial equation should be usually homogenous in r. A good approach is: $R_(r) = r^n$. Insert & cancel all r's provides (mostly) a quadratic polynomial in n. Let $n_\pm$ the roots of the polynomial, then the solution is given by: $R_l(r) = ar^{n_+} + br^{n_-}$

3. **simple boundary conditions:**
   More simple conditions, i.e something with $= 0$ or regularity in the origin, has some easy implications.
   For instance: let $n_+ > 0$. Assuming that $\lim_{|x|\to\infty} u(x) = 0$. It follows that $a = 0$, because r^{n_+} explodes for $|x| \to \infty$.

4. **difficult boundary condition:**
   The difficult boundary condition has usually the form of a function, which is defined on the sphere. We insert the power series ansatz into the boundary condition. We then see directly which coefficients $C_{l,m}$ vanish and which $C_{l,m} \neq 0$. This provides the solution.

5. **some tipps**
   In the case that the PDE with the Laplace operator is inhomogeneous, we determine the solution $u(r,\theta,\phi)$ as follows: $u = u_h + u_i$ where

   $$\begin{aligned} u_h &: \text{solution of the homogneous PDE with inhomogneous boundary condition} \\ u_i &: \text{solution of the inhomogeneous PDE with homogenous boundary condition} \end{aligned}$$

   From the linearity of the Laplace operator follows that $u = u_h + u_i$ is indeed a solution. Furthermore we sometimes can assume that u_i is invariant under rotation (since the boundary condition is homogeneous), meaning: $u_i(r,\theta,\phi) = u_i(r)$.

Let's do two examples to get a better understanding.

Example. Let $u \in C^2(\overline{\Omega})$ with $\Omega = \{x \in \mathbb{R}^3 | 1 < \|x\| < 2\}$.
We solve the boundary value problem:

$$\begin{cases} \Delta u(x) & = c, & x \in \Omega, \\ u(x) & = 0, & \|x\| = 1. \\ u(x) & = (x_3)^2, & \|x\| = 2. \end{cases}$$

The strategy goes like the following: first we determine the homogenous solution u_{hom} with inhomogeneous boundary conditions. Afterwards we determine the inhomogeneous solution u_{in} with homogeneous boundary value conditions. Due to the linearity of the Laplacian the solution of the boundary value problem as the sum of $u = u_{\text{hom}} + u_{\text{in}}$.

First we compute a solution $u = u_{\text{hom}}$ for the homogeneous problem ($c = 0$) with inhomogeneous boundaries:

$$\begin{cases} \Delta u(x) & = 0, & x \in \Omega, \\ u(x) & = 0, & \|x\| = 1. \\ u(x) & = (x_3)^2, & \|x\| = 2. \end{cases}$$

We make the separation ansatz:

$$u_{\text{hom}}(r,\theta,\phi) = R_l(r) Y_{l,m}(\theta,\phi).$$

Inserting this ansatz in the PDE yields to:

$$\Delta R_l(r) Y_{l,m}(\theta,\phi) = Y_{l,m}(\theta,\phi) \left(\partial_r^2 + \frac{2}{r}\partial_r\right) R_l(r) + \frac{R_l(r)}{r^2} \Delta_{S^2} Y_{l,m}(\theta,\phi) = 0$$

After the separation of variables we obtain:

$$\frac{r^2}{R_l(r)} \left(\partial_r^2 + \frac{2}{r}\partial_r\right) R_l(r) = \lambda = -\frac{\Delta_{S^2} Y_{l,m}(\theta,\phi)}{Y_{l,m}(\theta,\phi)}$$

If we set $\lambda = l(l+1)$ we obtain the two simpler problems:

$$\left(\partial_r^2 + \frac{2}{r}\partial_r\right) R_l(r) = \frac{l(l+1)}{r^2} R_l(r) \qquad \text{(radial equation)}$$

and

$$\Delta_{S^2} Y_{l,m}(\theta,\phi) = -l(l+1) Y_{l,m}(\theta,\phi) \qquad \text{(angular equation)}$$

In order to solve the radial equation, we notice that it is homogeneous in r and thus we insert the ansatz r^n into the (radial equation). This yields to the equation $n(n+1) = l(l+1)$. This equation is solved for $n = l$ and $n = -(l+1)$. Accordingly the (radial equation) is solved by the ansatz $R_l(r) = ar^l + br^{-(l+1)}$. The (angular equation) is solved by the spherical harmonics (this is the reason why we chose $\lambda = l(l+1)$). Since the spherical harmonics are an orthonormal base on $L^2(S^2)$ and the Laplacian equation is linear, the general solution is described by:

$$f(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{l,m} R_l(r) Y_{l,m}(\theta,\phi). \tag{20}$$

We can use this fact now to obtain $R_l(r) = ar^l + br^{-(l+1)}$. The boundary condition at $r = 1$ yields to:

$$u_{\text{hom}}(1,\theta,\phi) = R_l(1) Y_{l,m}(\theta,\phi) = 0, \quad \text{for } \phi \in [0, 2\pi), \theta \in [0,\pi)$$

This implies that $a = -b$. Thus we are searching a solution as a liner combination:

$$u_{\text{hom}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{l,m} \left(r^l - r^{-(l+1)}\right) Y_{l,m}(\theta,\phi)$$

Consider the second boundary expression. We can expand the boundary condition in spherical harmonics. After a coordinate transformation in spherical coordinates:

$$(x_3)^2 = r^2 \cos^2\theta = \frac{r^2}{3}(3\cos^2\theta - 1 + 1) = \frac{r^2}{3}\left(\sqrt{\frac{16\pi}{5}} Y_{2,0} + \sqrt{4\pi} Y_{0,0}\right)$$

From this we can deduce the coefficients in (20), namely $C_{0,0}, C_{2,0} \neq 0$ and $C_{l,m} = 0$ for $\{l,m\} \neq (0,0)$ and $(2,0)$.

$$u_{\text{hom}}(r,\theta,\phi) = C_{2,0}\left(r^2 - r^{-3}\right) Y_{2,0} + C_{0,0}(r^0 - r^{-1}) Y_{0,0}.$$

Now we insert the boundaries at $r = 2$ and obtain:

$$u_{\text{hom}}(2,\theta,\phi) = C_{2,0}\left(4 - \frac{1}{8}\right) Y_{2,0} + C_{0,0}\left(1 - \frac{1}{2}\right) Y_{0,0}.$$

Hence,

$$C_{2,0} = \left(4 - \frac{1}{8}\right)^{-1} \frac{4}{3}\sqrt{\frac{16\pi}{5}} = \frac{128}{93}\sqrt{\frac{\pi}{5}}$$
$$C_{0,0} = \left(1 - \frac{1}{2}\right)^{-1} \frac{4}{3}\sqrt{4\pi} = \frac{16\sqrt{\pi}}{3}$$

All in all:

$$u_{\text{hom}}(r,\theta,\phi) = \frac{16\sqrt{\pi}}{3}(1 - r^{-1}) Y_{0,0} + \frac{128}{93}\sqrt{\frac{\pi}{5}}\left(r^2 - r^{-3}\right) Y_{2,0}.$$

Now we solve the inhomogeneous problem ($c = 1$) with homogenous boundaries, i.e:

$$\begin{cases} \Delta u(x) &= 1, \qquad x \in \Omega, \\ u(x) &= 0, \qquad \|x\| = 1. \\ u(x) &= 0, \qquad \|x\| = 2. \end{cases}$$

The solution u_{in} is invariant under rotation, because the Laplacian equation is invariant under rotation and now also the boundary conditions. **Therefore $u_{\mathbf{in}}(r, \theta, \phi) = u_{\mathbf{in}}(r)$ is only dependent of the radius.** The differential equation becomes then:

$$u''_{\text{in}} + \frac{2}{r} u'_{\text{in}} = 1$$

This is ordinary differential equation which can be solved with tools from analysis 1. We first determine the the homogeneous solution and eventually the particular.
The homogeneous equation is:

$$u''_{\text{in}} + \frac{2}{r} u'_{\text{in}} = 0$$

The equation is homogeneous in r, so we insert the ansatz r^n and obtain $n(n+1) = 0$. The roots are given by $n = 0$ and $n = -1$, whereby the general solution ansatz is given by: $cr^0 + dr^1 - 1 = 0$. A particular solution is given by $\frac{1}{6}r^2$. Inserting the boundaries provides $d = 1$ and $c = -\frac{7}{6}$.
Summarized we have:

$$u_{\text{in}}(r) = \frac{1}{6}r^2 + \frac{1}{r} - \frac{7}{6}.$$

As explained in the beginning the solution of the original boundary value problem result from the sum of u_{in} and u_{hom}:

$$\begin{aligned} u(r, \theta, \phi) &= u_{\text{in}}(r, \theta, \phi) + u_{\text{hom}}(r, \theta, \phi) \\ &= \underbrace{\frac{1}{6}r^2 + \frac{1}{r} - \frac{7}{6}}_{u_{\text{in}}} + \underbrace{\frac{16\sqrt{\pi}}{3}(1 - r^{-1})Y_{0,0} + \frac{128}{93}\sqrt{\frac{\pi}{5}}\,(r^2 - r^{-3})\,Y_{2,0}}_{u_{\text{hom}}}. \end{aligned}$$

With this ansatz we cannot only solve the Laplacian equation $\Delta u = 0$, but also problems with similar character. As long as the differential equation adds more radial components and the angular dependent equation stays the same, we only have to adapt the radial equation.

Let us consider another example.

Example. Lets consider for $u \in C^2(\overline{\Omega})$ with $\Omega = \{x \in \mathbb{R}^3 : \|x\| < 1\}$ the boundary value problem:

$$\begin{cases} \Delta u(x) = \frac{1}{r}\partial_r u(x), & x \in \Omega \backslash \{0\}, \\ u(x) = (x_1), & |x| = 1. \end{cases}$$

As before we solve the problem in spherical coordinates. As usual we begin with the separations ansatz $u_{l,m} = R_l(r)Y_{l,m}(\theta, \phi)$. Inserting this into the PDE yields to,

$$Y_{l,m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)R_l + \frac{R_l}{r^2}\Delta_{S^2}Y_{l,m} = \frac{Y_{l,m}}{r}\frac{\partial}{\partial r}R_l.$$

We separate after variables:

$$\frac{r^2}{R_l}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)R_l - \frac{r}{R_l}\frac{\partial}{\partial r}R_l = \lambda = -\frac{\Delta_{S^2}Y_{l,m}}{Y_{l,m}}.$$

We again set $\lambda = -l(l+1)$, such that the angular dependent equation is solved by the spherical harmonics. As radial equation we obtain:

$$R_l''(r) + \frac{1}{r}R_l'(r) = \frac{l(l+1)}{r^2}R_l(r).$$

The above equation is homogenous in r, hence we the ansatz r^n is a good choice. Inserting yields,

$$\begin{aligned} n(n-1)r^{n-2} + nr^{n-2} - l(l+1)r^{n-2} &= 0 \\ \Leftrightarrow n(n-1) + n - l(l+1) &= 0 \\ \Leftrightarrow n^2 - l(l+1) &= 0 \\ \Rightarrow n_\pm(l) &= \pm\sqrt{l(l+1)} \end{aligned}$$

Then we get the formal solution,

$$R_l(r) = ar^{\sqrt{l(l+1)}} + br^{-\sqrt{l(l+1)}}.$$

From the regularity condition in the origin, it follows that $b = 0$. Therefore our formal solution is given by,

$$U(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm} r^{\sqrt{l(l+1)}} Y_{l,m}.$$

Now we represent the boundary condition $u(1,\theta,\phi) = x_1$ in spherical harmonics in order to determine the coefficients C_{lm}. Transforming it into spherical coordinates yields,

$$x_1 = r\sin\theta\cos\phi = \frac{r}{2}\sin\theta\left(e^{i\phi} + e^{-i\phi}\right) = -\frac{r}{2}\sqrt{\frac{8\pi}{3}}(Y_{1,1} - Y_{1,-1}).$$

From this we can deduce that $C_{1,\pm1} \neq 0$, all other coefficients are zero. We now can set $r = 1$, in order to compute $C_{1,\pm1}$ explicitly. In the end our solution is,

$$u(r,\theta,\phi) = -\sqrt{\frac{8\pi}{3}}\frac{r}{2}\left(Y_{1,1} - Y_{1,-1}\right).$$

4.8 multipole expansion*

This subsection treats a physical application of spherical harmonics, which was not made in the MMP1 lecture. It is an example from the electrodynamics course in the spring semester and therefore it has a slightly different notation (as warning).

Theorem 4.18 (addition theorem of spherical harmonics)**.**
Consider two points in space (in $\mathbb{R}^3$) as $\vec{x} = (r,\theta,\phi)$ and $\vec{y} = (r',\theta',\phi')$. Let denote γ as the angle between $\vec{x}$ and $\vec{y}$. Then $\cos(\gamma) = \frac{\vec{x}\cdot\vec{y}}{|\vec{x}|\,|\vec{y}|}$. It holds that,

$$P_l(\cos(\gamma)) = P_l(\vec{x}\cdot\vec{y}) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{l,m}(\theta,\phi)Y^*_{l,m}(\theta',\phi'). \tag{21}$$

This theorem has many applications in physics. As an example consider a charge distribution $\rho(\vec{x})$ which occupies on a volume V. We now interested in the electric potential that this distribution creates at a distance r far outside of the region V. Mathematically expressed $|\vec{r}| > |\vec{x}|\ \forall\vec{x}\in V$. The potential is given by

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3\vec{x}\,\frac{\rho(\vec{x})}{|\vec{x}-\vec{r}|}$$

.

Since $r > x$ we can expand the inverse distance in the integrand in $\frac{x}{r}$, obtaining:

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_V d^3\vec{x}\,\rho(\vec{x})x^l P_l(\cos(\gamma),$$

where γ is the angle formed by the vectors $\vec{x}$ and $\vec{r}$. We write the vectors in spherical coordinates: $\vec{x} = (x, \theta_x, \phi_x)$ and $\vec{r} = (r, \theta, \phi)$. We can write

$$\cos\gamma = \frac{\vec{x}\cdot\vec{r}}{xr} = \cos\theta\cos\theta_x + \sin\theta\sin\theta_x\cos(\phi - \phi_x)$$

The integration over $d^3\vec{x} = x^2 dx d\Omega_x$ is rather complicated. In order to solve it, we use the addition theorem for spherical harmonics 4.18. Inserting the identity we get:

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{1+2l}\frac{1}{r^{l+1}}\sum_{m=-l}^{m=l}\left[\int_V x^2 dx d\Omega_x\, Y^*_{l,m}(\theta_x, \phi_x)\rho(\vec{x})x^l\right]\frac{Y_{l,m}(\theta, \phi)}{r^{l+1}}$$

or in a more compact form:

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{1+2l}\frac{1}{r^{l+1}}\sum_{m=-l}^{m=l}\frac{q_{lm}Y_{l,m}(\theta, \phi)}{r^{l+1}},$$

where as q_{lm} are the so-called **multiploe moments**,

$$q_{lm} = \int_V d^3\vec{x}\, Y^*_{l,m}(\theta_x, \phi_x)\rho(\vec{x})x^l$$

. The moments characterise the geometry of the charge distribution. For example:

$$q_{00} = \int_V d^3\vec{x}\,\rho(\vec{x})Y_{0,0}(\theta_x, \phi_x) = \frac{1}{\sqrt{4\pi}}\int_V d^3\vec{x}\,\rho(\vec{x}) = \frac{Q}{\sqrt{4\pi}},$$

which is proportional to the total charge in the distribution. For the next higher moment we find:

$$q_{11} = -\sqrt{\frac{3}{8\pi}}(p_1 - ip_2),$$

with $\vec{p} = (p_1, p_2, p_3) = \int d^3\vec{x}\,\vec{x}\rho(\vec{x})$ the dipole moment. We leave out higher moments since they are described by the quadrupole tensor and they are anyway suppressed by powers of $1/r$ in their contribution to the potential.

5 Distributions

5.1 motivation

The theory of distribtions allows to define a type of solutions which are not sufficiently many times differentiable or which are not defined in the classical sense. Thus, we can see distribution as generalised functions. There are partial differential equations which do not posses a classical solution (function), but a solution in the distributional sense.

As an example let us model the density of a mass point. We start with a sphere of radius ϵ and consider the behaviour of te limit $\epsilon \to 0$.
We assume a homogeneous mass distribution:

$$\rho_\epsilon(x) = \begin{cases} 0 \quad , & \text{for} |x| > \epsilon, \\ \frac{3}{4\pi\epsilon^3}, & \text{for} |x| \leq \epsilon. \end{cases}$$

As mass we have then:

$$m_\epsilon = \int_{\mathbb{R}^3} \rho_\epsilon(x)\, dx = \int_{B_\epsilon} \frac{3}{4\pi\epsilon^3}\, dx = 1.$$

Therefore $\lim_{\epsilon\to 0} m_\epsilon = 1$ and

$$\rho_\epsilon(x) \to \delta(x) = \begin{cases} 0, & \text{for} x \neq 0, \\ \infty, & \text{for} x = 0. \end{cases}$$

On the other hand we have in the Lebesque's sense: $\int_{\mathbb{R}^3} \delta(x)\, dx = 0$. This is in contradiction to $m_\epsilon \to 1$. We thus see that δ in the sense of a real function is not possible to model. A good approach is to see ρ_ϵ respectively δ as distributions. We choose C_0^∞ as the space of test functions. Let us consider the action of ρ_ϵ on all test functions $\phi \in C_0^\infty(\mathbb{R}^3)$. We define:

$$\rho_\epsilon(\phi) = \int_{\mathbb{R}^3} \rho_\epsilon(x)\phi(x)\, dx.$$

5.2 tempered distributions

Definition 5.1 (tempered distribution)**.**
A tempered distribution is a continuous, linear mapping $\omega : \mathscr{S}(\mathbb{R}^n) \to \mathbb{C}$, $\varphi \mapsto \omega[\varphi]$.
In other words:

1. (linearity) $\omega[\lambda\varphi + \psi] = \lambda\omega[\varphi] + \omega[\psi]$, $\forall \varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$
2. (continuity) $\mathscr{S}-\lim_{n\to\infty} \varphi_n = \varphi \implies \mathscr{S}-\lim_{n\to\infty} \omega[\varphi_n] = \omega[\varphi]$ as a consequence of linearity we also have equivalently: $\varphi_n \overset{\mathscr{S}}{\to} 0 \implies \omega[\varphi_n] \overset{\mathscr{S}}{\to} 0$ because $\varphi_n \overset{\mathscr{S}}{\to} \varphi \implies \varphi_n - \varphi \overset{\mathscr{S}}{\to} 0$

Definition 5.2 (regular distribution)**.** A regular distribution is a distribution of the form: $f \in \mathscr{S}(\mathbb{R}^n)$:

$$\omega_f[\varphi] = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx$$

We call distributions, which are not regular, singular. One of the most important singular distribution is the following:

Definition 5.3 (Dirac δ-distribution)**.** The Delta distribution is defined by:

$$\delta[\varphi] := \varphi(0), \qquad \forall \varphi \in \mathscr{S}(\mathbb{R}^n). \tag{22}$$

The linearity is clear. To show continuity, we define $\varphi_j \in \mathscr{S}(\mathbb{R}^n)$ with $\varphi_j \xrightarrow{\mathscr{S}} 0$. In particular, $\varphi_j(0) \to 0$. So $\delta[\varphi_j] \to 0$.
Alternatively, the continuity also follows from: $|\delta[\varphi]| = |\varphi(0)| \leq \|\varphi\|_{0,0}$.
There does not exist a function in the conventional way, which can represent the δ-distribution. But we can define it in a heuristic way, namely:

$$\delta(x) = \begin{cases} \infty, & \text{for } x = 0 \\ 0, & \text{else} \end{cases}$$

and

$$\int_{\mathbb{R}^n} \delta^n(x)\, dx = 1$$

In the above equation the notation δ^n just means that the distribution is n-dimensional. We casually call the Dirac δ-distribution also δ-function.

Proposition 5.4 (properties of the Dirac δ-function)**.**

$$\begin{aligned}
\delta(-x) &= \delta(x) \\
x\delta(x) &= 0 \\
\delta(ax) &= \frac{1}{|a|}\delta(x), && a \in \mathbb{R} \\
\delta(x^2 - |a|^2) &= \frac{1}{2|a|}\left[\delta(x - |a|) + \delta(x + |a|)\right], \\
f(x)\delta(x-a) &= f(a)\delta(x-a), && \forall f \in L^1(\mathbb{R}^n) \\
\delta(f(x)) &= \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, && \text{for } f \in C^1(\mathbb{R}) \text{ and } f'(x) \neq 0 \forall x \in \mathbb{R},
\end{aligned}$$

where as the sum in the last property extends over all roots of f, which are assumed to be simple. The above equations are rules of manipulation for algebraic rearrangements involving δ-functions. As already mentioned, the equations give the same results as factors in a integrand.

Proof. The first two can be easy be proved by the properties of the δ-function. The fifth one can also be verified by the definition of the δ-function.

Let $f \in L^1(\mathbb{R}^n)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x^2 - |a|^2) \cdot f(x)\, dx &= \int_0^{\infty} \delta(x^2 - |a|^2) \cdot f(x)\, dx + \int_{-\infty}^{0} \delta(x^2 - |a|^2) \cdot f(x)\, dx \\
&\overset{\substack{x \to \sqrt{u} \\ x \to -\sqrt{u}}}{=} \int_0^{\infty} \frac{f(\sqrt{u})}{2\sqrt{u}} \delta(u - |a|^2)\, dx - \int_{-\infty}^{0} \frac{f(-\sqrt{u})}{2\sqrt{u}} \delta(u - |a|^2)\, dx \\
&= \frac{1}{2|a|}(f(a) + f(-a)) = \int_{-\infty}^{\infty} \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right) \cdot f(x)\, dx
\end{aligned}$$

Let us now proof the last property. Let $f \in C^1(\mathbb{R})$. Let $g \in L^1(\mathbb{R})$ and set $u = f(x)$.

$$\int_{-\infty}^{\infty} \delta(f(x))g(x)\,dx = \int_{-\infty}^{\infty} \frac{\delta(u)}{|f'(f^{-1}(u))|} g(f^{-1}(u))\,du = \sum_i \frac{g(x_i)}{|f'(x_i)|}$$

where as the sum extends over the zeros $f(x_i) = 0$. Assuming that f has a finite number of zeros we can write,

$$\int_{-\infty}^{\infty} \delta(f(x))g(x)\,dx = \sum_i \int_{-\infty}^{\infty} \delta(x - x_i) \frac{g(x_i)}{|f'(x_i)|}\,dx,$$

which provides the desired result.THe fourth property is a special case of the last property with $f(x) = x^2 - |a|^2$. □

5.3 operations on distributions

We now want to extend the usual operations on tempered distributions. For regular distributions the operations are the same as in the classical meaning. We now define for general (tempered distributions). So let $\omega \in \mathscr{S}'(\mathbb{R}^n)$ and $\varphi, f \in \mathscr{S}(\mathbb{R}^n)$.

- **translation:**

  $$(T_a\omega)[\varphi] = \omega[T_{-a}\varphi],$$

  because $(T_a f)[\varphi] = \int_{\mathbb{R}^n} f(x-a)\varphi(x)\,dx = \int_{\mathbb{R}^n} f(x)\varphi(x+a)\,dx = f[T_{-a}\varphi]$.

- **linear coordinate transformation:**
  $\forall A \in GL(n, \mathbb{R})$ we have,

  $$(U_A\omega)[\varphi] = |\det A|\,\omega[U_{A^{-1}}\varphi],$$

  because $(U_A f)[\varphi] = \int_{\mathbb{R}^n} f(A^{-1}x)\varphi(x)\,dx = \int_{\mathbb{R}^n} |\det A| f(y)\varphi(Ay)\,dy = f[U_{A^{-1}}]\,|\det A|$.

- **multiplication with a function:**
  Let $g \in \mathscr{S}(\mathbb{R}^n)$. Then we have:

  $$(g\omega)[\varphi] = \omega[g\varphi].$$

- **derivative:**

  $$(\partial^\alpha\omega)[\varphi] = (-1)^{|\alpha|}\omega[\partial^\alpha\varphi].$$

  This follows from partial integration, since
  $(\partial^\alpha f)[\varphi] = \int_{\mathbb{R}^n} (\partial^\alpha f(x))\varphi(x)\,dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)(\partial^\alpha\varphi(x))\,dx = (-1)^{|\alpha|} f[\partial^\alpha\varphi]$.

- **Fourier transform:**

  $$\hat{\omega}[\varphi] = \omega[\widehat{\varphi}], \quad \check{\omega}[\varphi] = \omega[\widecheck{\varphi}]$$

- **convolution:**
  Let $g \in \mathscr{S}(\mathbb{R}^n)$ and $\tilde{g}(x) = g(-x)$. Then we have:

  $$(g * \omega)[\phi] = \omega[\tilde{g} * \varphi], \quad (\omega * g)[\phi] = \omega[\tilde{g} * \varphi],$$

  because,
  $(g*f)[\varphi] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x-y)f(y)\,dy\varphi(x)\,dx = \int_{\mathbb{R}^n} f(y)\left(\int_{\mathbb{R}^n} g(x-y)\varphi(x)\,dx\right)\,dy = f[\tilde{g}*\varphi]$.

- **convolution + Fourier transform:**
  Let $g \in \mathscr{S}(\mathbb{R}^n)$. Then,
$$\widehat{g * \omega}[\varphi] = \hat{g}\hat{\omega}[\varphi], \quad \widecheck{g * \omega}[\varphi] = (2\pi)^n \check{g}\check{\omega}[\varphi]$$

- **convolution + derivative:**
  Let $g \in \mathscr{S}(\mathbb{R}^n)$. Then,
$$\partial^\alpha(\omega * g)[\varphi] = (\partial^\alpha \omega) * g[\varphi] = \omega * (\partial^\alpha g)[\varphi] = \partial^\alpha(g * \omega)[\varphi].$$

We notice that through this definition every distribution is smooth, i.e. infinite times differentiable.
Now let us make a lot of useful examples.

Example.

- $\frac{\mathrm{d}}{\mathrm{d}x}\theta = \delta$, because $\frac{\mathrm{d}}{\mathrm{d}x}\theta[\varphi] = -\theta[\frac{\mathrm{d}}{\mathrm{d}x}\varphi] = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\,dx = \varphi(0) = \delta[\varphi]$.
- $\hat{1} = (2\pi)^n\delta$, since $\hat{1}[\varphi] = 1[\widehat{\varphi}] = \int_{\mathbb{R}^n} \widehat{\varphi}(k)\,dk = (2\pi)^n\varphi^{\vee\wedge}(0) = (2\pi)^n\delta[\varphi]$.
- $\hat{\delta} = 1$, because $\hat{\delta}[\varphi] = \delta[\widehat{\varphi}] = \widehat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)e^{-ik\cdot 0}\,dx = 1[\varphi]$.
- $g * \delta = g, \forall g \in \mathscr{S}(\mathbb{R}^n)$, since
  $(g * \delta)[\varphi] = \delta[\tilde{g} * \varphi] = \int_{\mathbb{R}^n} \tilde{g}(0-y)\varphi(y)\,dy = \int_{\mathbb{R}^n} g(y)\varphi(y)\,dy = g[\varphi]$.
- $x\delta = 0$, because $x\delta[\varphi] = \delta[x\varphi] = 0$.
- $\widehat{\delta_a} = e^{-iax}$, since $\widehat{\delta_a} = \delta_a[\widehat{\varphi}] = \widehat{\varphi}(a) = \int_{\mathbb{R}^n} \varphi(x)e^{-iax}\,dx = e^{-iax}[\varphi]$.
- $\partial_x\widehat{\delta_a} = -iae^{-ax}$, since
  $\partial_x\widehat{\delta_a}[\varphi] = -\widehat{\delta_a}[\partial_x\varphi] = -\delta_a[(\varphi')^\wedge] = -\delta_a[ix\widehat{\varphi}] = -ia\widehat{\varphi}(a) - iae^{-iax}[\varphi]$.
- $(x\partial_x\delta) = -\delta$, since $(x\partial_x\delta)[\varphi] = (\partial_x\delta)[x\varphi] = -\delta[\partial_x(x\varphi)] = -\delta[\varphi + x\varphi'] = -\delta[\varphi]$.

Remark. In physicist notation we often write the delta function as:
$$\delta^n(k) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ik\cdot x}\,dx.$$

5.4 Fundamental solutions

The goal of this lecture is to solve partial differenital equations (PDE). We show now how distributions can help. For that we introduce the notion of fundamental solution. Let L be a differential operator
$$L = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|\le N}} a_\alpha \partial^\alpha$$
with constant complex coefficients. The tempered distributions $E \in \mathscr{S}'(\mathbb{R}^n)$ is a fundamental solution of L if E fulfills the equation
$$LE = \delta \tag{23}$$
where δ is the Dirac-delta. The equality holds in the sense of distributions, i.e.,
$$LE[\phi] = \delta[\phi]$$
for all test functions $\phi \in \mathscr{S}(\mathbb{R}^n)$.

Proposition 5.5. If the fundamental solution is known of an differential operator L then we have a solution of the inhomogeneous differential equation

$$Lu = f \tag{24}$$

via convolution of of the fundamental solution E with the interference function f

$$u = (E * f). \tag{25}$$

This equality again holds in the sense of distributions. We know how convolutions and differential operators behave,

$$L(F * g) = LF * g.$$

Therefore,

$$Lu = L(E * f) = LE * f = \delta * f = f.$$

Once a fundamental solution has been found, it is relatively easy to determine the classical solution. Finding the fundamental solution, however, can be difficult. Nevertheless, let us consider a simpler example which shows us the weakness of distributional solutions compared to classical solutions.

Example. Consider the one-dimensional Laplacian $L = \partial_x^2$. We look for a tempered distribution such that $\partial_x^2 E = \delta$. This is an inhomogneous differential equation. Hence we are looking for an homogneous solution E_h and a particular solution E_p. Our solution is then the sum of the two latter solutions. A homogeneous solutions is given by $E_h(x) = a + bx$. We note that $\partial_x \theta = \delta$ where θ describes the Heavyside function. We therefore have that

$$\partial_x E_p = \theta.$$

We integrate the latter and obtain

$$E_p = x\theta.$$

We verify if it is really a solution. Let $\phi \in \mathscr{S}(\mathbb{R})$. Then

$$\begin{aligned} \partial_x^2 E_p[\phi] &= \partial_x^2 (x\theta)[\phi] = (x\theta)[\phi''] = \theta[x\phi''] \\ &= \int_{\mathbb{R}} \theta(x) x \phi''(x)\, dx = \int_0^\infty x\phi''(x)\, dx \\ &= x\phi'(x)\Big|_0^\infty - \int_0^\infty \phi'(x)\, dx = \phi(0) = \delta[\phi]. \end{aligned}$$

Hence we have a particular solution in the sense of distributions. We get the set of all fundamental solutions of the 1-dimensional laplacian

$$\{E \in \mathscr{S}'(\mathbb{R}) | E'' = \delta\} = \{x\theta(x) + bx + a | a, b \in \mathbb{C}\}.$$

The constants can be further be determined with e.g. boundary conditions.

Let us now consider the differential equation,

$$\partial_x^2 u(x) = \sin(x).$$

We choose the constants to be $a = 0$ and $b = -\frac{1}{2}$, such that

$$E = x\theta - \frac{1}{2}x = \frac{1}{2}|x|.$$

The distributional solution is thus of the form

$$u = E * \sin = \int_{\mathbb{R}} \frac{1}{2}|x-y| \sin(y)\, dy.$$

This shows now that we have to be careful, when we are working wiht functions which are not "nice" enough (compact support, integrability, etc.), since the solution of eq. (5.4) is known as $u(x) = -\sin(x)$. On the other hand the above integrand diverges for all x. The both expression are equal in the sense of distributions. If we for example choose $1_{[0,1]}$ as interference function we obtain

$$u = E * 1_{[0,1]} = \int_{x-1}^{x} \frac{1}{2}|y|\, dy.$$

Through case distinctions it can be checked that the above expression provides a solution in the classical case.

Remark. A classical solution to be understand as a distribution provides always a distributional solution. However, in order for a distributional solution to induce a classical solution, we need certain regularity conditions.

5.5 Fundamental solution and Fourier transform

The fundamental solution for a lot of PDE's can be found with the help of the Fourier transform. That is also the reason why we are restricting ourself to tempered distributions, since the Fourier transform $\mathcal{F}$ is well defined on $\mathscr{S}$. We often use the property that $\hat{\delta} = 1$. Let

$$L = \sum_{|\alpha| \leq N} a_\alpha \partial^\alpha \tag{26}$$

be again a linear differntial operator with constant coefficients $a_\alpha \in \mathbb{R}$. After applying the Fourier transform to $LE = \delta$ we obtain

$$P(k)\hat{E}(k) = 1, \quad \text{where } P(k) = \sum_\alpha a_\alpha (ik)^\alpha. \tag{27}$$

As a solution we thus get

$$E = \left(\frac{1}{P(k)}\right)^\vee$$

This a well defined fundamental solution as long as $\frac{1}{P(k)}$ has integrable singularities.

We first consider an example where we are execissively detailed with the determination of $P(k)$. In that way, it gets obvious that the above formula for $P(k)$ is correct.

Example. The fundamental solution of the Laplacian in n dimensions is given by

$$E = -\left(\frac{1}{\|k\|^2}\right)^\vee.$$

The fundamental solution E fulfills $\nabla^2 E = \delta$. We apply now the Fourier transformation to the equation. The constant function 1 is to be understand as a distribtuion and it is the

Fourier transformed of δ. For $\nabla^2 E$ we obtain

$$\begin{aligned}
(\nabla^2 E)^\wedge[\phi] &= (\nabla^2 E)[\hat{\phi}]| && |\mathcal{F} \text{ in the sense of distributions} \\
&= E[\nabla^2 \hat{\phi}] && | \text{ distribut. derivative} \\
&= E[(\partial_1^2 + \dots \partial_n^2)\hat{\phi}] && |\nabla^2 = (\partial_1^2 + \dots \partial_n^2) \\
&= E[((-i)^2 x_1^2 + \dots + (-i)^2 x_n^2)\, \hat{\phi}] && | \text{ derivatives rules for FT} \\
&= E[-\left((x_1^2 + \dots + x_n^2)\phi\right)^\wedge] && \\
&= \hat{E}[-\|x\|^2 \phi] && |\|x\|^2 = x_1^2 + \dots x_n^2 \\
&= -\|x\|^2 \hat{E}[\phi] && | \text{ mutlipli. in the of distribtuions}
\end{aligned}$$

We thus have

$$\nabla^2 E = \delta \xrightarrow{\mathcal{F}} -\|k\|^2 \hat{E} = 1.$$

and with the inverse Fourier transfom we get

$$\hat{E} = -\frac{1}{\|k\|} \xrightarrow{\mathcal{F}^{-1}} E = -\left(\frac{1}{\|k\|^2}\right)^\vee .$$

We are often not that rigorous when calculaating such things. Often our claculation looks similar to:

Example. Let us consider the differnetial operator

$$L = -\partial_x^2 + a^2.$$

A fundamental solution fulfills ,

$$(-\partial_x^2 + a^2)E = \delta.$$

Fourier transfomring the above equation yields

$$(k^2 + a^2)\hat{E} = 1.$$

Since $\frac{1}{k^2+a^2}$ has integrable singularities, we can apply the inverse Fourier transform. We obtain

$$E = \left(\frac{1}{k^2 + a^2}\right)^\vee = \frac{1}{2a} e^{-a|x|}.$$

Example (heat equation)**.**
The heat equation contains the linear differnetial operator

$$L = \partial_t - a\nabla^2,$$

where $a > 0$ is the diffusion constant. We consider the problem in $1+1$ dimension (one time dim and 1 space dim) with $a = 1$. The equation for the fundamental equation thus reads

$$(\partial_t - \partial_x^2)E = \delta.$$

If we find E we can determine the solution of the heat equation. We note that we have two variables, that means we consider $\mathscr{S}(\mathbb{R}^2)$ as space of test functions and it holds that $\delta[\phi] = \phi(0, 0)$. We can now Fourier transform the heat equation w.r.t to space coordinate x

$$(\partial_t + k^2)\hat{E} = \delta_t,$$

where δ_t is to be understand as one dimensional function namely, $\delta_t[\phi] = \phi(0,x)$ for all x. For a fixed k the above equation is a ordinary (distributional) differential equation in t. we can now check that the solution is given by

$$\hat{E} = e^{-k^2 t}\theta_t.$$

Let $\phi \in \mathscr{S}(\mathbb{R})$ (here it is enough to only consider the t variable since k can be seen as constant). It holds that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}[\phi] = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-k^2 t}\theta_t\right)[\phi] = -\int_{\mathbb{R}} e^{-k^2 t}\theta(t)\phi'(t)\,dt \\
&= -\int_0^\infty e^{-k^2 t}\phi'(t)\,dt \overset{\text{p.I.}}{=} -e^{-k^2 t}\phi(t)\Big|_0^\infty - k^2\int_0^\infty e^{-k^2 t}\phi(t)\,dt \\
&= \phi(0) - k^2\int_{\mathbb{R}} e^{-k^2 t}\theta(t)\phi(t)\,dt = \delta_t[\phi] - k^2 e^{-k^2 t}\theta_t[\phi] \\
&= \delta_t[\phi] - k^2 E[\phi].
\end{aligned}$$

This expression is integrable, and with the inversion theorem (3.8) we get

$$\begin{aligned}
E = E^{\wedge\vee} = \left(e^{-k^2 t}\theta_t\right)^\vee &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-k^2 t}\theta(t)e^{ikx}\,dk \\
&= \theta(t)\frac{1}{2\pi}\int_{\mathbb{R}} e^{-k^2 t}e^{ikx}\,dk = \theta(t)\left(e^{-k^2 t}\right)^\vee \\
&= \sqrt{4\pi t}e^{-\frac{x^2}{4t}}\theta(t).
\end{aligned}$$

Strictly speaking we have to do the above calculation in the sense of distributions but it would yield the same result.

For $t > 0$, E is exactly the heat kernel. This looks similar to the solution of the heat equation in $C^\infty(\mathbb{R}\times(0,\infty))$ with intitial condition $g(x)$. By theorem ref it holds that the solution is given by

$$u(x,t) = K_t * g.$$

We also see that the inhomogneous as well as the homogeneous heat equation can be solved now. The general solution with interference function f and initial condition can then be written as

$$u(x,t) = (K_t * f)(x,t) + \int_0^t (\theta_s K_s * g)(x,s)\,ds.$$

6 Dirichlet problem & harmonic functions

The Laplacian equation for the function $u \in C^2(D) \subset \mathbb{R}^n$

$$\Delta u(x) = 0, \qquad x \in D.$$

Solutions of the above Laplacian equation are called **harmonic functions**. Let D be bounded region in $\mathbb{R}^n$, i.e., a non-empty connected open set with smooth boundary ∂D. Typical boundary value problems (BVP) are the Dirichlet-problem

$$\begin{cases} \Delta u(x) = 0, & x \in D \\ u(x) = f(x), & x \in \partial D \end{cases} \tag{D}$$

and the Neumann-problem

$$\begin{cases} \Delta u(x) = 0, & x \in D \\ \frac{\partial u}{\partial n} = g(x), & x \in \partial D \end{cases} \tag{N}$$

The functions f and g are prespecified boundary conditions and the goal is to find $u \in C^2(D)$ which fulfills (D) or (N). In electrostatics the electric potential Φ in neutral medium fulfills the Laplacian equation. The boundary condition (D) is given if the value of the potential on the boundary is prescribed and (N) if the charge density on the surface is.

Remark. In the boundary condition of the Neumann problem (N) we have the expression $\frac{\partial u(x)}{\partial n}$. This is just a abbreviation and has to be understand as

$$\frac{\partial u(x)}{\partial n} = n(x) \cdot \nabla u(x),$$

where $n(x)$ is the unit normal vector field on ∂D.

Theorem 6.1 (uniqueness)**.** Let u_1, u_2 be two solutions of (D). Then $u_1 = u_2$.
Let u_1, u_2 be two solutions of (N). Then $u_1 = u_2 + \text{const}$.

Proof. Let $u = u_1 - u_2$. Then $\Delta u(x) = 0$ and $u(x) = 0$ (respectively $\frac{\partial u}{\partial n} = 0$) for $x \in \partial D$. Due to the Gaussian divergence theorem we have

$$\begin{aligned} 0 \le \int_D \|\nabla u\|^2 \, dx = \int_D \nabla \cdot (u \cdot \nabla u) \, dx &\overset{\text{(Gauss)}}{=} \int_{\partial D} u(x) n(x) \cdot \nabla u \, d\Omega(x) \\ &= \int_{\partial D} u(x) \frac{\partial u}{\partial n} \, d\Omega(x) = 0. \end{aligned}$$

In the first equality we used that $\nabla u \cdot \nabla u = \nabla \cdot (u \nabla u) - u \underbrace{\Delta u}_{=0}$. Hence it follows now that $\nabla u = 0 \implies u = \text{const}$. In the case of (D) $u(x) = 0$ since it is 0 for all $x \in \partial D$. $\square$

Lemma 6.2 (Green's identity)**.** Let D be a bounded region in $\mathbb{R}^n$, with smooth boundary ∂D and with outwards pointing unit vectors $n(x)$, $x \in \partial D$. For all $u, v \in C^2(\overline{D})$ it holds that

$$\int_D (\Delta u v - u \Delta v) \, dx = \int_{\partial D} \left(\frac{\partial u}{\partial n} v - u \frac{\partial v}{\partial n} \right) d\Omega(x). \tag{Green id}$$

where $\frac{\partial}{\partial n} = \sum_{i=1}^n n_i(x) \frac{\partial}{\partial x_i} = n \cdot \nabla$ denotes the derivative in direction of the unit vector and $d\Omega(x)$ is the surface measure on ∂D.

Proof. Due to Gauss' theorem (Gauss) we have

$$\int_D (\Delta u)v - u\Delta v \, dx = \int_D \nabla \cdot ((\nabla u)v - u\nabla v) \, dx$$
$$\stackrel{\text{(Gauss)}}{=} \int_{\partial D} (n \cdot \nabla u)v - u(n \cdot \nabla v) \, d\Omega(x).$$

□

6.1 Green's functions

Green's functions were originally introduced as functions (hence the name) before they were transferred by Schwartz to the context of distributional theory. There they are often referred to as fundamental solutions. In this lecture we see Green's functions as analogous to fundamental solutions for initial and boundary value problems. This means that in addition to the fact that Green's functions satisfy $LG = \delta$, they now also fulfill boundary conditions. In the lecture we consider only the Green's function for the Laplace operator. Let therefore E be the fundamental solution of the Laplace operator.

Definition 6.3. Let $D \subset \mathbb{R}^n$ be open with smooth boundary. A continuous function $G(x, y)$ on $\{(x, y) \in \overline{D} \times D | x \neq y\}$ is called **Green's function** of the region D (for the Laplacian) if

1. $G(x, y) = E(x - y) + v(x, y)$, with $v \in C^2(\overline{D} \times D)$ and $\Delta_x v(x, y) = 0$.
2. $G(x, y) = 0$, for $x \in \partial D, \quad y \in D$.

We denote Δ_x the Laplacian in the variable x. Additionaly, $\Delta_x G(x, y) = 0$ for $x \neq y$.

Lemma 6.4. Let $u \in C^2(\overline{D})$. Then it holds for all $x \in D$

$$u(x) = \int_D E(x - y)\Delta u(y) \, dy - \int_{\partial D} \left(E(x - y)\frac{\partial u}{\partial n_y} - u(y)\frac{\partial}{\partial n_y}E(x - y) \right) d\Omega(y).$$

Proof. Let $B_\epsilon(x)$ be the circle with radius ϵ around x, i.e., $B_\epsilon(x) = \{y \in D | |y - x| \leq \epsilon\}$. Then

$$\int_D E(x - y)\Delta u(y) \, dy = \lim_{\epsilon \to 0} \int_{D \setminus B_\epsilon(x)} E(x - y)\Delta u(y) \, dy$$
$$= \lim_{\epsilon \to 0} \int_D E(x - y)\Delta u(y) \, dy - \int_{B_\epsilon(x)} E(x - y)\Delta u(y) \, dy$$
$$\stackrel{\text{(Green id)}}{=} \lim_{\epsilon \to 0} \Bigg(\underbrace{-\int_{B_\epsilon(x)} \Delta E(x - y)u(y) \, dy}_{=0} + \int_{\partial B_\epsilon(x)} \frac{\partial E(x - y)}{\partial n_y}u(y) - E(x - y)\frac{\partial u(y}{\partial n_y} \, d\Omega(y)$$
$$+ \int_D \underbrace{\Delta E(x - y)}_{\delta(x-y)} u(y) \, dy + \int_{\partial D} \frac{\partial E(x - y)}{\partial n_y}u(y) - E(x - y)\frac{\partial u(y)}{\partial n_y} \, d\Omega(y) \Bigg)$$
$$= u(x) + \int_{\partial D} \frac{\partial E(x - y)}{\partial n_y}u(y) - E(x - y)\frac{\partial u(y)}{\partial n_y} \, d\Omega(y).$$

□

With the Green's function we can now solve the Dirichlet problem (D).

Theorem 6.5. If u is a soultion of (D), we have that for all $x \in D$

$$u(x) = \int_{\partial D} \frac{\partial G}{\partial n_y} f(y)\, \Omega(y).$$

Proof. With the above lemma 6.4 and the fact that u is harmonic we have

$$u(x) \stackrel{\Delta u=0}{=} \int_{\partial D} \left(-E(x-y) \frac{\partial u}{\partial n_y} + u(y) \frac{\partial}{\partial n_y} E(x-y) \right) d\Omega(y).$$

On the other hand we have due to the Green's identity

$$0 = \int_{\partial D} \left(v(y,x) \frac{\partial u}{\partial n_y} - u(y) \frac{\partial}{\partial n_y} v(y,x) \right) \Omega(y).$$

Subtraction the first equation from the second one we get

$$u(x) = \int_{\partial D} \left(-\underbrace{G(y,x)}_{=0} \frac{\partial u}{\partial n_y} + u(y) \frac{\partial G}{\partial n_y} \right) d\Omega(y) = \int_{\partial D} u(y) \frac{\partial G}{\partial n_y}\, d\Omega(y).$$

□

6.1.1 Lippmann Schwinger equation and scattering amplitude *

Now we give an exmaple where we apply Green's functions in physics. This exmaple is usually a topic discussed in the course *quantum mechanis 2*.
For that we consider a scattering process on a fixed target and want to find a solution of stationary Schroedinger equation far away from the scattering centre. We describe our incoming beam by a plane wave. Hence the wave function far away from the scattering centre can be described by

$$\psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}, \qquad \text{for } |\vec{x}| \to \infty.$$

The equation to solve is

$$\left(-\frac{\hbar^2}{2m} + V(\vec{x}) \right) \psi_{\vec{k}}(\vec{x}) = E_{\vec{k}} \psi_{\vec{k}}(\vec{x}).$$

First we rearrange the equation

$$\underbrace{\left(\frac{\hbar^2}{2m} + E_{\vec{k}} \right)}_{L} \underbrace{\psi_{\vec{k}}(\vec{x})}_{w} = \underbrace{V(\vec{x})\psi_{\vec{k}}(\vec{x})}_{u}.$$

Now comes the Green's function into play. The general form of w is given by

$$w = Gu + v \tag{28}$$

where v is a homogeneous soltuion of L, i.e., $Lv = 0$. We can check that this form of w indeed solves the equation

$$Lw = \underbrace{LG}_{\text{id}}\, u + \underbrace{Lv}_{0} = u. \tag{29}$$

Our next task is to determine v and G. The solution of $Lv =$ is given by a plane wave

$$\varphi_{\vec{k}}(\vec{x}) = v(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}$$

with $\vec{k} \in \mathbb{R}^3$ such that

$$E_{\vec{k}} = \frac{|\vec{k}|^2\hbar^2}{2m}.$$

we impose this form of $E_{\vec{k}}$ such that the above form of v is really a homogeneous solution of L.

Now let us determine the Green's function of the operator L, i.e., we want to find the function $G_{\vec{k}}(\vec{x}, \vec{y})$ such that

$$\left(\frac{\hbar^2}{2m} + E_{\vec{k}}\right) G_{\vec{k}}(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}). \tag{30}$$

Note that all terms excepts G are invariant under translations, thus G is too, i.e., $G_{\vec{k}}(\vec{x}, \vec{y}) = G_{\vec{k}}(\vec{x} - \vec{y})$.

Using G as an integral kernel we get the solution w as

$$\psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} + \int d^3y\; G_{\vec{k}}(\vec{x}, \vec{y}) V(\vec{y}) \psi_{\vec{k}}(\vec{y}) \tag{LS}$$

This equation is called the **Lippmann-schwinger equation**. We converted the problem of finding a solution of the stationary Schroedinger equation into determining a solution of the above integral equation (LS). At first sight, the above equation looks much more difficult to solve since it is an integral equation for $\psi_{\vec{k}}(\vec{x})$. With the aid of he Fourier transfom and the Residue theorem A.1 we can solve the Lippmann-Schwinger equation.

Let us use the Fourier transform of the Green's function

$$G_{\vec{k}}(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3q\, \tilde{G}_{\vec{k}}(\vec{q}) e^{i\vec{q}\cdot\vec{x}}. \tag{31}$$

Inserting this equation into eq. (30), yields

$$\frac{1}{(2\pi)^3} \int d^3q \left(-\frac{\hbar^2}{2m} q^2 + E_{\vec{k}}\right) \tilde{G}_{\vec{k}}(\vec{q}) e^{i\vec{q}\cdot\vec{x}} = \delta(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3q\, e^{i\vec{q}\cdot\vec{x}}$$

We then can read off

$$\tilde{G}_{\vec{k}}(\vec{q}) = \frac{2m}{\hbar^2} \frac{1}{k^2 - q^2}.$$

Now we take the inverse Fourier transform

$$\begin{aligned}
G_{\vec{k}}(\vec{x}) &= \frac{2m}{(2\pi)^3\hbar^2} \int d^3q\, \frac{e^{i\vec{q}\cdot\vec{x}}}{k^2 - q^2} \\
&= \frac{2m}{(2\pi)^3\hbar^2} \int_0^\infty dq\, q^2 \int_0^\pi d\theta\; \sin\theta \int_0^{2\pi} d\varphi\, \frac{e^{i\vec{q}\cdot\vec{x}}}{k^2 - q^2} \\
&= \frac{2m}{(2\pi)^3\hbar^2} \int_0^\infty dq\, q^2 \int_0^\pi d\theta\; \sin\theta \int_0^{2\pi} d\varphi\, \frac{e^{iqr\cos\theta}}{k^2 - q^2} \\
&\overset{z=\cos\theta}{=} \frac{2m2\pi}{(2\pi)^3\hbar^2} \int_0^\infty dq\, q^2 \int_{-1}^1 dz\, \frac{e^{iqrz}}{k^2 - q^2} \\
&= \frac{2m2\pi}{(2\pi)^3\hbar^2} \int_0^\infty dq\, q^2 \frac{1}{iqr} \left[e^{iqr} - e^{-iqr}\right] = \frac{m}{2\pi^2\hbar^2 ir} \int_{-\infty}^\infty dq\, q \frac{e^{iqr}}{k^2 - q^2}.
\end{aligned}$$

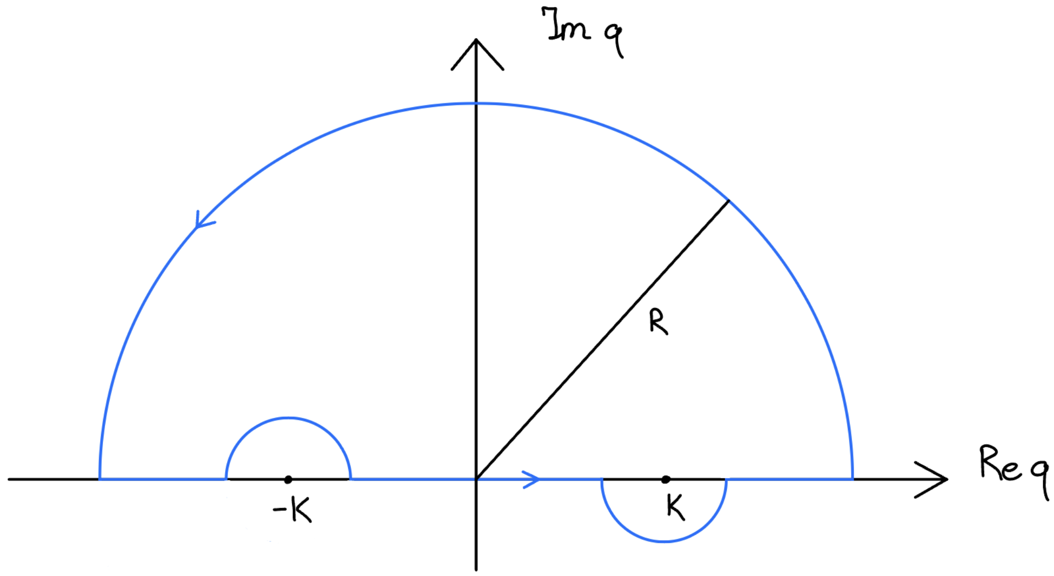

Figure 4: Contour used for the residue theorem. The contour is chosen such that only the pole $q = +k$ is included in the contour. For $R \to \infty$ the contribution from the large arc vanishes, since the large arc lies within the half plane $\text{Im}(q) > 0$ we have $e^{iqr} = e^{i\text{Re}(qr)}e^{-\text{Im}(qr)} \to 0$ for $R = |q| \to \infty$.

The integrand above is a holomorphic function with poles at $q = \pm k$ We evaluate the integral by extending the integral to the complex plane and choose the contour in form of a semicircle (see figure 6.1.1).
The integral along the semicircle vanishes for $R \to \infty$ because the integrand decays fast enough. A rigorous description can be found under *Jordan's lemma*.
Thus in the limit of $R \to \infty$ the contour integral approaches the integral of interest over the real line.

The residue of the pole $q = k$ can be found with eq A.2 and it is given by

$$\text{Res}(q = k) = -\frac{e^{ikr}k}{2k}.$$

Thus we are getting the Green's function

$$G_{\vec{k}}(\vec{x}) = \frac{m}{2\pi^2\hbar^2 ir}\left(-\frac{2\pi i e^{ikr}k}{2k}\right) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}.$$

We often assume that our potential is short-ranged, i.e., $\lim_{|\vec{x}|\to\infty} V(\vec{x}) = 0$. We then can Taylor expand the Green's function and come to the result

$$\psi_{\vec{k}}(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ik|\vec{x}|}}{|\vec{x}|}f(\vec{k},\vec{k}') \quad \text{with} \quad f(\vec{k},\vec{k}') = -\frac{m}{2\pi\hbar^2}\int d^3y\, e^{-i\vec{k}'\cdot\vec{y}}V(\vec{y})\psi_{\vec{k}}(\vec{x}) \tag{32}$$

where $\vec{k}' := k\vec{x}/|\vec{x}|$. So the above expression of $\psi_{\vec{k}}$ is a solution of the Lippmann-Schwinger equation (LS) which we found with the help of the Green's function. Note that $|\vec{k}'| = |\vec{k}|$, this embodies the fact that the scattering is elastic. Furthermore the above result can be used to deduce an expression for the differnetial cross section namely,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|f(\vec{k},\vec{k}')\right|.$$

This result is of great use in particle physics, where the scattering cross section plays an essential role in the experiments.

A residue computation

Theorem A.1.
Let $D \subset \mathbb{C}$ be a domain and $E \subset D$ a finite subset. Assume that $f : D \backslash E \to \mathbb{C}$ is holomorphic. Let $\gamma \subset D \backslash E$ a piecewise continuously differentiable, closed path which is contractible in D. Then:

$$\int_\gamma f(z)\, dz = 2\pi i \cdot \sum_{z \in E} \eta(\gamma, z) \cdot \mathrm{Res}_z(f)$$

where $\eta(\gamma, z)$ stands for the winding number of γ around the point z.

Proposition A.2 (Residue formula for poles of order n)**.**
Let $f : D^*(z_0, r) \to \mathbb{C}$ be a holomorphic function with a pole of order n at z_0. The Residue is then given by:

$$\mathrm{Res}_{z_0}(f) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left((z - z_0)^n \cdot f(z)\right) \tag{33}$$

B Integral identities

Theorem B.1 (divergence theorem, Gauss' theorem)**.**
Let $V \subset \mathbb{R}^n$ a compact set with smotth boundary $S = \partial V$ which is oriented by an outward pointing normalised unit vector field n. Further let F be a continuous differentiable vector field defined on an open set U with $V \subset U$. Then

$$\int_V \nabla \cdot F d^n V = \oint_S F \cdot n \, d^{n-1} S \tag{Gauss}$$